Financial Mathematics

An Introduction

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(iv) Now what happens to the option price in (iii) when $T \to \infty$?
(v) What accounts for the difference in (ii) and (iv)?

5
Portfolio Optimization-I

5.1 Introduction

The readers have already come across certain terms like *portfolio*, *portfolio return*, and *portfolio risk* in Chapter 1. There, while giving example of a simple *portfolio optimization problem*, it was remarked that we need to minimize the portfolio risk for given aspiration level of portfolio returns and thereby do a trade-off between the two. This has precisely been the approach of the celebrated *mean-variance theory* of Markowitz [90] for the single period portfolio optimization problem. The aim of this chapter is to continue this discussion in greater detail for a general n-asset problem and discuss other related results, in particular the *capital asset pricing model (CAPM)*.

5.2 Risk and Return of an Asset

Let us recollect the usual definition of *asset return* and *asset risk* from Chapter 1.

Definition 5.2.1 (Return of an Asset) *The return of an asset is an indicator of gain/loss in the investment of an asset in the financial market. It is determined by the following formula*

$$return = \frac{amount\ received - amount\ invested}{amount\ invested}.$$

In the context of a single period model, let at the initial time $t = 0$ the price of an asset be $B(0)$, and at the final time $t = T$ its price be $A(T)$. Then its return r is given by

$$r = \frac{B(T) - B(0)}{B(0)},$$

and is normally expressed in percentage.

The positive value of return on an asset signifies gain while the negative value of return signifies loss. The zero value of return means neither gain nor loss from the investment.

Remark 5.2.1 *It is important to mention here that Definition 5.2.1 is actually the definition of the rate of return on an asset. However, we shall continue to call it as return on an asset for consistency with the financial market glossary.*

Definition 5.2.2 (Risk of an Asset) *The risk is often defined as the degree of uncertainty of return on an asset. It signifies the possibility of loss in the investment.*

The risk can either be zero, implying that the asset is risk-free, or positive, implying the asset is risky. If the asset is risk-free then the future value of the asset is known with certainty otherwise the future value of the risky asset is uncertain. Any financial asset can thus be classified either as *risk-free asset* (like a fixed deposit) or as a risky asset (like share of a stock or share of a mutual fund).

Unfortunately there is no unique measure of risk of an asset. In practice we most often measure risk of an asset in terms of variance of returns on that asset, but that may not be the best thing to do. However, unless otherwise stated, the variance will always be used as a risk measure. But we shall have a much deeper discussion on various risk measures in the next chapter.

In order to obtain the mathematical model of a typical portfolio optimization problem we make certain assumptions. These are as follows :

(i) The prices of all assets at any time are strictly positive.
(ii) The return r of an asset is a random variable.
(iii) An investor can own a fraction of an asset. This assumption is known as *divisibility*. Also, we assume that this fraction can be positive or negative, i.e. short selling is allowed. We shall mention specifically if short selling is not allowed.
(iv) An asset can be bought or sold on demand in any quantity at the market price. This assumption is known as *liquidity*.
(v) No arbitrage principle holds.
(vi) *Unless stated otherwise*, there are no commissions/transaction costs.

5.3 Portfolio Optimization Problem

We are now in a position to move towards our main goal, i.e. to study the *portfolio optimization problem*. We first define a portfolio and other associated terms.

Definition 5.3.1 (Portfolio) *A portfolio is a collection of two or more assets, say, $a_1, \ldots, a_n$, represented by an ordered n-tuple $\Theta = (x_1, \ldots, x_n)$, where $x_i \in \mathbb{R}$ is the number of units of the asset a_i $(i = 1, \ldots, n)$ owned by the investor.*

We consider only a single period model, that is, in between the initial time taken as $t = 0$ and the final transaction time taken as $t = T$, no transaction ever takes place.

Let $V_i(0)$ and $V_i(T)$ be the values of the i-th asset at $t = 0$ and $t = T$, respectively. Let $V_\Theta(0)$ and $V_\Theta(T)$ denote the values of the portfolio $\Theta = (x_1, \ldots, x_n)$ at $t = 0$ and $t = T$, respectively. Then, we have

$$V_\Theta(0) = \sum_{i=1}^{n} x_i V_i(0), \quad \text{and} \quad V_\Theta(T) = \sum_{i=1}^{n} x_i V_i(T).$$

The quantity $r_\Theta(T) = \dfrac{V_\Theta(T) - V_\Theta(0)}{V_\Theta(0)}$ is referred as the *return of the portfolio* Θ.

Definition 5.3.2 (Asset Weights) *The weight w_i of the asset a_i is the proportion of the value of the asset in the portfolio at $t = 0$, i.e.*

$$w_i = \frac{x_i V_i(0)}{\sum_{j=1}^{n} x_j V_j(0)} \qquad (i = 1, \ldots, n).$$

It can be observed that $w_1 + \cdots + w_n = 1$.

In view of Definition 5.3.2, a portfolio can now be represented by the weights w_i, $(i = 1, 2, \ldots, n)$ such that $w_1 + \cdots + w_n = 1$. Thus we can imagine that we are having one unit of money (say Rs 1) and if its allocation in the ith asset is w_i, $(i = 1, 2, \ldots, n)$ then the resulting portfolio is $(w_1, w_2, \ldots, w_n)$.

Remark 5.3.1 *In a portfolio, if $w_i < 0$, for some i, it indicates that the investor has taken a short position on the i-th asset a_i.*

Let r_i be the return on the i-th asset. Then

$$r_i = \frac{V_i(T) - V_i(0)}{V_i(0)} \qquad (i = 1, \ldots, n).$$

Definition 5.3.3 (Mean of the Portfolio Return) *Let $(w_1, w_2, \ldots, w_n)$ be a portfolio of n assets $a_1, a_2, \ldots, a_n$. Let r_i $(i = 1, 2, \ldots, n)$, be the return on the ith asset a_i and $E(r_i) = \mu_i$ $(i = 1, 2, \ldots, n)$, be its expected value. Then the mean of the portfolio return is defined as*

$$\mu = E(\sum_{i=1}^{n} w_i r_i) = \sum_{i=1}^{n} w_i E(r_i) = \sum_{i=1}^{n} w_i \mu_i.$$

Definition 5.3.4 (Variance of the Portfolio) *Let $(w_1, w_2, \ldots, w_n)$ be a portfolio of n assets $a_1, a_2, \ldots, a_n$. Let r_i be the return on the ith asset a_i and $E(r_i) = \mu_i$, $(i = 1, 2, \ldots, n)$. Let $Cov\ (r_i, r_j) = \sigma_{ij}$, for $i, j = 1, 2, \ldots n$. Then the variance of the portfolio return is defined as*

$$\sigma^2 = Var(\sum_{i=1}^{n} w_i r_i) = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}.$$

Here the covariance term σ_{ij} equals $\rho_{ij}\sigma_i\sigma_j$, where $Var(r_i) = \sigma_i^2$, $Var(r_j) = \sigma_j^2$ and ρ_{ij} is the correlation coefficient between r_i and r_j.

In practice, the mean of the portfolio return is simply referred as the *return of the portfolio*. Also the variance or rather the standard deviation of the portfolio return is referred as the *risk of the portfolio*. Therefore given a portfolio A : $(w_1, w_2, \ldots, w_n)$, we can compute its mean μ_A and standard deviation σ_A and therefore get the point $A : (\sigma_A, \mu_A)$ in (σ, μ)-plane. Thus irrespective of the number of assets, a portfolio can always be identified as a point in the (σ, μ)-plane. This representation is called (σ, μ)-*diagram* or (σ, μ)-graph (see Fig. 5.1) and is very convenient for further discussion.

The *portfolio optimization problem* refers to the problem of determining weights w_i $(i = 1, 2, \ldots, n)$, such that the return of the portfolio is maximum and the risk of the portfolio is minimum. Thus we aim to solve the following optimization problem

$$\begin{aligned} &\text{Min } \textstyle\sum_{i,j=1}^{n} w_i w_j \sigma_{ij} \\ &\text{and} \\ &\text{Max } \textstyle\sum_{i=1}^{n} w_i \mu_i \\ &\text{subject to} \\ &\qquad w_1 + w_2 + \ldots + w_n = 1. \end{aligned}$$

The above problem, is a bi-criteria optimization problem which has been studied extensively in the mathematical programming literature. As in general, the two

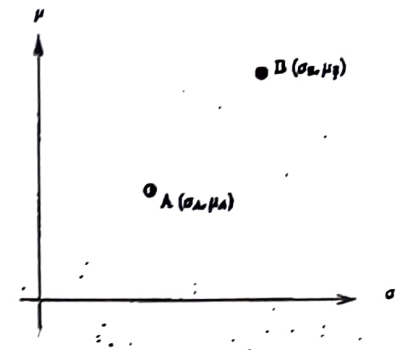

Fig. 5.1 (σ, μ)-diagram

objective functions may not attain their optimum at the same point, it is not possible to define the meaning of optimization in the usual way. Therefore we consider either (i) to minimize risk for a given level of return or (ii) to maximize return for a given level of risk. From the algorithmic point of view the first approach is more convenient because it results into a linearly constrained quadratic programming problem. We shall first discuss the two asset case and then move to the multi-asset case.

5.4 Two Assets Portfolio Optimization

Consider a portfolio with two assets, say a_1, a_2 with weights w_1, w_2, returns r_1, r_2 and standard deviations σ_1, σ_2, respectively. Then the portfolio expected return μ and portfolio variance σ^2 are respectively given by

$$\mu = E(w_1 r_1 + w_2 r_2) = w_1\mu_1 + w_2\mu_2, \tag{5.1}$$

$$\sigma^2 = Var(w_1 r_1 + w_2 r_2) = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2\rho w_1 w_2 \sigma_1 \sigma_2. \tag{5.2}$$

Here ρ is the coefficient of correlation between r_1 and r_2, and the value of ρ lies in $[-1, 1]$. We pause here to analyze the effect of ρ on the risk involved in a portfolio. What we shall be observing is that the value of ρ provides a measure of the extent of diversification of portfolio so as to reduce risk. The more negative the value of ρ, the greater are the benefits of the portfolio diversification.

As w_1 and w_2 are weights representing the proportions of total investment in two assets a_1 and a_2, respectively, we have $w_1 + w_2 = 1$. Moreover, in case of short

selling, the weights can be negative. Subsequently, we write $w_1 = 1 - s$, and so, $w_2 = s$, $s \in \mathbb{R}$. Now, it follows from relations (5.1) and (5.2) that

$$\mu = (1-s)\mu_1 + s\mu_2, \tag{5.3}$$

$$\sigma^2 = (1-s)^2\sigma_1^2 + s^2\sigma_2^2 + 2\rho(1-s)s\sigma_1\sigma_2. \tag{5.4}$$

Relation (5.4) can be simplified as

$$\sigma^2 = (\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)s^2 - 2\sigma_1(\sigma_1 - \rho\sigma_2)s + \sigma_1^2. \tag{5.5}$$

Without loss of generality we assume that $0 < \sigma_1 \le \sigma_2$. We discuss the following two independent cases

$$(i)\quad \rho = \pm 1 \qquad\qquad (ii)\quad -1 < \rho < 1.$$

Case (i) $\rho = \pm 1$. From relations (5.3) and (5.5), the portfolio expected return and standard deviation are respectively given by

$$\mu = (1-s)\mu_1 + s\mu_2.$$
$$\sigma = |(1-s)\sigma_1 \pm s\sigma_2|.$$

For $s \in [0,1]$ both weights are non-negative and thus the portfolio has no short positions. If $s > 1$ then $w_1 < 0$, and therefore that asset a_1 is held short. If $s < 0$ then $w_2 < 0$ and therefore it indicates that asset a_2 is held short. It may be noted that an investor can not take short position on both the assets simultaneously.

For $\mu = (1-s)\mu_1 + s\mu_2$, $\tilde{\sigma} = (1-s)\sigma_1 + s\sigma_2$, we plot the points $(\tilde{\sigma}, \mu)$ in $(\tilde{\sigma}, \mu)$-plane, and get the first graph of Fig. 5.2. The second graph in Fig. 5.2 corresponds to the case when $\mu = (1-s)\mu_1 + s\mu_2$, $\tilde{\sigma} = (1-s)\sigma_1 - s\sigma_2$. These graphs are essentially straight lines with the bold parts corresponding to $s \in [0,1]$.

Subsequently, we plot the standard deviation-mean diagram of the portfolio, i.e. the (σ,μ)-graph, for $\rho = \pm 1$. Observe that $\sigma = |\tilde{\sigma}|$ and therefore to get the required graphics we simply need to flip the portion of the line that lies in the left half-plane over the μ axis. The two graphs so obtained are depicted in Fig. 5.3. From Fig. 5.3 we have the following observations to share.

Remark 5.4.1 *When the returns of the two assets are perfectly positively or negatively correlated, i.e. $\rho = \pm 1$, the expected portfolio return increases with the gradual decrease in the overall risk of the portfolio (point A to point P Fig. 5.3) till the risk is completely eliminated (point P). After that the higher return comes with higher risk as weight of riskier asset increases. For the case $\rho = +1$, the point P corresponds to a portfolio in which a short position is taken on asset a_2, where as for the case $\rho = -1$, the point P corresponds to a portfolio in which weights of both assets are non-negative.*

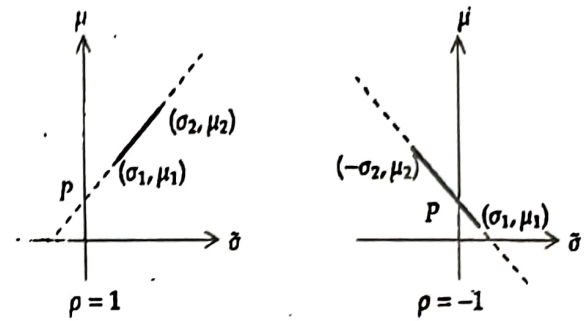

Fig. 5.2. $(\tilde{\sigma},\mu)$-graph for $\rho = \pm 1$.

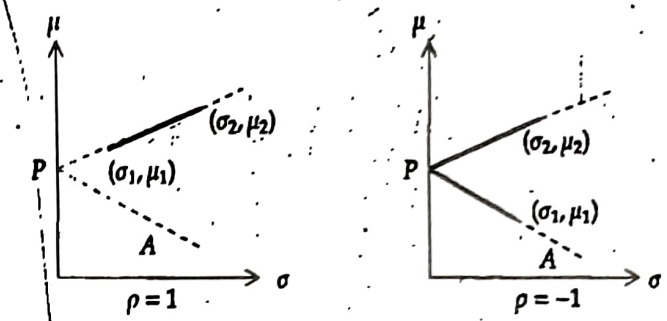

Fig. 5.3. (σ,μ)-graph for $\rho = \pm 1$.

Below we provide a mathematical justification of the above observations. For the case when $\rho = 1$ and $\sigma_1 = \sigma_2$, then $\sigma_{min} = \sigma_1$ for all $s \in \mathbb{R}$. Therefore we need to consider following two cases only.

(a) $\rho = 1$ and $\sigma_1 < \sigma_2$. In this case we have

$$\sigma^2 = (1-s)^2\sigma_1^2 + s^2\sigma_2^2 + 2(1-s)s\sigma_1\sigma_2.$$

Our aim is to minimize σ, or equivalently minimize σ^2. Now

$$\frac{d\sigma^2}{ds} = 2[s(\sigma_1 - \sigma_2)^2 - \sigma_1(\sigma_1 - \sigma_2)], \tag{5.6}$$

and

$$\frac{d^2\sigma^2}{ds^2} = 2(\sigma_1 - \sigma_2)^2 > 0. \tag{5.7}$$

Thus, to minimize the risk σ, we must choose the weight s such that $\frac{d\sigma^2}{ds} = 0$. Thereby (5.6) yields s_{min} (the value of s for which σ^2 is minimum) as

$$s_{min} = \frac{\sigma_1}{\sigma_1 - \sigma_2} < 0.$$

Hence $(1 - s_{min}) = \frac{\sigma_2}{\sigma_2 - \sigma_1} > 0$. Let μ_{min} and σ^2_{min} respectively denote the expected return and variance of the portfolio with $(w_1 = 1 - s_{min}, w_2 = s_{min})$ then

$$\mu_{min} = \frac{\sigma_1\mu_2 - \sigma_2\mu_1}{\sigma_1 - \sigma_2} \quad \text{and} \quad \sigma^2_{min} = 0.$$

Since $s_{min} < 0$ (i.e. $w_2 < 0$), an investor can eliminate risk in the portfolio by taking a short position with asset a_2.

(b) When $\rho = -1$ and $\sigma_1 \le \sigma_2$. In this case we have

$$\sigma^2 = (1-s)^2\sigma_1^2 + s^2\sigma_2^2 - 2(1-s)s\sigma_1\sigma_2.$$

Thus, we have

$$s_{min} = \frac{\sigma_1}{\sigma_1 + \sigma_2} > 0$$

$$1 - s_{min} = \frac{\sigma_2}{\sigma_1 + \sigma_2} > 0$$

$$\mu_{min} = \frac{\sigma_1\mu_2 + \sigma_2\mu_1}{\sigma_1 + \sigma_2}$$

$$\sigma^2_{min} = 0.$$

Since $s_{min} > 0$ and $1 - s_{min} > 0$ hence the investor can eliminate the risk in the portfolio without restoring to short selling.

Case (ii) We now consider the second case when $-1 < \rho < 1$. Recalling relations (5.3) and (5.5), we have

$$\mu = (1-s)\mu_1 + s\mu_2,$$

and

$$\sigma^2 = (\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)s^2 - 2\sigma_1(\sigma_1 - \rho\sigma_2)s + \sigma_1^2,$$

which represents the parametric equation of a parabola in (σ^2, μ)-plane. Now

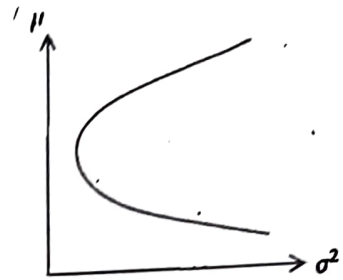

Fig. 5.4. (σ^2, μ)-graph for $-1 < \rho < 1$.

$$\frac{d\sigma^2}{ds} = 0 \Rightarrow s = \frac{\sigma_1^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}.$$

Also

$$\frac{d^2\sigma^2}{ds^2} = 2((\sigma_1 - \rho\sigma_2)^2 + \sigma_2^2(1 - \rho^2)) > 0.$$

Consequently $s_{min} = \frac{\sigma_1^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$, and the minimum value of σ^2 is given by

$$\sigma^2_{min} = \frac{\sigma_1^2\sigma_2^2(1-\rho^2)}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}.$$

Further, the corresponding expected portfolio return equals

$$\mu_{min} = (\mu_2 - \mu_1)s_{min} + \mu_1.$$

Remark 5.4.2 *It is important to take note of the following points.*

(i) The condition $-1 \le \rho < \frac{\sigma_1}{\sigma_2}$ *is equivalent to* $0 < s_{min} < 1$. *Thus the minimum risk can be achieved without short selling. Also, in this case*

$$\sigma^2_{min} = 0 \Leftrightarrow \rho = -1.$$

(ii) $\rho = \frac{\sigma_1}{\sigma_2} \Leftrightarrow s_{min} = 0 \Leftrightarrow \sigma^2_{min} = \sigma_1^2.$

(iii) The condition $\frac{\sigma_1}{\sigma_2} < \rho \le 1$ *is equivalent to* $s_{min} < 0$. *In this case the investor has taken a short position on asset* a_2 *in order to minimize the portfolio risk. Further, in this case*

$$\sigma^2_{min} = 0 \Leftrightarrow \rho = 1.$$

The entire theory of this section is summarized in Fig. 5.5.

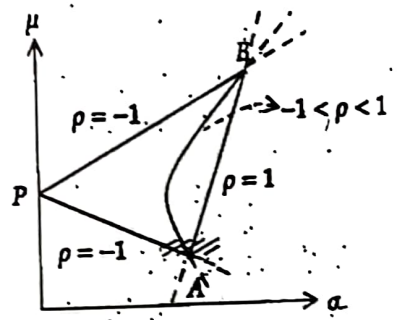

Fig. 5.5. Feasible region for two-asset problem.

The risk-return relation of two assets for various values of ρ provides us with a triangle APB. The points A and B signify undiversified portfolios. Since $-1 \le \rho \le 1$, ΔAPB specifies the limit of diversification. The risk-return relation for all values of ρ except ± 1 lie within this triangle. Here the bold portions of the graphs represent the case $0 \le s \le 1$. We verify the two-assets portfolio theory by considering the below given example.

Let A and B be two assets with expected returns 12% and 16% and standard deviation 16% and 20%, respectively. Let x_A and x_B denote the number of units of asset A and asset B, respectively, in a portfolio. Then we have the following table

x_A	x_B	μ	σ				
			$\rho = 1$	$\rho = 0{\cdot}5$	$\rho = 0$	$\rho = -0{\cdot}5$	$\rho = -1$
100	0	12·00	16·00	16·00	16·00	16·00	16·00
90	10	12·40	16·40	15·50	14·54	13·51	12·40
80	20	12·80	16·80	15·20	13·41	11·34	8·80
70	30	13·20	17·20	15·12	12·71	9·71	5·20
60	40	13·60	17·60	15·26	12·50	8·91	1·60
50	50	14·00	18·00	15·62	12·81	9·4	2·00
40	60	14·40	18·40	16·18	13·60	10·40	5·60
30	70	14·80	18·80	16·92	14·80	12·32	9·20
20	80	15·20	19·20	17·82	16·32	14·66	12·80
10	90	15·60	19·60	18·85	18·07	17·26	16·40
0	100	16·00	20·00	20·00	20·00	20·00	20·00

From the above table we observe the following.

(i) The two assets can be combined in such a way that the portfolio risk is less than the individual risks. For instance, when the assets are taken in the ratio 80 : 20 and $\rho = 0{\cdot}5$ then the risk in the portfolio is 15.20% whereas if the entire investment is put in asset A only or in asset B only then the risks involved are 16% and 20%, respectively. While if $\rho = -0 \cdot 5$ then 60 : 40 ratio can bring down the risk to 8.91%. It signifies that in many circumstances diversification of the portfolio is advisable for reducing the risk. The underline principle is thus that 'do not put all the eggs in one basket'.

(ii) For a given weight combination the risk reduces as ρ moves from 1 to -1.

(iii) When $\rho < 1$ then certain combinations of the assets are better than the others. For example, for $\rho = 0$ the 60:40 combination with $\mu = 13.60\%$ and $\sigma = 12.50\%$ is better than the 70:30 combination with $\mu = 13.20\%$ and $\sigma = 12.71\%$. For $\rho = 0.5$, the 70 : 30 combination yields return 13 · 20% and risk 15.12% which is better than the 80 : 20 ratio that gives lower return 12.80% with higher risk 15.20%

From the above discussion we thus conclude that by choosing appropriate ratio of investment between two assets, the risk can be reduced considerably.

5.5 Multi Asset Portfolio Optimization

The weights of the various assets $a_1, \ldots, a_n$ in the portfolio are written in the vector form $w^T = (w_1, \ldots, w_n)$. Let $e^T = (1, \ldots, 1) \in R^n$. Then $w_1 + \cdots + w_n = 1$

can be expressed as $e^T w = 1$. Let $m^T = (\mu_1, \ldots, \mu_n)$ be the expected return vector of the portfolio, where, $\mu_i = E(r_i)$ $(i = 1, \ldots, n)$, and $C = [c_{ij}]$ denotes the $n \times n$ variance-covariance matrix with entries $c_{ij} = Cov(r_i, r_j)$ $(i, j = 1, \ldots, n)$. Note that $c_{ii} = \sigma_i^2$ $(i = 1, \ldots, n)$. Obviously C is a symmetric matrix. Now the expected return μ of the portfolio is given by

$$\mu = E\left(\sum_{i=1}^{n} w_i r_i\right) = \sum_{i=1}^{n} w_i \mu_i = m^T w,$$

and the risk σ^2 of the portfolio is

$$\sigma^2 = Var\left(\sum_{i=1}^{n} w_i r_i\right) = \sum_{i,j=1}^{n} c_{ij} w_i w_j = w^T C w. \tag{5.8}$$

Here C is certainly positive semidefinite. In practice, it is also assumed to be positive definite (and hence invertible) because the minimum risk of a general n-asset portfolio is rarely zero.

During a portfolio selection, every investor is faced with a choice of either minimizing a risk with respect to certain value of return or maximizing a return with respect to certain value of risk. Now, from (5.8), we observe that the portfolio risk σ^2 depends on three factors, viz.,

(i) risk of each individual asset;
(ii) coefficient of correlation between assets returns;
(iii) weights of the assets.

Out of these contributing factors, the only factor that an investor can control is the weights of the assets. Our main aim is to examine the optimal choice of these weights.

The Feasible Region of a Portfolio Problem

Let $W = \{w \in \mathbb{R}^n : e^T w = 1\}$ be the collection of all portfolios. We have earlier observed that every portfolio $w \in W$ corresponds to a point in the (σ, μ)-plane, say $(\sigma^{(w)}; \mu^{(w)})$. The set $\{(\sigma^{(w)}, \mu^{(w)}) \mid w \in W\}$ is called the *feasible region* or *feasible set* of the given portfolio optimization problem. It should be of interest to know the geometry of the feasible region and that is the point of our discussion now.

Consider the n-dimensional hyperplane $e^T w = 1$ in which the weight vector w resides. Let f be the mapping that takes each weight vector in the weight hyperplane to the corresponding portfolio point in the (σ, μ)-plane. We try and

find the image of any straight line in the weight hyperplane $e^T w = 1$ under the mapping f. For this we note that the parametric equation of any line in the weight hyperplane is of the form

$$\begin{aligned} l(\xi) &= (s_1\xi + b_1, \ldots, s_n\xi + b_n)^T \\ &= \xi s + b, \qquad -\infty < \xi < \infty, \end{aligned}$$

where $s = (s_1, \ldots, s_n)^T$ and $b = (b_1, \ldots, b_n)^T$. Let w be any point on this line. Then

$$\begin{aligned} \mu &= m^T w \\ &= m^T(\xi s + b) \\ &= \xi(m^T s) + (m^T b). \end{aligned}$$

Let $\alpha = (m^T s)^{-1}$, $\beta = -(m^T b)(m^T s)^{-1}$. Then, $\xi = \alpha\mu + \beta$. Moreover,

$$\begin{aligned} \sigma^2 &= w^T C w \\ &= (\xi s + b)^T C(\xi s + b) \\ &= (s^T C s)\xi^2 + (s^T C b + b^T C s)\xi + b^T C b \\ &\equiv \gamma\xi^2 + \delta\xi + \eta. \end{aligned}$$

Substituting the value of ξ we get

$$\sigma^2 = \gamma(\alpha\mu + \beta)^2 + \delta(\alpha\mu + \beta) + \eta. \tag{5.9}$$

As ξ varies from $-\infty$ to ∞, the ordered pair (σ^2, μ) traces out a parabola given by (5.9) which lies in (σ, μ)-plane with axis parallel to σ-axis and sides open on the right.

We are actually interested in (σ, μ)-graph. Taking the square root of σ^2, the resulting curve is

$$\sigma = \sqrt{\gamma(\alpha\mu + \beta)^2 + \delta(\alpha\mu + \beta) + \eta}. \tag{5.10}$$

This curve is called a *Markowitz curve*. Thus, each line in the weight hyperplane is mapped onto a Markowitz curve. This phenomena is depicted in Fig. 5.6.

Remark 5.5.1 *Here it is important to note that the Markowitz curve (5.10) is not a parabola. In fact the main difference between the parabola (5.9) and the Markowitz curve (5.10) in (σ, μ)-graph is that a tangent can be drawn to the parabola (5.9) from any point on the μ-axis, whereas the Markowitz curve behaves almost as a straight line as $\mu \to \infty$, thereby, it is not possible to draw a tangent to.*

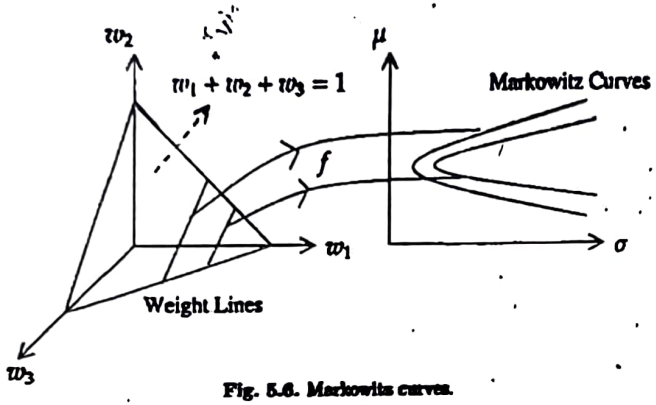

Fig. 5.6. Markowitz curves.

the Markowitz curve as $\mu \to \infty$. This difference may not sound significant right now but it plays a vital role when the portfolio consists of one risk-free asset as well. We shall be addressing to this type of portfolio in the next section. For the current discussion we have assumed that all the assets in the portfolio are risky.

Remark 5.5.2 *As we cover the weight hyperplane by taking weight lines, we trace a family of Markowitz curves in the (σ, μ)-plane. It is not difficult to get convinced that this region in the (σ, μ)-plane is going to be a solid region and its shape will be like a bullet, which is appropriately called the Markowitz bullet.*

There is another much simpler way to get convinced that the feasible set in the (σ, μ) plane should be of the type as described above. Suppose we have three assets a_1, a_2 and a_3. Let these be represented by three points $A : (\sigma_{a_1}, \mu_{a_1})$, $B : (\sigma_{a_2}, \mu_{a_2})$ and $C : (\sigma_{a_3}, \mu_{a_3})$ in the (σ, μ)-plane. As already seen in the case of two asset portfolio diagram, any two assets when combined to form portfolios give rise to a curved line or a straight line between them. Whether the line will be straight or curved depends upon the correlation coefficient between them. The three (curved) lines between the possible three pairs are shown in the left hand side figure of Fig. 5.7. Now if we take any asset D formed by the combination of B and C, then again combination of assets A and D will give us a (curved) line connecting A and D and the process continues. Eventually we get the feasible region which is a solid two dimensional region in the (σ, μ)-plane. This region is depicted in the right hand side figure of Fig. 5.7. Further the feasible region is convex to the left, i.e. given any two points in the region, the straight line connecting them does not cross the left boundary.

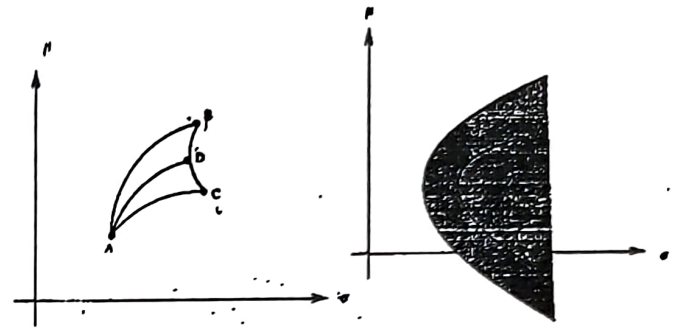

Fig. 5.7. Feasible region for three asset problem.

In the feasible region as depicted in Fig. 5.7, the outer region corresponds to the case when shorting is allowed and the inner region corresponds to the case when shorting is not allowed.

5.6 The Minimum Variance Set, The Minimum Variance Point and the Efficient Frontier

Consider the feasible set of a given portfolio optimization problem as depicted in (σ, μ)-diagram (Fig. 5.8). Its left boundary is called the *minimum variance set* because for any level of the mean rate return, the feasible point with the smallest variance (or standard deviation) is the corresponding left boundary point. This situation is illustrated in Fig. 5.8.

Here when the return is at level μ_0, the feasible point with the smallest variance is P_0. Similarly for return levels μ_1 and μ_2 we get the respective points P_1 and P_2. Amongst all such points (σ, μ) lying on the left boundary of the feasible region, there is a point P_{min} which has the least variance. This point is called the *minimum variance point*.

Mathematically, to find the minimum variance point we need to solve the following risk minimization problem

$$\begin{aligned} \text{Min} \quad & \sigma^2 = w^T C w \\ \text{subject to} \quad & \\ & e^T w = 1. \end{aligned}$$

The following theorem gives a closed form solution of the above problem

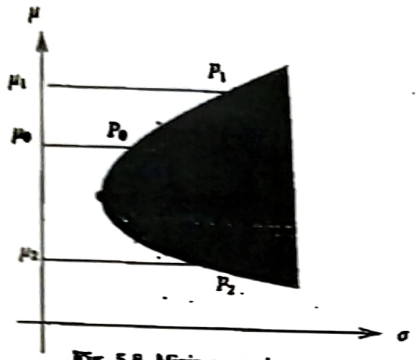

Fig. 5.8. Minimum variance set.

Theorem 5.6.1 *A portfolio with minimum risk has weights given by*

$$w = \frac{C^{-1}e}{e^T C^{-1} e}.$$

Proof. We desire to solve the following optimization problem

$$\text{Min} \quad \sigma^2 = w^T C w$$

subject to

$$e^T w = 1. \tag{5.11}$$

Using the Lagrange multiplier $\lambda \in \mathbb{R}$, we minimize the Lagrangian

$$L(w, \lambda) = w^T C w + \lambda(1 - e^T w). \tag{5.12}$$

Note that λ is unrestricted in sign because the constraint in the risk minimization problem is an equation $e^T w = 1$. Now, differentiating (5.12) with respect to w, we obtain

$$2w^T C - \lambda e^T = 0 \implies w = \frac{\lambda}{2} C^{-1} e.$$

Using (5.11), we get

$$e^T \left(\frac{\lambda}{2} C^{-1} e\right) = 1 \implies \frac{\lambda}{2} = \frac{1}{e^T C^{-1} e}.$$

Thus the requisite result follows. □

Markowitz Efficient Frontier

Looking at the minimum variance set in the (σ, μ)-diagram, we observe that for a given level of risk, (say σ_0), there are two values of returns ($\mu_0^{(L)}$ and $\mu_0^{(U)}$ with $\mu_0^{(U)} > \mu_0^{(L)}$). Since our aim in portfolio optimization is to maximize return for a given level of risk, we shall obviously choose the larger of these two returns, i.e $\mu_0^{(U)}$. Therefore in the minimum variance set, it is only the upper half which is of importance for investment. This upper half portion of the minimum variance set is called the *Markowitz efficient frontier*. A typical efficient frontier is illustrated in Fig. 5.9.

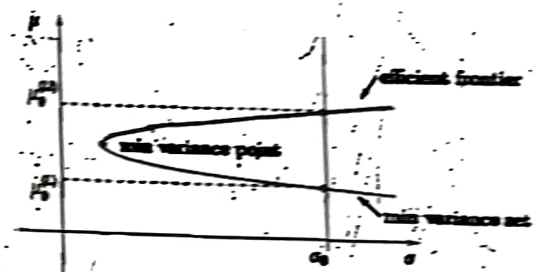

Fig. 5.9. Efficient frontier.

We now formalize the definition of efficient frontier and outline a procedure to determine the same.

Definition 5.6.1 (Dominating Point and Efficient Frontier) *Let $P : (\sigma_1, \mu_1)$ and $Q : (\sigma_2, \mu_2)$ be two points in the feasible region of a given portfolio optimization problem. Then the point P is said to dominate Q if $\sigma_1 \le \sigma_2$ and $\mu_1 \ge \mu_2$.*

Definition 5.6.2 (Non-dominated Point and Efficient Frontier) *Let A be a point in the feasible region. The point A is said to be a non-dominated point if there is no point P in the feasible region which dominates A. The set of all non-dominated points in the feasible region is called its efficient frontier.*

Geometrically the efficient frontier is the upper portion of the minimum variance set as depicted in Fig. 5.9. Obviously for an investor it is only the efficient frontier which will be of interest and therefore we should explore methodologies to determine the same.

Before we proceed in that direction we notice that in many cases, it is more likely that an investor provides a fixed value of the expected return, say μ, that

he/she desires to achieve. Therefore the investor's problem is to decide the right investment strategy to obtain the return μ with the minimum risk. We look at this scenario in the result to follow.

Theorem 5.6.2 *For a given expected return μ, the portfolio with minimum risk has weights given by*

$$w = \frac{det\begin{pmatrix} \mu & m^TC^{-1}e \\ 1 & e^TC^{-1}e \end{pmatrix} C^{-1}m + det\begin{pmatrix} m^TC^{-1}m & \mu \\ e^TC^{-1}m & 1 \end{pmatrix} C^{-1}e}{det\begin{pmatrix} m^TC^{-1}m & m^TC^{-1}e \\ e^TC^{-1}m & e^TC^{-1}e \end{pmatrix}}. \tag{5.13}$$

Proof. We wish to solve the following quadratic programming problem

$$\text{Min} \quad \sigma^2 = \frac{1}{2}w^TCw$$

subject to

$$\begin{aligned} m^Tw &= \mu \\ e^Tw &= 1. \end{aligned} \tag{5.14}$$

This is a convex quadratic programming problem with unrestricted variable vector w. Therefore we define the Lagrangian

$$L(w, \alpha, \beta) = \frac{1}{2}w^TCw + \alpha(\mu - m^Tw) + \beta(1 - e^Tw)$$

where $\alpha, \beta \in \mathbf{R}$ are Lagrange multipliers. Now solving $\frac{\partial L}{\partial w} = 0$ gives

$$w^TC - \alpha m^T - \beta e^T = 0$$

i.e.

$$w = C^{-1}(\alpha m + \beta e). \tag{5.15}$$

Substituting the value of w in (5.14), we get

$$\begin{aligned} (m^TC^{-1}m)\alpha + (m^TC^{-1}e)\beta &= \mu \\ (e^TC^{-1}m)\alpha + (e^TC^{-1}e)\beta &= 1. \end{aligned}$$

Solving the above two equations for α and β, we obtain

$$\alpha = \frac{det\begin{pmatrix} \mu & m^TC^{-1}e \\ 1 & e^TC^{-1}m \end{pmatrix}}{det\begin{pmatrix} m^TC^{-1}m & m^TC^{-1}e \\ e^TC^{-1}m & e^TC^{-1}e \end{pmatrix}}, \quad \beta = \frac{det\begin{pmatrix} m^TC^{-1}m & \mu \\ e^TC^{-1}m & 1. \end{pmatrix}}{det\begin{pmatrix} m^TC^{-1}m & m^TC^{-1}e \\ e^TC^{-1}m & e^TC^{-1}e \end{pmatrix}}.$$

Substituting these values in the expression (5.15) for w, we get the required expression (5.13).

□

Now to generate the entire efficient frontier we need to solve problems of type (5.14) for all values of $\mu \in \mathbf{R}$. This is almost impossible. But then an extremely interesting observation is made here. Recall from (5.14) and (5.15) that, for a given value of return μ, the points of minimum variance must satisfy the following system of $(n+2)$ linear equations in $(n+2)$ unknowns $w \in \mathbf{R}^n$, $\alpha \in \mathbf{R}$, $\beta \in \mathbf{R}$

$$\begin{aligned} w^TC - \alpha m^T - \beta e^T &= 0 \\ m^Tw &= \mu \\ e^Tw &= 1. \end{aligned} \tag{5.16}$$

Suppose we solve the system (5.16) for two distinct values of expected return μ, say $\mu^{(1)}$ and $\mu^{(2)}$. Let the two solutions be $(w^{(1)})^T = \left((w_1^{(1)}, \ldots, w_n^{(1)}), \alpha^{(1)}, \beta^{(1)}\right)^T$ and $(w^{(2)})^T = \left((w_1^{(2)}, \ldots, w_n^{(2)}), \alpha^{(2)}, \beta^{(2)}\right)^T$, respectively. Then it is simple to verify that the combination portfolio, $\lambda(w^{(1)}, \alpha^{(1)}, \beta^{(1)})^T + (1-\lambda)(w^{(2)}, \alpha^{(2)}, \beta^{(2)})^T$, $\lambda \in \mathbf{R}$, is also a solution of the system (5.16) corresponding to the return $\lambda\mu^{(1)} + (1-\lambda)\mu^{(2)}$. Therefore, in order to solve (5.16) for every value of μ, one is only required to solve it for two distinct values of μ and then form the combination of the two solutions. Thus, the knowledge of two distinct portfolios yielding the minimum variances is sufficient to generate the entire minimum variance set. This result is significant from investor's point of view. Also, it demonstrates a very good application of Karush-Kuhn-Tucker optimality conditions. The result is known as the *two-fund theorem*.

Theorem 5.6.3 (Two Fund Theorem) *Two efficient portfolios can be established so that any other efficient portfolio can be duplicated, in terms of mean and variance, as a linear combination of these two assets. In other words, it says that, an investor seeking an efficient portfolio need to invest only in the combination of these two assets.*

The most convenient way to get two solutions of (5.16) is to assign two distinct values to α and β, and then work out the solutions. The most convenient choices are $\alpha = 1$, $\beta = 0$ and $\alpha = 0$, $\beta = 1$. The above discussion is illustrated through the below given example.

Example 5.6.1 *Consider three risky assets with the variance-covariance matrix and expected returns as follows.*

variance - covariance matrix(C)			return(M)
2	1	0	0·4
1	2	1	0·8
0	1	2	0·8

Find two portfolios yielding the minimum variance. Also, determine the expected returns from these two portfolios. Using the two fund theorem, construct the portfolio giving the return of 33·4% with minimum risk.

Solution Taking $\alpha = 0$, $\beta = 1$ in (5.16), we need to solve: $\sum_{j=1}^{3} \sigma_{ij} v_j^{(1)} = 1$ $(i = 1, 2, 3)$, resulting in the following system of linear equations

$$\begin{aligned} 2v_1^{(1)} + v_2^{(1)} &= 1 \\ v_1^{(1)} + 2v_2^{(1)} + v_3^{(1)} &= 1 \\ v_2^{(1)} + 2v_3^{(1)} &= 1. \end{aligned}$$

The solution is $V^{(1)} = (0 \cdot 5, 0, 0 \cdot 5)^T$. We next take $\alpha = 1$, $\beta = 0$ in (5.16), to solve $\sum_{j=1}^{3} \sigma_{ij} v_j^{(2)} = \mu_i$ $(i = 1, 2, 3)$, i.e.

$$\begin{aligned} 2v_1^{(2)} + v_2^{(2)} &= 0 \cdot 4 \\ v_1^{(2)} + 2v_2^{(2)} + v_3^{(2)} &= 0 \cdot 8 \\ v_2^{(2)} + 2v_3^{(2)} &= 0 \cdot 4. \end{aligned}$$

The solution of the above system is $V^{(2)} = (0 \cdot 1, 0 \cdot 2, 0 \cdot 3)^T$.

Note that $\sum_{j=1}^{3} v_j^{(1)} = 1$, thus we take $w^{(1)} = V^{(1)} = (0.5, 0, 0.5)$. Normalizing $V^{(2)}$, we get, $w^{(2)} = (1/6, 1/3, 1/2)^T$ (so that, $\sum_{j=1}^{3} w_j^{(2)} = 1$). The corresponding returns from the two portfolios with weights $w^{(1)}$ and $w^{(2)}$ are $\mu^{(1)} = m^T w^{(1)} = 0 \cdot 6$ and $\mu^{(2)} = m^T w^{(2)} = 0 \cdot 733$, respectively.

Next, we consider the case when the investor desired a return of $\mu = 0 \cdot 334$ at minimum risk. It is simple to check that for $\lambda = 3$, $\lambda\mu^{(1)} + (1 - \lambda)\mu^{(2)} = 0 \cdot 334$.

Thus, by the two fund theorem the requisite portfolio is given by $w = \lambda w^{(1)} + (1 - \lambda) w^{(2)} = (7/6, -2/3, 1/2)$. Observe that the second asset has a short position in this portfolio. The variance corresponding to this portfolio is

$$w^T C w = \begin{pmatrix} 7/6 & -2/3 & 1/2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 7/6 \\ -2/3 \\ 1/2 \end{pmatrix} = 2/9.$$

□

5.7 Capital Asset Pricing Model (CAPM)

So far we have assumed that all assets in the portfolio are risky assets. So it is natural to query as to what would be the scenario if one risk-free asset is included in the portfolio? In this section we make an attempt to study this aspect of portfolio selection.

Consider a portfolio with n risky assets, $a_1, \ldots, a_n$ with weights $w_1, \ldots, w_n$ and one risk-free asset a_{rf} with weight w_{rf}. Then

$$w_{risky} + w_{rf} = \sum_{i=1}^{n} w_i + w_{rf} = 1. \tag{5.17}$$

Also, the expected return and the variance associated with this portfolio are respectively given by

$$\mu = \sum_{i=1}^{n} w_i \mu_i + w_{rf} \mu_{rf} = \mu_{risky} + w_{rf} \mu_{rf},$$

and

$$\sigma^2 = Var\left(\sum_{i=1}^{n} w_i r_i + w_{rf} r_{rf}\right) = Var\left(\sum_{i=1}^{n} w_i r_i\right) = \sigma^2_{risky}.$$

If we remove the risk-free asset from the portfolio and readjust the weights of the risky assets so that their sum remain 1, the resultant portfolio so obtained is referred to as the *derived risky portfolio*. We use μ_{der} and σ^2_{der} to denote the derived risky portfolio expected return and risk, respectively. Then,

$$
\begin{aligned}
\mu &= \sum_{i=1}^{n} w_i\mu_i + w_{rf}\mu_{rf} \\
&= w_{risky}\left(\sum_{i=1}^{n} \frac{w_i}{w_{risky}}\mu_i\right) + w_{rf}\mu_{rf} \qquad (5.18)\\
&= w_{risky}\mu_{der} + w_{rf}\mu_{rf} \\
&= w_{risky}\mu_{der} + (1 - w_{risky})\mu_{rf} \\
&= w_{risky}(\mu_{der} - \mu_{rf}) + \mu_{rf}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
\sigma^2 &= Var\left(\sum_{i=1}^{n} w_i r_i\right) \\
&= w_{risky}^2 Var\left(\sum_{i=1}^{n} \frac{w_i}{w_{risky}} r_i\right) \qquad (5.19)\\
&= w_{risky}^2 \sigma_{der}^2.
\end{aligned}
$$

which gives $w_{risky} = \frac{\sigma}{\sigma_{der}}$. From (5.18) and (5.19) we get

$$\mu = \mu_{rf} + \left(\frac{\mu_{der} - \mu_{rf}}{\sigma_{der}}\right)\sigma, \qquad (5.20)$$

which is an equation of the line joining $(0, \mu_{rf})$ and $(\sigma_{der}, \mu_{der})$ in the (σ, μ)-graph.

Now, for a given risk σ, if we choose various weight combinations of risk-free asset and risky assets satisfying (5.17), we generate different lines represented by (5.20) in (σ, μ)-graph. Obviously, among all such lines, the line that produces the point with highest expected return for a given risk is tangent to the upper portion of the Markowitz bullet. This is illustrated in Fig. 5.10.

Definition 5.7.1 (Capital Market Line) *Among all the lines (5.20) for various weight combinations of risk-free asset and risky assets, the line giving the highest return for a given risk is called the capital market line.*

Definition 5.7.2 (Market Portfolio) *The point on the Markowitz bullet where the capital market line is tangential is said to represent the market portfolio.*

Theoretically, the market portfolio must contain all risky assets, for if some asset is not in it then it will wither and die. Since the market portfolio contains all risky assets, it is a completely diversified portfolio with no unsystematic risk.

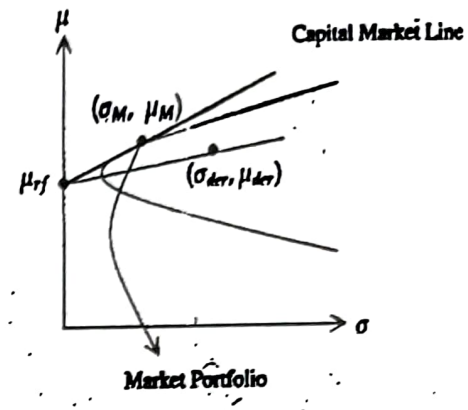

Fig. 5.10. Capital market line.

The basic idea of the capital asset pricing model (CAPM) is that an investor can improve the risk-expected return balance by investing partially in a portfolio of risky assets and partially in a risk-free asset. All investors will end up with portfolios along the capital market line as all *efficient portfolios* lie along this line while any other combination of risk-free asset and risky assets, except those which are efficient, lies below the capital market line. It is thus important to observe that all investors will hold combinations of only two assets, viz. the market portfolio M and a risk-free asset. This fund scenario is summarized in the following theorem.

Theorem 5.7.1 (One-Fund Theorem) *There exists a single portfolio, namely the market portfolio M, of risky assets such that any efficient portfolio can be constructed as a linear combination of the market portfolio M and the risk-free asset.*

Unlike with the two-fund theorem where any two efficient portfolios are sufficient, in this case, the tangent portfolio is a specific portfolio.

Theorem 5.7.2 *For any expected risk-free return μ_{rf}, the weight vector $\underline{w}_M$ of the market portfolio is given by*

$$w_M = \frac{C^{-1}(m - \mu_{rf}e)}{e^T C^{-1}(m - \mu_{rf}e)}.$$

Proof. From Fig. 5.10, we observe that for any point (σ, μ) in the Markowitz bullet, the slope of the line joining $(0, \mu_{rf})$ and (σ, μ) is

$$s = \frac{\mu - \mu_{rf}}{\sigma} = \frac{\sum_{i=1}^{n} \mu_i w_i - \mu_{rf}}{(\sum_{i,j=1}^{n} c_{ij} w_i w_j)^{\frac{1}{2}}}.$$

For the line joining $(0, \mu_{rf})$ to (σ, μ) to be a tangent line to the Markowitz bullet, we need to solve the following optimization problem

$$\begin{aligned} \text{Max} \quad & \frac{m^T w - \mu_{rf}}{(w^T C w)^{1/2}} \\ \text{subject to} \quad & e^T w = 1. \end{aligned} \tag{5.21}$$

The Lagrange function $L : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ for the problem (5.21) is described as

$$L(w, \lambda) = \frac{m^T w - \mu_{rf}}{(w^T C w)^{1/2}} + \lambda(1 - e^T w).$$

Now, solving (5.21) is same as maximizing $L(w, \lambda)$. So, $\nabla_w L(w, \lambda) = 0$, giving

$$\frac{1}{w^T C w}\left((w^T C w)^{1/2} m - (m^T w - \mu_{rf})(w^T C w)^{-1/2} C w\right) = \lambda e.$$

The above expression can be rewritten as

$$\sigma m - (\mu - \mu_{rf})\frac{Cw}{\sigma} = \lambda \sigma^2 e.$$

Multiplying by σ, we obtain

$$\sigma^2 m - (\mu - \mu_{rf}) C w = \lambda \sigma^3 e, \tag{5.22}$$

which in turn yields

$$\sigma^2 w^T m - (\mu - \mu_{rf}) w^T C w = \lambda \sigma^3 w^T e.$$

Since $e^T w = 1$, $\mu = w^T m$, and $\sigma = w^T C w$, we get

$$\lambda = \frac{\mu_{rf}}{\sigma}. \tag{5.23}$$

The requisite value of weight vector w_M now follows from (5.22) and (5.23). □

Remark 5.7.1 *Suppose the market portfolio (σ_M, μ_M) is known. Then, from (5.20), the equation of the capital market line is given by*

$$\boxed{\mu = \mu_{rf} + \left(\frac{\mu_M - \mu_{rf}}{\sigma_M}\right)\sigma.}$$

If the investor is willing to take a positive risk σ, he/she can earn an additional return $\left(\frac{\mu_M - \mu_{rf}}{\sigma_M}\right)\sigma$ over and above the risk-free return μ_{rf} to compensate the risk taken by him/her. Therefore sometimes the quantity $\left(\frac{\mu_M - \mu_{rf}}{\sigma_M}\right)$ is called the price of risk.

Example 5.7.1 *Suppose a portfolio comprises of one risk-free asset with return 0.5, and three mutually independent risky assets with expected returns 1, 2, 3 and variances 1, 1, 1, respectively. Determine the equation of the capital market line.*

Solution The given information gives $m^T = (\mu_1, \mu_2, \mu_3) = (1, 2, 3)$, $\mu_{rf} = 0.5$, $C = [\sigma_{ij}] = I_{3\times 3}$, $e^T = (1, 1, 1)$. Therefore, the weight vector of the market portfolio is given by

$$w_M = \frac{C^{-1}(m - \mu_{rf} e)}{e^T C^{-1}(m - \mu_{rf} e)} = \begin{pmatrix} 1/9 \\ 1/3 \\ 5/9 \end{pmatrix}.$$

Consequently, the expected return and variance of the market portfolio are

$$\mu_M = m^T w_M = \frac{22}{9}, \quad \sigma_M = ((w_M)^T C w_M)^{1/2} = \frac{\sqrt{35}}{9}.$$

Thus, the equation of the capital market line is

$$\begin{aligned} \mu &= \mu_{rf} + \left(\frac{\mu_M - \mu_{rf}}{\sigma_M}\right)\sigma \\ &= \frac{1}{2} + \frac{\sqrt{35}}{2}\sigma. \end{aligned}$$

□

In practice there are certain assets which are listed in the stock called *index stocks*. These limited assets are significant ones that can capture the pulse of the whole market. The most regularly quoted market indices are broad-base indices comprising of the stocks of large companies listed on a nation's largest stock exchanges, such as the American Dow Jones Industrial Average and S&P 500 Index, the British FTSE 100, the French CAC 40, the Japanese Nikkei 225. The

Bombay Stock Exchange is the largest in India, with over 6000 stocks listed and it accounts for over two thirds of the total trading volume in the country. The index stocks finally help us to compute the market portfolio (σ_M, μ_M). The knowledge of the market portfolio yields the equation of capital market line, see Remark 5.7.1. Now suppose an investor P is willing to take risk σ_P. Then for this risk, the expected return μ_P is maximum if the point (σ_P, μ_P) lies on the capital market line. Thus,

$$\mu_P = \mu_{rf} + \left(\frac{\mu_M - \mu_{rf}}{\sigma_M}\right)\sigma_P.$$

If we let $w_P = \dfrac{\sigma_P}{\sigma_M}$ then

$$\mu_P = w_P\mu_M + (1 - w_P)\mu_{rf}.$$

Remark 5.7.2 *The above relation suggests that if an investor is willing to take a risk σ_P, then he/she should invest $w_P = \dfrac{\sigma_P}{\sigma_M}$ proportion of investment in index fund and $(1 - w_P)$ proportion of investment in the risk-free investment schemes.*

We now aim to examine how an individual asset behaves with respect to the market portfolio. For this, we attempt to build a relationship between the expected return along with the risk of an individual asset with the market portfolio. This gives the CAPM formula (5.24).

Theorem 5.7.3 *Suppose the market portfolio is (σ_M, μ_M). The expected return of an asset a_i is given by*

$$\mu_i = \mu_{rf} + \beta_i(\mu_M - \mu_{rf}), \quad \text{where} \quad \beta_i = \frac{\mathrm{Cov}(r_i, r_M)}{\sigma_M^2}. \tag{5.24}$$

Proof. Suppose an investor portfolio comprises of asset a_i with weight w and the market portfolio M with weight $1 - w$. Then the expected return and risk of the investor portfolio are respectively given by

$$\begin{aligned} \mu &= w\mu_i + (1 - w)\mu_M \\ \sigma^2 &= w^2\sigma_i^2 + (1 - w)^2\sigma_M^2 + 2\rho w(1 - w)\sigma_i\sigma_M \end{aligned} \tag{5.25}$$

where ρ is the coefficient of correlation between the returns of asset a_i and the market portfolio M.

As w varies, these values trace out a curve in the (σ, μ)-graph. It can be observed from Fig. 5.11 that as w passes through zero, the capital market line becomes tangent to the curve at M. This tangency condition can be translated into the

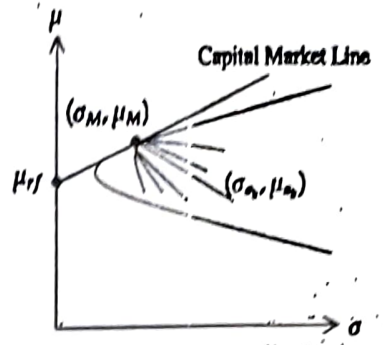

Fig. 5.11. Market portfolio.

condition that the slope of the curve is equal to the slope of the capital market line at M (corresponding to $w = 0$).

Now the slope of the curve at M is given by

$$\left.\frac{d\mu}{d\sigma}\right|_{(w=0)} = \left.\frac{d\mu}{dw}\frac{dw}{d\sigma}\right|_{(w=0)}$$
$$= (\mu_i - \mu_M)\left.\frac{dw}{d\sigma}\right|_{(w=0)}$$

Differentiating (5.25) with respect to w and computing its value at $w = 0$, we get

$$\left.\frac{d\sigma}{d\mu}\right|_{(w=0)} = \left.\frac{w\sigma_i^2 - (1 - w)\sigma_M^2 + \rho\sigma_i\sigma_M(1 - 2w)}{\sigma}\right|_{(w=0)}$$
$$= \frac{\sigma_{iM} - \sigma_M^2}{\sigma_M}, \qquad \sigma_{iM} = \rho\sigma_i\sigma_M.$$

Consequently,

$$\left.\frac{d\mu}{d\sigma}\right|_{(w=0)} = \frac{(\mu_i - \mu_M)\sigma_M}{\sigma_{iM} - \sigma_M^2} \tag{5.26}$$

As discussed above, the slope of the curve needs to be equal to the slope of the capital market line at M, thereby yielding that

$$\frac{\mu_M - \mu_{rf}}{\sigma_M} = \left.\frac{d\mu}{d\sigma}\right|_{(w=0)}.$$

The above relation along with (5.26), on simplification, yields

$$\mu_i = \mu_{rf} + \left(\frac{\mu_M - \mu_{rf}}{\sigma_M^2}\right)\sigma_{iM}$$
$$= \mu_{rf} + \beta_i(\mu_M - \mu_{rf}).$$

□

Remark 5.7.3 *Here, $\beta_i = \frac{\sigma_{iM}}{\sigma_M^2}$ is called the beta of an asset. Note that, for the market portfolio, $\beta_M = 1$. Beta is generally calculated for individual assets using regression analysis. As can be observed, beta measures an asset's volatility or risk in relation to the rest of the market. It is thus appropriately referred to as financial elasticity or correlated relative volatility, and it is all what is required to be known about the asset's risk characteristics in CAPM formula. In other words, an investor ready to bear some systematic risk gets rewarded for it. For instance, if $\beta_i = 2$, it indicates that the ith asset return is expected to increase (decrease) by 2% when the market increases (decreases) by 1%. Equivalently, if the market return fluctuates over a specific range of values, the asset returns will fluctuate over a larger range of values. Thus, the market risk is magnified in the asset risk.*

Remark 5.7.4 *If we take μ_{rf} as a base, then the above theorem states that the expected return of a particular asset over the risk-free asset (namely $(\mu_i - \mu_{rf})$) is proportional to that of the market (namely $(\mu_M - \mu_{rf})$) and the constant of proportionality is β_i. The CAPM essentially tells that the expected excess rate of return of an assets is directly proportional to its covariance with the market. So it is important to understand this covariance or equivalently the beta of the asset.*

If $\beta_i = 0$, the given asset is completely uncorrelated with the market and therefore CAPM gives $\mu_i = \mu_{rf}$. At first this looks surprising because it tells that no matter how risky the given asset is (even if σ_i is very large) the expected rate of return of this asset cannot be improved over the base point μ_{rf} and therefore there is no premium for risk. The main reason being that as the given asset is uncorrelated with the market its risk can be diversified away by having a small amount of this asset in the portfolio.

If $\beta_i < 0$ then CAPM gives $\mu_i < \mu_{rf}$. This again looks odd because it says that even though asset risk σ_i may be large, its expected rate of return is less than the base μ_{rf}. But then if, this asset is combined with the market it reduces the overall portfolio risk. Therefore the investor may be willing to have this asset in the portfolio as it has the capability of reducing its overall risk. Such an asset is expected to do well even when other assets are not doing that well and therefore it is called an insurance.

The above discussion suggests that though for a portfolio an appropriate measure of risk is σ but for an individual asset the proper measure of risk is its beta. Thus there is a paradigm shift in understanding the risk of an asset.

Example 5.7.2 *Let the risk-free rate μ_{rf} be 8% and the market has $\mu_M = 12\%$ and $\sigma_M = 15\%$. Let an asset a be given which has covariance of 0.045 with the market. Determine the expected rate of return of the given asset.*

Solution From the given data we have $\beta_a = \frac{0 \cdot 045}{(0 \cdot 015)^2} = 2$. Then CAPM gives

$$\mu_a = \mu_{rf} + \beta(\mu_M - \mu_{rf}) = 0.08 + 2(0.12 - 0.08) = 0 \cdot 16.$$

Therefore the expected rate of return of the given asset is 16%.

□

Definition 5.7.3 (Beta of the Portfolio) *The overall β of the portfolio is the weighted average of the betas of the individual assets in the portfolio, with the weights being those that define the portfolio, i.e. $\beta = \sum_{i=1}^{n} w_i\beta_i$.*

The above formula is immediate because for the portfolio $w = (w_1, w_2, \ldots, w_n)$, the rate of return is $\sum_{i=1}^{n} w_i r_i$ and $Cov(r, r_M) = \sum_{i=1}^{n} w_i Cov(r_i, r_M) = \sum_{i=1}^{n} w_i\sigma_{iM} = \sum_{i=1}^{n} \rho\sigma_i\sigma_M$.

Definition 5.7.4 (Security Market Line) *A linear equation*

$$\mu = \mu_{rf} + \beta(\mu_M - \mu_{rf}), \quad \text{where } \beta = \frac{Cov(r, r_M)}{\sigma_M^2}$$

that describes the expected return for all assets in the market is called the security market line.

The security market line highlights the essence of CAPM formula. It says that under the equilibrium conditions assumed by CAPM, all portfolio investments lie along the security market line in the beta-return space. It emphasizes that the risk of an asset is a function of its covariance with the market, or equivalently a function of its beta. The security market line is depicted in bold in Fig. 5.12

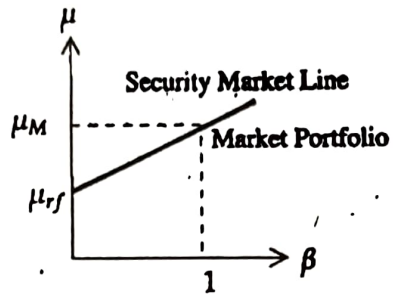

Fig. 5.12. Security market line.

CAPM as a Pricing Formula

We now give another interpretation of CAPM as a pricing formula. Let an asset be purchased at price P and later sold at price Q. Then the rate of return is $r = \frac{(Q-P)}{P}$. Here P is known but Q is random. If we write $E(Q) = \bar{Q}$, then the CAPM formula gives

$$\frac{\bar{Q}-P}{P} = \mu_{r_f} + \beta(\mu_M - \mu_{r_f})$$

i.e.

$$P = \frac{\bar{Q}}{1+\mu_{r_f}+\beta(\mu_M-\mu_{r_f})}. \tag{5.27}$$

Here β is the beta of the given asset.

The formula (5.27) can be viewed as a pricing formula. Here for the random pay-off Q, $\bar{Q}$ is known and the aim is to determine the price P.

For the deterministic scenario, it is well known that $P = \frac{Q}{1+\mu_{r_f}}$. Therefore, the quantity $\mu_{r_f} + \beta(\mu_M - \mu_{r_f})$ can be interpreted as risk adjusted interest rate. This gives another interpretation of formula (5.27).

Example 5.7.3 *Consider a mutual fund that invests 10% it funds at a risk free rate of 7%. The remaining 90% is invested in a widely diversified portfolio (resembling the market portfolio) which is expected to give a return of 15%. Further it is known that the beta of the fund is 0.90, and one share of the mutual fund costs Rs 100. Is this price of a share of the mutual fund fair? Justify your answer.*

Solution: The value of a share after one year will be $(10\times1.07)+(90\times1.15) = 114.20$. Thus $\bar{Q} = 114.2$. Therefore

$$P = \frac{114.20}{(1.07)+(0.90)(0.15-0.07)} = Rs\ 100.$$

This shows that the price of the share, namely *Rs* 100, represents *Rs* 100 of assets in the fund, and therefore CAPM tells that the *price is right*.

□

The CAPM as a Factor Model

The CAPM can be derived as a special case of a single factor model. Let us assume that the asset return r_i and market return r_M (taken as a factor) are related as follows

$$(r_i - \mu_{rf}) = \alpha_i + \beta_i(r_M - \mu_{rf}) + \epsilon_i. \tag{5.28}$$

Here μ_{rf} is the risk-free interest rate and $E(\epsilon_i) = 0$. Also ϵ_i is uncorrelated with the market return r_M and also with other ϵ_j's. Further α_i and β_i are the usual coefficients appearing in a single factor model.

Taking expectation in (5.27) gives

$$(\mu_i - \mu_{rf}) = \alpha_i + \beta_i(\mu_M - \mu_{rf}). \tag{5.29}$$

Here we note that (5.29) is identical with CAPM except that in CAPM, $\alpha_i = 0$. If we further take the covariance of both sides in (5.29) we get

$$\sigma_{iM} = \beta_i\sigma_M^2.$$

Hence,

$$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2},$$

which is the same expression as used in CAPM. The equation (5.29) represents a line between the quantities $(\mu_M - \mu_{rf})$ and $(\mu_i - \mu_{rf})$. This line is called the *characteristic equation or characteristic line*.

The characteristic line in a sense is more general than CAPM because here α_i need not be zero. In fact α_i can have a very nice economic interpretation. A stock with non zero α_i can be regarded as mispriced. If $\alpha_i > 0$ then in view of CAPM, the asset is performing better than it should. Similarly if $\alpha_i < 0$, then it is performing worse than it should.

Though we have tried to explain CAPM as a single factor model, we must note that the two are not equivalent. In CAPM we assume that the market is efficient, but in a single factor model we have taken arbitrary covariance matrix σ_{iM} and made no assumption on market efficiency.

The CAPM and β_i can be understood from a different angle if we take the following model

$$r_i = r_{rf} + \beta_i(r_M - r_{rf}) + \epsilon_i,$$

where ϵ_i is a random variable. Taking expectation in the above equation and using CAPM we get $E(\epsilon_i) = 0$. Further taking the correlation with r_M in the above equation we get $Cov(\epsilon_i, r_M) = 0$. Therefore we have

$$\sigma_i^2 = \beta_i^2\sigma_M^2 + Var(\epsilon_i).$$

This equation tells that σ_i^2 is the sum of two expressions. The first expression $\beta_i^2\sigma_M^2$ is called the *systematic risk*. This is the risk associated with the market as a whole. There is no chance of reducing this risk by diversification because all assets with nonzero beta have this risk. The second expression $Var(\epsilon_i)$ is uncorrelated with the market and therefore can be reduced by diversification. The quantity $Var(\epsilon_i)$ is called *unsystematic* risk of the asset. Therefore the systematic risk measured by β becomes more important because it directly combines with the systematic risk of other assets.

5.8 Summary and Additional Notes

- In this chapter we analyzed the advantages of diversifying the total investment among several assets optimally so as to get a 'decent' return with minimum risk. The theory described here is mainly based on the work of Markowitz [90].
- The requisite terminologies were introduced in Section 5.2, followed by a simple case of two assets portfolio optimization in Section 5.3, to get the feel of the subject. The ideas of Section 5.3 lead us to extend the analysis to the multi asset scenario in Section 5.6.
- Section 5.7 continues with the multi asset portfolio optimization with the difference that one risk-free asset is included in the portfolio. This inclusion results in the Capital Asset Pricing model (CAPM) and a new concept of market portfolio. It was shown that the market portfolio can guide the investor to determine the advantage of taking more risk with his investment.
- The derivation of CAPM assumes the *efficient market hypothesis*. This hypothesis tells that in an efficient market, asset prices fully reflect all available

information. Therefore no relevant information is ignored, and systematic errors are not made. As a consequence, prices are always at levels consistent with fundamentals.
- The contents of the chapter are kept very simple with the intention to familiarize the readers with the basics behind portfolio optimization. However things can be very complex in financial circuits. One needs to realize that any study related to financial problems requires extra skills and high level of understanding. Several interesting books covering various aspects of portfolio theory can be referred to, like, Bartholomew-Biggs [7], Cepinski and Zastwawniak [25], Cornuejols and Tütüncü [32], Elton et al. [41], Luenberger [85], Roman [113], Ross [114], to name a few.
- A prominent area where portfolio optimization has gained momentum in recent years is the asset-liability management of the institutional investors, like, insurance companies, pension funds or mutual funds. The institutional investors make huge investments in the markets and simultaneously repay the maturity amounts to the other investors who had invested with them. For this reasons, they need to constantly rebalance their portfolio after every time frame, which is generally very small. An institutional investor will get some inflow of money at t instance of time as the return from various investments that had been made in the market earlier and which had subsequently matured at t time, and also the institutional investor needs to pay the maturity amount to all those investors who had invested with him and whose funds have matured at the end of $t-1$ time. The remaining amount is reinvested in the market. The time scale involved in such asset-liability problems has been captured by using stochastic linear programming models (see Kall [74] for stochastic LPP). Number of research papers can be found on asset-liability management, like Sodhi [125], Yu et al. [148] and Steiner et al. [130] and references therein.
- There are several commercial packages, for instance, CPLEX, LINGO, MATLAB, SAS that provide lot of inbuilt functions for Portfolio Analysis. The major disappointment with all the commercial packages is that they can best generate only the approximate piecewise linear representation of the efficient frontier in portfolio optimization. With large number of assets involved, say 600-800, the performance of these softwares in computing the efficient frontier deteriorates. The MPQ (multi-parametric quadratic programming), programmed in Java and available in public domain, performs exceptionally well on large-scale applications in a reasonable time and yields the exact efficient frontier. For more on MPQ, we refer to Steiner et al. [130].
- The effect of introduction of transaction costs and/or different lending and

borrowing rates in portfolio optimization theory has also been analyzed in literature. Some of the books mentioned above contain subject matter on this issue.

- Some researchers have re-visited Markowitz's mean- variance model so as to simplify the analysis and computations in the determination of efficient frontier. The interested readers may refer to Chawla [27] and Steinbach [129].

5.9 Exercises

Exercise 5.1 *Suppose there are three financial market scenarios $\Omega = \{w_1, w_2, w_3\}$ with different probabilities of occurrence. Consider the following table showing the returns on two different stocks in these three scenarios*

scenario	prob	return k_1%	return k_2%
w_1	0·2	−10	−30
w_2	0·5	0	20
w_3	0·3	20	15

(a) What are the expected returns on the stocks?
(b) Suppose 60% of the available fund is invested in stock 1 and the remaining is invested in stock 2, then what is the expected return of the portfolio?
(c) Compute the weights if the expected return on a portfolio is 20%.

Exercise 5.2 *Consider the following data for two different stocks*

scenario	prob	return k_1%	return k_2%
w_1	0·4	−10	20
w_2	0·2	0	20
w_3	0·4	20	10

Suppose a portfolio comprises of 40% of total investment in stock 1 and 60% in stock 2. Compare the risk of the portfolio with the risks of its individual components. What will be the risk situation if a portfolio is designed with investment of 80% in stock 1 and the remaining in stock 2.

Exercise 5.3 *Prove that if short sales are not allowed then the risk of the portfolio can not exceed the greater of the risks of the individual components of the portfolio.*

Exercise 5.4 *Let a portfolio be designed with investment of 50% in stock 1 and the remaining 50% in stock 2. Further let short sale be allowed in stock 1 and all the other data bring the same as in Exercise 5.2. Does the conclusion of Exercise 5.3 hold.*

Exercise 5.5 *Suppose the portfolios are constructed using three securities a_1, a_2, a_3 with expected returns, $\mu_1 = 20\%$, $\mu_2 = 13\%$, $\mu_3 = 4\%$, standard deviations of returns, $\sigma_1 = 25\%$, $\sigma_2 = 28\%$, $\sigma_3 = 20\%$, and the correlation between returns, $\rho_{12} = 0 \cdot 3$, $\rho_{13} = 0 \cdot 15$ and $\rho_{23} = 0 \cdot 4$. Among all the attainable portfolios, find the one with minimum variance. What are the weights of the three securities in this portfolio? Also compute the expected return and standard deviation of this portfolio.*

Exercise 5.6 *Among all attainable portfolios with expected return 20% constructed using the data provided in Exercise 5.5, find the portfolio with minimum variance. Compute the weights of individual assets in this portfolio.*

Exercise 5.7 *Consider the following data*

	μ	σ
asset 1	10%	5%
asset 2	8%	2%

For each correlation coefficient $\rho = -1, -0 \cdot 5, 0, 0 \cdot 5, 1$, what is the combination of the two assets that yields the minimum standard deviation and what is the minimum value of the standard deviation?

Exercise 5.8 *Compute the minimum risk portfolio for the following rate return (%) data*

	Jan	Feb	Mar	Apr	May	June
asset 1	12	10	5	7	15	12
asset 2	7	12	10	10	12	15

Also compute the expected return for the optimal portfolio.

Exercise 5.9 *Consider three risky assets with the variance-covariance matrix and expected returns (all data in %) as follows.*

variance - covariance matrix(C)			return(M)
10	4	0	5
4	12	6	6
0	6	10	1

Find two efficient portfolios. Also construct the portfolio giving the return of 2·8% with minimum risk. Will this portfolio be also efficient?
(Hint : use two fund theorem).

Exercise 5.10 *Suppose an investor is interested in constructing a portfolio with one risk-free asset a_1, and three risky assets a_2, a_3 and a_4. Let the expected returns of a_1, a_2, a_3 and a_4 be 6%, 10%, 12% and 18% respectively. Let the variance-covariance matrix C of the three risky assets be*

$$C = \begin{pmatrix} 4 & 20 & 40 \\ 20 & 10 & 70 \\ 40 & 70 & 14 \end{pmatrix}.$$

Determine all efficient portfolios for the investor.

Exercise 5.11 *Consider the data of two risky assets a_1, a_2 with $\mu_1 = 12{\cdot}5\%$, $\mu_2 = 10{\cdot}5\%$, $\sigma_1 = 14{\cdot}9\%$, $\sigma_2 = 14\%$, $\rho = 0{\cdot}33$.*

(a) Is it advisable to diversify the investment? If so then what composition of the assets will minimise the risk?
(b) What is the minimum value of the risk?
(c) If the risk-free rate of return is 5% then derive the equation of the capital market line?

Exercise 5.12 *Given the following information about the one risk-free asset and three risky assets, find the expected return and standard deviation of the market portfolio. Also determine the equation of the capital market line.*

$$\mu_{rf} = 5\%,\ \mu_1 = 14\%,\ \mu_2 = 8\%,\ \mu_3 = 20\%;$$
$$\sigma_1 = 6\%,\ \sigma_2 = 5\%,\ \sigma_3 = 15\%;\ \sigma_{12} = 0{\cdot}5\%,\ \sigma_{13} = 0{\cdot}2\%,\ \sigma_{23} = 0{\cdot}4\%.$$

Exercise 5.13 *Assume that the following assets are correctly priced according to the security market line. Derive the security market line.*

$$\mu_1 = 6\%,\ \beta_1 = 0{\cdot}5;\quad \mu_2 = 12\%,\ \beta_2 = 1{\cdot}5.$$

What is the expected return on an asset with $\beta = 2$?

Exercise 5.14 *If the following two assets are correctly priced according to the security market line, what is the return of the market portfolio? What is the risk-free return?*

$$\mu_1 = 9{\cdot}5\%,\ \beta_1 = 0{\cdot}8;\quad \mu_2 = 13{\cdot}5\%,\ \beta_2 = 1{\cdot}3.$$

Exercise 5.15 *Let the expected rate of return on the market portfolio be 23% and that of the risk free asset be 7%. Also let the standard deviation of the market portfolio be 32% and let us assume that the market is efficient.*

(a) What is the equation of capital market line?
(b) If Rs 300 is invested in the risk free asset, and Rs 700 in the market portfolio then what is the expected return at the end of the year?
(c) If an investor has Rs 1000 to invest and he/she desires a return of 39%, then what should be his/her portfolio?

6
Portfolio Optimization-II

6.1 Introduction

We have presented Markowitz's mean variance model for portfolio optimization in the last chapter. But contrary to its theoretical reputation, this model in its original form has not found much favor with the practitioners to construct large scale portfolios. There are several theoretical and practical reasons for not using this model extensively in practice - particularly when the number of assets in the portfolio is large. This chapter aims to understand these reasons and then present some other models which have been developed to improve Markowitz's model both theoretically and computationally.

6.2 Markowitz's Model: Some Theoretical and Computational Issues

Theoretically, Markowitz's model is known to be valid if returns r_is are multivariate normally distributed and the investor is *risk averse* in the sense that he/she prefers less standard deviation of the portfolio to more. But one is not fully convinced of the validity of the standard deviation as a measure of risk. An investor is certainly unhappy to have small or negative profit, but feels happy to have larger profit. In other words, this means that the investor's perception about risk is not symmetric about the mean. There are several empirical studies, which reveal that most r_i are not normally or even symmetrically distributed. In this scenario, one possible approach seems to be to consider the skewness and kurtosis of the distribution in addition to the mean and variance and extend the Markowitz's model to generate the efficient frontier in (mean, variance, skewness, kurtosis)-space. This has been the approach of some of the models, e.g. Konno and Suzaki [79], and

Joro and Na [71], but has not found much favor because the resulting optimization problem is not easy to handle. An alternative and popular approach is to introduce certain new measures of risk which carry information about the possible portfolio losses implied by the tail of the return distribution, even in the case when the distribution is not symmetric. This takes care of those situations where the return distribution is heavily tailed. These risk measures are called *downside or safety-first risk measures* which aim to maximize the probability that the portfolio loss is below a certain acceptable level, commonly referred as the benchmark or the disaster level. Thus these risk measures are *quantile based risk measures* and are different from standard deviation or other *moment based risk measures*. Some of the most popular quantile based risk measures are value at risk (VaR) and conditional value at risk (CVaR). Since downside risk measures of individual securities cannot be easily aggregated into portfolio downside risk measures (we need the entire joint distribution of security returns), their application in practice requires computationally intensive non parametric estimation, simulation and optimization techniques.

There is another major problem associated with the classical Markowitz's model. This model gives us an optimal portfolio assuming that we have perfect information about μ_i's and σ_{ij}'s for the assets that we are considering. Therefore an important practical issue is the estimation of the μ_i's and σ_{ij}'s. A reasonable approach for estimating these data is to use time series of past returns r_{it} which represents the return of ith asset from time $(t-1)$ to t, where $t = 1, 2, \ldots, T$. However, it has been observed that small changes in the time series r_{it} lead to changes in the μ_i's and σ_{ij}'s that often lead to significant changes in the optimal portfolio. This is a fundamental weakness of the Markowitz model, no matter how cleverly μ_i's and σ_{ij}'s are computed. This is because the optimal portfolio construction is very sensitive to small changes in the data. Only one small change in one μ_i may produce a totally different portfolio. In fact recent research (Chopra and Ziemba [30]) has revealed that errors in the estimation of means μ_i can be more damaging than errors in other parameters. This has motivated researchers to employ *robust optimization techniques* in Markowitz's model, e.g. Ben-Tal [9], and Tütüncü and Koenig [139]. A much simpler approach is to consider portfolio optimization under a minimax rule (Cai et al. [23]) and provide some flexibility by allowing μ_i to lie in some interval $a_i \le \mu_i \le b_i$ (Deng et al. [36]).

The mean-variance model of Markowitz, in general, results in a dense quadratic programming problem. If the number of assets in the portfolio is large then it becomes very difficult to obtain an optimal solution of such large-scale dense quadratic programming problem on a real time basis. This has motivated researchers to consider mean-absolute deviation of the portfolio as a measure of risk (Konno and Yamazaki [80]) so that the resulting optimization problem reduces to a linear programming problem. These models are called *L_1-risk models* because these are based on L_1 metric on $\mathbb{R}^n$. In this terminology, Markowitz's model can be termed as a *L_2-risk model* since it uses variance as a risk measure which is based on the notion of L_2 metric. In the same spirit, the models based on the minimax rule portfolio selection strategies can be termed as *L_∞-risk models*. It is simple to note that L_1 and L_∞ risk models will result in a linear programming formulation. This is in contrast to an L_2- risk model which results in a quadratic programming formulation. Therefore computationally, L_1 and L_∞ risk models are easier to handle in comparison to L_2 risk model. The L_1, L_2 and L_∞ are three standard moment based risk models which have been studied in the literature.

The above discussion suggests that the choice of a proper risk measure is very crucial to study portfolio selection problems. But then what would be the guiding principles for this choice? The concept of *coherent risk measures* (Artzner et al. [5]) is an important contribution in this regard. This stipulates an axiomatic approach to the study of risk measure by presenting certain desirable properties.

Some other issues associated with Markowitz's model are very natural and these have been incorporated in the original model. These are with regard to its extension for the multi-period scenario and also to incorporate transaction costs. We shall discuss some of the models in the subsequent sections which addresses the above issues.

6.3 Mean Absolute Deviation Based Portfolio Optimization: A L_1-Risk Model

We first introduce the L_1-risk measure and formulate the corresponding portfolio optimization problem with this risk measure. Let an investor has initial wealth M_0 which is to be invested in n assets a_i (i=1,2,...,n). Let r_i be the return of the asset a_i which is a random variable. Also let x_i be the amount of money to be invested in the asset a_i out of the total fund M_0.

We now define $\mu_i = E(r_i)$ and $q_i = E(|r_i - \mu_i|)$, i=1,2,...,n. Then μ_i denotes the expected return rate of the asset a_i and q_i denotes the expected absolute deviation of r_i from its mean. Then the expected return of a portfolio $(x_1, x_2, \ldots, x_n)$ is given by

$$\mu = E\left(\sum_{i=1}^{n} r_i x_i\right) = \sum_{i=1}^{n} \mu_i x_i.$$

We now define L_1-risk measure or the mean absolute deviation of the portfolio $(x_1, x_2, \ldots, x_n)$.

Definition 6.3.1 (L_1-Risk Measure of a Portfolio) *Let $(x_1, x_2, \ldots, x_n)$ be the given portfolio. Then its L_1-risk measure or mean absolute deviation is defined as*

$$w_{L_1}(x_1, x_2, \ldots, x_n) = E\left[\left|\sum_{i=1}^{n} r_i x_i - E\left(\sum_{i=1}^{n} r_i x_i\right)\right|\right].$$

In terms of the L_1-risk measure $w_{L_1}(x_1, x_2, \ldots, x_n)$, the L_1-risk model of the portfolio optimization problem is formulated as

$$\text{Min} \quad w_{L_1}(x_1, x_2, \ldots, x_n) = E\left[\left|\sum_{i=1}^{n} (r_i - \mu_i) x_i\right|\right]$$

subject to

$$\begin{aligned} &\sum_{i=1}^{n} \mu_i x_i \geq \alpha M_0 \\ &\sum_{i=1}^{n} x_i = M_0 \\ &0 \leq x_i \leq u_i \qquad (i = 1, 2, \ldots, n), \end{aligned} \tag{6.1}$$

where $(\alpha > 1)$ is a parameter representing the minimum rate of return required by the investor. Also u_i is the maximum amount of the money which can be invested in the asset a_i.

In (6.1) it may be noted that by defining $w_i = x_i/M_0$, $(i = 1, 2, \ldots, n)$, the portfolio $(x_1, x_2, \ldots, x_n)$ can also be represented in terms of its weight vector $(w_1, w_2, \ldots, w_n)$. Traditionally in the portfolio optimization literature, a portfolio has been represented by its weight vector $(w_1, w_2, \ldots, w_n)$ but it can also be equivalently represented in terms of its amount allocation vector $(x_1, x_2, \ldots, x_n)$.

In practice, the historical data is used to estimate the parameters in the optimization problem (6.1). Let r_{it} be the realisation of the random variable r_i during the period t $(t = 1, 2, \ldots, T)$. We assume that r_{it} is available through the historical data and the expected value of r_i can be approximated by the average derived from the data. This gives

$$\mu_i = E(r_i) = \frac{1}{T} \sum_{t=1}^{T} r_{it}. \tag{6.2}$$

Also $w_L(x_1, x_2, \ldots, x_n)$ can be approximated by

$$E\left(\left|\sum_{i=1}^{n} (r_i - \mu_i) x_i\right|\right) = \frac{1}{T} \sum_{t=1}^{T} \left|\sum_{i=1}^{n} (r_{it} - \mu_i) x_i\right|. \tag{6.3}$$

In (6.3) it may be noted that, due to the absolute value function, the expression on the right hand side becomes a nonlinear and non smooth function of $(x_1, x_2, \ldots, x_n)$.

Using (6.2) and (6.3), problem (6.2) can be reformulated as

$$\text{Min} \quad \frac{1}{T} \sum_{t=1}^{T} \left|\sum_{i=1}^{n} (r_{it} - \mu_i) x_i\right|$$

subject to

$$\begin{aligned} &\sum_{i=1}^{n} \mu_i x_i \geq \alpha M_0 \\ &\sum_{i=1}^{n} x_i = M_0 \\ &0 \leq x_i \leq u_i \qquad (i = 1, 2, \ldots, n). \end{aligned} \tag{6.4}$$

If we now denote $\left|\sum_{i=1}^{n} (r_{it} - \mu_i) x_i\right|$ by y_t and employ the definition of the absolute function then we get

$$\begin{aligned} y_t &= \left|\sum_{i=1}^{n} (r_{it} - \mu_i) x_i\right| \\ &= Max\left(\sum_{i=1}^{n} (r_{it} - \mu_i) x_i, \quad -\sum_{i=1}^{n} (r_{it} - \mu_i) x_i\right). \end{aligned} \tag{6.5}$$

Therefore problem (6.4) gets transformed to

$$\text{Min} \quad \frac{1}{T}\sum_{t=1}^{T} y_t$$

subject to

$$\begin{aligned}
&\sum_{i=1}^{n}(r_{it}-\mu_i)x_i \le y_t \\
&-\sum_{i=1}^{n}(r_{it}-\mu_i)x_i \le y_t \\
&\sum_{i=1}^{n}\mu_i x_i \ge \alpha M_0 \\
&\sum_{i=1}^{n} x_i = M_0 \\
&0 \le x_i \le u_i \qquad (i=1,2,\ldots,n),
\end{aligned} \tag{6.6}$$

where the first two constraints in (6.6) follow from (6.5) and the definition of maximum.

Denoting $(r_{it}-\mu_i)$ by c_{it} $(i=1,2,\ldots,n;\ t=1,2,\ldots,T)$, we can rewrite (6.6) as

$$\text{Min} \quad \frac{1}{T}\sum_{t=1}^{T} y_t$$

subject to

$$\begin{aligned}
&y_t - \sum_{i=1}^{n} c_{it}x_i \ge 0 \qquad (t=1,2,\ldots,T) \\
&y_t + \sum_{i=1}^{n} c_{it}x_i \ge 0 \qquad (t=1,2,\ldots,T) \\
&\sum_{i=1}^{n}\mu_i x_i \ge \alpha M_0 \\
&\sum_{i=1}^{n} x_i = M_0 \\
&0 \le x_i \le u_i \qquad (i=1,2,\ldots,n).
\end{aligned} \tag{6.7}$$

Remark 6.3.1 *The formulation (6.4) is essentially nonlinear and non smooth. But since this nonlinearity and non smoothness occurs due to the presence of absolute value function only, it can be handled in a reasonably simple manner as explained above. The resulting linear programming problem (6.7) can be solved efficiently even when n is large.*

One obvious question at this stage is to enquire if there is any relationship between L_1 and L_2-risk models. The below given theorem answers this question.

Theorem 6.3.1 *Let $(r_1, r_2, \ldots, r_n)$ be multivariate normally distributed. Then for a given portfolio $x = (x_1, x_2, \ldots, x_n)$*

$$w_{L_1}(x) = \sqrt{\frac{2}{\pi}}\,\sigma(x),$$

where the standard deviation $\sigma(x)$ is given by

$$\sigma(x_1, x_2, \ldots, x_n) = \sqrt{E\left[\left\{\sum_{i=1}^{n} r_i x_i - E\left(\sum_{i=1}^{n} r_i x_i\right)\right\}^2\right]}.$$

Proof. Let $(\sigma_{ij}) \in \mathbb{R}^{n\times n}$ be the variance-covariance matrix of $(r_1, r_2, \ldots, r_n)$. Then under the given hypothesis, $\sum_{i=1}^{n} r_i x_i$ is normally distributed with mean $\sum_{i=1}^{n} r_i \mu_i$ and standard deviation

$$\sigma(x) = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j}.$$

Therefore

$$\begin{aligned}
w_{L_1}(x) &= \frac{1}{\sqrt{2\pi}\sigma(x)}\int_{-\infty}^{\infty} |u| \exp\left(-\frac{u^2}{2\sigma^2(x)}\right)du \\
&= \frac{2}{\sqrt{2\pi}\sigma(x)}\int_{0}^{\infty} u \exp\left(-\frac{u^2}{2\sigma^2(x)}\right)du,
\end{aligned}$$

which on substitution $(u^2/2\sigma^2) = s$ gives

$$w_{L_1}(x) = \sqrt{\frac{2}{\pi}}\,\frac{\sigma^2(x)}{\sigma(x)}\int_0^{\infty} e^{-s}ds = \sqrt{\frac{2}{\pi}}\,\sigma(x).$$

□

In view of Theorem 6.3.1, for the multivariate normal case, both L_1 and L_2-risk models are equivalent. In other words, Markowitz's mean-variance portfolio

selection strategy will be the same as the one given by mean-absolute deviation selection strategy. Even in the case when normality assumption does not hold, through certain case studies, it has been shown that minimizing the L_1-risk produces portfolios which are comparable to Markowitz's mean-variance model which minimizes L_2-risk, i.e. standard deviation. However as one expects the variance of mean-absolute deviation portfolio is always at least as large as the corresponding mean-variance portfolio. But in actual applications, this difference is small.

In fact Konno and Yamazaki [80] applied both L_1 and L_2-risk models in Tokyo Stock Market by using historical data of 224 stocks in NIKKEL 225 index. They generated efficient frontiers and observed that the difference of the standard deviation of the optimal portfolio generated by L_2 and L_1-risk models is at most 10% for what ever value of α. Of course two frontiers will coincide if r_i's are multivariate normally distributed. Thus this difference can be largely attributed to the non normality of the data. Therefore irrespective of the distribution scenario, L_1-risk model provides a good alternative to Markowitz's L_2-risk model.

Advantages of the Formulation (6.7)

We now list some of the advantages of L_1-risk model (6.7) over the classical Markowitz's model which minimizes the L_2-risk.

(i) The formulation (6.7) is a linear programming problem and hence can be solved much more efficiently in comparison to quadratic programming problem which is obtained in Markowitz's model. This is particulary significant for portfolios having large number of assets.

(ii) We do not need to calculate the variance-covariance matrix to set up the L_1-risk model.

(iii) In the formulation (6.7) there are always (2T+2) constraints regardless of the number of assets included in the model. This allows to handle very large portfolios on a real time basis.

(iv) An optimal solution $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$ of problem (6.7) contains at most (2T+2) positive components if $u_i = +\infty$ for $i = 1, 2, \ldots, n$. This means that an optimal portfolio consists of at most (2T+2) assets regardless of the size of n. Therefore we can use T as a control variable when we wish to restrict the number of assets in the portfolio.

6.4 Minimax Rule Based Portfolio Optimization: An L_∞-Risk Model

We discuss the *minimax rule based portfolio optimization model* due to Cai et al. [23]. We first introduce the L_∞-risk measure which is based on L_∞-metric on R^n. We then discuss the corresponding portfolio optimization model, namely the L_∞-risk model, and compare the same with Markowitz's mean-variance model.

We continue with the notations introduced in Section 6.3. Thus M_0 denotes the initial wealth to be invested in n assets a_i $(i = 1, 2, \ldots, n)$. Further for $(i = 1, 2, \ldots, n)$, r_i denotes the return of the asset a_i and x_i denotes the allocation from M_0 in this asset. Then $\mu_i = E(r_i)$ and $q_i = E(|r_i - \mu_i|)$ respectively denote the expected return rate of the asset a_i and the expected absolute deviation of r_i from its mean μ_i.

Let us now introduce the L_∞-risk measure. For this let

$$F = \{x = (x_1, x_2, \ldots, x_n) : \sum_{i=1}^{n} x_i = M_0;\ x_i \geq 0 (i = 1, 2, \ldots, n)\}. \tag{6.8}$$

Then $x \in F$ is a portfolio of the assets $a_1, a_2, \ldots, a_n$ and F is called the *feasible region* of the portfolio optimization problem. As explained earlier, the expected return of the portfolio $x : (x_1, x_2, \ldots, x_n)$ is given by $\mu = \sum_{i=1}^{n} \mu_i x_i$. We now have the following definition.

Definition 6.4.1 (L_∞-Risk Measure) *Let $x : (x_1, x_2, \ldots, x_n)$ be the given portfolio. Then its L_∞-risk measure is defined as*

$$w_{L_\infty}(x_1, x_2, \ldots, x_n) = \underset{1 \leq i \leq n}{Max}\, E(|r_i x_i - \mu_i x_i|) = \underset{1 \leq i \leq n}{Max}\, (q_i x_i).$$

In terms of the L_∞-risk measure $w_{L_\infty}(x_1, x_2, \ldots, x_n)$, the L_∞-risk model of the portfolio optimization problem, denoted by (POL_∞) is formulated as

$$\underset{x \in F}{Min} \left(\underset{1 \leq i \leq n}{Max}\, (q_i x_i),\ -\sum_{i=1}^{n} \mu_i x_i \right). \tag{6.9}$$

The problem (6.9) is a bi-criteria optimization problem as there are two objectives involved in the optimization. These two objectives are the portfolio risk and the portfolio return. This scenario is again same as for the L_1 and L_2-risk

models studied earlier. There we had minimized the portfolio risk for a given level of portfolio return. Following the same strategy, we get the following problem

$$\underset{x\in F}{\text{Min}} \quad \left(\underset{1\le i\le n}{\text{Max}} \ (q_i x_i)\right)$$

subject to

$$\sum_{i=1}^{n} \mu_i x_i \ge \alpha M_0. \tag{6.10}$$

Writing $y = \underset{1\le i\le n}{\text{Max}} \ (q_i x_i)$, we rewrite (6.11) as

$$\underset{x\in F}{\text{Min}} \quad y$$

subject to

$$\sum_{i=1}^{n} \mu_i x_i \ge \alpha M_0$$
$$q_i x_i \le y \qquad (i = 1, 2, \ldots, n). \tag{6.11}$$

For each α, (6.11) is a linear programming problem and hence can be solved efficiently. However to generate the entire efficient frontier we need to solve (6.11) for every α, which is practically not possible. In the case of Markowitz's L_2-risk model, the famous two fund theorem comes to our rescue which allows to generate the entire efficient frontier from the knowledge of only two efficient points. In the case of L_∞-risk model, we do not have any analogue of two fund theorem, so we look to some other suitable option which does not require simplex algorithm to solve (6.11) for every α, but rather gives a solution in close form for every level of return. This is the approach which we shall be following now and describe the details below.

The solution of the bi-criteria optimization problem (6.9) is to be understood in terms of an efficient point which in this context is termed as an *efficient portfolio*.

Definition 6.4.2 (Efficient Portfolio) *A feasible portfolio $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n) \in F$ is said to be an efficient portfolio if there does not exist any other portfolio $x = (x_1, x_2, \ldots, x_n) \in F$ such that*

$$(i) \qquad \underset{1\le i\le n}{Max} \ (q_i \bar{x}_i) \le \underset{1\le i\le n}{Max} \ (q_i x_i),$$

and

$$(ii) \qquad \sum_{i=1}^{n} \mu_i \bar{x}_i \ \ge \ \sum_{i=1}^{n} \mu_i x_i.$$

The collection of all *efficient portfolios is called the efficient frontier*. As explained in the context of Markowitz's model, an investor is always interested in determining the efficient frontier so as to select that portfolio which gives maximum return for the chosen level of risk. This principle remains valid for L_∞-risk model as well.

The problem (POL_∞) for the L_∞-risk model can be rewritten as

$$\text{Min} \quad (y, \ -\sum_{i=1}^{n} \mu_i x_i)$$

subject to

$$q_i x_i \le y \qquad (i = 1, 2, \ldots, n)$$
$$x \in F. \tag{6.12}$$

Now borrowing the standard technique from the multi-objective literature, we introduce the following parametric optimization problem $PO(\lambda)$ for $0 < \lambda < 1$

$$\text{Min} \quad \lambda y + (1-\lambda)(-\sum_{i=1}^{n} \mu_i x_i)$$

subject to

$$q_i x_i \le y \qquad (i = 1, 2, \ldots, n)$$
$$x \in F. \tag{6.13}$$

We now have the following lemma connecting problems (POL_∞) and $PO(\lambda)$.

Lemma 6.4.1. *The pair $(\bar{x}, \bar{y})$ is an efficient solution of (POL_∞) if and only if there exists $0 < \lambda < 1$ such that $(\bar{x}, \bar{y})$ is an optimal solution of $PO(\lambda)$.*

Here λ can be considered as an investor's risk tolerance parameter which helps to do appropriate trade-off between the risk and the return.

In view of Lemma 6.4.1, finding efficient frontier of (POL_∞) is equivalent to finding solution of $PO(\lambda)$ for all $0 < \lambda < 1$. The discussion given below illustrates that given an arbitrary $\bar{\lambda} \in (0,1)$, the solution $(\bar{x}, \bar{y})$ of $PO(\bar{\lambda})$ can be obtained explicitly in a close form and there is no need of applying any simplex like numerical optimization algorithm. This greatly reduces the computational burden in determining the efficient frontier.

Analytical Solution of $PO(\lambda)$

Without any loss of generality we can assume that (i) $\mu_1 \le \mu_2 \le \ldots \le \mu_n$ and (ii) there do not exist two assets a_i and a_j, $i \ne j$, such that $\mu_i = \mu_j$ and $q_i = q_j$. There

is obviously no problem with the ordering of assel returns μ_i and therefore the first assumption can always be met. The second assumption is also valid because if $\mu_i = \mu_j$ and $q_i = q_j$ for two assets i, j $(i \neq j)$, then we may treat them as a single aggregated asset. We first consider the case when all assets are risky assets.

Theorem 6.4.1 *Let all asset $a_1, a_2, \ldots, a_n$ be risky assets. Then for any $0 < \lambda < 1$, an optimal solution (x^*, y^*) of the problem $PO(\lambda)$ is given by*

$$x_i^* = \begin{cases} y^*/q_i & i \in T^*(\lambda) \\ 0 & \text{otherwise,} \end{cases}$$

where

$$y^* = M_0 \left(\sum_{i \in T^*(\lambda)} (1/q_i) \right)^{-1},$$

and the set $T^*(\lambda)$ is identified by the following rule.

(a)

$$\frac{(\mu_n - \mu_{n-1})}{q_n} < \frac{\lambda}{1-\lambda}$$

$$\frac{(\mu_n - \mu_{n-2})}{q_n} + \frac{(\mu_{n-1} - \mu_{n-2})}{q_{n-1}} < \frac{\lambda}{1-\lambda}$$

$$\vdots$$

$$\frac{(\mu_n - \mu_{n-k})}{q_n} + \frac{(\mu_{n-1} - \mu_{n-k})}{q_{n-1}} + \ldots + \frac{(\mu_{n-k+1} - \mu_{n-k})}{q_{n-k+1}} < \frac{\lambda}{1-\lambda}$$

and

$$\frac{(\mu_n - \mu_{n-k-1})}{q_n} + \frac{(\mu_{n-1} - \mu_{n-k-1})}{q_{n-1}} + \ldots + \frac{(\mu_{n-k} - \mu_{n-k-1})}{q_{n-k}} \geq \frac{\lambda}{1-\lambda}.$$

Then

$$T^*(\lambda) = \{n, n-1, \ldots, n-k\}, \tag{6.14}$$

(b) Otherwise

$$T^*(\lambda) = \{n, n-1, \ldots, 1\}. \tag{6.15}$$

Proof. We apply the Karush Kuhn-Tucker (KKT) conditions to $PO(\lambda)$. For this, we first introduce the Lagrangian of $PO(\lambda)$ as

$$L(x, y, \beta, \delta, \gamma) = \lambda y + (1-\lambda)\left(-\sum_{i=1}^{n} \mu_i x_i\right) + \sum_{i=1}^{n} \beta_i (q_i x_i - y) + \delta\left(\sum_{i=1}^{n} x_i - M_0\right) - \sum_{i=1}^{n} \gamma_i x_i,$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, $\beta \in \mathbb{R}^n$, $\delta \in \mathbb{R}$ and $\gamma \in \mathbb{R}^n$. We now write the KKT conditions that an optimal solution of $PO(\lambda)$ must satisfy. These are

$$\frac{\partial L}{\partial y} = \lambda - \sum_{i=1}^{n} \beta_i = 0 \tag{6.16}$$

$$\frac{\partial L}{\partial x_i} = -(1-\lambda)\mu_i + \beta_i q_i + \delta - \gamma_i = 0 \quad (i = 1, \ldots, n) \tag{6.17}$$

$$\sum_{i=1}^{n} x_i = M_0 \tag{6.18}$$

$$(q_i x_i - y)\beta_i = 0 \quad (i = 1, \ldots, n) \tag{6.19}$$

$$\gamma_i x_i = 0 \quad (i = 1, \ldots, n) \tag{6.20}$$

$$\beta_i \geq 0 \quad (i = 1, \ldots, n) \tag{6.21}$$

$$\gamma_i \geq 0 \quad (i = 1, \ldots, n). \tag{6.22}$$

Our aim now is to obtain a solution of the above system. Define $T^*(\lambda) = \{i : \beta_i > 0\}$. This means that for (6.19) to be satisfied we must have

$$q_i x_i - y = 0, \quad \text{for } i \in T^*(\lambda),$$

i.e.

$$x_i = \frac{y}{q_i}, \quad \text{for } i \in T^*(\lambda). \tag{6.23}$$

We let

$$x_i = 0, \quad \text{for } i \notin T^*(\lambda). \tag{6.24}$$

But $M_0 = \sum_{i=1}^{n} x_i = \sum_{i \in T^*(\lambda)} x_i + \sum_{i \notin T^*(\lambda)} x_i = \sum_{i \in T^*(\lambda)} x_i$, and hence (6.23) gives

$$M_0 = y \sum_{i \in T^*(\lambda)} (1/q_i) \tag{6.25}$$

i.e.

$$y = M_0\left(\sum_{i\in T^*(\lambda)} \frac{1}{q_i}\right)^{-1}. \tag{6.26}$$

Therefore, we have

$$x_i^* = \begin{cases} \dfrac{M_0}{q_i}\left(\sum_{i\in T^*(\lambda)} \dfrac{1}{q_i}\right)^{-1}, & i \in T^*(\lambda) \\ 0, & \text{otherwise.} \end{cases} \tag{6.27}$$

From (6.20), for $x_i > 0$ we have $\gamma_i = 0$. Hence $\gamma_i = 0$ whenever $i \in T^*(\lambda)$. Thus from (6.17), we have for $i \in T^*(\lambda)$

$$\beta_i = \frac{1}{q_i}[(1-\lambda)\mu_i - \delta]. \tag{6.28}$$

From (6.16) and (6.28), we get

$$\lambda = \sum_{i\in T^*(\lambda)} \frac{1}{q_i}[(1-\lambda)\mu_i - \delta] = (1-\lambda)\sum_{i\in T^*(\lambda)} (\mu_i/q_i) - \delta \sum_{i\in T^*(\lambda)} (1/q_i).$$

Therefore we get

$$\delta = \left(\sum_{i\in T^*(\lambda)} \frac{1}{q_i}\right)^{-1}\left((1-\lambda)\sum_{i\in T^*(\lambda)} \frac{\mu_i}{q_i} - \lambda\right). \tag{6.29}$$

Thus, from (6.28) for $i \in T^*(\lambda)$

$$\beta_i = \frac{1}{q_i}\left[(1-\lambda)\mu_i - \left(\sum_{i\in T^*(\lambda)} \frac{1}{q_i}\right)^{-1}\left((1-\lambda)\sum_{i\in T^*(\lambda)} \frac{\mu_i}{q_i} - \lambda\right)\right]. \tag{6.30}$$

Also from (6.17) and noting that for $i \notin T^*(\lambda)$, $\beta_i = 0$, we have for $i \notin T^*(\lambda)$,

$$\gamma_i = -(1-\lambda)\mu_i + \delta. \tag{6.31}$$

Clearly, if it is possible to determine the set $T^*(\lambda)$ which ensures that β_i and γ_i as given by (6.30) and (6.31) are all non negative, then y and x_i as given by (6.26) and (6.27), respectively, will be a solution of KKT conditions (6.16) to (6.22).

Now to prove our theorem, it is required that if there exists an integer $k \in [0, n-2]$ such that inequalities specified in (a) hold, then $T^*(\lambda)$ as given by (6.14) will ensure that $\mu_i \geq 0$ and $\gamma_i \geq 0$. The following analysis proves the above arguments.

By (6.30), it follows that, for any $i \in T^*(\lambda) = \{n, n-1, \ldots, n-k\}$,

$$\begin{aligned}\beta_i &= \left(\sum_{i\in T^*(\lambda)} \frac{q_i}{q_i}\right)^{-1}\left[(1-\lambda)\sum_{i\in T^*(\lambda)} \frac{\mu_i - \mu_i}{q_i} + \lambda\right] \\ &= \left(\sum_{i\in T^*(\lambda)} \frac{q_i}{q_i}\right)^{-1}(1-\lambda)\left[\left(-\sum_{i=i+1}^{n} \frac{\mu_i - \mu_i}{q_i} + \frac{\lambda}{(1-\lambda)}\right) + \sum_{i=n-k}^{i} \frac{\mu_i - \mu_i}{q_i}\right],\end{aligned}$$

which by inequalities in (a), is strictly positive. On the other hand, for $i = 1, 2, \ldots, (n-k-1)$, from (6.31) and (6.29), we have

$$\begin{aligned}\gamma_i &= -(1-\lambda)\mu_i + \left(\sum_{i\in T^*(\lambda)} \frac{1}{q_i}\right)^{-1}\left((1-\lambda)\sum_{i\in T^*(\lambda)} \frac{\mu_i}{q_i} - \lambda\right) \\ &= \left(\sum_{i\in T^*(\lambda)} \frac{1}{q_i}\right)^{-1}\left((1-\lambda)\sum_{i\in T^*(\lambda)} \frac{\mu_i - \mu_i}{q_i} - \lambda\right) \\ &\geq \left(\sum_{i\in T^*(\lambda)} \frac{1}{q_i}\right)^{-1}\left((1-\lambda)\sum_{i\in T^*(\lambda)} \frac{\mu_i - \mu_{n-k-1}}{q_i} - \lambda\right),\end{aligned}$$

which by (6.14), is nonnegative. The above details show that the KKT conditions (6.21) and (6.22) are satisfied. In the case when there does not exist any integer $k \in [0, n-2]$ such that inequalities in (a) are satisfied, we need to show that the solution given by (6.26) and (6.27) with $T^*(\lambda) = \{n, n-1, \ldots, 2, 1\}$, will satisfy all KKT conditions. To do this, we introduce a dummy asset a_0 with $\mu_0 = -L$ and $q_0 = L$, where L is a sufficiently large positive number. Now following in the same manner as above, we can show that all KKT conditions are satisfied.

Thus above analysis shows that the KKT conditions (6.16) to (6.22) are satisfied, if we select $T^*(\lambda)$ by the selection procedure (6.14) to (6.15) and solutions y and x_i are given by (6.26) and (6.27). Now since PO(λ) is a convex programming problem the KKT conditions become necessary and sufficient for optimality. So solution given by (6.12) to (6.15) is an optimal solution. This proves the theorem. □

We next consider the case when one risk-free asset is included in the portfolio. To be specific, let the asset a_1 be risk-free. It is natural to assume that the risk-free asset has the lowest return. This is because choosing assets having lower returns than that of the risk-free asset in the portfolio can not make it optimal.

Now for the risk-free asset a_1, we have $q_1 = 0$. Hence Theorem 6.4.1 can not be directly applied in this scenario. However, we can take $q_1 = \epsilon$ where $\epsilon > 0$ is a small number. Now we can apply Theorem6.4.1 and get the desired result by taking $\epsilon \to 0^+$.

Taking $q_1 = \epsilon(> 0)$, when we apply Theorem 6.4.1 to determine the set $T(\lambda)$,there are two possibilities. These are (i) $1 \notin T(\lambda)$ and (ii) $1 \in T(\lambda)$.

When $1 \notin T(\lambda)$, the situation is exactly same as discussed earlier and so the optimal solution for $PO(\lambda)$ as given in Theorem 6.4.1 remains unchanged.

When $1 \in T(\lambda)$. Then by Theorem 6.4.1, the optimal solution of $PO(\lambda)$ is given by

$$x_i^* = \begin{cases} \dfrac{y^*}{q_i}, & i \in T(\lambda) \\ 0, & i \notin T(\lambda), \end{cases} \tag{6.32}$$

where

$$y^* = M_0 \left(\frac{1}{\epsilon} + \sum_{i=1}^{n} \frac{1}{q_i} \right)^{-1}. \tag{6.33}$$

In (6.32) and (6.33), if we now take the limit as $\epsilon \to 0^+$, we obtain $x_i^* = 0$ for all $i > 1$, $x_1^* = M_0$ and $y^* = 0$.

An obvious question now is to state certain condition so that the two cases, namely $1 \notin T(\lambda)$ and $1 \in T(\lambda)$, can be verified easily. In this context, we state the following condition

$$\frac{(\mu_n - \mu_1)}{q_n} + \frac{(\mu_{n-1} - \mu_1)}{q_{n-1}} + \ldots + \frac{(\mu_2 - \mu_1)}{q_2} < \frac{\lambda}{(1-\lambda)}. \tag{6.34}$$

Here μ_1 is the risk-free return. For the risk-free asset a_1, $q_1 = 0$ but it does not enter in (6.34).

Form the statement of Theorem 6.4.1 and related condition for the determination of $T(\lambda)$ for $k = (n-2)$ it is clear that if the condition (6.34) is not satisfied then $1 \notin T(\lambda)$. Otherwise if condition (6.34) is satisfied then $1 \in T(\lambda)$. Therefore the above discussion leads to the following theorem

Theorem 6.4.2 *Let $\lambda \in (0,1)$ be given. If the condition (6.34) is not satisfied then $T(\lambda)$ and associated (x^*, y^*) should be determined as per Theorem 6.4.1. Otherwise if (6.34) is satisfied, then all wealth M_0 should be invested in the risk free asset, i.e. $x_1^* = M$, $x_i^* = 0$, $i \neq 1$ and $y^* = 0$.*

Certain Observations about the Solution of Problem $PO(\lambda)$

(i) Consider the solution where

$$\frac{(\mu_n - \mu_{n-1})}{q_n} < \frac{\lambda}{(1-\lambda)},$$

and

$$\frac{(\mu_n - \mu_{n-2})}{q_n} + \frac{(\mu_{n-1} - \mu_{n-2})}{q_{n-1}} \geq \frac{\lambda}{(1-\lambda)}.$$

Then Theorem 6.4.1 tells that the optimal portfolio should consist of assets a_n and a_{n-1} only. Further, the actual amounts of investments for these assets should be

$$x_n^* = \frac{M_0}{q_n}\left(\frac{1}{q_{n-1}} + \frac{1}{q_n}\right)^{-1} = M_0\left(1 + \frac{q_n}{q_{n-1}}\right)^{-1},$$

and

$$x_{n-1}^* = \frac{M_0}{q_{n-1}}\left(\frac{1}{q_{n-1}} + \frac{1}{q_n}\right)^{-1} = M_0\left(1 + \frac{q_{n-1}}{q_n}\right)^{-1}.$$

Therefore, if q_n is much larger than q_{n-1}, then it is possible that x_n^* is nearly zero while x_{n-1}^* is nearly equal to M_0.

The above discussion suggests that the process of constructing an optimal portfolio is a *two phase decision process*. In the first phase, the assets are selected according to their rates of return, thereby giving the set of *investable assets*. In the second phase, the actual amounts allocated to the investable assets (those assets which have been selected in Phase-1) are determined based on their risk levels. Here if the risk of any investable asset is very high, then its wealth allocation is very small and therefore it may be neglected in the optimal portfolio. Thus in Phase-1, an asset may be eliminated if its return is very low, while in Phase-2, it may be eliminated if its risk is very high.

(ii) For the optimal portfolio x^* as given by Theorem 6.4.1, we have

$$x_i^* q_i = \begin{cases} y^*, & i \in T(\lambda) \\ 0, & \text{otherwise.} \end{cases} \tag{6.35}$$

Since $q_i x_i^*$ is the risk associated with the ith asset when the amount x_i^* is investable in it, (6.35) tells that for the assets selected for investment, we invest them with the amounts such that they have the same risk y^*. Thus the optimal solution of $PO(\lambda)$ gives an investment plan in which we invest a small amount for assets having high risk, and invest a large amount for assets having low risk. This strategy will not increase the maximum risk but certainly increase the overall expected return.

(iii) The amounts x_i^* $(i \in T(\lambda))$ as given in Theorem 6.4.1 does not depend on μ_i as long as the set $T(\lambda)$ is selected. Thus the information about the expected returns μ_i is used to determine the set of investable assets only. Later only risks of the investable assets are used to determine the actual allocation of wealth in the investable assets.

(iv) In Theorem 6.4.1, the inequalities are used to define the ranking rule for selecting the investable assets. Because of the presence of these inequalities, this model has some robustness and is not that much sensitive against errors in the parameters μ_i $(i = 1, 2, \ldots, n)$. In fact Cai et al. [23] gave an explicit result for allowable perturbation δ_i in μ_i so that the optimal portfolio as obtained by Theorem 6.4.1 remains unchanged.

We now present some illustrative examples to verify the above points.

Example 6.4.1 *Consider a six asset $(a_1, a_2, a_3, a_4, a_5, a_6)$ portfolio optimization problem $PO(\lambda)$ with the following data.*

$$\mu = col(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6) = (1, 3, 5, 6, 7, 9)^T$$
$$q = col(q_1, q_2, q_3, q_4, q_5, q_6) = (5, 10, 6, 7, 400, 8)$$
$$M_0 = Rs\ 60{,}000$$
$$\lambda = 0 \cdot 35.$$

Use Theorem 6.4.1 to determine the optimal portfolio.

Solution Here the asset returns μ_i are already in the ascending order. Next we note that

$$\frac{\lambda}{(1-\lambda)} = 0.5384,$$

and

$$\frac{(\mu_6 - \mu_5)}{q_6} = \frac{(9-7)}{8} = 0 \cdot 25 < 0 \cdot 5384$$

$$\frac{(\mu_6 - \mu_4)}{q_6} + \frac{(\mu_5 - \mu_4)}{q_5} = \frac{(9-6)}{8} + \frac{(7-6)}{400} = 0 \cdot 3775 < 0 \cdot 5384$$

$$\frac{(\mu_6 - \mu_3)}{q_6} + \frac{(\mu_5 - \mu_3)}{q_5} + \frac{(\mu_4 - \mu_3)}{q_4} = \frac{(9-5)}{8} + \frac{(7-5)}{400} + \frac{(6-5)}{7} = 0 \cdot 647 > 0 \cdot 5384.$$
$$= 0 \cdot 647 > 0 \cdot 5384.$$

Hence $(n-k) = 4$, which gives $k = 6 - 4 = 2$. This gives

$$T(0.35) = \{n, n-1, \ldots, n-k\} = \{6, 5, 4\},$$

and therefore the investable assets are a_4, a_5 and a_6.

Next we determine the allocation in these investable assets using the formulae given in Theorem 6.4.1, we have:

$$x_4^* = \frac{60{,}000}{7}\left(\frac{1}{7} + \frac{1}{400} + \frac{1}{8}\right)^{-1} = 31704 \cdot 09$$

$$x_5^* = \frac{60{,}000}{400}\left(\frac{1}{7} + \frac{1}{400} + \frac{1}{8}\right)^{-1} = 554 \cdot 82$$

$$x_6^* = \frac{60{,}000}{8}\left(\frac{1}{7} + \frac{1}{400} + \frac{1}{8}\right)^{-1} = 27741 \cdot 08,$$

and

$$y^* = 60{,}000\left(\frac{1}{7} + \frac{1}{400} + \frac{1}{8}\right)^{-1} = 221928 \cdot 66.$$

Therefore the optimal portfolio consists of the allocation $x_1^* = 0$, $x_2^* = 0$, $x_3^* = 0$, $x_4^* = 31704 \cdot 09$, $x_5^* = 554 \cdot 82$ and $x_6^* = 27741 \cdot 08$ from the given wealth $M_0 = 60{,}000$. Further the minimum value of the L_∞-risk measure for the optimal portfolio x^*, i.e. $w_{L_\infty}(x^*)$ is $y^* = 221928 \cdot 66$. In fact $y^* = q_4 x_4^* = q_5 x_5^* = q_6 x_6^* = 221928 \cdot 66$ and therefore all investable assets with the chosen optimal allocation have the same risk y^*. Further the maximum value of the portfolio expected return is

$$\sum_{i=1}^{n} \mu_i x_i^* = \mu_4 x_4^* + \mu_5 x_5^* + \mu_6 x_6^* = 443778 \cdot 0.$$

Here we see that the returns of the asset a_1, a_2 and a_3 are small in comparison to a_4, a_5 and a_6 and they are not selected for the investment. This is precisely the

Phase-1 of the investable assets selection procedure. Further, as risk of the asset a_5 is very large, it is therefore allocated a very small amount of investment. This is the Phase-2 of the investable selection procedure.

□

Some More Observations on L_∞-Risk Model

(i) Looking at the definition of the L_∞ risk measure $w_{L_\infty}(x)$ or the formula for the optimal allocation x^*, one gets the impression that none of these depend on the covariances between the assets. Further it seems that only the risks of the individual assets, rather than the risk of the entire portfolio, are taken care off in this development. But this is not true as being explained below.

Let us recall that the total portfolio risk is given by $E\left(\left|\sum_{i=1}^{n} r_i x_i - \sum_{i=1}^{n} \mu_i x_i\right|\right)$.

Let ξ be a given positive number. We are interested in making $P\left(\left|\sum_{i=1}^{n} r_i x_i - \sum_{i=1}^{n} \mu_i x_i\right| \geq \xi\right)$ as small as possible because this ensures that the deviation of the actual total return from the expected total return is as small as possible. But by Markov inequality we have

$$\begin{aligned} P\left(\left|\sum_{i=1}^{n} r_i x_i - \sum_{i=1}^{n} \mu_i x_i\right| \geq \xi\right) &\leq \frac{1}{\xi} E\left(\left|\sum_{i=1}^{n} r_i x_i - \sum_{i=1}^{n} \mu_i x_i\right|\right) \\ &\leq \frac{1}{\xi} \sum_{i=1}^{n} E(|r_i - \mu_i| x_i) \\ &\leq \frac{n}{\xi} w_{L_\infty}(x). \end{aligned} \tag{6.36}$$

The inequality (6.36) illustrates that the total portfolio risk is small if $w_{L_\infty}(x)$ is kept small. But in the expression for the total portfolio risk, the covariances among various assets are certainly involved. This explains that in the L_∞-risk model, the covariances are not ignored and minimizing $w_{L_\infty}(x)$ indirectly attempts minimizing the total portfolio risk.

(ii) In analogy with Markowitz's mean-variance model, it is natural to ask certain questions with regard to the efficient frontier and the status of CAPM type model for L_∞-risk model. Cai et al. [23] developed an explicit procedure for tracing the efficient frontier for L_∞-risk model but did not present any CAPM type result.

(iii) For Markowitz's model we can generate efficient frontier easily provided short selling is allowed (we refer to the *two fund theorem* in this regard). For L_∞-risk model discussed here, generation of efficient frontier becomes easier provided short selling is *not* allowed (we refer to Theorem 6.4.1 in this regard).

(iv) Cai et al. [23] L_∞-risk model can be made more robust if more flexibility is allowed with regard to the parameters μ_i ($i = 1, 2, \ldots, n$). Deng et al. [36] presented a minimax type model for portfolio optimization where μ_i's are allowed to be in the interval, $a_i \leq \mu_i \leq b_i$ ($i = 1, 2, \ldots, n$). They used the celebrated *minimax theorem* along with the results of Cai et al. [23] to solve the resulting optimization problem.

So far we have considered certain variations of Markowitz's mean-variance model where the portfolio risk is taken different from the standard deviation (e.g. L_1-risk or L_∞-risk) but it is still essentially based on moments of the portfolio return. In the coming sections, we shall discuss few variations of Markowitz's model where portfolio risk is quantile based. In particular, we discuss VaR and CVaR based portfolio optimization problems.

6.5 Value-at-Risk of an Asset

Before defining *value-at-risk* of an asset (denoted by VaR), we consider the below given example for the sake of motivation.

Consider two time instances $t = 0$ and $t = T$. Let $S(0) = 100$ and assume that we buy a share of stock at $t = 0$ to sell it at $t = T$. As $S(T)$ is a random variable, we cannot predict the quantum of profit/loss. Obviously we shall suffer a loss if $S(T) < 100\, e^{rT}$, where r is the risk-free rate under continuous compounding.

A natural question at this stage could be to determine the probability that loss is less than or equal to a specified amount, say Rs 20, i.e. $P[(100\, e^{rT} - S(T)) \leq 20]$. This probability can be easily be computed once the probability distribution of $S(T)$ is known. But it becomes more interesting if this question is reversed. Here we fix the probability, say 95%, and seek the *amount* such that the probability of loss not exceeding *this amount* is more than or equal to 95%. The amount so obtained is essentially called the value-at-risk of the given asset. Thus

$$P[(100\, e^{rT} - S(T)) \leq VaR] \geq 0 \cdot 95\,, \tag{6.37}$$

and VaR represents the predicted maximum loss with specified probability (0.95 in our example) over a certain period of time which is T in our case.

The expression (6.37) suggests that VaR is a measure related to percentiles of loss distributions. In certain sense, it tries to answer the basic question which every investor seems to ask at some point in time, namely, what is the most he/she can lose on his/her investments? Value-at-Risk tries to answer this question within a reasonable bound. VaR and also other downside risk measures are very useful in assessing the risk for securities with asymmetric return distributions, such as call and put options.

VaR has been developed by JP Morgan, and made available through Risk Metrics software in October 1994. We now give the general definition of VaR for a random variable X. This random variable may represent the loss distribution of the asset return with $-X$ being represented as gain.

Definition 6.5.1 (Value at Risk) *Let X be the given random variable and α be the given probability level. Then the VaR of X with confidence level $(1-\alpha)$, $0 < \alpha < 1$, denoted by $VaR_{(1-\alpha)}(X)$ is defined as*

$$VaR_{(1-\alpha)}(X) = Min\{z : F_X(z) \geq (1-\alpha)\} = F_X^{-1}(1-\alpha), \tag{6.38}$$

where F_X denotes the cumulative distribution function of the random variable X.

In view of the above definition $VaR_{(1-\alpha)}(X)$ is a lower $(1-\alpha)$ percentile of the random variable X. If we substitute the expression for $F_X(z)$ in (6.38), we obtain

$$VaR_{(1-\alpha)}(X) = \text{Min}\{z : P(X \leq z) \geq (1-\alpha)\}. \tag{6.39}$$

Thus $VaR_{(1-\alpha)}(X)$ for an asset is the value z such that the probability that the maximum loss X is at most z, is at least $(1-\alpha)$.

The use of VaR involves two chosen parameters. These are confidence level $(1-\alpha)$ and the holding period T of the asset. The choice of α, and hence $(1-\alpha)$, depends on the purpose to which our risk measure is utilized. In practice α is typically taken as 10%, 5% *and* 1%, so that the typical confidence levels are 90%, 95% *and* 99%. The usual holding periods are one day or one month, but it can be even one quarter or more. Given the confidence level $(1-\alpha)$ and horizon T, VaR is a bound such that the loss over the horizon is less than this bound with probability equal to the confidence coefficient. For example, if horizon is one week, the confidence level is 99% (so $\alpha = 0.01$) and VaR is Rs 50,000, then there is only a 1% chance of loss exceeding Rs 50,000 over the next week.

From (6.39) we note that $VaR_{(1-\alpha)}(X) = F_X^{-1}(1-\alpha)$ and therefore for a continuous loss distribution $VaR_{(1-\alpha)}(X)$ is simply the loss such that

$$P(X \leq VaR_{(1-\alpha)}(X)) = (1-\alpha).$$

Let us recall that if $X \sim N(\mu, \sigma^2)$ then the q-percentile of X is $\mu + \sigma\Phi^{-1}(q)$, where Φ is the standard normal density function. Therefore

$$VaR_{(1-\alpha)}(X) = \mu + \sigma\Phi^{-1}(1-\alpha),$$

where $(1-\alpha)$-percentile of Φ, namely $\Phi^{-1}(1-\alpha)$, is that value of the standard normal variate for which the area in the left is $(1-\alpha)$.

The above relation shows that for normally distributed random variables, VaR is proportional to the standard deviation.

Example 6.5.1 *Suppose that the stock price is lognormal with mean 12% and standard deviation 30%. Let the interest rate r for the period $t = 0$ to $t = 1$ be 8% and stock price $S(0)$ at time $t=0$ be Rs 100. Determine the VaR for the given stock at 95% confidence level.*

Solution We are given that

$$ln\left(\frac{S(1)}{S(0)}\right) \sim N(0.12, (0.30)^2),$$

and our aim is to find VaR for $(1-\alpha) = 0.95$, i.e. for $\alpha = 0.05$. By definition, loss equals $(100e^{0.08} - S(1))$ and

$$P[(100e^{0.08} - S(1)) \leq VaR] = 0.95. \tag{6.40}$$

Now from the given information

$$\frac{ln\left(\frac{S(1)}{S(0)}\right) - 0.12}{0.30} \sim N(0,1). \tag{6.41}$$

Therefore (6.40) gives

$$P\left[\frac{S(1)}{S(0)} \geq \frac{(100e^{0.08} - VaR)}{100}\right] = 0.95$$

i.e.

$$P\left[ln\left(\frac{S(1)}{S(0)}\right) \geq ln\frac{(100e^{0.08} - VaR)}{100}\right] = 0.95$$

i.e.

$$P\left[\frac{ln\left(\frac{S(1)}{S(0)}\right)-0.12}{0.30} \geq \frac{ln\left(\frac{(100e^{0.08}-VaR)}{100}\right)-0.12}{0.30}\right] = 0.95. \qquad (6.42)$$

But from (6.41) and (6.42), we get

$$\frac{ln\left(\frac{100e^{0.08}-VaR}{100}\right)-0.12}{0.30} = -1.645,$$

which determines

$$VaR = 100(e^{0.08} - e^{-0.3750}) = Rs\ 39.50.$$

□

Example 6.5.2 *Let an investment A return a gain of Rs 100 with probability 0.96 and a loss of Rs 200 with probability 0.04. Obtain VaR of the given investment with 95% confidence level.*

Solution We note that our definition of VaR is based on the probability distribution of the random variable X, where X represents the loss of the given investment. Therefore from the given information we have

$$X = \begin{cases} -100, & \text{with probability } 0\cdot96 \\ 200, & \text{with probability } 0\cdot04. \end{cases}$$

For determining the value of VaR of the given investment we need to use the relation

$$VaR_{(1-\alpha)}(X) = \text{Min}\{z : P(X \leq z) \geq (1-\alpha)\}$$

with $(1-\alpha) = 0\cdot95$. Since X is a discrete random variable we have

$$P(X \leq -100) = P(X = -100) = 0\cdot96 \geq 0\cdot95,$$

and $P(X \leq 200) = 1 \geq 0\cdot95$. But $-100 = \text{Min}(-100,\ 200)$ and therefore the required VaR is $Rs\ -100$.

□

Example 6.5.3 *Let there be two identical bonds A and B. Each of these defaults with probability 0.04 giving a loss of Rs 100. Further there is no loss if default does not occur. Let default occur independently and C be a portfolio consisting of these two bonds A and B. Obtain VaR of bonds A and B; and also of the portfolio C at 95% confidence level.*

Solution Following on lines similar to Example 6.5.2, it is simple to get VaR(A)=0 and VaR(B)=0. Since C is the portfolio consisting of bonds A and B, its return R becomes the sum of returns of bonds A and B. Writing returns in terms of the loss function, we obtain the distribution of loss X of investment C as

$$X = \begin{cases} 0, & \text{with probability } (0\cdot96)^2 = 0\cdot9216 \\ 200, & \text{with probability } (0\cdot04)^2 = \cdot0016 \\ 100, & \text{with probability } (1-(0\cdot96)^2-(0\cdot04)^2) = 0\cdot0768. \end{cases}$$

Therefore at 95% confidence level,

$$VaR(C) = Rs\ 100,$$

because P(loss of investment $C \leq 100$) $= 0.9216 + 0.0768 = .9984 > 0.95$, and P(loss of investment $C \leq 200$) $= 1 > 0\cdot95$, but $100 = \text{Min}(100, 200)$. Here we may note that $P(X \leq 0) = P(X = 0) = 0\cdot9216 < 0\cdot95$.

□

Some Theoretical and Computational Difficulties with VaR

(i) In Example 6.5.3, at 95% confidence level VaR(C)=100 but VaR(A)+VaR(B)=0. Thus $VaR(C) > VaR(A) + VaR(B)$. But this violates the principle that *diversification reduces risk*. We expect a *good* risk measure to respect this principle. Unfortunately VaR does not do so. Mathematically it means that *VaR is not a subadditive risk measure*. We call a risk measure f to be *subadditive* if for two different investments A and B,

$$f(A+B) \leq f(A) + f(B),$$

i.e. the total risk of two different investment portfolios does not exceed the sum of individual risks.

In Example 6.5.3, diversification has actually increased the risk if VaR is used as a risk measure.

(ii) VaR does not pay any attention to the magnitude of losses beyond the VaR value. For example it is very unlikely that an investor will take a neutral view for two portfolios with identical expected return and VaR, but return distribution of one portfolio having short left tail and other having a long left tail.

(iii) There is additional difficulty with VaR in its computation and optimization. When VaR is computed by generating scenarios, it turns out to be a non smooth and non convex optimization problem is required to be solved.

The above shortcomings of VaR has motivated researchers to look for other quantile based risk measures and CVaR (conditional value at risk) is one such risk measure.

Before we proceed with the discussion of CVaR, we remark that inspite of the difficulties outlined above, VaR is still very popular in the market. Therefore we need to discuss some non-parametric and parametric methods for its estimation.

Nonparametric Estimation of VaR

Here we describe a procedure for the nonparametric estimation of VaR. Since it is a nonparametric estimation, the loss distribution is not assumed to be in any parametric family of distributions such as normal distributions.

To motivate the procedure, let us consider a simple example. Suppose that we hold a Rs 20,000 position in an index fund, e.g. NIFTY, so that our returns are those of this index. Suppose we require a 24-hour VaR by using 1000 daily returns on NIFTY for the period ending on December 31,2010. These are approximately the last four years of daily returns in the time series of NIFTY.

Let the confidence level $(1-\alpha)$ be 95%, i.e. $\alpha = 5\%$. Since 5% of 1000 is 50 and VaR is a loss which is minus of the revenue, we can estimate the required VaR by taking the 50th smallest daily return. Let this be $r_{(50)}$, the 50th *order statistic* of the sample of historic returns. This means that a daily return of $r_{(50)}$ or less occurred only 5% time in the historic data. Equivalently this means that a daily loss of $-r_{(50)}$ or more occurred only 5% time in the historic data. A return of $r_{(50)}$ on a $Rs\ 20,000$ investment yields a revenue of $20,000r_{(50)}$ which can be taken as an estimate of VaR at 95% confidence level. Specifically if $r_{(50)} = -0 \cdot 0227$, then $-r_{(50)} = 0 \cdot 0227$ and the estimated VaR is $20,000(0 \cdot 0227)$, i.e. $Rs\,454$.

The above procedure can be generalized easily. We first note that $(1-\alpha)^{th}$ percentile of loss equals the αth percentile of the returns. Suppose that there are n returns $r_1, r_2, \ldots, r_n$ in the historic sample and let k be equal to $(n\alpha)$ rounded to the nearest integer. Then α percentile of the sample of returns is the kth smallest return, that is, the kth order statistic $r_{(k)}$ of the sample of returns. If S is the size of the initial investment then the required estimate of VaR is $-Sr_{(k)}$. Here minus sign converts revenue (return times initial investment) to a loss.

Remark 6.5.1 *Estimation using sample percentiles is only feasible if the sample size is large. If T(holding period) is taken a quarter rather than a day, then in a four years duration we will have only 16 observations. In such a situation, increasing the number of years for historical data would also not increase sample size substantially, and also because of volatility there may be bias in our estimate.*

Parametric Estimation of VaR

For small sample size, VaR can be best estimated by parametric technique. We discuss only the case of normal distribution because it is extremely simple.

Since, in general, the historical data or returns is given, we have

$$VaR_{(1-\alpha)}(loss) = -VaR_{(\alpha)}(return).$$

Therefore

$$VaR_{(1-\alpha)}(X) = -S(\overline{\mu} + \Phi^{-1}(\alpha)\sigma),$$

which can be estimated by $-S(\overline{X} + \Phi^{-1}(\alpha)\hat{\sigma})$. Here $\overline{X}$ and $\hat{\sigma}$ are the mean and standard deviation of sample of returns and S is the size of the initial investment.

Let us assume that from the given historical data, $\overline{X} = -3 \cdot 107 \times 10^{-4}$ and $\hat{\sigma} = 0 \cdot 0151$. Also $\Phi^{-1}(\alpha) = -1 \cdot 645$ for $\alpha = 0 \cdot 05$. Then the estimate of the required VaR is $((-3 \cdot 107 \times 10^{-4}) + (-1 \cdot 645)(0 \cdot 0151)(20,000)) = Rs\ 471$.

In this example, the estimate of the expected return, is negative. This may be because the particular years used include a prolonged bear market. However, we certainly do not expect average future returns to be negative, otherwise we would not have invested Rs 20,000 in the market.

Since in actual practice, normality assumption is not going to hold, we need to apply historical simulation and Monte Carlo strategies to generate a large number of possible scenarios. Also we need to understand VaR for a portfolio of assets which could include options along with certain number of stocks. The estimation of VaR in this situation also requires Monte Carlo simulation. We shall discuss Monte carlo simulation in Chapter 14.

6.6 Conditional Value-at-Risk

One major criticism of VaR is that it pays no attention to the magnitude of losses beyond the VaR value. Also VaR is not subadditive, which not only violates the principle of diversification, but also creates computational difficulty in the resulting portfolio optimization problem. These undesirable features of VaR led to the development of *conditional value-at-risk*, denoted by CVaR, which is obtained by computing the expected loss given that the loss exceeds VaR. Some other names for CVaR are *expected shortfall*, *expected tail loss* and *tail VaR*. CVaR has been introduced by Rockafellar and Uryasev [111] who studied many mathematical properties of CVaR in detail.

To define CVaR we first define the occurrence of a tail event. We say that a *tail event* occurs if the loss exceeds the VaR. Then CVaR is the conditional expectation of loss given that the tail event occurs. We now proceed to define CVaR and discuss its minimization mathematically. To be specific, we define CVaR in context of portfolio optimization.

We consider a portfolio of assets with random returns. Let $f(w,r)$ denote the loss function when we choose the investment w from a set of feasible portfolios and r is the realization of random returns. In the context of our portfolio optimization problem, $f(w,r) = -r^T w$ where $r = (r_1, r_2, \ldots, r_n)^T$ is the vector of random returns and $w = (w_1, w_2, \ldots, w_n)^T$ with $e^T w = 1$, e being the vector $(1,1,\ldots,1)^T$. Here in the expression of $f(w,r)$, minus sign has been taken to express return $r^T w$ as loss function. We assume that the return vector r has a probability density function $p(r)$, e.g., the random returns may have a multivariate normal distribution.

For a fixed weight vector w, we define

$$\psi(w,q) = \int_{f(w,r)\le q} p(r)dr, \tag{6.43}$$

which represents the cumulative distribution function of the loss associated with the weight vector w. In fact $\psi(w,q) = P(\text{loss} \le q)$. Therefore for a given confidence level $(1-\alpha)$, the $VaR_{(1-\alpha)}(w)$ associated with the portfolio is given by

$$VaR_{(1-\alpha)}(w) = \text{Min}\{q \in \mathbf{R} : \psi(w,q) \ge (1-\alpha)\}.$$

Definition 6.6.1 (Conditional Value-at-Risk) *Let w be a given portfolio and $(1-\alpha)$ be the given confidence level. Then $CVaR_{(1-\alpha)}(w)$ associated with the portfolio w is defined as*

$$CVaR_{(1-\alpha)}(w) = \frac{1}{\alpha}\int_{f(w,r)\ge VaR_{(1-\alpha)}(w)} f(w,r)p(r)dr. \tag{6.44}$$

Remark 6.6.1 *In special case of returns having discrete probability distribution, we have*

$$CVaR_{(1-\alpha)}(w) = \frac{1}{\alpha}\left(\sum_{j\in J_1} p_j f(w,r^{(j)})\right), \tag{6.45}$$

where $J = \{1,2,\ldots,m\}$, $J_1 = \{j \in J : f(w,r^{(j)}) \ge VaR(w)\}$ and the vector $r^{(j)}$ is the jth realization of the return vector r with probability p_j.

Remark 6.6.2 *In certain sense, (6.44) and (6.45) tell that $CVaR_{(1-\alpha)}(w)$ is the average of the outcomes greater than $VaR_{(1-\alpha)}(w)$. This is certainly true for continuous distribution functions; but for general distributions this is not exactly true. There are certain subtle aspects which need to be explained. We shall refer to Sriboonchitta et el [128] in this regard.*

Example 6.6.1 *Let the loss function $f(w,r)$ be given by $f(w,r) = -r$ where $r = 75 - j$, $(j = 0,1,2,\ldots,99)$ with probability 1%. Evaluate $VaR(w)$ and $CVaR(w)$ at 95% confidence level.*

Solution The loss function $f(w,r) = -r$, where $r = 75 - j$, $(j = 0,1,2,\ldots,99)$, takes 100 values given by $(-75, -74, -73, \ldots, 0, 1, \ldots, 19, 20, 21, 22, 23$ and $24)$ with equal probability $p = 0\cdot01$. Therefore $VaR(w) = 20$ for $1-\alpha = 0\cdot95$.
We next evaluate $CVaR_{(1-\alpha)}(w)$ for $(1-\alpha) = 0.95$ by using formula (6.45). This gives

$$CVaR(w) = \frac{1}{0\cdot05}(20+21+22+23+24)(0\cdot01) = 22.$$

Thus at 95% confidence level VaR is 20 and CVaR is 22.

□

In the above example for the same confidence level CVaR is more than VaR. We have the following result in this regard.

Lemma 6.6.1. *For the given confidence level $(1-\alpha)$, $CVaR_{(1-\alpha)}(w) \ge VaR_{(1-\alpha)}(w)$.*

Proof. We have

$$\begin{aligned} CVaR_{(1-\alpha)}(w) &= \frac{1}{\alpha}\int_{f(w,r)\ge VaR_{(1-\alpha)}(w)} f(w,r)p(r)dr \\ &\ge \frac{1}{\alpha}\int_{f(w,r)\ge VaR_{(1-\alpha)}(w)} VaR_{(1-\alpha)}(w)p(r)dr \\ &= \frac{VaR_{(1-\alpha)}(w)}{\alpha}\int_{f(w,r)\ge VaR_{(1-\alpha)}(w)} p(r)dr \\ &= VaR_{(1-\alpha)}(w). \end{aligned}$$

□

Remark 6.6.3 *As CVaR of a portfolio is always more than or equal to VaR for same $(1-\alpha)$, portfolios with small CVaR will also have small VaR. But this does not mean that the minimization of CVaR is equivalent to the minimization of VaR.*

Minimization of CVaR

We now discuss Rockafellar and Uryasev's [111] procedure to minimize CVaR. Since the definition of CVaR involves the VaR function explicitly, it is not very convenient to optimize CVaR directly. Therefore we introduce the auxiliary function

$$F_{(1-\alpha)}(w,q) = q + \frac{1}{\alpha}\int_{f(w,r)\geq q}(f(w,r)-q)p(r)dr. \tag{6.46}$$

If we denote $a^+ = Max(a,0)$, $a \in \mathbb{R}$, then from (6.46) we get

$$F_{(1-\alpha)}(w,q) = q + \frac{1}{\alpha}\int(f(w,r)-q)^+p(r)dr. \tag{6.47}$$

We refer to Rockafellar and Uryasev [111] for below given results.

Lemma 6.6.2. *The auxiliary function $F_{(1-\alpha)}(w,q)$ is a convex function of q.*

Lemma 6.6.3. *$VaR_{(1-\alpha)}(w)$ is a minimizer of $F_{(1-\alpha)}(w,q)$ over q.*

Lemma 6.6.4. *The minimum value of $F_{(1-\alpha)}(w,q)$ over q is $CVaR_{(1-\alpha)}(w)$.*

In view of the above Lemmas, we have for a given w,

$$CVaR_{(1-\alpha)}(w) = \mathrm{Min}_q(F_{(1-\alpha)}(w,q)) = F_{(1-\alpha)}(w, VaR_{(1-\alpha)}(w)). \tag{6.48}$$

The left equality of (6.48) tells that we can minimize CVaR directly without computing VaR first. Since for portfolios, the loss function $f(w,r) = -r^Tw$ is a linear and hence also convex function of w, the auxiliary function $F_{(1-\alpha)}(w,q)$ is a convex function of w. Therefore the problem

$$\begin{aligned} &\mathrm{Min} \quad CVaR_{(1-\alpha)}(w) \\ &\text{subject to} \\ &\qquad e^Tw = 1, \end{aligned} \tag{6.49}$$

is equivalent to

$$\begin{aligned} &\underset{w\in A}{\mathrm{Min}} \quad F_{(1-\alpha)}(w,q) \\ &\text{subject to} \\ &\qquad e^Tw = 1, \end{aligned} \tag{6.50}$$

which is a smooth convex optimization problem. Though problem (6.50) can be solved by using standard convex optimization techniques, the formulation (6.50)

still needs the computation/determination of joint density function $p(r)$ of random return vector r. In practice it is not simple or even possible, so we present another approach which is based on generation of scenarios via computer simulation. An added advantage of this approach is that it results in a linear programming formulation.

Suppose we have scenarios $r^{(s)}$, $(s=1,2,\ldots,n_s)$, which may represent historical values of the random vector of returns or obtained via computer simulation. We shall assume that all scenarios have equal probability and define the following empirical distribution of the random returns based on the available scenarios

$$\widetilde{F}_{(1-\alpha)}(w,q) = a + \frac{1}{\alpha n_s}\sum_{s=1}^{n_s}(f(w,r^{(s)})-q)^+. \tag{6.51}$$

Using (6.51) to approximate $F_{(1-\alpha)}(w,q)$, we get an approximation to the problem (6.50) as

$$\begin{aligned} &\underset{w\in A}{\mathrm{Min}} \quad \widetilde{F}_{(1-\alpha)}(w,q) \\ &\text{subject to} \\ &\qquad e^Tw = 1. \end{aligned} \tag{6.52}$$

Now writing $(f(w,r^{(s)})-q)^+$ as z_s and using the definition of a^+, we obtain

$$\begin{aligned} &\underset{w\in A, z_s}{\mathrm{Min}} \quad q + \frac{1}{\alpha n_s}\sum_{s=1}^{n_s} z_s \\ &\text{subject to} \\ &\qquad z_s \geq f(w,r^{(s)}) - q, \quad s = 1,2,\ldots,n_s \\ &\qquad e^Tw = 1 \\ &\qquad z_s \geq 0, \quad s = 1,2,\ldots,n_s. \end{aligned} \tag{6.53}$$

In the context of portfolio optimization, $f(w,r^{(s)}) = -w^Tr^{(s)}$ is linear and so the problem (13.5) becomes a linear programming problem.

Most often we try to optimize a suitable performance measure (e.g, expected return) while making sure that certain risk measures do not exceed a threshold value. It could be variance or absolute deviation as has been discussed earlier. When the risk measure is CVaR, the resulting optimization problem is

$$\text{Max} \quad \mu^T w$$

subject to

$$\begin{aligned} CVaR_{(1-\alpha)}(w) &\le u_\alpha \\ e^T w &= 1, \end{aligned} \tag{6.54}$$

which can be approximated as

$$\underset{w,z,q}{\text{Max}} \quad \mu^T w$$

subject to

$$\begin{aligned} u_\alpha &\ge q + \frac{1}{\alpha n_s}\sum_{s=1}^{n_s} z_s \\ z_s &\ge f(w, r^{(s)}) - q \qquad (s = 1, 2, \ldots, n_s) \\ e^T w &= 1 \\ z_s &\ge 0 \qquad (s = 1, 2, \ldots, n_s). \end{aligned}$$

In (6.54), we can have more than one CVaR constraint for different levels α. Also similar to Markowitz's model, we can have a trade off between return and CVaR. We can refer to Mansini et al. [89] for more details in this regard.

6.7 Preference Relation, Utility theory and Decision Making

The mean-variance criterion used in the Markowitz portfolio model can also be explained in terms of the expected utility maximization principle in decision making. To understand this aspect we first wish to motivate the readers to the celebrated *Von Neumann- Morgenstern's* model for decision making. This model tries to explain how a *rational person* makes his/her *optimal choice* in a given situation. This requires some discussion on *preference relation* and related results in utility theory.

Choice under certainty

We first consider the situation when there is no uncertainty present, i.e. the alternatives are certain. Let this set of alternatives be denoted by Z. As the choice is being made under certainty, we shall get the element of our choice for sure.

In order to perform our choice, i.e. the selection procedure, we prescribe our own preferences, and use them to select the desired *optimal* element of Z. These preferences are expressed as a *preference relation*, say $\succ$ on Z. Thus we say that

for $x, y \in Z$, $x \succ y$ states that we *strictly prefer* x to y. Here our preference relation should reflect our *valuation* of different alternatives, i.e. different elements of Z. These valuations give a quantitative representation of our preference relation.

The above discussion can now be formalized mathematically. A preference relation $\succ$ is a binary relation on Z i.e. a subset A of $Z \times Z$ where $(x, y) \in A$ when $x \succ y$. We demand the following properties with our preference relation $\succ$

(i) The preference relation $\succ$ is *asymmetric* i.e. $x \succ y \Rightarrow y \not\succ x$. Thus if x is strictly preferred to y then y is not strictly preferred to x.

(ii) The preference relation $\succ$ is *negative transitive* i.e. $x \not\succ y$ and $y \not\succ z \Rightarrow x \not\succ z$. Thus if x is not strictly preferred to y and y is not strictly preferred to z then x is not strictly preferred to z.

We write $x \simeq y$ if $x \not\succ y$ and $y \not\succ x$. The relation $\simeq$ is called an *indifference* or *equivalence* relation because it is reflexive, symmetric and transitive. Thus given $x, y \in Z$, exactly one of $x \succ y$, $y \succ x$, or $x \simeq y$ holds. We further write $x \succeq y$ when $y \not\succ x$, i.e. either $x \succ y$ or $x \simeq y$. The relation $\succeq$ is called the *weak preference relation*. It is simple to verify that $x \succeq y$ when $y \not\succeq x$. Also $x \simeq y$ when $x \succeq y$ and $y \succeq x$. For making an *optimal choice* we need to prescribe a numerical representation of our strict preference relation $\succ$. This is done via the utility function.

Definition 6.7.1 (Utility Function) *A function $u : Z \to R$ is called a utility function if*

$$x \succ y \Leftrightarrow u(x) > u(y).$$

An obvious question at this stage is about the existence of such a function u. We have the following theorem in this regard (Sriboonchitta et al. [128]).

Theorem 6.7.1 *Let Z be finite or countably infinite. Then a utility function u as defined above certainly exists.*

Choice Under Uncertainty

We next discuss the more realistic situation of making an *optimal* choice under uncertainty. A typical example of this scenario could be possible losses in investment portfolio which are real-valued random variables. Here the range space Z is the real line R. Taking motivation from the earlier case of choice under certainty, we attempt to give a numerical representation of a preference relation $\succ$ on a class of random variables $\mathfrak{X}$. We say that for $X, Y \in \mathfrak{X}$ and a given preference relation $\succ$, there exists a utility function $u : Z \to R$ (unique up to a positive linear transformation) such that

$$X > Y \Leftrightarrow E(u(X)) > E(u(Y)),$$

provided that the expectations exist. Thus for making their *optimal* choices under uncertainty, decision makers try to maximize their expected utilities. This principle is called *Von Neumann- Mangenstern expected utility maximization principle*. Is this principle always applicable? Does there always exist a utility function as described above for the random variables in the class $\mathcal{X}$? Apart from various mathematical conditions for its existence it was argued that the existence of utility function is restricted to "rational" people. This latter part has been a topic of debate among economists and psychologists. For this as well as other aspects of Von Neumann - Mangenstern theory we shall refer to Fishburn [47].

Utilities and Risk Attitudes

We all know that different people react differently in risky situations. They make decision based upon their own *attitudes* towards risk. Thus, making an investment choice in risky and risk free assets, each person will act differently depending upon his/her risk attitude. Roughly there are three main risk attitudes of an investor. Let the random variable X be a risky prospect, and u be the utility function of the decision maker. We say that the investor is

(i) *risk neutral* if, when facing two risky prospects with same expected value will feel indifferent. In terms of the utility function u, this means that $u(E(X)) = E(u(Y))$. This will happen for example if u is linear, say $u(x) = x$.

(ii) *risk averse* if, when facing two risky prospects with the same expected value, will prefer the less risky one. In terms of the utility function u, this means that $u(E(X)) > E(u(X))$. This will happen for example if u is concave, say $u(x) = \log x$ or $u(x) = -x^2$.

(iii) *risk seeking* if, when facing two risky prospects with the same expected value, will prefer the more risky one. In terms of the utility function u, this means that $u(E(X)) < E(u(X))$. This will happen for example if u is convex, say $u(x) = x^2$.

It is obvious that given a utility function $u(x)$, any function of the form $v(x) = au(x) + b$ with $a > 0$ is a utility function equivalent to $u(x)$. This is because equivalent utility functions give identical rankings in terms of the principle of expected utility maximization.

To explain the meaning of risk averse scenario, let us consider an investor who has two alternatives for future wealth. The first alternative is based on the outcome of tossing of a coin. If the coin turns up 'head' the investor gets *Rs* x, otherwise he/she gets *Rs* y. The second alternative is a sure event of getting *Rs* $\left(\frac{x+y}{2}\right)$ with certainty. Let the investor's utility function be given by a strictly concave function

$u(x)$. Then the expected utility of first alternative is $\frac{u(x)+u(y)}{2}$, whereas the expected utility of the second alternative is $u\left(\frac{x+y}{2}\right)$. Since u is strictly concave we have

$$u\left(\frac{x+y}{2}\right) > \frac{u(x)+u(y)}{2}.$$

Hence by the principle of expected utility maximization we shall prefer the sure wealth of *Rs* $\frac{(x+y)}{2}$ to a 50-50 chance of x or y. Here we must note that both alternatives have the same expected value of *Rs* $\frac{(x+y)}{2}$, but the one without risk (certainty of getting) is preferred. This explains the risk averse attitude of the investor.

The below given result is useful in this regard.

Theorem 6.7.2 *An investor is risk averse if and only if his/her utility function is concave.*

Remark 6.7.1 *Although it has not been mentioned in the statement of above theorem, u is non-decreasing by the definition of utility function. We refer to Sriboonchitta et al. [128] for the proof of the above theorem. The proof essentially uses Jensen's inequality and the defining condition $u(E(X)) > E(u(X))$ for a risk averse investor.*

Let us take another example to understand the risk attitudes. For an investor let there be three possible scenarios: a 5% chance of loosing *Rs* 20,000, a 10% chance of loosing *Rs* 10,000, and a sure chance of loosing *Rs* 1,000. A risk neutral investor will not find any of these situations worse than the other. For a risk averse investor, the first situation is worse than the second one, and the second one is worse than the last one. This is because risk averse investors dislike uncertainty about the size of losses, so they do not prefer even a small possibility of large amount of loss.

Certainty Equivalent

The *certainty equivalent* of a random wealth X is defined to be the amount of a certain (i.e. risk-free) wealth that has a utility level equals to the expected utility of X. Thus if C is the certainty equivalent of a random wealth X then by definition

$$u(C) = E(u(X)).$$

The certainty equivalent of a random variable is same for all equivalent utilities and is measured in units of wealth.

6.8 Mean-Variance Model Revisited in Utility Frame Work

We now discuss Markowitz's mean-variance model in the frame work of expected utility maximization principle. Here we need to solve the following optimization problem

$$\text{Max} \quad E\left(u(r^T w)\right)$$

subject to

$$e^T w = 1, \tag{6.55}$$

where $r^T = (r_1, r_2, \ldots r_i, \ldots, r_n)$ is the vector of asset returns, u is investor's utility function and $w^T = (w_1, w_2, \ldots w_i, \ldots, w_n)$ is the decision vector of asset weights.

Earlier we had agreed to represent a portfolio by $w : (w_1, w_2, \ldots w_i, \ldots, w_n)$ such that $e^T w = 1$, where $e^T = (1, 1, \ldots, 1)$. Therefore $X = r^T w$ denotes the return (random) of the portfolio w. The mean-variance rule of Markowitz states that the portfolio $w^{(1)}$ is preferred to the portfolio $w^{(2)}$ if.

(i) $E(r^T w^{(1)}) \geq E(r^T w^{(2)})$,

and

(ii) $\sigma^2(r^T w^{(1)}) \leq \sigma^2(r^T w^{(2)})$.

If we denote the random variable $r^T w^{(1)}$ by X and the random variable $r^T w^{(2)}$ by Y then the mean-variance rule says that X is preferred to Y when $E(X) \geq E(Y)$ and $Var(X) \leq Var(Y)$. The essence of Markowitz's theory is that the above mean-variance rule is constrained with the principle of expected utility maximization for the utility function $u(x) = ax - \frac{1}{2}bx^2$ with $a > 0$ and $b \geq 0$. This utility function is really meaningful only in the range $x \leq a/b$, where the function is increasing. Also as $b > 0$, the function u is strictly concave every where and therefore corresponds to the risk aversion scenario.

Let the given portfolio have a random wealth value of y. Then

$$\begin{aligned} E(u(y)) &= E\left(ay - \frac{1}{2}b\, y^2\right) \\ &= aE(y) - \frac{1}{2}b(E(y))^2 - \frac{1}{2}bVar(y). \end{aligned} \tag{6.56}$$

Now as per the expected utility maximization principle the optimal portfolio y is the one that maximizes (6.56) with respect to all feasible choices of the random wealth variable y. Thus $E(u(\hat{y})) \geq E(u(y))$ for all feasible y. Let $E(\hat{y}) = M$ and $S = \{y : y \text{ is feasible and } E(y) = M\}$. We shall now show that $Var(\hat{y}) \leq Var(y)$ for all $y \in S$. If possible let there exists $\bar{y} \in S$ such that $Var(\bar{y}) \leq Var(\hat{y})$. Then as $\bar{y} \in S$ it is feasible and $E(\bar{y}) = M$. Further

$$aM - \frac{1}{2}bM^2 - \frac{1}{2}bVar(\hat{y}) \leq aM - \frac{1}{2}bM^2 - \frac{1}{2}bVar(\bar{y}). \tag{6.57}$$

But then (6.57) contradicts that $\hat{y}$ is optimal. Therefore $Var(\hat{y}) \leq Var(y)$ for all $y \in S$, i.e. $\hat{y}$ has the minimum variance with respect to all feasible y's with $E(y) = M$. Hence $\hat{y}$ corresponds to a mean-variance efficient point. Different mean-variance efficient points are obtained by providing different values for the parameters a and b. The readers may identify $y = r^T w$, $E(y) = \mu^T w$ and $Var(y) = w^T C w$ with C as the variance-covariance matrix.

Now we have a very natural question. What happens if we take a general utility function u? How different are the mean-variance efficient portfolios and portfolios that maximize the expected utility i.e. those which are the solution of the problem (6.55) for a general u? In this context we have the following theorem

Theorem 6.8.1 *Let X and Y be two random variables, normally distributed with means μ_1, μ_2 and variances σ_1^2, σ_2^2 respectively. Then following are equivalent*

(i) $E(u(X)) \geq E(u(Y))$ for any $u : R \to R$ which is non decreasing and concave.
(ii) $\mu_1 \geq \mu_2$ and $\sigma_1^2 \leq \sigma_2^2$.

Proof. See details in Sriboorchitta et al. [128]. □

Remark 6.8.1 *In view of the above theorem, if the returns are normally distributed and the utility function is concave, then the portfolio that maximizes the expected utility is on Markowitz's efficient frontier. Exactly which portfolio on the efficient frontier will be obtained, will depend on the choice of utility function.*

6.9 Risk Modeling and Financial Risk Measures

Financial risk management is a broad concept involving various perspectives. It is about managing exposure to risk, such as market risk and credit risk. For these purposes we need a *quantitative concept of risk* in its own right.

Mathematically, risk management is a procedure for shaping a loss function or loss distribution. We have already studied several moment based risk measures (e.g. L_1, L_2 and L_∞-risk measures) and also some quantile based risk measures (e.g.

VaR and CVaR). Here we attempt to understand risk measures in a broader mathematical framework and introduce the concept of *coherent risk measures*(Artzner [5]). An excellent tutorial on this topic is due to Rockafellor [112].

Basically, we wish to answer the following question: *How can we discuss the risk of a financial position mathematically*? For this we note that a financial position is captured by a random variable X, e.g. loss. Information about loss is in its cumulative distribution function F. In the financial context where the *risk* is about the risk of loosing money, we wish to define numerical risk for X as some appropriate number $\rho(X)$ and call it as a risk measure. It seems natural to choose $\rho(\cdot)$ in such a manner that it captures our risk perception and it is also suitable for financial management purposes.

Domain and Range of Risk Measures

As stated above, we wish to assign a numerical value $\rho(X)$ to each random variable X to describe its risk. Here X stands for the loss of an investment portfolio, the loss of a financial position, or capital needed to hold for an insurance company to avoid insolvency.

It is very clear that the range of ρ has to be $\mathbb{R}$. But what about its domain? Let $\mathcal{U}$ denote the real vector space of all possible real valued random variables. Mathematically we do not need the entire vector space $\mathcal{U}$ as the domain of ρ. But rather we need a particular subset of $\mathcal{U}$ namely a convex cone $\mathfrak{X}$.

Definition 6.9.1 (Cone) *A subset $\mathfrak{X}$ in $\mathcal{U}$ is called a cone if for $X \in \mathfrak{X}$ and $\lambda > 0$, we have $\lambda X \in \mathfrak{X}$.*

Definition 6.9.2 (Convex Cone) *A subset $\mathfrak{X}$ in $\mathcal{U}$ is called a convex cone if it is a cone and also a convex set. Thus $\mathfrak{X}$ is a convex cone if for $X, Y \in \mathfrak{X}$, $\lambda > 0$, we have $X + Y \in \mathfrak{X}$ and $\lambda X \in \mathfrak{X}$.*

It is economically meaningful to take the domain of risk measures as a convex cone of the vector space £. This is because we know that diversification should reduce risk. But for this we need to evaluate our risk measure on an investment portfolio of type $X = \sum_{i=1}^{k} \lambda_i Y_i$, where $\lambda_i > 0$, $\sum_{i=1}^{k} \lambda_i = 1$ and Y_i denotes the rate of return of ith asset in the portfolio. Thus if Y_i are in a domain $\mathfrak{X}$ of £, we also require that $\sum_{i=1}^{n} \lambda_i Y_i$, ($\lambda_i > 0$, $\sum_{i=1}^{k} \lambda_i = 1$), should also be in $\mathfrak{X}$; i.e. $\mathfrak{X}$ is a convex cone.

Desirable properties of Risk Measures

Let $\mathfrak{X}$ denote the convex cone of *loss* random variable X. Then, as per Artzner [5], we consider the following desirable properties for a risk measure $\rho : \mathfrak{X} \mapsto \mathbb{R}$.

(A1) (Monotonicity) *If $X \leq Y$ almost surely, i.e. $P(X \leq Y) = 1$, then $\rho(X) \leq \rho(Y)$.*
(A2) (Positive Homogeneity) $\rho(\lambda X) = \lambda \rho(X)$ for $\lambda \geq 0$.
(A3) (Translation Invariance) $\rho(X + a) = \rho(X) + a$, $a \in \mathbb{R}$.
(A4) (Subadditivity) $\rho(X + Y) \leq \rho(X) + \rho(Y)$.

Definition 6.9.3 (Coherent Risk Measure) *A functional $\rho : \mathfrak{X} \mapsto \mathbb{R}$ is called a coherent risk measure if it satisfies axioms (A1)-(A4) listed above.*

Remark 6.9.1 *Though we are calling above properties as axioms, our approach is not axiomatic as these do not characterize a risk measure. These are simply the desirable properties and therefore even if a functional satisfies these axioms, it may not have any meaning in terms of risk. This can be verified for $\rho(X) = E(X)$.*

The above axioms are based on our perception about risk. In case of *loss* random variables, it is natural to believe that smaller losses have smaller risks. This is precisely the axiom (A1). Also the risk of a loss of a financial position should be proportional to the size of the loss of the position. This leads to the axioms (A2) and (A3). The principle that *diversification should reduce risk* leads to the axiom (A4). Also axioms (A2) and (A4) imply that the risk measure is convex, which is certainly a desirable property.

Before we analyze some of already studied risk measures, we need to introduce the notion of *stochastic dominance*. Stochastic dominance rules refer to criteria to order distributions of random variables modelling uncertain quantities such as future returns or investments. Thus these are various partial orders on the space of distribution functions of random variables and they are based on the principle of maximization of expected utility (Sriboonchitta et al. [128]). While stochastic dominance rules shed light on which future prospects are better than others, they do not provide us with any numerical value which we need to manage further our financial positions. This numerical value is provided by risk measures. Therefore, for further analysis of risk measures, apart from axioms (A1)-(A4), we also need to check consistency of risk assessment with respect to stochastic dominance.

Definition 6.9.4 (First Order Stochastic Dominance) *Let X, Y be two random variables with distribution functions F, G respectively. Then X is said to dominate Y in the first order stochastic dominance (FSD), denoted by $X \geq_1 Y$, if $F(.) \leq G(.)$.*

Remark 6.9.2 *If $X \geq_1 Y$ then $E(X) \geq E(Y)$ but the converse is not true. This is because*

$$E(X) = \int_0^{\infty} (1 - F(t))\, dt + \int_{-\infty}^{0} F(t) dt.$$

As preference relations on the space of distribution functions, stochastic dominance rules are essentially *stochastic orders* which have been introduced much earlier in statistics. These were rediscovered and made use in Economics/Finance. We may refer to the survey article by Levy [83] in this regard. An obvious question at this stage is: How can we rank distributions when they intersect? In this situation FSD cannot be applied and therefore we need to weaken the criteria which is consistent with FSD.

Definition 6.9.5 (Second Order Stochastic Dominance) *Let X, Y be two random variables with distribution functions F, G respectively. Then X is said to dominate Y in the second order stochastic dominance (SSD), denoted by $X \geq_2 Y$, if*

$$\int_{-\infty}^{t} (G(u) - F(u))du \geq 0,$$

for all $t \in \mathbf{R}$.

To analyze any proposed risk measure we have to take care of two main criteria. These are being coherent and being consistent with the stochastic dominance rules. In general we say that X *precedes* Y when $X \preceq Y$, and a risk measure ρ is said to be *consistent* with respect to the partial order $\preceq$ if

$$(X \preceq Y) \Rightarrow (\rho(X) \leq \rho(Y)).$$

Here $\preceq$ could be $\preceq_1$ or if need be $\preceq_2$ or some other higher order stochastic dominance rule. Since risks are related to preferences, such a consistency with respect to a suitable partial order $\preceq$ is important.

Remark 6.9.3 *First order stochastic dominance (FSD) implies the Second order stochastic dominance (SSD). Thus $(X \geq_1 Y) \Rightarrow (X \geq_2 Y)$. Also if $(X \geq_2 Y)$, then $E(X) \geq E(Y)$. Further if $(X \geq_2 Y)$ and $E(X) = E(Y)$, then $Var(X) \leq Var(Y)$.*

Though higher order stochastic dominance rules have also been introduced in the literature, in context of portfolio optimization only FSD and SSD have been popular.

Some Already Studied Popular Risk Measures

(i) **Variance** In the early theory of portfolio selection, Markowitz proposed the use of variance as a risk measure. Thus the risk measure ρ is given by

$$\rho(X) = Var(X) = E(X - E(X))^2,$$

and portfolio with small variance is considered less risky.
Here we note that variance as a risk measure is not a coherent risk measure as it is not subadditive. Further it is not consistent with the first order stochastic dominance because $(X \geq_1 Y)$ does not imply $Var(X) \leq Var(Y)$. Due to these reasons, and also to take care of non symmetry in the loss distributions, risk measures based on downside tail distribution of returns have been introduced.

(ii) **Value-at-Risk** For a loss random variable X with the cumulative distribution function F, the value-at-risk of X with confidence level $(1-\alpha)$, $0 < \alpha < 1$ is defined as

$$VaR_{(1-\alpha)}(X) = F^{-1}(1-\alpha) = \text{Min}\{x \in \mathbf{R} : F(x) \geq (1-\alpha)\}.$$

Here $(1-\alpha)$-quantile is the position such that $P(X \leq F^{-1}(1-\alpha)) \geq (1-\alpha))$. Thus, $P(X > F^{-1}(1-\alpha)) \leq \alpha$.
It can now be shown that VaR is again not a coherent risk measure because it is not subadditive. We have already given such an example but below given another example which is probably more general.

Example 6.9.1 *Let X, Y be two independent and identically distributed Bernoulli random variables with parameter p, i.e. $P(X = 0) = (1-p)$ and $P(X = 1) = p$. Let $(1-p)^2 < \alpha < 1$. Show that VaR is not sub additive in this situation.*

Solution As $(1-p)^2 < \alpha < 1$, we have $0 \leq (1-\alpha) < 2p(1-p)$. Therefore $F_X^{-1}(1-\alpha) = 0 = F_Y^{-1}(1-\alpha)$. But $F_{X+Y}^{-1}(1-\alpha) > 0$. Therefore for this choice of α, the subadditive law does not hold. □

Let us recall the definition of SSD. We say that $X \geq_2 Y$ if

$$\int_{-\infty}^{t} (F_Y - F_X)(u)du \geq 0, \quad \forall\, t \in \mathbf{R}.$$

With respect to SSD, $\geq_2$ we define $\geq_{2'}$ as follows: $X \geq_{2'} Y$ if

$$\int_{t}^{\infty} (F_Y - F_X)(u)du \geq 0, \quad \forall\, t \in \mathbf{R}.$$

The order $\geq_{2'}$ is called *risk seeking stochastic dominance* (RSSD), also called *stop-loss-order* which is very popular in actuarial science.

Definition 6.9.6 (Distortion Function) *A function $g : [0,1] \to [0,1]$ is called a distortion function if (i) it is nondecreasing and (ii) $g(0)=0$, $g(1)=1$.*

Definition 6.9.7 (Risk Measure ρ_g) *Let g be the given distortion function. A risk measure ρ_g (risk measure with respect to the given distortion function g) is defined as*

$$\rho_g(X) = \int_0^{\infty} g(1-F(t))\,dt + \int_{-\infty}^{0} [g(1-F(t))-1]\,dt,$$

where F is the cumulative distribution function of the random variable X (say loss random variable).

Remark 6.9.4 *VaR is a risk measure of type ρ_g. Some readers may like to obtain the distortion function g for VaR.*

Lemma 6.9.1. *If the distortion function g is concave then*

$$X \succeq_{2'} Y \Rightarrow \rho_g(X) \le \rho_g(Y).$$

Thus any risk measure ρ_g defined with respect to a concave distortion function g is $\preceq_{2'}$ always consistent.

Using the above lemma it can be shown that VaR is consistent with respect to the stochastic dominance rule induced by the partial order $\preceq_{2'}$. Thus $(X \preceq_{2'} Y) \Rightarrow VaR_{(1-\alpha)}(X) \le VaR_{(1-\alpha)}(Y)$ for all $\alpha \in (0,1)$.

(iii) **Conditional Value-at-Risk** Let us recall that $CVaR_{(1-\alpha)}$ is the conditional expectation of loss given that the loss exceeds $VaR_{(1-\alpha)}$. It can be shown that unlike other risk measures CVaR is a coherent risk measure. Further using Lemma 6.9.1, it is coherent with respect to the stochastic dominance rules induced by the partial order $\preceq_{2'}$. Thus $(X \preceq_{2'} Y)$ implies that $CVaR_{(1-\alpha)}(X) \le CVaR_{(1-\alpha)}(Y)$ for all $\alpha \in (0,1)$.

Remark 6.9.5 *The proofs of most of results in this section have been omitted. This has been done deliberately because the proofs are involved and need a good knowledge of measure theory, in particular the theory of Choquet integrals. Interested readers are encouraged to refer to Sriboonchitta et al. [128] and references cited therein.*

6.10 Stochastic Dominance and Portfolio Optimization

Let $r^T = (r_1, r_2, \ldots, r_n)$ be the vector of random returns for n assets and $w = (w_1, w_2, \ldots, w_n)$ be the given portfolio. Then the portfolio return $R(w)$ is given by

the random variable $r^T w$ where $w^T = (w_1, w_2, \ldots, w_n)$. In the context of portfolio optimization we consider the stochastic dominance relations between the random returns $r^T w^{(1)}$ and $r^T w^{(2)}$ corresponding to the two portfolios $w^{(1)}$ and $w^{(2)}$. Thus we say that portfolio $w^{(1)}$ dominates portfolio $w^{(2)}$ under the first order stochastic dominance, written as $w^{(1)} \succeq_1 w^{(1)}$ if $R(w^{(1)}) \succeq_1 R(w^{(2)})$: Thus $w^{(1)} \succeq_1 w^{(2)}$ if

$$F\left(R(w^{(1)}); \eta\right) \le F\left(R(w^{(2)}); \eta\right) \qquad \forall\, \eta \in \mathbf{R}. \tag{6.58}$$

Here $F\left(R(w^{(1)}), \cdot\right)$ is the cumulative distribution function of the random variable $R(w)$.

We say that $w^{(1)} \succ_1 w^{(2)}$ if $R(w^{(1)}) \succeq_1 R(w^{(2)})$ and $R(w^{(2)}) \not\succeq_1 R(w^{(1)})$. This amounts to saying that at least one inequality is strict at (6.58).

Similarly $w^{(1)} \succeq_2 w^{(2)}$ if $R(w^{(1)}) \succeq_2 R(w^{(2)})$, i.e.

$$F_2\left(R(w^{(1)}); \eta\right) \le F_2\left(R(w^{(2)}); \eta\right) \quad \forall\, \eta \in \mathbf{R}, \tag{6.59}$$

where

$$F_2(V; \eta) = \int_{-\infty}^{\eta} F(V, \xi)\,d\xi, \quad \forall\, \eta \in \mathbf{R}.$$

We say that $w^{(1)} \succ_2 w^{(2)}$ if $R(w^{(1)}) \succeq_2 R(w^{(2)})$ and $R(w^{(2)}) \not\succeq R(w^{(1)})$. This amounts to saying that at least one inequality is strict at (6.59).

Since stochastic dominance rules also try to capture preference between two random alternatives, it is natural that there should be some connection between these rules and associated utility functions. We have following results in this regard (Sriboonchitta et al. [128])

Theorem 6.10.1 *Let $w^{(1)}$ and $w^{(2)}$ be two portfolios. Then $R(w^{(1)}) \succeq_1 R(w^{(2)})$ if and only if*

$$E\left(u(R(w^{(1)}))\right) \ge E\left(u(R(w^{(2)}))\right),$$

for any nondecreasing function $u : \mathbf{R} \to \mathbf{R}$ for which these expectations are finite.

Theorem 6.10.2 *Let $w^{(1)}$ and $w^{(1)}$ be two portfolios. Then $R(w^{(1)}) \succeq_2 R(w^{(2)})$ if and only if*

$$E\left(u(R(w^{(1)}))\right) \ge E\left(u(R(w^{(2)}))\right),$$

for any nondecreasing function $u : \mathbf{R} \to \mathbf{R}$ for which these expectations are finite.

Definition 6.10.1 (FSD-efficient portfolio) *A feasible portfolio $\overline{w}$ is called FSD- efficient if there is no other feasible portfolio w such that $R(w) \succ_1 R(\overline{w})$.*

Definition 6.10.2 (SSD-efficient portfolio) *A feasible portfolio $\overline{w}$ is called SSD- efficient if there is no other feasible portfolio w such that $R(w) \succ_2 R(\overline{w})$.*

6.11 Portfolio Optimization with Stochastic Dominance Constraints

In view of the above discussion it makes sense to study a typical portfolio optimization problem having stochastic dominance constraints. Let us assume that a reference random return Y having a finite expected value is available. For example this could be the (random) return of a reference portfolio like an index portfolio or the current portfolio itself. Then we introduce the following problem

$$\begin{aligned} &\text{Max} \quad E(R(w)) \\ &\text{subject to,} \\ &\qquad R(w) \geq_2 Y \\ &\qquad e^T w = 1, \end{aligned} \tag{6.60}$$

where $R(w) = r^T w$.

The problem (6.60) aims to determine those portfolios whose return dominate the reference return under second order stochastic dominance rule. This problem has continum of constraints because of the presence of the constraints $R(w) \geq_2 Y$, and as such is difficult to handle. Never the less, the application of stochastic dominance in portfolio selection problem has been an important recent contribution in the literature. Some notable contributions in this direction are due to Dentcheva and Ruszczynski [37], Ruszczynski and Vanderbei [116], and Fábián, Mitra and Roman [44].

6.12 Summary and Additional Notes

- This chapter presents certain developments post Markowitz's mean-variance theory with regard to other risk measures.
- Section 6.2 discusses certain theoritical and computational issues pertaining to Markowitz's model and need to discuss other risk measures.
- Section 6.3 presents mean absolute deviation based portfolio optimization whereas Section 6.4 discusses minimax rule based portfolio optimization.
- Sections 6.5 and 6.6 are devoted to the study of value at risk and conditional value at risk respectively. These are quantile based risk measures, in contrast to moment based risk measures (L_1, L_2 and L_∞) which are discussed in earlier sections.
- The expected utility maximization principle of Von Neumann - Morgestern's is important for rational decision making. This requires some discussion on preference relation. After presenting a very brief introduction of these topics in Section 6.7, Markowitz's model is revisited in Section 6.8 in utility theory frame work.
- Sections 6.9 and 6.10 respectively introduce coherent risk measures and stochastic dominance. The application of stochastic dominance in portfolio optimization is relatively new. These two topics provide a very basic theoretical frame work of portfolio optimization. Much of the current research on portfolio optimization is devoted to the application of this theoretical frame work to study the real life portfolio selection problems.
- We have not discussed certain other interesting portfolio optimization problems. For example we have not included transaction costs in our study. Also we have not looked into multi period and continuous portfolio optimization problems. We may refer to Yu et al. [148], Zhou and Li [150] and references cited therein in this regard. An interesting application of dynamic programming to study multiperiod portfolio optimization problem is presented in Sun et al. [131].
- Is there any tool which can help us to describe the possible dependency of several random variables? Let us first try to understand the question itself. We explain it through a simple example. Suppose X_1 and X_2 are two real-valued random variables that can take values from $\{1, 2, \ldots, 6\}$. Assume that we know the value of X_1 and we wish to infer the value of X_2 from it. An important thing to know here is that how much information we gained from knowing the value of X_1! In other words, it is same if we ask what is the interrelation or dependence of these two random variables? In certain simple situations we can indeed find the answer. For instance, match the above situation with the procedure of throwing a fair die twice and the outcome on the first throw is X_1 and the one of the second is X_2. The variables are independent; the knowledge of X_1 gives us no information about X_2. The situation will be quite different if it is given that X_1 is always the smaller number of the two throws and X_2 is the larger one. We then have a strict monotonic relation $X_1 \leq X_2$. Now, if $X_1 = 6$, then we also know $X_2 = 6$. If $X_1 = 5$, we would guess that X_2 is either 5 or 6, both with a probability 1/2, and so on. We need some mathematical tool to depict the latter dependency. It is precisely what we ask for in the beginning. The answer lies in the notion called *Copulas*. Copulas are the tools for modeling and describing dependence of several random variables. The name first appeared, in context of finance, in the work of Sklar (1959) and is derived from the Latin word 'copulare', meaning 'to connect or to join'.

The concept of copula has received growing attention in finance and economics

in recent years. In fact one of the main issues of risk management is the aggregation of individual risks. Copulas have been used to aggregate risk in [42] and [18] and many more references cited therein. They are often used to identify market risk, credit risk and operational risk. Though in this book we have used correlation coefficient to compute interdependence of returns of two or more assets in a portfolio, yet it is equally important to inform readers about a relatively complex but highly useful notion of copula. Useful because, correlation parameter works well with normal distributions, while distributions in financial markets are mostly skewed and copulas are very handy to deal with the skewness. For this reason, they have been applied in option pricing, portfolio value-at-risk, and interest rates derivatives, to name a few. There are various types of copulas, like, Frank copula, normal (Gaussian) copula, student copula, singular copula, and many more. For more details on how copulas are used in risk management, we can refer to [17, 18, 29], and numerous excellent articles on web. We encourage you to hunt for this concept and its applications in finance.

6.13 Exercises

Exercise 6.1 *Consider the data of Exercise 1.1 of Chapter 1. Let there be Rs 50,000 to be invested on these three assets. Formulate and solve the resulting portfolio optimization problem if*

(i) L_1 risk measure is used.
(ii) L_2 risk measure is used.
(iii) L_∞ risk measure is used.

Exercise 6.2 *Suppose that the portfolios are constructed using three securities a_1, a_2, a_3 with expected returns $\mu_1 = 20\%, \mu_2 = 15\%$ and $\mu_3 = 4\%$. Further, let expected absolute deviations be $q_1 = 50, q_2 = 400$ and $q_3 = 10$ respectively. Let there be M_0 = Rs 50,000 to be invested. Then for the risk tolerance parameter $\lambda = 0.5$, obtain the minimax rule based optimal portfolio.*

Exercise 6.3 *Solve the above problem if $q_3 = 0$ and there is no change in the other data.*

Exercise 6.4 *An investor has utility function $u(x) = x^{1/4}$ for his/her salary. He/she has a new job offer which pays Rs 80,000 with a bonus. The bonus will be Rs. 0, Rs 10,000, Rs 20,000, Rs 30,000, Rs 40,000, Rs 50,000, or Rs 60,000 each with equal probability. Find the certainty equivalent of this job offer.*

Exercise 6.5 *Let $X \sim N(\mu, \sigma^2)$. Show that*

(i) for each $\alpha \in (0,1)$, $F_X^{-1}(\alpha)$ is proportional to α.
(ii) if $X \le Y$, a.s. then $F_X^{-1}(\alpha) \le F_Y^{-1}(\alpha)$.
(iii) $F_{\lambda X}^{-1}(\alpha) = \lambda F_X^{-1}(\alpha)$ for $\lambda > 0$.
(iv) $F_{X+a}^{-1}(\alpha) = F_X^{-1}(\alpha) + a$ for any $a \in \mathbb{R}$.

Exercise 6.6 *Find $Var_{(1-\alpha)}(X)$ when*

(i) $X \sim uniform[a,b]$.
(ii) $X \sim N(\mu, \sigma^2)$.
(iii) $\log X \sim N(\mu, \sigma^2)$.

Exercise 6.7 *Compute $CVar_{(1-\alpha)}(X)$ when*

(i) $X \sim N(\mu, \sigma^2)$.
(ii) $\log X \sim N(\mu, \sigma^2)$.

Exercise 6.8 *Let $\mathfrak{X}$ be the class of profit or loss random variables with finite variances. Verify that*

(i) $\mathfrak{X}$ is a convex cone.
(ii) If $\rho : \mathfrak{X} \to \mathbb{R}^+$ given by $\rho(X) = Var(X)$, then ρ is not a coherent risk measure.

Exercise 6.9 *Let X and Y be two random variables such that*

$$P(X = 2) = 1 - P(X = 12) = 1/5,$$

and

$$P(Y = 8) = 1 - P(Y = 18) = 4/5.$$

(i) Verify that $E(X) = E(Y)$ and $V(X) = V(Y)$.
(ii) Consider the utility function $u(x) = \log x$ for $x > 0$. Does any risk averse investor feel the indifference between the two risky prospects X and Y? (Hint: Is $E(u(X)) = E(u(Y))$?

Exercise 6.10 *Let $\succ$ be a preference relation on a given set Z. Show that*

(i) $z \succ x$ *and* $x \approx y \Rightarrow z \succ y$.
(ii) $z \succ x$ *and* $z \approx y \Rightarrow y \succ x$.

Exercise 6.11 *Prove that*

(i) $X \succeq_1 Y \Rightarrow X \succeq_2 Y$.
(ii) $X \succeq_2 Y, E(X) = E(Y) \Rightarrow V(X) \leq V(Y)$.
(iii) $X \preceq_1 Y \Leftrightarrow F_X^{-1}(\cdot) \leq F_Y^{-1}(\cdot)$.

Exercise 6.12 *Let* $(t - X)^+ = Max(t - X, 0)$. *Show that* $X \succeq_2 Y$ *if and only if* $E((t - X)^+) \leq E((t - Y)^+)$ *for all* $t \in \mathbf{R}$.