FINANCIAL ENGINEERING
ASSIGNMENT-3

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1. Sample space $\Omega = \{(u,u), (u,d), (d,u), (d,d)\}$

Non trivial σ- Field :

$f_1 = \{\phi, \Omega, \{(u,u), (u,d)\}, \{(d,u), (d,d)\}\}$

Largest σ- Field on Ω:

$f_2 = \{\phi, \{(u,u)\}, \{(u,d)\}, \{(d,u)\}, \{(d,d)\}, \{(u,u), (u,d)\}, \{(u,u), (d,u)\}, \{(u,u), (d,d)\}, \{(u,d), (d,u)\}, \{(u,d), (d,d)\}, \{(d,u), (d,d)\}, \{(u,u), (u,d), (d,u)\}, \{(u,u), (u,d), (d,d)\}, \{(u,u), (d,u), (d,d)\}, \{(u,d), (d,u), (d,d)\}, \Omega\}$

2. X and Y are IID random variable each having uniform distribution on the interval $(-\pi, \pi)$

$E(X) = 0 \qquad Var(X) = 1$

$E(Y) = 0 \qquad Var(Y) = 1$

$E(XY) = E(X)E(Y) = 0$

$E(X^2) = 1 \qquad E(Y^2) = 1$

$Z(t) = \cos(Xt + Y)$

$E(Z(t)) = E(\cos(Xt+Y))$

Also $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots$

$$E(z(t)) = E\left[1 - \frac{(Xt+Y)^2}{2!} + \frac{(Xt+Y)^4}{4!} - \cdots\right]$$

$$= 1 - \frac{1}{2}E(X^2t^2 + Y^2 + 2XYt) + \frac{1}{4!}E[(Xt+Y)^4] - \cdots$$

$$= 1 - \frac{1}{2}(t^2+1) + \frac{1}{4!}E[(Xt+Y)^4] + \cdots$$

In a wide sense stationary process, $E(z(t))$ must be independent of t.
Here $E(z(t))$ depends on t^2
Hence this is not a wide sense stationary process.

(3.) Brownian motion is defined as following.
A stochastic Process $W(t)$ is said to be a Brownian Motion if it satisfies the following properties:

(i) $W(0) = 0$

(ii) for $t>0$ $W(t)$ is continuous

(iii) $W(t)$, $t \geq 0$ has independent & stationary increments

(iv) for $0 \leq s < t < \infty$ $W(t) - W(s)$ is normally distributed random variable with mean '0' and variance $(t-s)$.

To prove $\hat{W}(t) = tW(1/t)$ is brownian

i) $\lim_{t\to 0} \hat{W}(t) = \lim_{t\to 0} tW(1/t) = \lim_{s\to\infty} \frac{W(s)}{s} = 0$

ii) In order to show $\hat{W}(t)$ is continuous it is enough to show $\hat{W}(t)$ is continuous at $t=0$

It is given $(t\,W(1/t))_{t=0} = 0$.
Hence $\hat{W}(t)$ is continuous.

$$\mathrm{Cov}(\hat{W}(s), \hat{W}(t)) = \mathrm{Cov}(s \cdot W(1/s), t\,W(1/t))$$
$$= s \cdot t\ \mathrm{Cov}(W(1/s), W(1/t))$$
$$= s \cdot t \times \frac{1}{t} = s.$$

$$E\,(\hat{W}(t) - \hat{W}(s))(\hat{W}(t) - \hat{W}(s))$$
$$= t - s - t - s = 0.$$

Independent Increments

$\therefore$ $\hat{W}(t)$ is a Brownian Motion.

(5)
$$\Omega = \{a, b, c, d\}$$
$$F_1 \subset F_2 \subset F_3 \subset F_4$$

$$F_1 = \{\phi, \Omega\}$$
$$F_2 = \{\phi, \{a\}, \{b, c, d\}, \Omega\}$$
$$F_3 = \{\phi, \Omega, \{a\}, \{b, c, d\}, \{b\}, \{a, c, d\}, \{c, d\}, \{a, b\}\}$$

$$F_4 = \{\phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{b, c\}, \{c, d\}, \{a, c\}, \{a, d\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \{a, c, d\}, \{a, b, c, d\}\}$$

Hence F_1, F_2, F_3, F_4 satisfy given condition.

(6.) $Y_n = \exp\left\{\left(\sum_{i=1}^{n} X_n\right) - \frac{n}{2}\right\}$

$W(n) = \sum_{i=1}^{n} X_i$

$Y_n = \exp\left(W_n - n/2\right)$

Let $0 \le S \le n$, $X_1, X_2, \dots X_n$ are r.v. I.I.D.

$W(n) - W(s)$ is independent of F_s where F_s is filter where $s > 0$.

We have, $E\left(e^{W(n)} / F_s\right) = e^{W(n)} \sum \left(e^{W(n) - W(s)}\right)$

Given $W(t) - W(s)$ the normal distribution with mean 0 and variance $= 1$

We get $E\left(e^{W(n) - W(s)}\right)$

$E\left(e^{W(n)} \mid F_s\right) = e^{W(s)} e^{1/2}$

$E\left(e^{W(n) - n/2} \mid F_s\right) = e^{-n/2} E\left(e^{W(n)} / F_s\right)$

$= e^{W(s)} - S/2$

Hence Y_n is a Martingle.

(7.) $\lambda = 1$, $\{N(t), t > 0\}$ is a Poisson process with Parameter 1.

i) $\{N(t) - t \mid F_s, t \geq 0\}$

$E\left[N(t) - t \mid F_s\right] = E\left[N(t) - N(s) + N(s) - (t - s) - s \mid F_s\right]$

$= E(N(t)-N(s)\mid F_s) + E(N(s)/F_s) - E(t-s/F_s) - E(s\mid F_s)$

$= \cancel{\lambda(t-s)} + N(s) - \cancel{(t-s)} - s$

$\lambda = 1$

$= N(s) - s$

$\therefore E(N(t)-t\mid F_s)$ is a function of s.

Hence it is a Martingale.

8. Given $\{W(t), t \geq 0\}$ be a wiener process.

Let $0 < s < t$. Since $W(t)-W(s)$ is independent of $F(s)$ and $W(s)$ is F_s measurable.

$$E(e^{\sigma W(t)}\mid F_s) = E[e^{\sigma(W(t)-W(s))}\, e^{\sigma W(s)}\mid F_s]$$
$$= e^{\sigma W(s)}\, E[e^{\sigma(W(t)-W(s))}\mid F_s]$$
$$= e^{\sigma W(s)}\, E[e^{\sigma(W(t)-W(s))}]$$
$$= e^{\sigma W(s)} \cdot e^{\sigma^2(t-s)}$$

Now $$E[e^{\sigma W(t) - \frac{\sigma^2 t}{2}}\mid F_s] = e^{-\frac{\sigma^2 t}{2}}\, E[e^{\sigma W(t)}\mid F_s]$$
$$= e^{-\sigma^2 t}\, e^{\sigma^2(t-s)}\, e^{\sigma W(s)}$$
$$= e^{\frac{\sigma^2}{2}(t-s)} \exp(2\sigma W(s) - \sigma^2 s)$$

Hence $\exp\{\sigma W(t) - \frac{\sigma^2 t}{2}\}$ is a Martingale.

9. SDE of $W^2(t)$

Using Ito - Doebins formula we get

$$d\,W^2(t) = dt + 2W(t)\,dW(t)$$

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Using equivalent Integral equation with condition $W(0) = 0$

$$W^2(t) = t + 2\int_0^t W(t)\, dW(t)$$

$$\int_0^t W(t)\, dW(t) = \frac{1}{2} W^2(t) - \frac{1}{2} t$$

$\therefore$ $W(t)$ is Ito process.

(11) Given that $Z(t)$ is normally distributed random variable with

$$E(Z) = 0, \quad Var(Z) = 1$$

$$Var(Z) = E(Z^2) - (E(Z))^2$$

$$E(Z^2) = 1.$$

Now $X = \sqrt{t}\, Z$

$$E(X) = E(\sqrt{t}\, Z) = \sqrt{t}\, E(Z) = 0.$$

$$Var(X) = (\sqrt{t})^2 \Rightarrow Var(X) = t.$$

Hence $X(t)$ is a Brownian Motion

(13.) Given that stock price $S_0 = \$50$

Time T (in years) $= 2$

Expected return $\mu = 0.18$

volatility $= \sigma = 30\% = 0.3$

We use probability distribution of the stock price in 2 years using log normal distribution.

$\ln S_T = \phi\left(\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right) \times T, \sigma^2 \times T\right)$

$= \phi\left(\ln 50 + \left(0.18 - \frac{0.09}{2}\right) \times 2, 0.3^2 \times 2\right)$

$\ln S_T = \phi(4.18, 0.18)$

The mean of stock price $E(S_T)$ is given by:
$E(S_T) = S_0 e^{rT}$

$= 50 e^{0.18 \times 2} = \$71.67 = E(S_T)$

Standard deviation of stock price σ_{S_T} is given by:

$\sigma_{S_T} = S_0 e^{rT} \sqrt{e^{\sigma^2 \times T} - 1} = 50 e^{0.18 \times 2} \sqrt{e^{0.09^2 \times 2} - 1}$

$\sigma_{S_T} = \$31.83$

95% confidence interval for $\ln S_T$ are :-

By interval table for critical value at $\sigma/2 = \frac{0.05}{2}$
$= 0.025$ is 1.96

$4.18 \pm 1.96 \times 0.42$
$3.35, 5.01$

Corresponding 95% confidence interval for S_T are
$e^{3.75}$ & $e^{5.01}$
$= 28.52$ & 150.44

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(4.) X & Y are iid r.v. each having uniform distribution b/w the intervals 0 & 1

$Z = X + Y$

So $E\left(\frac{X}{Z}\right) = E\left(\frac{Y}{Z}\right)$ — (1)

$E\left(\frac{X}{Z}\right) = E\left(\frac{Z-Y}{Z}\right) = E\left(\frac{Z}{Z}\right) - E\left(\frac{Y}{Z}\right)$

$E\left(\frac{X}{Z}\right) + E\left(\frac{Y}{Z}\right) = Z \qquad E(Z/Z) = Z$

$2E\left(\frac{X}{Z}\right) = Z$

$E(X/Z) = \frac{Z}{2}$.