Definition 7.4.3 (Geometric Brownian Motion) *A stochastic process* $\{X(t), t \geq 0\}$ *is said to be a geometric Brownian motion (GBM) if* $X(t) = X(0)\, e^{W(t)}$ *where* $W(t)$ *is a standard Brownian motion.*

$X(0)\, e$

Result 7.4.1 *For any* $h > 0$, *we have*

$$
\begin{aligned}
X(t+h) &= X(0)\, e^{W(t+h)} \\
&= X(0)\, e^{W(t)+W(t+h)-W(t)} \\
&= X(t)\, e^{W(t+h)-W(t)}
\end{aligned}
$$

We note that, BM has independent increments. Hence given $X(t)$, *the future* $X(t+h)$ *only depends on the future increment of the BM. Thus future is independent of the past and therefore the Markov property is satisfied. Hence,* $\{X(t), t \geq 0\}$ *is a Markov process.*

Because a geometric Brownian motion is nonnegative, it provides for a more realistic model of stock prices. Also, the GBM model considers the ratio of stock prices to have the same normal distribution. Therefore, the percentage change in price as opposed to the absolute change in price is modeled by a GBM. Fig. 7.6 shows a sample path of geometric Brownian motion.

How does geometric Brownian motion relate to stock ptices? One possibility is to think of modeling the rate of return of the stock price as a Brownian motion. Suppose that the stock price $S(t)$ at time t is given by

$$S(t) = S(0)\, e^{H(t)},$$

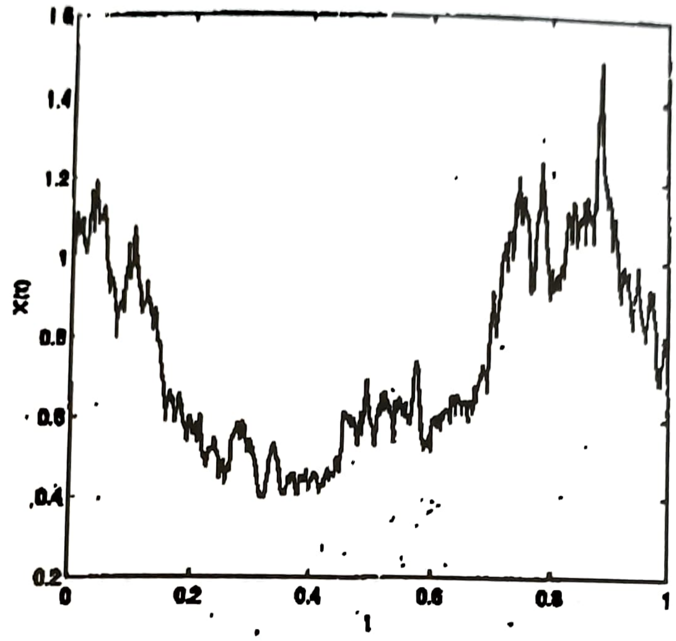

Fig. 7.6. Sample path of geometric Brownian motion

where $S(0)$ is the initial price and $H(t) = \mu t + \sigma W(t)$ is a Brownian motion with drift. In this case, $H(t)$ represents a continuously compounded rate of return of the stock price over the period of time $[0,t]$. Here, $H(t)$ refer to the logarithmic growth of the stock price, satisfies

$$H(t) = \ln\left(\frac{S(t)}{S(0)}\right).$$

This gives

$$\ln(S(t)) = \ln(S(0)) + H(t).$$

Therefore, $\ln(S(t))$ has a normal distribution with mean $\mu t + \ln(S(0))$ and variance $\sigma^2 t$. As we have seen, if a random variable X has the property that $\ln X$ has normal distribution, then the random variable X is said to have a lognormal distribution. Accordingly, $S(t)/S(0)$ is lognormal distributed random variable.

Example 7.4.1 *Suppose that the stock price $S(t)$ at time t is given by $S(t) = S(0)\, e^{H(t)}$ where $S(0)$ is the initial price and $H(t) = \mu t + \sigma W(t)$ is a Brownian motion with drift μ and volatility σ. Prove that*

(i) $E(S(t)) = S(0)\ \exp\left(\left(\mu + \frac{\sigma^2}{2}\right)t\right)$.

(ii) $Var(S(t)) = \left(S(0)\ \exp\left(\left(\mu + \frac{\sigma^2}{2}\right)t\right)\right)^2 (\exp(\sigma^2 t) - 1)$.

Solution We know that, if a random variable X is normally distributed with mean μ and variance σ^2, then the moment generating function of X is given by

$$M_X(\theta) = E\left(e^{\theta X}\right) = \exp\left(\mu\theta + \frac{1}{2}\sigma^2\theta^2\right). \qquad (7.1)$$

Since for every t, $W(t)$ is normally distributed with mean zero and variance t, $H(t) = \mu t + \sigma W(t)$ is normally distributed with mean μt and variance $\sigma^2 t$. Hence,

$$M_{H(t)}(\theta) = E\left(e^{\theta H(t)}\right) = \exp\left(\mu t\theta + \frac{1}{2}\sigma^2 t\theta^2\right). \qquad (7.2)$$

(i) Now

$$\begin{aligned} E(S(t)) &= E\left(S(0)e^{H(t)}\right) \\ &= S(0)E\left(e^{H(t)}\right) \end{aligned}$$

Using (7.2) and substituting $\theta = 1$, we obtain

$$E(S(t)) = S(0)\exp\left(\left(\mu + \frac{\sigma^2}{2}\right)t\right).$$

(ii)

$$\begin{aligned} Var(S(t)) &= E(S^2(t)) - (E(S(t)))^2 \\ &= E\left(S^2(0)e^{2H(t)}\right) - S^2(0)\exp\left(2\left(\mu + \frac{\sigma^2}{2}\right)t\right) \end{aligned}$$

Using (7.2) and substituting $\theta = 2$, we obtain

$$\begin{aligned} Var(S(t)) &= S^2(0)\exp\left(2\left(\mu + \sigma^2\right)t\right) - S^2(0)\exp\left(2\left(\mu + \frac{\sigma^2}{2}\right)t\right) \\ &= \left(S(0)\exp\left(\left(\mu + \frac{\sigma^2}{2}\right)t\right)\right)^2\left(\exp\left(\sigma^2 t\right) - 1\right). \end{aligned}$$

□

Remark 7.4.1 *Letting* $\bar{r} = \mu + \frac{1}{2}\sigma^2$, *we get*

$$E(S(t)) = S(0)\, e^{\bar{r}t}$$

$$Var(S(t)) = \left(S(0)\, e^{\left(r+\frac{\sigma^2}{2}\right)t}\right)^2 \left(e^{\sigma^2 t}-1\right).$$

Here we observe that, the expected stock price depends not only on the drift μ of $H(t)$ but also on the volatility σ. Further, it shows that, the expected price grows like a fixed-income security with continuously compounded interest rate $\bar{r}$. In real scenario, r is much lower than $\bar{r}$, the real fixed-income interest rate, that is why one invests in stocks. But the stock has variability due to the randomness of the underlying Brownian motion and hence a risk is involved here.

Example 7.4.2 *Suppose that stock price $\{S(t), t \geq 0\}$ follows geometric Brownian motion with drift $\mu = 0.12$ per year and volatility $\sigma = 0.24$ per annum. Assume that, the current price of the stock is $S(0) =$ Rs 40. What is the probability that a European call option having four years to exercise time and with a strike price $K =$ Rs 42, will be exercised?*

Solution We have

$$P(S(4) > 42) = P\left(\frac{S(4)}{40} > \frac{42}{40}\right)$$
$$= P\left(\ln\left(\frac{S(4)}{40}\right) > \ln\left(\frac{42}{40}\right)\right).$$

Since $\ln\left(\frac{S(4)}{40}\right)$ follows normal distribution with mean 0.48 and variance $(0.48)^2$, we get

$$P\left(\ln\left(\frac{S(4)}{40}\right) > \ln\left(\frac{42}{40}\right)\right) = P\left(\frac{\ln\left(\frac{S(4)}{40}\right) - 0.48}{0.48} > \frac{\ln\left(\frac{42}{40}\right) - 0.48}{0.48}\right)$$
$$= 1 - \Phi\left(\frac{\ln\left(\frac{42}{40}\right) - 0.48}{0.48}\right)$$
$$= 1 - \Phi(-0.8983)$$
$$= \Phi(0.8983) = 0.3133,$$

where Φ is the cumulative standard normal distribution function

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy.$$

□

Definition 7.4.4 **(Ornstein-Uhlenbeck Process)** *Let $\{W(t), t \geq 0\}$ be a Wiener process. Define*

$$X(t) = X(0)e^{-at} + be^{-at}W(e^{2at} - 1)\,,$$

where a and b are strictly positive real numbers and $X(0)$ is independent of $W(t)$. Then, we say $\{X(t), t \geq 0\}$ is an Ornstein-Uhlenbeck process.

We note that, $\{X(t), t \geq 0\}$ is a Markov process. It is also a Gaussian process if $X(0) = x(0)$ is fixed or $X(0)$ is Gaussian.

In the "real" world, we observe that asset price processes have jumps or spikes and risk-managers have to take them into account. We need processes that can describe the observed reality of financial markets in a more accurate way than models based on Brownian motion. Levy processes provide us with the appropriate framework to model both in the "real" and in the "risk-neutral" world.

Definition 7.4.5 **(Levy Process)** *A stochastic process $\{X(t), t \geq 0\}$ is said to be a Levy process if it satisfies the following properties*

(i) $X(0) = 0$,
(ii) for all n and for $0 \leq t_0 < t_1 < t_2 < \ldots < t_n$, increments $X(t_i) - X(t_{i-1})$, $i = 1, 2, \ldots, n$, are independent and stationary,
(iii) for $a > 0$, $P(|X(t) - X(s)| > a) \to 0$ when $t \to s$.

Remark 7.4.2 *Let b be a constant and $X(t) = bt$. Then $\{X(t), t \geq 0\}$ is a Levy process.*

Remark 7.4.3 *A Wiener process $\{W(t), t \geq 0\}$ defined in Definition 7.3.1, is a Levy process in $\mathbf{R}$ that has continuous paths and has the Gaussian distribution with mean zero and variance Δt for its increments $W(t + \Delta t) - W(t)$. The most general continuous Levy process in $\mathbf{R}$ has the form $X(t) = bt + cW(t)$, $t \geq 0$, where b and c are real constants.*

Remark 7.4.4 *A Poisson process $\{N(t), t \geq 0\}$ with parameter λ defined in Definition 7.2.19 is a Levy process that is a counting process having the Poisson distribution with mean $\lambda\Delta t$ for its increments $N(t + \Delta t) - N(t)$.*