Corollary 5.6. Denote significance level $\alpha > 0$, and for $1 \leq i \leq pd$, define interval $I_i = [\widehat{\theta}_i^u - \delta(\alpha, T-p), \widehat{\theta}_i^u + \delta(\alpha, T-p)]$. Here, $\delta(\alpha, T-p) = \Phi(1-\alpha/2)(\sigma/\sqrt{T-p})[\mathbf{M}\widetilde{\mathbf{\Sigma}}_n\mathbf{M}^\top]_{i,i}^{1/2}$. Then

$$\lim_{T-p\to\infty} \mathbb{P}(\theta_i^* \in I_i) = 1-\alpha.$$

By Corollary 5.6, the asymptotic coverage probability corresponds the the given confidence level. Note that we can replace σ with $\widehat{\sigma}$ by the Slutsky Theorem. Similarly, we confirm in the following corollary that the type I error for hypothesis test $\Psi_Z(\alpha)$, introduced in (4.7), matches the given significance level α. Furthermore, we prove that the CDF of the p-value P_i for $\Psi_Z(\alpha)$, which we introduced in (4.8), converges in distribution to a uniform distribution.

Corollary 5.7. With $\Psi_Z(\alpha)$ and P_i defined as above, and significance level $\alpha > 0$, we have:

$$\mathbb{P}(\Psi_Z(\alpha) = 1 | H_0^i) \xrightarrow{(T-p)\to\infty} \alpha \quad \text{and} \quad P_i \xrightarrow{D} U[0,1].$$

We now turn our attention to demonstrating the asymptotic validity of the FDR control method we present in Section 4.2. To control FDR we desire the following property:

$$\frac{\sum_{i\in\mathcal{H}_0} \mathbb{1}(|\widehat{Z_i}| \geq \widehat{\nu})}{2|\mathcal{H}_0|(1-\Phi(\widehat{\nu}))} \xrightarrow{P} 1. \tag{5.1}$$

Unfortunately, in this application, the test statistics $\widehat{Z_i}$ are correlated, rendering the convergence in (5.1) non-trivial. In order to prove (5.1), we will leverage martingale theory, empirical process theory, and the following assumption.

Assumption 5.8. For constant $c > 2$,

$$\sum_{i\in\mathcal{H}_1} \mathbb{1}\left(\frac{|\theta_i^*|}{\sigma\widetilde{\mathbf{\Sigma}}_{i,i}^{-1/2}} \geq \sqrt{\frac{c\log(pd)}{(T-p)}}\right) \to \infty,$$

as $(T-p, pd) \to \infty$.

Assumption 5.8 implies that the number of true alternative hypotheses approaches infinity. This property proves important because, as demonstrated by Liu et al. (2014), FDR control is impossible when the number of true alternative hypotheses is fixed. This assumption allows us to present the following theorem:

Theorem 5.9. Assume $pd \leq (T-p)^r$ and $\log(pd) = o(\sqrt{T-p})$ for some $r > 0$. Furthermore, suppose that Assumption 5.8 and the assumptions of Theorem 5.5 hold. Then at significance level α,

$$\lim_{(T-p,pd)} \frac{\mathrm{FDR}(\widehat{\nu})}{\alpha|\mathcal{H}_0|/(pd)} = 1 \text{ and } \frac{\mathrm{FDP}(\widehat{\nu})}{\alpha|\mathcal{H}_0|/(pd)} \xrightarrow{P} 1,$$

as $(T-p, pd) \to \infty$.

Theorem 5.9 establishes that the FDR control procedure we present in Section 4.2 asymptotically controls both FDR and FDP. Note that the upper bound rate imposed on pd is very mild and will pose no issues in the vast majority of applications. The assumptions of Theorem 5.5 guarantee the asymptotic normality of test statistic $\widehat{Z_i}$.

6 Numerical Experiments

In this section, we establish the effectiveness of our de-biased Lasso Granger estimator and our FDR control procedure via experimental results. We also demonstrate that our methods outperform existing techniques.

6.1 Synthetic Data

Table 1: Empirical type I errors when testing $H_0 : \theta_i^* = 0$ over 500 simulations for $\alpha = .05$ for "random" and "cluster" transition matrix patterns.

T	d	p	Random	Cluster
150	200	1	.046	.052
150	200	2	.054	.054
250	300	1	.050	.040
250	300	2	.046	.046

In this section, we corroborate our theoretical results and compare our contributions to existing methods with numerical experiments on synthetic data. The data for these experiments are generated by model (3.1). In order to satisfy the assumptions of Theorem 5.5, each $\boldsymbol{\theta}^{j*}$ is a sparse vector such that the probability of each element being non-zero is $\sqrt{T-p}/(2pd\log(pd))$ for $1 \leq j \leq d$. We use the R package"flare" (Li et al., 2012) to generate sparse transition matrices, and the "glmnet" package (Friedman et al., 2010) to compute the biased Lasso Granger estimate. We examine multiple different transition matrix patterns ("random" and "cluster", as generated by the "flare" package) and multiple different configurations of (T, d, p).

In Table 1, we see that the empirical type 1 error of hypothesis test $\Psi_Z(\alpha)$ (4.7) corresponds to the given significance level across multiple configurations of (T, d, p). Figure 1(a) corroborates Theorem 5.5 by demonstrating that the empirical distribution of test statistic $\widehat{Z_i}$ under the null hypothesis is the standard normal distribution. Figure 1(a) also illustrates that coefficient point estimates for the biased Lasso Granger estimator do not follow the standard normal distribution. Figure 1(b) validates Corollary 5.7 by demonstrating that the empirical CDF of p-value (4.8) for a true zero parameter is the uniform distribution. Furthermore, Figures 1(c) and 1(d) exhibit that hypothesis test $\Psi_Z(\alpha)$ (4.7) attains higher power than the biased Lasso Granger estimator when testing a single true non-zero parameter. Table 2 demonstrates the accuracy of the de-biased Lasso Granger estimator via computations of the ℓ_1 and ℓ_2 norms of the