Stochastic Calculus for Quants: A Primer

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1 Probability

Probability, in its simplest form, is a way of expressing how likely something is to happen. Think of it as a scale from 0 to 1. A probability of 0 means something won't happen, while a probability of 1 means it's certain to happen. Everything in between gives a sense of the likelihood.

In the world of finance, probability plays a crucial role. Financial markets are uncertain by nature. When investors, traders, or even everyday people think about money matters, they're often dealing with probabilities, whether they realize it or not.

For instance, consider a company's stock. There are numerous factors, both internal (like the company's earnings report) and external (like geopolitical events), that can influence its price. While it's impossible to predict the stock price with absolute certainty, analysts and traders use probability to gauge the potential directions the stock might take.

Example: Let's say an analyst believes there's a 60% chance (or a probability of 0.6) that a particular stock will go up tomorrow due to a positive earnings report that's expected to be released. This doesn't mean the stock will definitely rise. Instead, out of many days with similar circumstances, the stock would go up about 60% of the time. If the earnings report is better than expected, the stock might see a significant rise. If it's as expected, there might be a moderate rise. If the earnings are poor, the stock might even fall. The probability helps the analyst communicate the level of confidence in the prediction.

Moreover, probability isn't just about predicting stock movements. It's used in various financial instruments, from options pricing to risk assessment in loans. For instance, a bank might use probability to determine the likelihood of a borrower defaulting on a loan. If there's a high probability of default, the bank might charge a higher interest rate to compensate for the increased risk.

In essence, probability gives a structured way to think about uncertainty, allowing financial professionals to make more informed decisions in the unpredictable world of finance.

2 Random Walk

At its core, a random walk is like taking a stroll where you don't plan your steps in advance; instead, each step you take is random and independent of the previous one. Imagine walking blindfolded, taking steps forward or backward based on the flip of a coin. Over time, your path will zigzag in a seemingly unpredictable manner.

In the world of mathematics, this concept has been formalized as the "random walk" model, where each step or movement is determined randomly. It's a way to capture the essence of unpredictability in various scenarios.

When it comes to finance, the random walk theory is often associated with stock prices. Why? Well, stock prices are influenced by countless factors, from company-specific news like earnings reports to broader events like geopolitical tensions or economic policy changes. Predicting the exact movement of a stock price based on all these factors is incredibly challenging. The random walk theory suggests that stock prices move in a way where each day's price change is random and independent of the change on the previous day. Essentially, the theory posits that stocks take a "random walk," making it difficult, if not impossible, to consistently predict their future movements.

This idea has profound implications for investing. If stock prices truly follow a random walk, then it suggests that strategies based on predicting short-term price movements are doomed to fail in the long run. Instead, it would mean that the market is efficient, reflecting all available information in current prices.

Example: Let's dive deeper into our coin-flipping trader. Suppose this trader flips a coin every morning. If it lands on heads, he buys a share of a particular stock, pushing its price slightly up. If it lands on tails, he sells a share, causing a small drop in the stock price. Over days, weeks, and months, the stock's price chart would show a series of ups and downs, each one independent of the last. This price chart would resemble a random walk, highlighting the unpredictable nature of the stock's movements based solely on the trader's coin flips.

However, it's essential to note that in the real world, many believe that stock prices don't follow a pure random walk. There may be trends, patterns, or anomalies that can be exploited. But the random walk theory serves as a baseline model, reminding investors of the inherent unpredictability of financial markets.

3 Sigma Algebra

Imagine you're trying to organize a vast collection of books in a library. You'd create categories, subcategories, and maybe even sub-subcategories. Now, think of these categories as sets. In probability and statistics, we also deal with sets, especially when talking about events. However, we need a systematic way to organize and handle these sets, especially when we want to measure probabilities associated with them.

Enter the concept of a sigma algebra (or σ-algebra). It's like a rulebook for organizing these sets. This rulebook ensures three things:

1. The entire collection of possible outcomes (think of it as the complete library) is always included. 2. If a set (or category) is in our rulebook, its complement (all books not in that category) should also be in the rulebook. 3. If we have a sequence of sets (like a series of subcategories), their union (combination) should also be in our rulebook, even if we have infinitely many of them.

Why is this important? Well, when we're dealing with probabilities, we need to ensure that we can assign a probability to any set in our sigma algebra. It ensures consistency and completeness in our probability assignments.

4 Filtrations

In everyday life, our knowledge about things grows over time. Think of it as watching a movie: at the start, you know nothing about the plot, but as time progresses, the story unfolds, and your information about it accumulates.

Filtrations capture this idea of accumulating information in a formal mathematical way. It's a sequence of nested information sets, where each set contains all the information up to a specific time. As time moves forward, our set grows, including all the new information.

In finance, filtrations play a vital role, especially in the world of derivatives pricing and risk management. Financial professionals need to make decisions based on the information they have at a particular time. Filtrations help in modeling this available information.

Example: Let's talk about a stock analyst tracking a company. On the 1st of the month, she might only have the company's past performance data. By the 15th, she might get a press release about a new product launch. By the end of the month, she might have the company's quarterly earnings report. The information available to her keeps growing as the month progresses. If we were to capture all the information she has at each point in time in sets, we'd get a filtration. The set on the 15th would include the set from the 1st plus the new product information, and the set at the end of the month would include all that plus the earnings report.

5 Wiener Process

Imagine you're watching the erratic motion of a tiny pollen grain floating on water under a microscope. It darts around, making unpredictable movements in every direction. This phenomenon, first observed by the botanist Robert Brown, is called Brownian motion. Now, when mathematicians and physicists started studying this motion, they formalized it into what's known as the Wiener Process.

The Wiener Process is essentially a mathematical representation of this seemingly random motion. It's like taking our earlier concept of a random walk and making it continuous, so instead of discrete steps, you have a continuous curve.

In finance, this concept becomes crucial. Stock prices, for example, don't jump in fixed intervals; they move every moment the market is open. The Wiener Process helps model this continuous price movement, especially over short intervals. It's a foundation upon which many other financial theories and models are built.

Example: Let's say you're observing a stock's price movement over a single day. From opening to close, the stock's price moves up and down, reacting to countless factors. Over such a short duration, these movements might seem erratic and unpredictable, much like the Brownian motion of a pollen grain. This is where the Wiener Process comes in, providing a model that closely resembles the stock's behavior over that day.

6 Martingale

Imagine you're at a casino, playing a completely fair coin-toss game. Every time the coin lands heads, you win a dollar, and every time it's tails, you lose a dollar. If you were to track your winnings over time, the pattern you'd see could be described as a martingale. In a martingale, no matter how much you've won or lost so far, your expected winnings in the next round are always zero.

The idea behind a martingale is that you can't predict future outcomes based on past events. It's like saying, no matter what's happened before, the future remains unpredictable.

In finance, this concept is used to model certain types of asset prices or investment strategies, suggesting that past price movements or returns don't provide any useful information to predict future ones.

Example: Let's consider a hypothetical stock where any news or events affecting its price are completely random and unforeseeable. Today, the stock might go up due to positive unexpected news, and tomorrow it might drop due to some negative surprise. If you were to invest in this stock, your expected return, regardless of its past performance, remains constant over time. Such a stock's price behavior can be modeled as a martingale.

7 Types of Martingales

Apart from the standard martingale, there are variations that describe different types of random processes:

- **Submartingale:** Imagine the coin-toss game, but with a slight bias where heads (your wins) come up a bit more often. Over time, you're expected to have more winning days than losing ones. In this case, your winnings are likely to increase over time, on average. This scenario can be described as a submartingale.

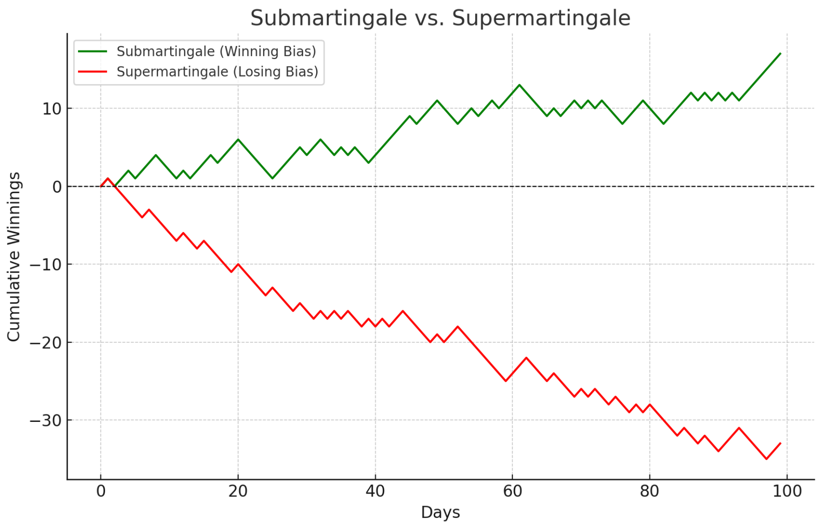

Figure 1: Visualisation of types of Martingales

- **Supermartingale:** Now, consider the opposite. The game is slightly biased against you. Over time, you're more likely to lose than win. Your expected winnings decrease over time. This situation is a supermartingale.

In the financial world, these concepts help professionals understand and model the behavior of assets or investment strategies under different assumptions about their expected returns.

8 Ito Process

Picture a boat gently floating down a river. The river's current (which is always in one direction) represents a steady force called the "drift," while the boat's bobbing due to waves and wind represents random, unpredictable movements, termed "diffusion." The combination of these two elements—the steady current and the random bobbing—captures the essence of the Ito Process.

Named after Kiyoshi Ito, the Ito Process is a mathematical model that describes such phenomena. It expands on the Wiener Process by adding a drift term, which represents a consistent trend, and a diffusion term, which captures the random fluctuations.

In finance, this model becomes particularly useful. For instance, a stock might have a general upward growth trend (thanks to the company's performance, sector growth, etc.), but day-to-day trading might cause random price fluctuations.

Example: Consider a technology stock that's been growing steadily over the years due to consistent innovation and market leadership. However, daily news, trader sentiment, or market events lead to price volatility. This stock's price movement—having a general upward trend but also experiencing random fluctuations—can be aptly described using the Ito Process.

9 Ito's Lemma

Calculus students are likely familiar with the chain rule—a method that helps differentiate a composite function. Now, what if our function is a bit more unpredictable, with random movements? That's where Ito's Lemma comes in. It's essentially the chain rule, but for stochastic processes.

Named after the same Kiyoshi Ito, this lemma is a cornerstone in stochastic calculus. It provides a way to differentiate functions of stochastic processes, paving the path for many advanced financial models, especially those concerning option pricing.

10 P-Q Measures

In the world of finance, understanding and measuring risk is crucial. To do this, professionals often switch between two views or "measures" of the world: the P measure and the Q measure.

1. The P measure (or physical measure) is the "real-world" view. It's about actual probabilities, representing how assets like stocks are expected to behave based on historical data and future predictions.

2. The Q measure (or risk-neutral measure) is more of a hypothetical view. Here, we assume that all assets grow at the risk-free rate (like the rate of a government bond). It's not about what we truly expect to happen but provides a simplified world that's crucial for pricing derivatives.

The switch between these measures is more than just a mathematical trick. It's rooted in a fundamental finance principle: there shouldn't be any arbitrage opportunities (free money) in the market.

Example: Imagine you're a quant trying to price an option (a financial derivative). In the real world, the stock associated with that option might have all sorts of expected growth rates, volatilities, and risks. However, to price the option, you'd switch to the Q measure, simplifying your calculations and ensuring the price you arrive at doesn't allow arbitrage.

11 Monte Carlo Simulation

Picture yourself in a casino, standing before a roulette table. As the wheel spins and the ball bounces, it seems almost impossible to predict where it will land. But what if you could spin that wheel thousands or even millions of times and record every outcome? Over time, you'd start to see patterns or probabilities emerge. This idea of understanding complex systems through repeated random sampling is at the heart of the Monte Carlo Simulation. Named after the famous Monaco casino town, the Monte Carlo Simulation is a computational technique used to estimate the probability of different outcomes.

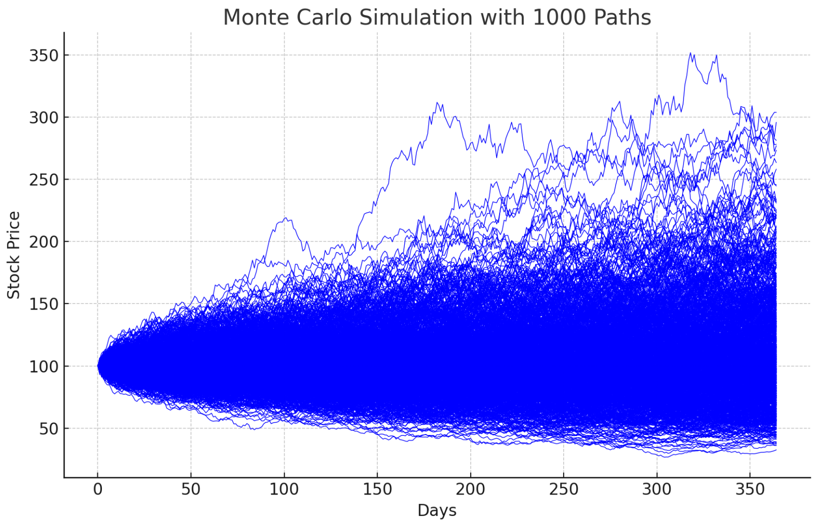

Figure 2: Visualisation of Monte Carlo Simulation

It's a bit like conducting a vast number of "what if" scenarios to understand potential future events. By repeatedly simulating a process with random inputs, we can obtain a distribution of outcomes, helping us understand the probabilities and risks associated with a particular system or decision.

In finance, Monte Carlo Simulation becomes invaluable. Financial markets are inherently complex and filled with uncertainties. Instead of trying to predict the future with a single deterministic model, the Monte Carlo method allows professionals to explore a myriad of potential scenarios, understanding not just the most likely outcomes but also the range of possibilities.

Example: Consider an investment portfolio comprising various assets: stocks, bonds, commodities, etc. Predicting its future value is challenging due to the countless factors influencing each asset. Using Monte Carlo Simulation, an investor can simulate thousands of potential future market scenarios, each with different market returns, interest rates, and economic conditions. After running these simulations, the investor won't get a single predicted value for the portfolio but rather a distribution of potential values, helping them assess the portfolio's risk and potential return.

12 Stochastic Differential Equations (SDEs)

At its core, a differential equation is like a puzzle or a riddle. It provides a relationship between something and its rate of change, and solving it gives us insight into the behavior of the system described by the equation. But what if this system isn't stable and predictable? What if, like the weather, it's subject to random and unforeseeable influences?

This is where Stochastic Differential Equations (SDEs) come into play. These are equations that, in addition to the usual terms, have components that behave unpredictably or "stochastically." The solutions to these equations aren't precise trajectories but rather a myriad of possible paths, each with a certain likelihood.

In finance, many assets and instruments are influenced by a multitude of unpredictable factors: sudden news, geopolitical events, market sentiment shifts, and more. SDEs provide a framework to model and understand these assets, taking into account both the deterministic trends and the random fluctuations.

Example: Consider a company's stock price. While it might generally grow due to solid performance, it's also subject to unexpected news—like a sudden merger or a regulatory hurdle. An SDE can help model this stock price by incorporating both its general growth trend and the random shocks it might experience.

13 Geometric Brownian Motion (GBM)

Imagine you're watching a tree grow. Over the years, the tree not only gets taller but also wider, its branches more spread out. This growth isn't linear; the bigger the tree gets, the more it grows each year. Now, add to this growth some randomness—like varying weather conditions affecting the tree's growth differently each year.

This combination of consistent growth and randomness is what Geometric Brownian Motion (GBM) captures. In the mathematical world, GBM is a model that describes a quantity that grows steadily and is simultaneously subject to random changes. In the

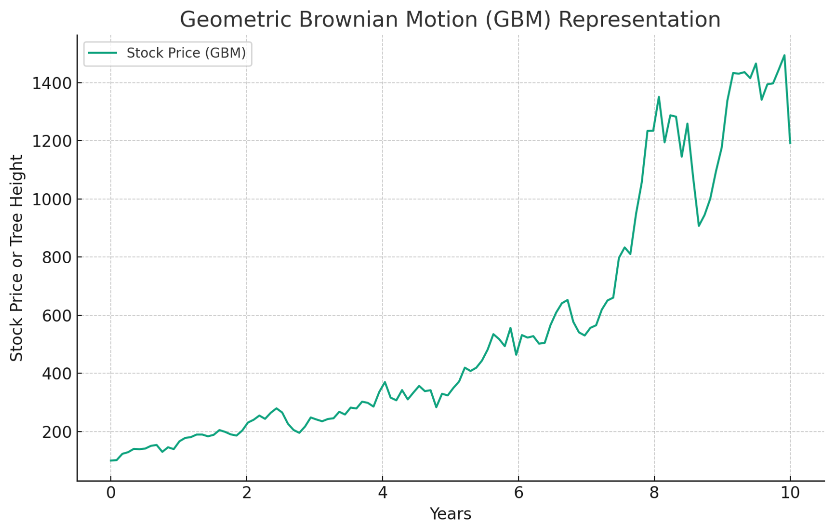

Figure 3: Visualisation of GBM

realm of finance, GBM becomes particularly relevant when talking about stock prices. While a company's stock might have a general trend—maybe due to company performance or overall market conditions—it also experiences random fluctuations based on countless factors.

Example: Take a booming tech company. Its stock might generally be on the rise due to consistent innovation and market demand. However, on any given day, the stock could go up or down based on various factors: a product announcement, a competitor's move, or global market conditions. The GBM model captures this behavior, reflecting both the stock's general growth trend and its day-to-day randomness.

14 Local Volatility

Imagine you're on a long road trip. As you drive through different terrains and weather conditions, the speed of your car varies. On a clear highway, you might speed up, but in a rainy mountain pass, you'd naturally slow down. This varying speed, depending on your location and conditions, can be likened to the concept of local volatility.

In finance, local volatility refers to the idea that volatility (or the rate at which a financial instrument's price moves) isn't constant but varies depending on factors like the current price level and time. Developed as a way to refine the Black-Scholes Model, which assumes constant volatility, local volatility models provide a more dynamic view of market behavior, adjusting volatility based on observed market conditions.

Example: Think of a stock that historically sees significant price swings whenever it approaches a particular price level, maybe due to psychological or historical reasons. A local volatility model would capture this behavior, adjusting the volatility higher when the stock is near that price level and possibly lower when it's far from it.

15 Stochastic Volatility

Now, instead of a road trip, imagine you're sailing on the open sea. The intensity of the waves (or the sea's volatility) isn't just determined by where you are (like near an island or in the deep ocean) but also by random and unpredictable factors like sudden wind gusts or distant storms. This unpredictability in the intensity of waves is akin to stochastic volatility.

Stochastic volatility models in finance embrace the idea that volatility itself is random and can change unpredictably over time. While local volatility models adjust volatility based on factors like price level and time, stochastic volatility models introduce an additional layer of randomness, acknowledging that markets can be influenced by unforeseen events or shifts in sentiment.

Example: Consider a global event like a sudden geopolitical conflict. Such an event might cause markets worldwide to become more volatile, not because of specific asset prices or historical patterns, but due to the uncertainty and unpredictability introduced by the event. A stochastic volatility model would capture this kind of random spike in volatility, providing a more comprehensive view of market risks.

16 Black-Scholes Model

When you're cooking a dish, knowing the recipe helps you predict the outcome. Similarly, in the world of finance, especially in options trading, the Black-Scholes Model acts as a recipe. Developed by Fischer Black, Myron Scholes, and Robert Merton in the early 1970s, this model provides a theoretical estimate of the price of European-style options.

Just as a recipe requires specific ingredients in precise amounts, the Black-Scholes Model considers various factors to estimate an option's price. These factors include the current stock price, the option's strike price, the time until the option expires, the stock's volatility, and the risk-free interest rate.

The beauty of the Black-Scholes Model lies in its ability to boil down these multiple factors into a single formula, offering traders and investors a standardized way to value options. However, it's essential to understand that like any model, it's based on assumptions, some of which might not hold in real-world scenarios.

Example: Imagine an investor trying to decide whether to buy an option on a tech company's stock. Using the Black-Scholes Model, they can input the current stock price, how long until the option expires, and other factors into the formula. The resulting value gives them an estimate of what the option should be worth, helping guide their decision.

17 Girsanov's Theorem

Translators help us understand one language in terms of another. In the realm of mathematical finance, Girsanov's Theorem plays a similar role, but for probability measures. When dealing with financial models, especially those involving stochastic processes, we often encounter different "views" or "measures" of probability. The most common ones are the P (physical or real-world) measure and the Q (risk-neutral) measure. Girsanov's

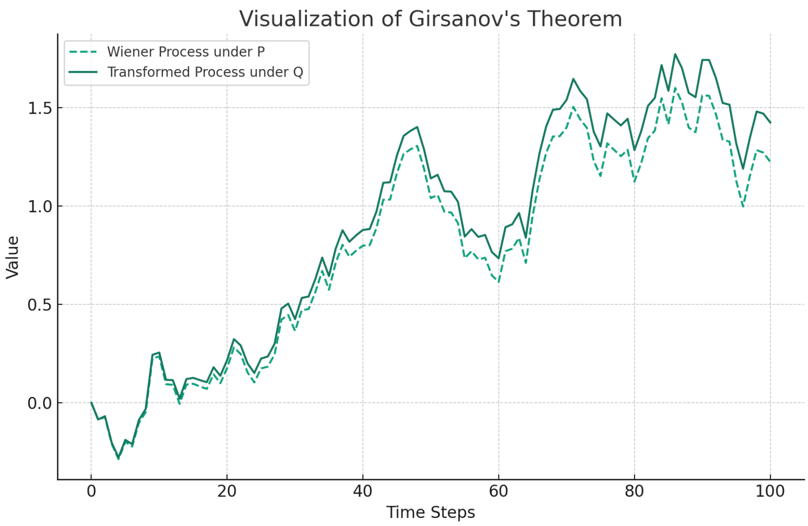

Figure 4: Visualisation of Girsanov

Theorem provides the mathematical framework to transition between these measures. It's like a bridge that ensures a smooth passage, preserving the structure and properties of our models.

18 Jump Diffusion Models

Nature is full of surprises. While things might seem calm and predictable, sudden events can disrupt the status quo. The same holds true for financial markets. Stock prices, for instance, might exhibit a steady trend but can experience abrupt changes due to unexpected news or events. Jump Diffusion Models capture this dual nature. They combine the usual random walk or diffusion process (the calm river flow) with sudden jumps (the big fish causing ripples). These jumps can represent sudden market reactions to major news, like mergers, regulatory changes, or geopolitical events.

Example: Consider a pharmaceutical company's stock. While its price might exhibit typical market fluctuations, the announcement of a breakthrough drug or, conversely, a

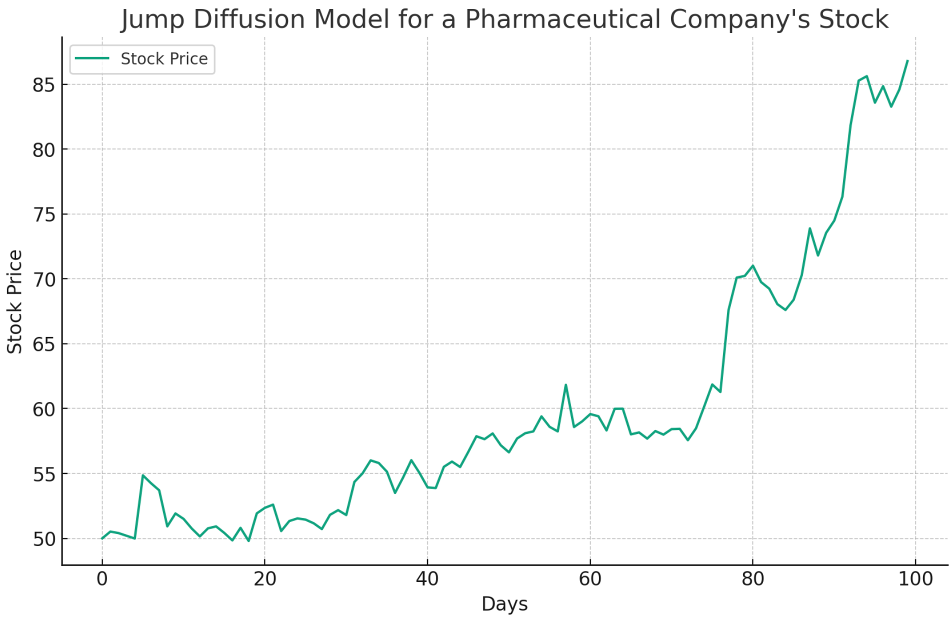

Figure 5: Jump Diffusion Model for a Pharmaceutical Company's Stock

failed clinical trial can cause a sudden and significant jump or drop in its stock price. Jump Diffusion Models can incorporate these abrupt changes, providing a more comprehensive view of the stock's price dynamics.

19 Lévy Models

Imagine you're watching a serene pond, where occasionally a stone is thrown, causing ripples to spread across the water. Most of the time, the pond is calm, but these random disturbances create sudden and noticeable effects. Lévy Models in finance are akin to this scenario, where asset prices mostly evolve smoothly but can be impacted by sudden and significant jumps.

Named after the French mathematician Paul Lévy, Lévy Models are a class of stochastic processes that incorporate both continuous paths (like the calm pond) and discontinuous jumps (the ripples from the thrown stones). While traditional models like Geometric Brownian Motion describe asset prices as smooth paths with some randomness, Lévy Models introduce the possibility of abrupt changes or jumps in these prices.

These jumps can be due to various reasons: sudden news releases, major geopolitical events, or any other unexpected occurrences that can drastically affect market sentiment. By incorporating these jumps, Lévy Models offer a more realistic representation of asset price dynamics, especially in markets known for their abrupt movements.

Example: Consider a biotechnology company awaiting regulatory approval for a new drug. For months, the stock might exhibit typical market fluctuations. However, the day the approval (or rejection) news is released, the stock might experience a significant jump (or drop) in price. A Lévy Model would be well-suited to describe this stock's behavior, accounting for both its usual price movements and the potential for sudden jumps based on significant news.