To the best of our knowledge, the only fuzzy extension of $\mathcal{EL}$ that has been studied so far is based on the Gödel t-norm [Mailis *et al.*, 2012].[6] In that paper, the authors describe a polynomial-time algorithm for deciding fuzzy subsumption between concepts. Beyond this tractable case, very little is known about the complexity of subsumption with general t-norms. If we restrict the set of membership degrees to be finite, then subsumption can be decided in exponential time [Borgwardt and Peñaloza, 2013; Bobillo and Straccia, 2013], but for the interval $[0,1]$ nothing is known, even for more expressive fuzzy DLs in which consistency is decidable [Borgwardt *et al.*, 2012b].

We consider fuzzy extensions of $\mathcal{EL}$ with general t-norm semantics and identify for which cases reasoning remains polynomial. As for the classical case, we are interested in deciding subsumption between concepts. However, the different membership degrees must also be taken into account. For that reason, we consider the *positive subsumption* problem: deciding whether the (fuzzy) implication between two concepts is always greater than 0. Intuitively, a positive subsumption between two fuzzy concepts expresses that they are causally related *to some degree*. We show that the complexity of this problem depends on the properties of the t-norm chosen: if the t-norm has zero divisors, then positive subsumption is co-NP-hard; otherwise, the problem is reducible in linear time to classical subsumption. We also consider the computation problem of finding the best lower bound for the subsumption degree and show that the corresponding decision problem is co-NP-hard if the t-norm contains the Łukasiewicz t-norm.

2 Fuzzy $\mathcal{EL}$

In this section we introduce the fuzzy Description Logic $\otimes$-$\mathcal{EL}$ and its reasoning tasks, along with some of the properties that will be used throughout the paper. The semantics of $\otimes$-$\mathcal{EL}$ depend on the choice of a t-norm $\otimes$.

A *t-norm* is an associative, commutative, and monotone binary operator $\otimes\colon [0,1]\times[0,1]\to[0,1]$ that has unit 1 [Klement *et al.*, 2000]. We consider only *continuous* t-norms, i.e. those that are continuous as a function. Every continuous t-norm defines a unique *residuum* $\Rightarrow\colon [0,1]\times[0,1]\to[0,1]$ where $x\Rightarrow y := \sup\{z \mid x\otimes z\le y\}$. From this it follows that (i) $x\Rightarrow y = 1$ iff $x\le y$, and (ii) $1\Rightarrow y = y$ hold for all $x,y\in[0,1]$. The *residual negation* $\ominus$ is defined as $\ominus x := x\Rightarrow 0$. Table 1 lists three important continuous t-norms and their residua. It is well known that all other continuous t-norms can be described as the ordinal sums of copies of these three t-norms, as described next.

Let $((a_i,b_i))_{i\in I}$ be a (possibly infinite) family of non-empty, disjoint open subintervals of $[0,1]$ and $(\otimes_i)_{i\in I}$ be a family of continuous t-norms over the same index set I. The *ordinal sum* of $(((a_i,b_i),\otimes_i))_{i\in I}$ is the t-norm $\otimes$, where

$$x\otimes y := a_i + (b_i-a_i)\left(\tfrac{x-a_i}{b_i-a_i}\otimes_i \tfrac{y-a_i}{b_i-a_i}\right)$$

if $x,y\in[a_i,b_i]$ for some $i\in I$, and $x\otimes y := \min\{x,y\}$ otherwise. This yields a continuous t-norm, whose residuum

Table 1: The three fundamental continuous t-norms.

Name	t-norm ($x\otimes y$)	residuum ($x\Rightarrow y$)
Gödel	$\min\{x,y\}$	$\begin{cases}1 & \text{if } x\le y\\ y & \text{otherwise}\end{cases}$
Product	$x\cdot y$	$\begin{cases}1 & \text{if } x\le y\\ y/x & \text{otherwise}\end{cases}$
Łukasiewicz	$\max\{x+y-1,0\}$	$\min\{1-x+y,1\}$

$x\Rightarrow y$ is given by

$$\begin{cases}1 & \text{if } x\le y,\\ a_i+(b_i-a_i)\left(\frac{x-a_i}{b_i-a_i}\Rightarrow_i \frac{y-a_i}{b_i-a_i}\right) & \text{if } a_i\le y<x\le b_i,\\ y & \text{otherwise,}\end{cases}$$

where $\Rightarrow_i$ is the residuum of $\otimes_i$, for each $i\in I$. Intuitively, this means that the t-norm $\otimes$ and its residuum "behave like" $\otimes_i$ and its residuum in each of the intervals $[a_i,b_i]$, and like the Gödel t-norm and residuum everywhere else.

Theorem 1 ([Mostert and Shields, 1957]). *Every continuous t-norm is isomorphic to the ordinal sum of copies of the Łukasiewicz and product t-norms.*

Let $\otimes$ be a continuous t-norm and $(((a_i,b_i),\otimes_i))_{i\in I}$ be its representation as ordinal sum given by Theorem 1.[7] Note that the only elements $x\in[0,1]$ that are *idempotent* w.r.t. $\otimes$, i.e. that satisfy $x\otimes x = x$, are those that are not in (a_i,b_i) for any $i\in I$. Thus, every continuous t-norm except the Gödel t-norm has infinitely many non-idempotent elements. We call $(((a_i,b_i),\otimes_i))_{i\in I}$ the *components* of $\otimes$. We further say that $\otimes$ *contains* a t-norm $\otimes'$ if it has a component of the form $((a_i,b_i),\otimes')$. It *starts with Łukasiewicz* if it has a component of the form $((0,b),\otimes_{Ł})$, where $\otimes_{Ł}$ is the Łukasiewicz t-norm; and is *product-free* if it does not contain the product t-norm.

A value $x\in(0,1]$ is called a *zero divisor* for a t-norm $\otimes$ if there is a $y\in(0,1]$ such that $x\otimes y=0$. It can be shown [Klement *et al.*, 2000] that for every t-norm without zero divisors, the residual negation corresponds to the Gödel negation. More precisely, if $\otimes$ has no zero divisors, then

$$\ominus x = \begin{cases}0 & \text{if } x>0,\\ 1 & \text{otherwise.}\end{cases}$$

Of the three continuous t-norms from Table 1, only the Łukasiewicz t-norm has zero divisors: every value $x\in(0,1)$ is a zero divisor for this t-norm since $1-x>0$ and $x\otimes(1-x)=0$. In fact, a continuous t-norm can only have zero divisors if it starts with the Łukasiewicz t-norm.

Lemma 2 ([Klement *et al.*, 2000]). *A continuous t-norm has zero divisors iff it starts with the Łukasiewicz t-norm.*

Every continuous t-norm $\otimes$ defines a fuzzy DL $\otimes$-$\mathcal{EL}$. The syntax of $\otimes$-$\mathcal{EL}$ is identical to the one of the classical DL $\mathcal{EL}$, which allows only for the top concept, conjunctions, and existential restrictions. Formally, from two disjoint sets $\mathsf{N_C}$

[6] Mailis *et al.* consider an extension of $\mathcal{EL}$ called $\mathcal{EL}^{++}$.

[7] For ease of presentation, we treat the isomorphism as equality.