Option Pricing Using Fractional Geometric Brownian Motion (fGBM)

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Introduction

This short note presents the detailed derivation of the option pricing formula using Fractional Geometric Brownian Motion (fGBM). Unlike the standard Geometric Brownian Motion (GBM), fGBM incorporates long-range dependencies and memory effects, making it more suitable for modeling financial markets.

Hurst Parameter

The Hurst parameter H is a measure of the long-term memory of a time series, named after Harold Edwin Hurst. It ranges between 0 and 1 and is used to characterize the behavior of a stochastic process.

- If $H = 0.5$, the process is a standard Brownian motion with no long-term memory. - If $H > 0.5$, the process exhibits positive long-term autocorrelation, meaning that high values are likely to be followed by high values and low values by low values (persistent behavior). - If $H < 0.5$, the process shows negative long-term autocorrelation, meaning that high values are likely to be followed by low values and vice versa (anti-persistent behavior).

Why Use fGBM?

Traditional GBM assumes that asset returns are normally distributed and do not exhibit memory effects. However, real financial markets often display long-range dependencies, volatility clustering, and other complex behaviors that GBM cannot capture. fGBM, with its incorporation of the Hurst parameter, provides a more realistic model by accounting for these effects, leading to better pricing of options and other financial derivatives.

Fractional Geometric Brownian Motion (fGBM)

A fractional Brownian motion $B^H(t)$ with Hurst parameter H (0 ¡ H ¡ 1) is used instead of the standard Brownian motion. The fGBM is defined by the stochastic differential equation (SDE):

$$dS(t) = \mu S(t)dt + \sigma S(t)dB^H(t) \tag{1}$$

where:

- $S(t)$ is the asset price at time t.
- μ is the drift term.
- σ is the volatility term.
- $B^H(t)$ is a fractional Brownian motion.

Risk-Neutral Valuation

In a risk-neutral world, we replace the drift term μ with the risk-free rate r:

$$dS(t) = rS(t)dt + \sigma S(t)dB^H(t) \tag{2}$$

Solution to the fGBM SDE

To solve the SDE for fGBM, we consider the following steps:

Change of Variables

Let $S(t) = e^{X(t)}$. Applying Itô's lemma for fractional Brownian motion, we need to find $dX(t)$.

Itô's Lemma for Fractional Brownian Motion

For a function $f(X) = e^X$, Itô's lemma states:

$$df(X) = f'(X)dX + \frac{1}{2}f''(X)(dX)^2$$

For $f(X) = e^X$, we have:

$$f'(X) = e^X \quad \text{and} \quad f''(X) = e^X$$

Substituting these into Itô's lemma, we get:

$$de^{X(t)} = e^{X(t)}dX(t) + \frac{1}{2}e^{X(t)}(dX(t))^2$$

Since $S(t) = e^{X(t)}$, we have:

$$dS(t) = e^{X(t)}dX(t) + \frac{1}{2}e^{X(t)}(dX(t))^2$$

Given the SDE for $S(t)$:

$$dS(t) = rS(t)dt + \sigma S(t)dB^H(t)$$

Substituting $S(t) = e^{X(t)}$, we get:

$$d(e^{X(t)}) = re^{X(t)}dt + \sigma e^{X(t)}dB^H(t)$$

Applying Itô's lemma, we get:

$$e^{X(t)}dX(t) + \frac{1}{2}e^{X(t)}(dX(t))^2 = re^{X(t)}dt + \sigma e^{X(t)}dB^H(t)$$

Dividing by $e^{X(t)}$:

$$dX(t) + \frac{1}{2}(dX(t))^2 = rdt + \sigma dB^H(t)$$

For fractional Brownian motion, the term $(dX(t))^2$ is negligible compared to dt, so we get:

$$dX(t) = \left(r - \frac{1}{2}\sigma^2\right)dt + \sigma dB^H(t)$$

Integral Form

Integrating both sides from t to T, we obtain:

$$X(T) = X(t) + \int_t^T \left(r - \frac{1}{2}\sigma^2\right) ds + \sigma \int_t^T dB^H(s) \tag{3}$$

Since $\int_t^T dB^H(s)$ represents the fractional Brownian motion increment, we can write:

$$X(T) = X(t) + \left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\left(B^H(T) - B^H(t)\right) \tag{4}$$

Exponentiation

Returning to the original variable $S(t)$, we exponentiate $X(t)$:

$$S(T) = S(t)\exp\left(\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\left(B^H(T) - B^H(t)\right)\right) \tag{5}$$

The term $B^H(T) - B^H(t)$ is normally distributed with mean 0 and variance $(T-t)^{2H}$, so:

$$S(T) = S(t)\exp\left(\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma(T-t)^H Z\right) \tag{6}$$

where Z is a standard normal variable $Z \sim N(0,1)$.

Pricing the European Call Option

The price of a European call option $C(t,S)$ with strike price K and maturity T is:

$$C(t,S) = e^{-r(T-t)}\mathbb{E}[(S(T) - K)^+] \tag{7}$$

Substituting the expression for $S(T)$:

$$C(t,S) = e^{-r(T-t)}\mathbb{E}\left[\left(S(t)\exp\left((r - \frac{1}{2}\sigma^2)(T-t) + \sigma(T-t)^H Z\right) - K\right)^+\right] \tag{8}$$

Transforming the Expectation

This expectation can be transformed by considering the properties of the normal distribution:

$$C(t,S) = e^{-r(T-t)}\int_{-\infty}^{\infty}\left(S(t)e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(T-t)^H x} - K\right)^+ \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx \tag{9}$$

Breaking Down the Integral

The integral is broken down into parts where the expression inside the max function is positive:

$$C(t,S) = e^{-r(T-t)}\int_{x^*}^{\infty}\left(S(t)e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(T-t)^H x} - K\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx \tag{10}$$

where x^* is defined as:

$$x^* = \frac{\ln\left(\frac{K}{S(t)}\right) - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma(T-t)^H} \tag{11}$$

Final Option Pricing Formula

After solving the integral using the cumulative distribution function of the normal distribution, we get the final formula for the European call option price under fGBM:

$$C(t,S) = S(t)\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) \tag{12}$$

where:

$$d_1 = \frac{\ln\left(\frac{S(t)}{K}\right) + (r + \frac{1}{2}\sigma^2(T-t)^{2H})}{\sigma(T-t)^H} \tag{13}$$

$$d_2 = d_1 - \sigma(T-t)^H \tag{14}$$

and Φ is the cumulative distribution function of the standard normal distribution.

Python Code for Option Pricing Comparison

The following Python code was used to compare the option pricing using the Geometric Brownian Motion (GBM) model and the Fractional Geometric Brownian Motion (fGBM) model with a higher Hurst parameter (H = 0.95) and an extended time to maturity (T = 5 years).

```
import numpy as np
import scipy.stats as stats
import pandas as pd
import matplotlib.pyplot as plt

# Function to calculate European call option price using Black-Scholes model
def black_scholes_call(S, K, T, r, sigma):
    d1 = (np.log(S / K) + (r + 0.5 * sigma ** 2) * T) / (sigma * np.sqrt(T))
    d2 = d1 - sigma * np.sqrt(T)
    call_price = S * stats.norm.cdf(d1) - K * np.exp(-r * T) * stats.norm.cdf(d2)
    return call_price

# Function to calculate European call option price using Fractional Black-Scholes model
def fractional_black_scholes_call(S, K, T, r, sigma, H):
    d1 = (np.log(S / K) + (r + 0.5 * sigma ** 2 * T ** (2 * H - 1))) / (sigma * T ** H)
    d2 = d1 - sigma * T ** H
    call_price = S * stats.norm.cdf(d1) - K * np.exp(-r * T) * stats.norm.cdf(d2)
    return call_price

# Parameters
S = 100  # Underlying asset price
K = 100  # Strike price
T = 5    # Time to maturity in years
r = 0.05 # Risk-free rate
H_single = 0.95 # Increased Hurst parameter for fGBM to highlight differences

# Volatility range
volatilities = np.linspace(0.1, 0.5, 10)
```

```
# Initialize results DataFrame
results_single_high_hurst_extended = pd.DataFrame(columns=['Volatility', 'GBM Call Price', 'fG

# Calculate option prices for different volatilities
for sigma in volatilities:
    gbm_price = black_scholes_call(S, K, T, r, sigma)
    fgbm_price = fractional_black_scholes_call(S, K, T, r, sigma, H_single)
    results_single_high_hurst_extended = results_single_high_hurst_extended.append({'Volatilit

# Plotting the updated option prices for GBM and fGBM with higher Hurst parameter
plt.figure(figsize=(10, 6))
plt.plot(results_single_high_hurst_extended['Volatility'], results_single_high_hurst_extended[
plt.plot(results_single_high_hurst_extended['Volatility'], results_single_high_hurst_extended[
plt.xlabel('Volatility')
plt.ylabel('Call Option Price')
plt.title(f'Comparison of GBM and fGBM Call Option Prices under Changing Volatility (H={H_sing
plt.legend()
plt.grid(True)
plt.savefig('option_prices_comparison.png')
plt.show()

results_single_high_hurst_extended
```

Results Table

The following table shows the comparison of the option prices calculated using the Geometric Brownian Motion (GBM) model and the Fractional Geometric Brownian Motion (fGBM) model under changing volatility with a higher Hurst parameter (H = 0.95) and extended time to maturity (T = 5 years).

Volatility	GBM Call Price	fGBM Call Price (H=0.95)
0.100000	23.421057	26.585239
0.144444	25.678750	32.277342
0.188889	28.414592	37.549337
0.233333	31.367435	42.247446
0.277778	34.417314	46.313664
0.322222	37.501400	49.749977
0.366667	40.582488	52.600463
0.411111	43.636192	54.936339
0.455556	46.645204	56.842459
0.500000	49.596495	58.406011

Table 1: Comparison of GBM and fGBM Call Option Prices under Changing Volatility

Results Plot

The following plot shows the comparison of the option prices calculated using the Geometric Brownian Motion (GBM) model and the Fractional Geometric Brownian Motion (fGBM) model under changing volatility with a higher Hurst parameter (H = 0.95) and extended time to maturity (T = 5 years).

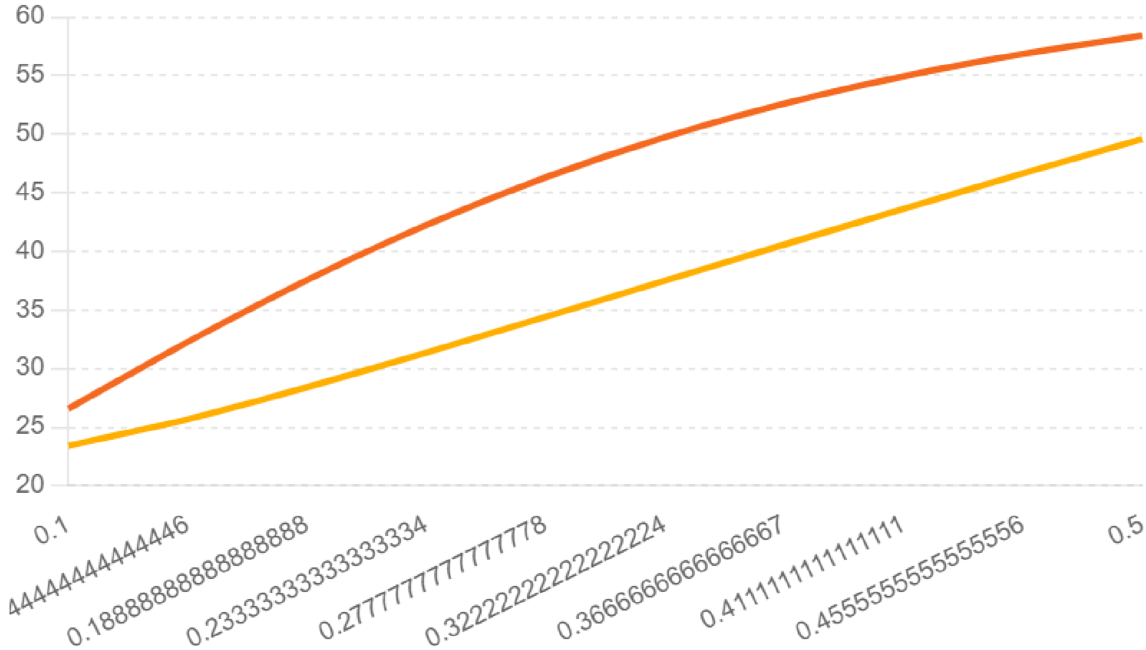

Figure 1: Comparison of GBM and fGBM Call Option Prices under Changing Volatility (H=0.95, T=5 years)

Conclusion

By incorporating the Hurst parameter H, the fGBM model provides a more realistic representation of financial markets, especially in capturing memory effects and long-range dependencies that are not addressed by the standard GBM. This detailed derivation demonstrates how the option pricing formula is adjusted to reflect these unique properties of fGBM.