1
Financial Mathematics: An Overview

1.1 Introduction

The word *financial mathematics* refers to the application of mathematics to study the problems arising in the area of finance. Here the word mathematics has been used in a wider sense so as to include subjects like *probability* and *statistics*, *stochastic processes*, *optimization*, *econometrics*, *numerical analysis* and *partial differential equations*. Also the scope of the word *finance* could be very broad but here it should be understood in terms of *investment* of money for the purpose of receiving more money (hopefully!!) at sometime in future. Thus *financial mathematics* could also be termed as *investment science* or *investment theory*.

Though investment as an art form has always been there, it developed as science/theory due to the pioneering work of Fisher Black, Myron Scholes, Robert Merton, Harry Markowitz, William Sharpe and John Lintner amongst many others. In fact, Merton and Scholes got Nobel Prize in Economics (in 1997) for their work on pricing of option carried out during late 60's and early 70's. Black could not share this prize as he died of illness before 1997. Earlier Markowitz along with Sharpe (Markowitz's Ph.D. student) were awarded Nobel Prize (in 1990) for their work on portfolio optimization carried out during late 50's and early 60's. They shared Nobel Prize with another famous economist Merton Miller.

The above extra ordinary theoretical developments had great impetus on research in finance. This together with globalization of investment activities and tremendous power of computing technology has made the financial mathematics area intellectually very fertile. There is a significant growth in theory as well as in applications in the areas of derivative pricing, portfolio management, interest rate modeling, credit risk methodologies and algorithmic trading.

The aim of this chapter is to introduce basic notions and assumptions of a simple financial market model so as to facilitate the presentation of some simple

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investment problems for illustration purposes. Our discussion here is motivated by some of the excellent texts. We shall particularly like to mention Capinski and Zastawniak [25], Hull [65] and Luenberger [85].

1.2 Some Basic Terminology

Any financial market model has to deal with the trading of financial assets. By a financial asset we mean a financial instrument that can be bought or sold. Here an asset will always mean a financial asset. It is common knowledge that some assets are *risk free* and some are *risky*. For example a bank deposit or a bond issued by a government are risk free where as stocks are risky. To introduce certain notations and terminology for a financial market we first assume a very simple scenario. We assume that only two assets are being traded. These are, one risk free asset (bond) and one risky asset (stock). The other parameter in modeling a financial market is the *time horizon*. The simplest of these is the *single period model* where we restrict the time scale to be two instants only, namely $t = 0$ and $t = 1$ and any transaction takes place at these two time instants only. Realistically it has to be a multi period or continuous time scenario. But here we restrict our discussion to the case of single period only, the multi period and continuous time models will be discussed later.

Let $t = 0$ and $t = 1$ be two time instants. Let $B(0)$ denote the price of one bond at $t = 0$ and $B(1)$ be the corresponding price at $t = 1$. Similarly let $S(0)$ denote the price of one share of stock at $t = 0$ and $S(1)$ be the corresponding price at $t = 1$. As bond is a risk free asset, both $B(0)$ and $B(1)$ are known at $t = 0$. But for the case of stock, which is a risky asset, though $S(0)$ is known, $S(1)$ remains uncertain. The stock price at $t = 1$, namely $S(1)$, may go up or down with certain probability, i.e. $S(1)$ is a random variable. Thus for a risk free asset both $B(0)$ and $B(1)$ are deterministic, but for a risky asset $S(0)$ is deterministic while $S(1)$ is a random variable. We next define the *rate of return of an asset*. For the case of bond it is defined as

$$r_B = \frac{B(1) - B(0)}{B(0)},$$

and for the case of stock it is defined as

$$r_S = \frac{S(1) - S(0)}{S(0)}.$$

Here we note that r_B is known exactly but r_S is a random variable because $S(1)$ is a random variable. In the finance literature, rate of return is called *return* only.

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Thus when we refer to the return of an asset we really mean the rate of return of that asset.

Let an investor hold x shares of stock and y units of bond at $t = 0$. The pair (x, y) is called a *portfolio* and it is denoted by $P : (x, y)$. The value of the portfolio $P : (x, y)$ at time $t = 0$ is given by

$$V_P(0) = x\ S(0) + y\ B(0).$$

Similarly the value of the portfolio at time instant $t = 1$ is given by

$$V_P(1) = x\ S(1) + y\ B(1).$$

The quantity

$$r_P = \frac{V_p(1) - V_p(0)}{V_p(0)},$$

is called the *rate of return of the portfolio* $P : (x, y)$. We note that r_P is a random variable because $V_P(1)$ is so.

The units of $B(0)$, $B(1)$, $S(0)$ and $S(1)$ are units of money and here they are always taken in Rupees. Thus $B(0) = 100$ etc. will always mean $B(0) = Rs\ 100$ etc. We shall follow this convention throughout the book.

Since the portfolio return r_P is a random variable we can talk of its expected value and variance. The quantity $E(r_P)$ is called the *expected return of the portfolio* $P : (x, y)$ and the *standard deviation* σ_P *of* r_P is called its *risk*. Obviously if we are given a choice between two portfolios with the same expected return, we should choose the one for which the risk is least. In a similar manner, if the risk levels of two portfolios are same we should choose the one for which return is maximum. This is a typical problem of portfolio optimization. We shall have opportunity of discussing a general n-asset problem of portfolio optimization in context of mean-variance theory of Markowitz in later chapters of the book. The concept of risk itself has been a topic of research in the recent past. Is standard deviation or variance the *correct* measure of risk? Are there some other *better* definitions of risk? All these questions need to be answered and we plan to discuss them in greater detail in a later chapter.

Example 1.2.1 *Let* $B(0) = 100$, $B(1) = 110$, $S(0) = 80$ *and*

$$S(1) = \begin{cases} 100, & \textit{with probability } 0 \cdot 8 \\ 60, & \textit{with probability } 0 \cdot 2. \end{cases}$$

What is the expected return on the stock? Consider the portfolio $P : (x = 50, y = 60)$. *Determine the expected return and the risk for the given portfolio.*

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Solution The return on bond is obviously 10%. But for the stock

$$r_S = \begin{cases} \dfrac{100-80}{80}, & \text{with probability } 0\cdot 8 \\[2ex] \dfrac{60-80}{80}, & \text{with probability } 0\cdot 2, \end{cases}$$

i.e.

$$r_S = \begin{cases} 0\cdot 25, & \text{with probability } 0\cdot 8 \\ -0\cdot 25, & \text{with probability } 0\cdot 2. \end{cases}$$

Therefore $E(r_S) = (0\cdot 25)(0\cdot 8) + (-0\cdot 25)(0\cdot 2) = 0\cdot 15$.

Further as $x = 50$ and $y = 60$, we have

$$V_P(0) = (50\times 80) + (60\times 100) = 10,000.$$

Also

$$V_P(1) = \begin{cases} (50\times 100) + (60\times 110), & \text{with probability } 0\cdot 8 \\ (50\times 60) + (60\times 110), & \text{with probability } 0\cdot 2 \end{cases}$$

$$= \begin{cases} 11,600, & \text{with probability } 0\cdot 8 \\ 9,600, & \text{with probability } 0\cdot 2. \end{cases}$$

Therefore

$$r_P = \frac{V_P(1) - V_P(0)}{V_P(0)}$$

$$= \begin{cases} 0\cdot 16, & \text{with probability } 0\cdot 8 \\ -0\cdot 04, & \text{with probability } 0\cdot 2. \end{cases}$$

This gives the expected return on the portfolio as

$$E(r_P) = (0\cdot 16)(0\cdot 8) + (-0\cdot 04)(0\cdot 2) = 0\cdot 12.$$

We have defined the standard deviation of the portfolio as its risk. Therefore the risk of given portfolio is

$$\sigma_P = \sqrt{E\left(r_p - E(r_p)\right)^2} = \sqrt{(0\cdot 16 - 0\cdot 12)^2\times(0\cdot 8) + (-0\cdot 04 - 0\cdot 12)^2\times(0\cdot 2)}$$
$$= 0\cdot 08. \qquad \square$$

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1.3 Certain Assumptions

We now state certain assumptions which help in building a simple mathematical model of a typical financial market. The real world financial market is extremely complex and any mathematical model has to trade off between reality and simplifications which make the model amenable to analytical or numerical solution. Apart from the time horizon, it is the dynamics of the assets which plays a very crucial role in the modeling aspect of a financial market. Thus, for a general case, we need to describe the dynamics of the stock price $S(t)$ as t varies over a continuum. Since dynamics generally refers to a differential equation, we expect that the stock price movement will be represented by a differential equation. But things are not that simple as for each t, $S(t)$ is a random variable i.e. it is a random process. So there has to be an appropriate stochastic process component in this differential equation leading to something called a *stochastic differential equation* (SDE). So for a proper understanding of this dynamics we need to study stochastic calculus and stochastic differential equations more deeply. Since more than one stock may be in the portfolio, one has to deal with more than one SDE and also to take into account their interaction as stocks may be correlated. But as agreed in Section 1.2, in this chapter we restrict our discussion to the case when only two assets (one bond and one stock) are being traded and the time horizon is only one period (from $t = 0$ to $t = 1$).

For the present we do not take the general case and state certain assumptions for our single period case only. We should take all these assumptions only as a first step towards modeling and try to modify them later so as to make the model more realistic.

Assumption 1 (Randomness)
The stock price $S(1)$ at $t = 1$ is a random variable with at least two different values.

Assumption 2 (Positivity of Prices)
All bond and stock prices are strictly positive, i.e. $B(0) > 0$, $B(1) > 0$, $S(0) > 0$ and $S(1) > 0$.

Assumption 3 (Divisibility, Liquidity and Short Selling)
Let the investor hold x shares of stock and y units of bond. Then $x \in \mathbf{R}$ and $y \in \mathbf{R}$.

Thus the investor is permitted to hold any units of bond and any number of shares of stock. These numbers may be zero, positive or negative reals. This property is referred to as *divisibility* property. In a real market this property is expected to hold when the volume of transactions is large compared to unit prices.

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As there are no bounds on x and y they can be as large or as small as we wish. In other words, an investor can buy or sell arbitrary quantities at the market price on demand. This property is called *liquidity*. In reality there is always a restriction on the volume of trading so this property may not hold in general.

The next thing to understand is the interpretation of sign restriction on x and y. If $x > 0$ then it obviously means that at $t = 0$, the investor is buying x shares of stock and this we state as "The investor has taken a *long position* on stock". Similarly we interpret $y > 0$. However, if $x < 0$ (respectively $y < 0$), we say that "the investor has taken a *short position* on stock (respectively bond)". Thus if $x > 0$ and $y < 0$, we say that investor has taken a long position on stock and a short position on bond. If the investor has taken a short position on an asset we say that the asset is *shorted*.

But what is the physical interpretation of a short position ? A short position in a risk free asset (bond) means borrowing cash, where the interest rate is determined by the bond price. Thus if $y = -10$, $B(0) = 100$ and $B(1) = 110$, then it means that the investor is borrowing Rs 1000 from the bank where the interest rate is 10%. We say that the short position in the bond is closed when the borrowed cash along with interest is repaid to the bank or lender.

A short position in the stock is slightly more complicated. This is best understood in terms of *short selling*. Short selling is a way to make profit if price goes down. Suppose that the investor feels that the price of a stock is overvalued, so it makes sense in selling the stock and purchasing it when the price goes down. In this situation the investor can resort to short selling. Here the investor borrows the stock from someone (say broker), sells it and uses the amount received to make some other investment. But the original owner of the stock has all the rights vested to him/her. Thus he/she receives all dividends due and has the full right to sell the stock any time. In this situation, the investor has to return the stock to the original owner. If he/she does not have stock with him/her, he/she may have to buy from the market at the prevailing market price. Therefore it becomes imperative for the investor to keep sufficient resources (cash) with him/her to fulfil this obligation. We say that the short position on the stock is closed when the stock is returned to the original owner by the investor. The investor may return the stock of his/her own or when the owner asks for the same. In practice the investor is also asked to give a security deposit with the broker from which the stock is borrowed. But in theory we do not consider this.

The above discussion suggests that to meet his/her obligations as per Assumption 3, the wealth of the investor should never be negative. This is precisely the Assumption 4 given below.

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Assumption 4 (Solvency)
The wealth of the investor should never be negative, i.e. $V_P(t) \geq 0$ at all times t. For the single period case it simply means that $V_P(0) \geq 0$ and $V_P(1) \geq 0$.

A portfolio (x, y) satisfying the *solvency* property is termed as an *admissible portfolio*.

Short selling is generally considered to be extremely risky because in short selling there are possibilities of unlimited losses. Let the borrowed asset be sold for an amount X_0 at $t = 0$ and later at $t = 1$ it be purchased for an amount X_1 so as to close the short position. If X_1 is less than X_0 then a profit of $(X_0 - X_1)$ is made. In other words short selling is profitable if the asset price goes down. But if the asset value increases then the loss is $(X_1 - X_0)$, which can be arbitrary large since X_1 can increase arbitrary. Because of this not all financial institutions allow short selling. In some institutions it may be allowed only on some assets and that too with certain restrictions.

Let us consider an example in this regard. Let $S(0) = Rs\ 10$ and let an investor decide to short 100 shares of this stock at $t = 0$. For this he/she borrows 100 shares from a broker and sells these in the stock market to receive an amount of Rs 1000. Let at $t = 1$, the stock price $S(1)$ become Rs 9, so the investor buys back 100 shares from the market at Rs 900 and gives these shares to the broker to close the short position. This has been a favorable position to the investor as stock price has gone down. In fact the investor has made a profit of Rs 100 due to short selling. If the price had gone up, the investor would obviously had made losses.

Now imagine another investor who has not gone for short selling but rather has purchased 100 shares in the beginning at $t = 0$. Imagine that this investor has to sell these shares at $t = 1$ (the problem being a single period only). This will entail a loss of Rs 100 to this investor. The rate of return on the asset for this investor will be

$$r = \frac{900 - 1000}{1000} = -0 \cdot 10$$

i.e. -10%.

This example clearly illustrates that short selling allows an investor to convert a negative rate of return into a profit. This is because the original investment of the investor is also negative. It gives a profit of (original investment)×(rate of return), i.e. $(-1000)(-0 \cdot 10) = 100$.

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Assumption 5 (Discrete Unit Prices)
By Assumption 1, the stock price $S(1)$ at time $t = 1$, is a random variable taking at least two distinct values. What we assume here is that $S(1)$ is a random variable taking only finitely many values.

This assumption is again not realistic as $S(t)$ at any time t can take values in a continuum. But again it is taken only for the sake of simplicity and will be generalized later.

We next present *no arbitrage principle* and *the law of one price.* In fact the law of one price is a consequence of no arbitrage principle. We say that there is an arbitrage opportunity in the market when someone, with no invested capital, has a positive probability of achieving a positive return with no risk of loss. So in our common usage of English language, arbitrage opportunity really means *free lunch.* It is certainly not desirable in a fair market, and therefore we make the assumption given below.

Assumption 6 (No Arbitrage Principle)
There is no portfolio with initial value $V(0) = 0$ such that $V(1) \geq 0$ with probability 1 and $V(1) > 0$ with non-zero probability. Thus if the initial value of a portfolio is zero, i.e. $V(0) = 0$ (no initial investment), and $V(1) \geq 0$ (no risk of loss), then $V(1) = 0$ (no profit) with probability 1.

As a consequence of *no arbitrage principle*, no investor can lock in a profit without risk of loss and no initial endowment. In other words, there are no zero cost investment strategies that allow an investor to make a profit without taking some risk of a loss. In this context, we may note that investing in a Government bond (or fixed deposit) is not an arbitrage though it guarantees a positive net return because capital is invested initially. Some readers may like to analyze if short selling is an arbitrage opportunity? The answer is obviously in negative.
Assumption 6 can also be stated mathematically as follows
There is no portfolio such that for some t

(i) $V(0) = 0$,
(ii) $V(t) \geq 0$ with probability 1,
(iii) $P[V(t) > 0] > 0$,

where $V(t)$ is the wealth of the portfolio at time t. For the single period case it means that there is no portfolio such that

(i) $V(0) = 0$,
(ii) $V(1) \geq 0$ with probability 1,
(iii) $P[V(1) > 0] > 0$.

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The following is rather an artificial example of arbitrage opportunity. Consider two banks (say Bank 1 and Bank 2) that offer to loan money or accept deposits at the same rate of interest but these are different in the two banks. Let the rate at Bank 1 for loan and deposits be 10% and at Bank 2 be 12%. So an investor can go to Bank 1 and borrow say *Rs* 10,000 at 10% and then deposit that *Rs* 10,000 at 12% interest at Bank 2. At $t = 1$, the investor will get *Rs* 11,200 and then he/she can return *Rs* 11,100 (borrowed money and 10% interest on it) to Bank 1. In this process he/she has earned *Rs* 100 (guaranteed) without making any investment of his/her own. Obviously such opportunities are rare and instantaneously both banks will adjust the same rate of interest and there will be no arbitrage.

The above discussion suggests that in a financial market, arbitrage opportunity cannot prevail for extended periods of time and therefore it justifies Assumption 6. This assumption is extremely important for any meaningful study in financial mathematics and we shall have opportunity to see its application throughout the book.

As a consequence of no arbitrage principle we have the *law of one price* or also called the *comparison principle*. It essentially tells that if two financial instruments have exactly the same pay-offs, then they have the same price.

To have a better idea of the law of one price, let us imagine two portfolios, say Portfolio P and Portfolio Q. Let us consider the case of single period only and refer to the time instants as $t = 0$ and $t = 1$. The two portfolios P and Q have known (determinate) value at $t = 0$ but their values at $t = 1$ (also called pay-offs) are random variables. Thus $V_P(0)$ and $V_Q(0)$ are deterministic, but $V_P(1)$ and $V_Q(1)$ are random variables, whose values will depend upon the state of economy $\omega \in \Omega$. Therefore,

$$V_P(1) \in \{V_{P,1}(\omega) : \omega \in \Omega\} ,$$

where $V_{P,1}(\omega)$ refers to the value taken by the random variable $V_P(1)$ when the state of the economy is ω. In particular, if there are m possible states of economy, i.e. $\Omega = \{\omega_1, \omega_2, \ldots, \omega_m\}$ then

$$V_P(1) \in \{V_{P,1}(\omega_1), \ldots, V_{P,1}(\omega_m)\} ,$$

where $V_{P,1}(\omega_i)$ refers to the value of the random variable $V_P(1)$ when the state of economy is $\omega_i (i = 1, 2, \ldots, m)$. In a similar manner

$$V_Q(1) \in \{V_{Q,1}(\omega) : \omega \in \Omega\}.$$

The law of one principle states that if $V_{P,1}(\omega) = V_{Q,1}(\omega)$, for all $\omega \in \Omega$, then $V_P(0) = V_Q(0)$.

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To see that it should be true, suppose that $V_P(0) > V_Q(0)$. Then at $t = 0$ an investor can sell the costlier portfolio (Portfolio P) and purchase the cheaper portfolio (Portfolio Q), pocketing the difference. At $t = 1$, no matter what the state of economy is in, the investor receives the common final value of the portfolios. Thus the investor looses nothing at the end and made profit. But this violates the principle of no arbitrage. Hence $V_P(0)$ cannot be more than $V_Q(0)$. In a similar manner we can show that $V_Q(0)$ cannot be more than $V_P(0)$. Hence $V_P(0) = V_Q(0)$.

The law of one price has proved to be a very effective tool in pricing *derivative securities*, which we shall discuss in detail in later chapters.

Apart from the above main assumptions it also assumed that *there are no commissions or transaction costs and the lending rate is equal to the borrowing rate.*

1.4 Derivative Securities

Some financial securities have the property that their values depend upon the value of another security. In this case, the former security is called a *derivative security* of the later security which is called the *underlying security* for the given derivative security. We often call a derivative security simply as *derivative* and the corresponding underlying security simply as *underlying*. Derivative securities are also called *contingent claims*.

Our aim here is to introduce some simple derivatives and illustrate them through examples. In particular we shall describe a *forward contract, a call option* and a *put option*. These derivative securities in their pure form have stock as underlying security. But sometimes it is possible to have a derivative whose underlying is another derivative. So we can have options on *futures contracts*. Thus a given financial entity can be derivative under some circumstances and an underlying in other circumstances. In fact sometimes underlying may even be interest rate and currency exchange rate which do not look like typical financial securities. Our discussion here remains valid even when some other asset, other than stock, is the underlying security. We shall also illustrate the application of no arbitrage principle (Assumption 6) for their *pricing*. All these topics will be further continued in the later chapters of the book.

Forward Contracts

A *forward contract* is a derivative security whose underlying is a stock and it has the below given basic characteristics.

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(i) It is an agreement to buy or sell the stock at a specified future time, called *delivery date*, for a fixed price F, called the *forward price* which is agreed at $t = 0$. Obviously for the single period model, delivery date has to be $t = 1$.
(ii) An investor who agrees to buy the stock is said to enter into a *long forward contract* or take a *long forward position*.
(iii) An investor who agrees to sell the stock is said to enter into a *short forward contract* or take a *short forward position*.
(iv) No money is paid at $t = 0$ when a forward contract is exchanged.
(v) A forward contract guarantees that the stock will be bought (for long position) or sold (for short position) for forward price F at the delivery date.

In general, the party holding the long forward contract will benefit if $S(1) > F$ and suffer a loss if $S(1) < F$. Therefore the pay-off of a long forward contract is $S(1) - F$, which could be zero, positive or negative. For a short forward position, the pay-off is obviously $F - S(1)$.

The question which we have to answer is as follows: What should be the forward price F? Though a formal theoretical formula will be derived in Chapter 2, here we explain the logic through an example.

Example 1.4.1 *Let $B(0) = 100$, $B(1) = 110$, $S(0) = 50$. Determine the forward price F of a forward contract on the given stock.*

Solution Here delivery is one period (from $t = 0$ to $t = 1$) which let us take as one year. As $B(0) = 100$ and $B(1) = 110$, the nominal rate of interest is 10%. Since $S(0) = 50$, the forward price F should at least be the amount which the holder of the forward contract would have got by depositing *Rs* 50 in the bank for one year with rate of interest 10%. Thus F should be at least *Rs* 55.

We shall now argue that F should be exactly *Rs* 55. If possible let $F > 55$. Then at $t = 0$, we construct the portfolio $P : (x = 1, y = -\frac{1}{2}, z = -1)$, where z denotes the units of forward contracts in the portfolio. The interpretation of the portfolio P so constructed is that at time $t = 0$ we borrow *Rs* 50 as $(y = -\frac{1}{2})$, buy the stock for $S(0) = 50$ as $(x = 1)$ and enter into a short forward contract as $(z = -1)$ with forward price F and delivery date $t = 1$. This is the interpretation of $x = 1, y = -\frac{1}{2}$ and $z = -1$.This gives

$$V_P(0) = x\,S(0) + y\,B(0) = (1 \times 50) + (-\frac{1}{2} \times 100) = 0.$$

Here it may be noted that in $V_P(0)$ there is no term corresponding to the short forward contract $(z = -1)$ because in forward contract there is no exchange of money at the time when the contract is initialized, i.e. $t = 0$.

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Now at $t = 1$, we close the short forward position by selling the asset for $Rs\ F$ and close the risk free position by paying $\frac{1}{2}B(1) = \frac{1}{2} \times 110 = Rs\ 55$. Therefore at $t = 1$, $V_P(1) = F - 55$, which is strictly positive as we have assumed that $F > 55$. This clearly violates the no arbitrage principle. Hence $F = 55$.

□

Call and Put Options

Similar to forward contracts, the call and put options in their basic form are also derivative securities whose underlying is stock. An option has the below given characteristics.

(i) The owner of a call option gets the right to buy stock at a future date (called *maturity* or *expiration date*) at a predetermined price (called *strike* or *exercise price*). In a similar manner, the owner of a put option gets the right to sell stock at maturity at a predetermined strike price.

(ii) The owner of the option (whether call or put) has the right without any obligation. Thus at maturity the owner may not exercise the option if he/she so wishes.

(iii) The person who has bought the option is called its *holder* and the person who has sold the option is called its *writer*.

(iv) As explained in (ii) above, the holder of the option has the right but no obligation whereas the writer of the option does have a potential obligation. Thus in case of a call option, the writer is duty bound to sell the stock should the holder choose to exercise the call option. Similarly in case of a put option, the writer is duty bound to purchase the stock should the option is exercised by the holder.

What we have described above are basic characteristics of a *European call/European put option*. Here the option can be exercised at maturity only and not any time before that. Thus for a single period model, maturity has to be $t = 1$. In contrast, if the option can be exercised at any time before or at maturity, then it is called an *American option*. Thus for a single period model, an American option can be exercised at $t = 1$ or at $t = 0$. Unless we mention it specifically, for us an option shall always mean a European option.

Since an option confers on its holder a right without any obligation, the holder needs to pay some amount at $t = 0$ to have this right. This amount which is paid at $t = 0$ when option is bought is called the *premium* or *price* of the option. Thus we need to determine the price of a European call, American call, European put and American put etc.

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Remark 1.4.1 *At first, a call option resembles very much with a long forward contract because both involve buying stock at a future date for a fixed price in advance. But the holder of long forward contract is committed for buying the asset for the fixed price whereas the holder of a call option has the right but no obligation to do so. For this reason there is no premium in case of a long forward contract but there is a premium in the case of an option.*

Remark 1.4.2 *In the case of derivatives e.g. forward contracts, options etc., there are two parties which get together and set a rule by which one of two parties will receive a payment from the other party depending upon the value of some financial variables. Also profit of one party is the loss of other party. This is a typical situation of a two person zero sum game. Therefore one can visualize many of the derivative pricing and other finance related problems as game theoretic problems and use the rich theory available there.*

Example 1.4.2 *Let* $B(0) = 100$, $B(1) = 110$, $S(0) = 100$ *and*

$$S(1) = \begin{cases} 120, & \textit{with probability } 0 \cdot 8 \\ 80, & \textit{with probability } 0 \cdot 2. \end{cases}$$

Consider a European call option C *on the given stock for which the strike price* $K = 100$ *and expiration date is 1 year. Determine* $C(0)$, *i.e. the call price of this option.*

Solution Here $S(1)$, the stock price at $t = 1$, takes two values. Denoting these by $u\,S(0)$ and $d\,S(0)$, we obtain $u = 1.2$ and $d = 0.8$. Such a stock dynamics is called a *single period binomial model.* Here

$$S(1) = \begin{cases} u\,S(0), & \text{with probability } p \\ d\,S(0), & \text{with probability } (1-p), \end{cases}$$

with $0 < p < 1$.

Also as $B(1) = 110$, $B(0) = 100$ we have nominal rate of interest $r = 10\% = 0 \cdot 1$. It is important to note here that $d < (1 + r) < u$ which in the context of binomial model is a requirement for no arbitrage principle to hold. This result is proved analytically in Chapter 3.

Though this dynamics looks extremely simple and probably unrealistic, it forms the basis of other multi period and continuous pricing models.

To determine the call price $C(0)$ we shall make use of the law of one price. For this we shall first determine the pay-off of the given call, i.e its value $C(1)$ at $t = 1$;

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and then construct a portfolio $P : (x, y)$ of stock and bond such that $V_P(1)$ equals $C(1)$. Then the law of one price tells that the required price of the call has to be the value of the portfolio at $t = 0$; i.e. $C(0) = V_P(0)$.

To compute the pay-off $C(1)$ we argue that the holder will exercise the call option only when the stock price $S(1)$ is more than K and receive the amount $S(1) - K$. If $S(1) \leq K$, then it is not beneficial for the holder and therefore he/she will use his/her right not to exercise the option. Thus

$$C(1) = \begin{cases} S(1) - K\,, & \text{if } S(1) > K \\ 0\,, & \text{if } S(1) \leq K\,, \end{cases}$$

i.e.

$$C(1) = \max((S(1) - K),\ 0) = (S(1) - K)^+ \,.$$

Here for $x \in R, x^+$ denotes $\max(x, 0)$.
For our problem

$$C(1) = \begin{cases} 20, & \text{with probability } 0 \cdot 8 \\ 0, & \text{with probability } 0 \cdot 2\,. \end{cases}$$

Next we need to determine x and y such that for the portfolio $P : (x, y)$, $V_P(1) = C(1)$. This gives

$$x\ S(1) + y\ B(1) = C(1)\,,$$

i.e.

$$\begin{aligned} 120x + 110y &= 20 \\ 80x + 110y &= 0. \end{aligned}$$

Solving above two equations we get $x = \frac{1}{2}$ and $y = -\frac{4}{11}$. Hence by the law of one price we obtain

$$\begin{aligned} C(0) = V_P(0) &= x\ S(0) + y\ B(0) \\ &= \frac{1}{2}(100) - \frac{4}{11}(100) \\ &= 13 \cdot 6364. \end{aligned}$$

Thus the buyer (holder) of the call option has to pay a premium of *Rs* 13.64 to the seller (writer) of the option. □

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Here it must be noted that the above arguments are not valid if the inequality $d < (1 + r) < u$ is not true. This is because if this inequality is not true, then no arbitrage principle does not hold, and hence the law of one price is not valid.

Remark 1.4.3 *At the time of writing the option $S(1)$ is unknown. Therefore the first problem of option pricing is to find how much the holder should pay the writer for an asset (stock) worth $(S(1)-K)^+$ at time $t = 1$? From the writers point of view it is imperative to use the premium $C(0)$ in constructing a portfolio (x, y) such that it generates an amount $(S(1) - K)^+$ at $t = 1$. This is the problem of hedging the option and the portfolio (x, y) is called the replicating portfolio because in terms of pay-off at $t = 1$ it replicates the given option. Thus while pricing a derivative, two problems are being solved simultaneously namely the pricing problem for the holder and hedging problem for the writer. Here again a game theoretic interpretation could be given so as to interpret these two problems as primal-dual pair.*

Remark 1.4.4 *On first reading, option may look superfluous in a market because we are replicating in terms of two basic securities of stock and bond. This is certainly true for a single period model. But for multi period and continuous time models, this is not that simple because it involves re-balancing of the replicating portfolio at every time instant. Therefore options and other derivatives are treated as separate and independent financial instruments and not something which can be expressed in terms of stock and bond.*

Example 1.4.3 *Let $B(0)$, $S(0)$, $B(1)$, $S(1)$, K and T be same as in Example 1.4.2. Let the option P be a put option. Determine $P(0)$.*

Solution It is simple to argue that for the case of put option the pay-off at time $t = 1$ is Max $((K - S(1)),\ 0) = (K - S(1))^+$.
Thus

$$P(1) = \begin{cases} 0, & \text{with probability } 0 \cdot 80 \\ 20, & \text{with probability } 0 \cdot 20. \end{cases}$$

This gives

$$\begin{aligned} 120x + 110y &= 0 \\ 80x + 110y &= 20\ , \end{aligned}$$

i.e. $x = -\dfrac{1}{2}$ and $y = \dfrac{6}{11}$. Therefore,

$$P(0) = x\ S(0) + y\ B(0) = -\frac{1}{2}(100) + \frac{6}{11}(100) = 4.54. \qquad \square$$

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Remark 1.4.5 *In above examples we have seen that while determining* $C(0)$ *and* $P(0)$ *we have never used the actual probabilities* $p = 0 \cdot 8$ *and* $(1 - p) = 0 \cdot 2$. *It is probably not possible to explain its reason at this stage because it has to be explained in terms of risk neutral probability measure (RNPM) which will be discussed in Chapter 3 and also in other chapters on option pricing.*

1.5 Some More Terminology

We have already introduced terms like bonds, stocks, portfolios, securities and derivatives. In this section we first introduce some more terminology with respect to bond and stock and then proceed to discuss *time value of money* and *money market account*.

Security is a general term used for bonds, stocks and derivatives. Stocks are also referred as *equities*. We can treat bonds as *fixed income securities*, because at maturity they pay fixed amounts of money to their owners. Regular savings accounts, certificates of deposits are other examples of fixed income securities.

Thus a bond is a security that gives its owner the right to a fixed predetermined payment at a future predetermined date, called *maturity*. The amount of money that a bond will pay in future is called *nominal value*, *face value*, *par value* or *principal*. Thus bonds represent the general paradigm of risk free securities. In fact we can treat money (cash) also as bond with zero interest rate and zero (immediate) maturity.

There are bonds that involve only an initial payment (the initial price) and a final payment (the final value or principal). Such bonds are called *pure discount bonds* or *zero coupon bonds*. In contrast there are *coupon bonds* where the holder of the bond gets periodic payments from the issuing institution during the life of the bond. These periodic payments are specified percentage of the principal and are called *coupons*. If the price at which the bond is sold is same as its nominal value we say that the bond sells *at par*. The terms *above par* and *below par* are defined in an obvious manner.

Sometimes the institution issuing the bond does not meet the promise and it defaults in payment. This type of risk is called *credit risk* or *default risk*. The bonds issued by the Government are considered to be free of default risk. Even when there is no default risk, there is always *inflation risk* because future prices of goods are uncertain. If the rate of interest does not take into consideration the inflation risk then it is called *nominal interest rate*. If a bond guarantees a payment that takes care of inflation level, it is called *inflation indexed bond*.

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In context of stock, we call the owner of the stock as *stockholder*. The profit that the company distributes to the stockholders is called *dividend*. Dividend in general is not known in advance because it depends upon the company's profit and its policy. The difference between the selling price and the initial price is called *capital gain* or *loss*.

Time Value of Money

The way in which money changes its value in time is of utmost importance in finance. Though a general scenario may be very complex, the below given questions are simple to answer.

(i) What is the future value of an amount invested or borrowed today?
(ii) What is the present value of an amount to be paid or received at a certain time in future?

We all have studied simple and compound interest in our schools. Let an initial amount (called *principal*) P be deposited into a bank account where it earns interest. The future value of this investment consists of the principal P together with all the interest earned for the period the money remained deposited. Let the interest rate be $r\%$ per year and t be the time (expressed in years) for which the money was deposited. Let $V(t)$ denote the value of this investment at time t. Then under the assumption that the principal P attracts *simple interest rate* only, we have

$$V(t) = P(1 + rt).$$

We next consider the case of *periodic compounding*. Here again an amount P is deposited in a bank account attracting a constant interest rate $r\%$ per year. But in contrast to the simple interest case, here the interest earned is added to the principal periodically. This is called *periodic compounding* which could be annually, semi-annually, quarterly, monthly, weekly or daily basis. Therefore interest will be attracted not only on the principal P but also by all the interest earned so far. In this case we have

$$V(t) = P\left(1 + \frac{r}{m}\right)^{tm},$$

where m refers to the period used in the compounding. Thus if it is compounded semi-annually then $m = 2$ because there are two periods in an year. For quarterly compounding $m = 4$ and for monthly compounding $m = 12$, etc. Here the number $\left(1 + \frac{r}{m}\right)^{tm}$ is called the *growth factor*.

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Example 1.5.1 *How long will it take for a sum of Rs 800 attracting simple interest to became Rs 830 if the rate of interest is 9% per year? Also compute the return on this investment.*

Solution From the formula for the simple interest we have

$$830 = 800(1 + 0 \cdot 09\, t)\,,$$

which gives $t = 0 \cdot 417$ year which is approximately 152 days. Further the return on this investment will be

$$\frac{V(t) - V(0)}{V(0)} = \frac{830 - 800}{800} = 0 \cdot 0375\,,$$

i.e. $3 \cdot 75\%$.

□

Example 1.5.2 *Which will deliver a higher future value after one year, a deposit of Rs 1,000 attracting interest at 15% compounded daily, or at 15·5% compounded semi-annually?*

Solution At 15% compounded daily the deposit will grow to

$$V(t) = 1000\left(1 + \frac{0 \cdot 15}{365}\right)^{1\times 365} \cong Rs\ 1161 \cdot 80\,,$$

after one year. If the interest is compounded semi-annually at $15 \cdot 5\%$, the value after one year will be

$$V(t) = 1000\left(1 + \frac{0 \cdot 155}{2}\right)^{1\times 2} \cong Rs\ 1161 \cdot 01\,.$$

So interest at 15% compounded daily will give higher future value.

□

We next introduce the concept of *continuous compounding*. Let us recall the formula

$$V(t) = P\left(1 + \frac{r}{m}\right)^{tm}\,,$$

which can be written as

$$V(t) = P\left[\left(1 + \frac{r}{m}\right)^{\frac{m}{r}}\right]^{rt}\,.$$

Therefore if we take the limit as $m \to \infty$ we get $V(t) = Pe^{rt}$. This is known as continuous compounding. Here the growth factor is e^{rt}.

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Example 1.5.3 *It is given that the future value of Rs* 950 *subject to continuous compounding will be Rs* 1,000 *after half year. Find the interest rate.*

Solution The rate r satisfies

$$950\, e^{r\times 0.5} = 1000.$$

This gives $r = 0.1026$, i.e. 10.26%.

□

Let us recall the simple interest formula $V(t) = P(1 + rt)$. Denoting the initial deposit P as $V(0)$ (value of this investment at $t = 0$) we have $V(t) \equiv V(0)(1 + rt)$, which gives $V(0) = V(t)(1+rt)^{-1}$. This last formula gives the *present* or *discounted value* of $V(t)$ and $(1 + rt)^{-1}$ is the *discount factor*.

For the case of periodic compounding the *present* or *discounted value* $V(0)$ of $V(t)$ is given by $V(t)\left(1 + \frac{r}{m}\right)^{-tm}$ and $\left(1 + \frac{r}{m}\right)^{-tm}$ is the discount factor. In case of continuous compounding we have the present or discounted value as $V(0) = V(t)e^{-rt}$ and the discount factor as e^{-rt}. If the principal P is invested at time s rather than at time 0, then the modifications are obvious.

Example 1.5.4 *Find the present value of Rs* 1,00,000 *to be received after* 100 *years if the interest rate is assumed to be* 5% *throughout the whole period and* (*a*) *daily or* (*b*) *annual compounding applies.*

Solution Under daily compounding the present value is given by

$$V(0) = 100000\left(1 + \frac{0.05}{365}\right)^{-100\times 365} \cong Rs\ 674 \cdot 03.$$

Under annual compounding the present value is given by

$$V(0) = 100000(1 + 0.05)^{-100} \cong Rs\ 760 \cdot 45.$$

□

It is obvious that frequent compounding will produce a higher future value than less frequent compounding if the initial deposit P and the interest rate r are same. But for the general case, to compare two compounding methods we need to compare the corresponding growth factor over the same base, say one year. This motivates the below given definition.

Definition 1.5.1 (Comparison of compounding methods) *Two compounding methods are said to be equivalent if their corresponding growth factors over a period of one year are same. If one of the growth factor exceeds the other, then the compounding method having the higher growth factor is preferred over the other.*

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Example 1.5.5 *Is semi-annual compounding at 10% is preferable to monthly compounding at 9%?.*

Solution For semi-annual compounding at 10%, the growth factor over a period of one year is

$$\left(1+\frac{0\cdot 1}{2}\right)^2 = 1\cdot 1025.$$

But for monthly compounding at 9%, the growth factor is

$$\left(1+\frac{0\cdot 09}{12}\right)^{12} \cong 1\cdot 0938.$$

Therefore 10% semi annual compounding is preferable to 9% monthly compounding.

□

There is another equivalent convention to compare the two compounding methods. Here rather than comparing the corresponding growth factors, we compare the corresponding *effective rate* r_e as defined below:

Definition 1.5.2 (Effective Rate) *For a given compounding method with interest rate r, the* effective rate r_e *is the one which gives the same growth factor over a one year period under annual compounding.*

In view of Definition 1.5.2, for the case of periodic compounding, the effective rate r_e satisfies the equation

$$\left(1+\frac{r}{m}\right)^m = (1+r_e)\,.$$

If it is the case of continuous compounding then r_e satisfies the equation

$$e^r = 1 + r_e\,.$$

In terms of effective rates, two compounding methods are equivalent if the corresponding effective rates r_e and r_e' are equal. if $r_e > r_e'$, then the compounding method with effective rate r_e is preferred over the compounding method with effective rate r_e'. This follows directly from Definitions 1.5.1 and 1.5.2.

Example 1.5.6 *Find the effective rate r_e if it is semi-annual compounding at 10%.*

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Solution We have

$$\left(1+\frac{0.01}{2}\right)^2 = (1+r_e)\ ,$$

which gives $r_e = 10 \cdot 25\%$.

□

Example 1.5.7 *Is daily compounding at* 15% *preferable to semi-annual compounding at* $15 \cdot 5\%$*?*

Solution Let the corresponding effective rates be r_e and r_e'. Then

$$1 + r_e = \left(1+\frac{0.15}{365}\right)^{365} \cong 1 \cdot 1618\ ,$$

and

$$1 + r_e' = \left(1+\frac{0.155}{2}\right)^{2} \cong 1 \cdot 1610.$$

Thus $r_e \cong 16 \cdot 18\%$ and $r_e' \cong 16 \cdot 10\%$. Therefore daily compounding at 15% is preferable.

□

We can also express future value in terms of effective rate r_e because of its relationship with the growth rate. In fact we can show that

$$V(t) = P(1+r_e)^t$$

for all $t \geq 0$. This formula holds both for the periodic compounding and continuous compounding, but it cannot hold for the case of simple interest. This is because for the simple interest case, $V(t)$ is a linear function of t, whereas for the periodic/continuous compounding it is an exponential function of t.

Streams of Payments

An *annuity* is a sequence of finitely many payments of a fixed amount due at equal time intervals. Let payment of an amount C are to be made once a year for n years, the first one is due a year hence. If we assume that annual compounding applies, then the present value of such a stream of payments is given by

$$\frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \ldots + \frac{C}{(1+r)^n}.$$

If we denote

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$$PA(r,n) = \frac{1}{(1+r)} + \frac{1}{(1+r)^2} + \ldots + \frac{1}{(1+r)^n},$$

then the present value of the given stream of payments is $(C \times PA(r,n))$. Here the number $PA(r,n)$ is called the *present value factor for an annuity*. Obviously

$$PA(r,n) = \frac{1-(1+r)^{-n}}{r}.$$

We define *perpetuity* as an infinite stream of payments of a fixed amount C occurring at the end of each year. Therefore the *perpetuity* can be computed by $\lim_{n\to\infty} PA(r,n) \times C$ which equals $\frac{C}{r}$.

Example 1.5.8 *Consider a loan of Rs* $1,000$ *to be paid back in* 5 *equal instalments due at yearly intervals. The instalments include both the interest payable each year calculated at* 15% *of the current outstanding balance and the repayment of the fraction of the loan. Determine the amount of each instalment.*

Solution We can look the loan as equivalent to annuity from the point of view of the lender. Therefore if C is the amount of each instalment then

$$1000 = C \times PA(0 \cdot 15, 5).$$

But

$$PA(0 \cdot 15, 5) = \frac{1-(1+0 \cdot 15)^{-5}}{0 \cdot 15}.$$

This gives $C \cong Rs\ 298 \cdot 32$.

□

Example 1.5.9 *Let the interest rate be* 18% *and a person can afford to pay Rs* $10,000$ *at the end of each year for next* 10 *years. Determine the amount which the person can borrow?*

Solution The person can borrow the amount $PA(0 \cdot 18, 10) \times 10000 = Rs\ 44,941$.

□

Money Market

By the money market we mean default free securities. Thus money market consists of risk free bonds like treasury bills and notes and other financial securities which promise the holder a sequence of guaranteed future payments. Risk free here means default free i.e. these payments will be delivered with certainty. However in

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practice the risk, even for risk free securities, cannot be completely avoided since the market prices of such securities may fluctuate unpredictably and there may be default in payments. But in our discussion here we shall assume that it is a default free, and hence risk free situation.

As mentioned earlier, the simplest case of a bond is *zero coupon bond*. Here the issuing institution (a government, a bank, or a company) promises to exchange the bond for a certain amount of money F, called the *face value*, on a given day T, called the *maturity date*. A typical zero coupon bond has the face value of *Rs* 100 (or some other round figure) and maturity date as one year (or multiple of a year). If an investor buys a bond, he/she becomes its holder and its effect is that he/she is lending money to the bond writer (a government, a bank, or a company). Here it must be noted that investor need not always be an individual, it could also be a financial institution.

Given the interest rate, the present value of such a bond can be computed easily. Thus if the face value of a bond is $F = Rs\ 100$ and maturity date $T = 1$ year, then for 12% compounding annually we have $V(0) = F(1+r)^{-1} \cong Rs\ 89 \cdot 29$.

For simplicity, we shall assume that the face value F of the given bond is *Rs* 1. Such bonds are called *unit bonds*. Typically a bond can be sold any time prior to maturity at the prevailing market price. Let $B(t,T)$ denote the price of the given unit bond at time t when the maturity is T. Thus $B(T,T) = 1$ and $B(t,T) = e^{-r(T-t)}$. Here we have assumed that r remains constant throughout the period up to maturity.

But in reality the bonds are freely traded in the market and their prices are determined by market forces. Therefore the *market interest rate* (depending upon the market price of the given bond) may be different from the *implied interest rate* $r = \dfrac{F}{V(0)} - 1$ given by the formula $V(0) = F(1+r)^{-1}$. In fact the implied interest rate should not be constant as it should depend on the trading time t as well as on the maturity time T. Not only this, but also the bond prices $B(t,T)$ are determined by (random) market forces. Therefore the implied interest rate should be taken as a random process $r(t,T)$. We shall have occasion to discuss this aspect in a later chapter.

As our bond is a unit bond we can treat $B(0,T)$ as the discount factor and $B(0,T)^{-1}$ as the growth factor. This observation allows to compute time value of money without restoring to the corresponding interest rates.

Coupon Bonds

In zero coupon bonds, there is just a single payment at maturity. But in *coupon*

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bonds, there is a sequence of payments. These payments consist of the face value due at maturity, and *coupons* paid regularly, say annually, semi-annually or quarterly. The last coupon becomes due at maturity only. If the interest rate r is assumed to be constant throughout, then it is simple to compute the price of a coupon bond by discounting all the future payments. We explain this process with the help of the next example.

Example 1.5.10 *Find the price of a bond with face value Rs* 100 *and Rs* 5 *annual coupons that matures in* 4 *years, if the continuous compounding rate is* 8%.

Solution The price of the given bond is obtained as

$$5e^{-0.08} + 5e^{-2\times 0.08} + 5e^{-3\times 0.08} + 105e^{-4\times 0.08} \cong Rs\ 89 \cdot 06.$$

Here we should note that as maturity is 4 years, the payments have to be Rs 5, 5, 5 and105.

□

Money Market Account

An investment in the money market can be realized by means of an investment bank which buys and sells bonds on behalf of its customers. The risk free position of an investor is given by the level of his/her account with the bank. We can think of this account as a tradable asset, since the bond themselves are tradable. A long position in the money market involves investing the money and a short position means borrowing money.

Thus an investment of amount $A(0)$ in the money market amounts to purchasing $\frac{A(0)}{B(0,T)}$ bonds at $t = 0$. At time t, the value of each bond will be

$$B(t,T) = e^{-r(T-t)} = e^{rt}e^{-rT} = e^{rt}B(0,T).$$

As a result, at time t, $t \leq T$, the investment $A(0)$ will become

$$A(t) = \frac{A(0)}{B(0,T)}B(t,T) = A(0)e^{rt}.$$

The investment in bond has a finite time horizon, say T. To extend the position in the money market beyond T, the investor can invest the amount received at maturity into a newly issued bond at time T, but maturing at $T'(T' > T)$. We explain this by the below given example.

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Example 1.5.11 *Suppose Rs 1 is invested in zero coupon bonds maturing after one year. At the end of each year the proceeds are re-invested in new bonds of the same kind. How many bonds will be purchased at the end of year n $(n \geq 1)$? It may be assumed that the interest rate is constant at r % and continuous compounding is followed.*

Solution At $t = 0$, the investor buys $\frac{1}{B(0,1)} = e^r$ bonds. At time $t = 1$, increases his/her holdings to $\frac{e^r}{B(1,2)} = e^{2r}$ bonds. Therefore at the end of year n, the investor will purchase $e^{(n+1)r}$, one year bonds. □

1.6 Summary and Additional Notes

- The date of March 29, 1900 should be considered as the origin of Financial Mathematics. On that day, a French postgraduate student, Louis Bachelier successfully defended his thesis *Théorie de la spéculation*. Bachelier presented a model of Brownian motion (see Chapter7) while deriving the evolution of Paris asset prices. This work of Bachelier was neglected by the economic community for many years but probabilists took great interest in this development. Kolmogorov's analytical theory of Markov process has its origin in the theory proposed by Bachelier. The economic community also later recognized Bachelier's contribution and named *International Finance Society* after him.
  Bachelier's thesis topic at that time was out of ordinary. The appropriate topic of research in that era were on the theory of functions. Therefore this topic was not initially accepted as a research topic. For these and other historical details on Bachelier's work we may refer to Vassiliou [144].
- In this chapter we have presented an overview of the topic of *financial mathematics*. Here our main aim has been to introduce the basic terminology of finance and motivate the readers to the art of mathematical modeling to deal with the problems arising in this area. In the process, we have also introduced the readers to the three core topics of the book, namely *option pricing, portfolio optimization* and *interest rate theory* through elementary illustrative examples.
- Though the illustrative examples presented here require only an elementary knowledge of algebra, calculus and probability theory, it has been emphasized that real life problems in finance are extremely complex. Therefore we need to get ourselves equipped with certain advanced topics in mathematics, e.g.

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2
Forward and Futures Contracts

2.1 Introduction

A *forward contract* is probably the simplest of all derivative securities. It has a simple pricing mechanism and has wide applications, particularly in commodity and foreign exchange markets. We had already initiated some discussion on forward contracts in Chapter 1 where we had also explained the pricing methodology through a simple example.

Though *futures contracts* are very much in the spirit of forward contracts, they have specially been designed to standardize these contracts so as to eliminate the risk of default by the party suffering the loss. For this, a process called *marking to market* is conceptualized which requires an individual to open a *margin account* which is managed by an organized clearing house/exchange.

This chapter continues our earlier discussion on forward contracts and explains various concepts related with the working of futures contracts. We also describe very briefly another derivative security called *swap* which is again very popular in commodity and foreign exchange markets.

2.2 Forward Contract

Let us recall the definition of a *forward contract* from Chapter 1 and note that it is an agreement between two investors (also called parties) to buy or sell a risky asset at a specified future time (called the *delivery date*) for a price F (called the *forward price*) fixed at the present time (say $t = 0$). The party which agrees to buy the asset is said to enter into a *long forward contract* or to take a *long forward position*. The party which agrees to sell the asset is said to enter into a *short forward contract* or to take a *short forward position*. There is no payment of money by either party

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when the agreement for the forward contract is made. Some typical examples of forward contract could be a farmer wishing to fix the sale price of his/her crops in advance, an importer arranging to buy foreign currency at a fixed rate in future, a fund manager wishing to sell stock for a fixed price in future or a country wishing to import commodities (like wheat, sugar, oil etc.) from another country at a fixed rate at some specified future date. In these situations, forward contracts become very handy because they provide an opportunity to hedge against the unknown future price of the underlying risky asset.

Let $t = 0$ denote the time when the two parties enter into the specific forward contract agreement and $t = T$ be the delivery date. Let the agreed forward price be $F(0, T)$. Here the two arguments in F denote the time $t = 0$ and $t = T$ respectively. In case the context is clear and there is no ambiguity, we write $F(0, T)$ simply as F.

Let us now analyze the two scenarios, namely $F(0, T) < S(T)$ and $F(0, T) > S(T)$. Let us first consider the case $F(0, T) < S(T)$. In this case the party having the long forward contract will benefit because it can buy the asset for $F(0, T)$ and sell the same for the market price $S(T)$, making a profit of $S(T) - F(0, T)$. But the other party which has taken a short forward position will suffer loss of $S(T) - F(0, T)$ because it will have to sell below the market price. Fig. 2.1 clearly exhibits these two positions.

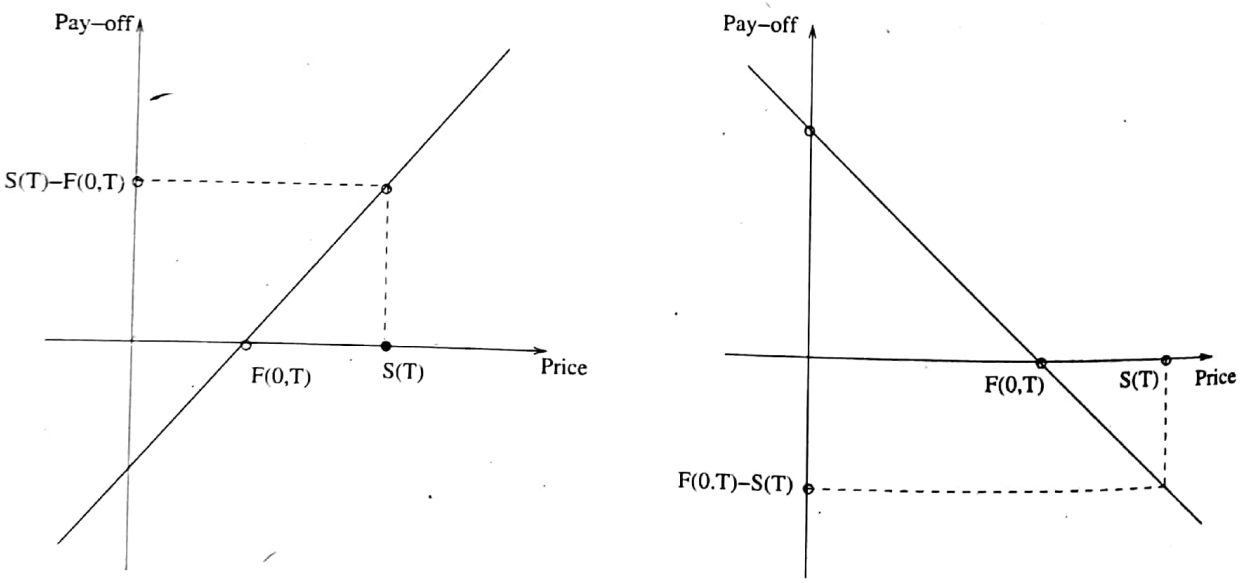

Fig. 2.1. Pay-off of a forward contract

If $F(0, T) > S(T)$ then these arguments will simply get reversed. In this case the

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party taking a short forward position will benefit and make a profit of $F(0,T)-S(T)$ whereas the other party (which has taken a long forward position) will make a loss of $F(0,T)-S(T)$ and analogue of Fig. 2.1 can be drawn for this case as well.

Therefore in either case the pay-off of a long forward contract at delivery is $S(T)-F(0,T)$ and that of a short forward contract is $F(0,T)-S(T)$. Sometimes the contract may be initiated at time $t<T$ rather than $t=0$. In this scenario the pay-offs of a long forward position and a short forward position are $S(T)-F(t,T)$ and $F(t,T)-S(T)$ respectively where $F(t,T)$ denotes the forward price.

2.3 Forward Price Formula

We now look into the problem of determining the price of the forward contract, i.e. the forward price $F(0,T)$. We shall make use of the *no arbitrage principle* and follow the methodology indicated in Chapter 1 to get the next theorem.

Theorem 2.3.1 (Forward Price Formula) *Let the price of an asset at $t=0$ be $S(0)$. Then the forward price $F(0,T)$ is given by*

$$F(0,T)=\frac{S(0)}{d(0,T)}, \tag{2.1}$$

where $d(0,T)$ is the discount factor between $t=0$ and $t=T$.

Proof. If possible let $F(0,T)>(S(0)/d(0,T))$. Then at $t=0$ we construct the portfolio P as follows

(i) borrow $S(0)$ amount of cash until $t=T$,
(ii) buy one unit of the underlying asset on the spot market at price $S(0)$ (if the underlying is the stock then it will mean buying one share of stock for amount $S(0)$),
(iii) take a short forward position with delivery at $t=T$ and forward price as $F(0,T)$.

This gives the value of the portfolio P at $t=0$ as

$$V_P(0)=-S(0)+S(0)=0.$$

Now at $t=T$, we sell the asset for $F(0,T)$ and pay the amount $(S(0)/d(0,T))$ to clear the loan with interest and thereby close the short position. This gives

$$V_P(1)=F(0,T)-(S(0)/d(0,T)),$$

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which is strictly positive under the stated assumption. Thus there is a strictly positive amount of risk free profit with zero net investment. This is contrary to the no arbitrage principle.

Next suppose that $F(0,T) < (S(0)/d(0,T))$. In this case, at $t = 0$, we construct the portfolio Q as follows

(i) short sell one unit of the underlying asset (thus if the underlying asset is stock, then we borrow one share of stock and sell the same in the market with all conditions and procedure as specified in short selling),
(ii) invest the proceeds (i.e. the amount $S(0)$ received by selling one unit of the asset) at the risk free rate up to $t = T$,
(iii) enter into a long forward contract with delivery as $t = T$ and forward price $F(0,T)$.

This gives

$$V_Q(0) = S(0) - S(0) = 0.$$

Now at $t = 1$, we

(i) cash the risk free investment with interest, i.e. collect cash $(S(0)/d(0,T))$,
(ii) buy the underlying asset for $F(0,T)$ using the long forward position,
(iii) close the short position on the underlying asset by returning the borrowed unit of asset to the owner.

This gives

$$V_Q(T) = (S(0)/d(0,T)) - F(0,T),$$

which is strictly positive. This again allows a strictly positive risk free profit with zero net investment contradicting the no arbitrage principle. Thus $F(0,T) = S(0)/d(0,T)$.

□

Remark 2.3.1 *For the case of constant interest rate r being compounded continuously, we have $d(0,T) = e^{-rT}$ and hence*

$$F(0,T) = S(0)\, e^{rT}. \tag{2.2}$$

Remark 2.3.2 *If the contract is initiated at some intermediate time t, $0 < t < T$, then $d(t,T) = e^{-r(T-t)}$ and*

$$F(t,T) = S(t)\, e^{r(T-t)}, \tag{2.3}$$

where r is same as in Remark 2.3.1.

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Remark 2.3.3 *In terms of zero coupon bond prices, the forward price formula 2.1 becomes*

$$F(0,T) = \frac{S(0)}{B(0,T)}, \tag{2.4}$$

which is more convenient to use as it does not require the assumption that the interest rate r is constant.

It is a matter of common knowledge that holding of physical assets like gold, sugar, oil, wheat etc. entail inventory carrying costs, such as rental for storage and insurance fees etc. These costs affect the theoretical forward price formula as given at (2.1). The standard forward price formula (2.1) can be generalized in several ways to take into consideration the costs of carry and also to include dividends. We have the below given result in this regard.

Theorem 2.3.2 (Forward Price Formula with Carrying Costs) *Let an asset carry a holding cost of $c(i)$ per unit in period i $(i = 0, 1, 2, \ldots (n-1))$. Also let at $t = 0$, the price of this asset be $S(0)$ and short selling be allowed. Then*

$$F(0,T) = \frac{S(0)}{d(0,T)} + \sum_{i=0}^{n-1} \frac{c(i)}{d(i,n)}, \tag{2.5}$$

where delivery date is $t = T$ and between $t = 0$ and $t = T$ there are n periods which have appropriately been identified as per the given context.

Next we consider the case when the underlying asset (e.g. stock) is dividend paying.

Theorem 2.3.3 (Forward Price Formula with Dividend) *Let an asset be stored at zero cost and also sold short. Let the price of this asset at $t = 0$ be $S(0)$ and a dividend of Rs div be paid at time τ, $0 < \tau < T$. Then*

$$F(0,T) = \frac{[S(0) - (div)\,(d(0,\tau))]}{d(0,T)}. \tag{2.6}$$

For the case of constant interest rate r being compounded continuously, the above formula becomes

$$F(0,T) = [S(0) - (div)\, e^{-r\tau}]\, e^{rT}. \tag{2.7}$$

Sometimes the asset (stock) may pay dividends continuously at a rate of $r_{div} > 0$. This is called *continuous dividend yield.* If the dividends are reinvested in the stock, then an investment in one share held at $t = 0$ will become $e^{(r_{div})T}$ shares at $t = T$. This is very similar to continuous compounding and therefore $e^{-(r_{div})T}$ shares at $t = 0$ will give one share at $t = T$. This observation leads to Theorem 2.3.4.

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Theorem 2.3.4 (Forward Price Formula with Continuous Dividend) *Let an asset be stored at zero cost and also sold short. Let the price of the asset at $t = 0$ be $S(0)$ and the asset pay dividends continuously at a rate of r_{div}. Then*

$$F(0,T) = S(0)\, e^{(r-r_{div})T}. \tag{2.8}$$

Theorems 2.3.2, 2.3.3 and 2.3.4 are not proved here as their proofs can be constructed analogous the proof of Theorem 2.3.1. Interested readers may refer to Luenberger [85] and Capinski and Zastawniak [25] in this regard.

Forward contracts in foreign currency market are very common and we wish to derive a formula for determining the price of the same. To be specific, let us consider the two currencies as British pound and US dollars with the latter as the underlying. The readers may imagine a British importer of US goods requiring US dollars after $t = T$. So this British importer may think of taking a forward contract on US dollars with delivery as $t = T$ to hedge against the fluctuating exchange rate of British pound versus US dollars.

Let at $t = \tau$, the buying and selling exchange rate be: 1 British pound = $P(t)$ US dollars. Also let the risk free interest rates for investments in British pounds and US dollars be r_{GBP} and r_{USD} respectively. Then the forward price is given by

$$F(0,T) = P(0)\, e^{(r_{USD}-r_{GBP})T}. \tag{2.9}$$

USD/GBP

The above formula gives the agreed exchange rate at $t = T$, i.e. at $t = T$, the British importer will be able to buy $F(0,T)$ US dollars for one British pound.

To prove formula (2.9), let us consider two strategies as described below.

Strategy A

Invest $P(0)$ US dollars at the rate of r_{USD} until $t = T$.

Strategy B

Buy one British pound for $P(0)$ US dollars, invest it until $t = T$ at the rate of r_{GBP}, and take a short position in $\exp(r_{GBP} \cdot T)$ British pound forward contract with delivery as $t = T$ and forward price as $F(0,T)$. This gives

$$P(0)\, e^{(r_{USD})T} = F(0,T)\, e^{(r_{GBP})T},$$

and hence the result.

Example 2.3.1 *The current price of gold is Rs 18,000 per 10gm. Assuming a constant interest rate of 8% per year compounded continuously, find the theoretical price of gold for delivery after 9 months.*

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Solution Here $S(0) = 18,000$, $r = 8\%$ per year and $T = 9$ months $= 3/4$ year. Hence by the forward price formula

$$\begin{aligned} F &= S(0)\, e^{rT} \\ &= (18000)\, e^{(0.08\times 3/4)} \\ &= (18000)\, e^{0.06} \\ &= Rs\ 19113.06. \end{aligned}$$

□

Example 2.3.2 *Find the forward price of a non-dividend paying stock traded today at Rs 100, with the continuously compounded interest rate of 8% per year, for a contract expiring seven months from today.*

Solution We have $S(0) = 100$, $r = 8\%$ per year and $T = 7/12$ year. This gives

$$F = 100\, e^{(0.08)(7/12)} = Rs\ 104.78.$$

□

Example 2.3.3 *The current price of sugar is Rs 60 per Kg and its carrying cost is 10 paisa per Kg per month to be paid at the beginning of each month. Let the constant interest rate r be 9% per annum. Find the forward price of the sugar (Rupees per Kg) to be delivered in 5 months.*

Solution The interest rate is $(.09)/12 = .0075$ per month. Therefore the reciprocal of one month discount rate for any month is (1.0075). Therefore we have

$$\begin{aligned} F(0,5) &= (1.0075)^5(60) + \Big(\sum_{i=1}^{5}(1.0075)^i\Big)(.1) \\ &= Rs\ 62.79. \end{aligned}$$

□

Example 2.3.4 *Let the price of a stock on 1st April 2010 be 10% lower than it was on 1st January 2010. Let the risk free rate be constant at $r = 6\%$. Find the percentage drop of the forward price on 1st April 2010 as compared to the one on 1st January 2010 for a forward contract with delivery on 1st October 2010.*

Solution It is convenient to take 1st January 2010 as $t = 0$. Then 1st October 2010 is 9 months, i.e. $3/4$ year. Thus $t = 3/4$. Also $t = \tau$ is 1st April 2010, i.e. $3/12 = 1/4$. Therefore using Theorem 2.3.1 we get

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$$F(0,3/4) = S(0)\, e^{(0.06\times 3/4)}$$
$$F(1/4,3/4) = S(0)(0.9)\, e^{(0.06\times 2/4)}.$$

Here it may be noted that on 1st April 2010 the price of the stock is 10% lower than that on 1st January 2010, i.e. $(0.9)S(0)$. Now

$$\frac{F(0,3/4) - F(1/4,3/4)}{F(0,3/4)} = \frac{e^{(0.06\times 3/4)} - (0.9)\, e^{(0.06\times 1/2)}}{e^{(0.06\times 3/4)}}$$
$$= 0.1134.$$

Therefore the percentage drop in the forward price is 11.34%.

Wrong

□

Example 2.3.5 *An Indian importer wants to arrange a forward contract to buy US dollars in half a year. The interest rates for investment in Indian rupees and US dollars are $r_{INR} = 6\%$ and $r_{USD} = 4\%$ respectively. Also the current rate of exchange is Rs 46 to a dollar, i.e. $\frac{1}{46}$ dollars to a rupee. Find the forward price of dollar-rupee exchange rate.*

Solution Here US dollars play the role of underlying. Therefore from (2.9) we get

USD/INR INR USD

$$F(0,1/2) = (1/46)\, e^{(0.5\times(0.06-0.04))} \times$$
$$= (1/46)\, e^{(0.5\times 0.02)}$$
$$= 0.022,$$

dollars to a rupee, i.e. $\frac{1}{(0\cdot 022)} = 45\cdot 54$ rupees to a dollar.

□

2.4 The Value of a Forward Contract

Let a forward contract be initiated at $t = 0$ with delivery $t = T$. Also let $F(0,T)$ be the forward price of this contract. Consider an intermediate time $0 < \tau < T$ and let $F(\tau,T)$ be the forward price of the contract initiated at $t = \tau$ with delivery $t = T$.

Thus we have two forward contracts; one initiated at $t = 0$ and other initiated at $t = \tau$, both having the same delivery date, namely $t = T$. Then $F(0,T)$ and $F(\tau,T)$ are the respectively the forward prices of these two contracts. Now as time

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progress, *the value of the forward contract* initiated at $t = 0$ will be changing. Let at $t = \tau$, its value be $f(\tau)$. Is there a relationship connecting $f(\tau)$, $F(0,T)$ and $F(\tau,T)$? The below given theorem is precisely the answer to this question.

Theorem 2.4.1 (Value of a Forward Contract) *Let $f(\tau)$, $F(0,T)$ and $F(\tau,T)$ be as explained above. Then*

$$f(\tau) = (F(\tau,T) - F(0,T))\ d(\tau,T), \tag{2.10}$$

where $d(\tau,T)$ is the risk free discount factor over the period $t = \tau$ to $t = T$.

Proof. If possible let

$$f(\tau) < (F(\tau,T) - F(0,T))\ d(\tau,T).$$

Now at $t = \tau$ we construct a portfolio P as follows

(i) borrow the amount $f(\tau)$ to enter into a long forward contract with forward price $F(0,T)$ and delivery $t = T$,
(ii) enter into a short forward position with forward price $F(\tau,T)$ for which there is no cost as per the definition of forward contract.

Then the value of the portfolio P at $t = \tau$ is given by

$$V_P(\tau) = 0.$$

Next at time $t = T$, we

(i) close the forward contracts by collecting (or paying, depending upon the sign of pay-offs) the amounts $S(T) - F(0,T)$ for the long forward position and $-S(T) + F(\tau,T)$ for the short forward position,
(ii) pay back the loan amount with interest, i.e. amount $(f(\tau)/d(\tau,T))$.

Therefore the value of the portfolio at $t = T$ is

$$V_P(T) = F(\tau,T) - F(0,T) - (f(\tau)/d(\tau,T)),$$

which is strictly positive and risk free. This violates no arbitrage principle.

Let us now consider the second case, namely $f(\tau) > [F(\tau,T) - F(0,T)]\,d(\tau,T)$. In this case our strategy is to construct a portfolio Q at $t = \tau$ as follows

(i) sell the forward contract which was initiated at $t = 0$ for the amount $f(\tau)$,
(ii) invest this amount $f(\tau)$ risk free from $t = \tau$ to $t = T$,

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(iii) enter into a long forward contract with delivery time $t = T$ and forward price as $F(\tau, T)$.

Then $V_Q(\tau) = 0$ and $V_Q(T)$ is given by

$$\begin{aligned} V_Q(T) &= \frac{f(\tau)}{d(\tau, T)} + (S(T) - F(\tau, T)) + (F(0, T) - S(T)), \\ &= \frac{f(\tau)}{d(\tau, T)} + F(0, T) - F(\tau, T), \end{aligned}$$

which is strictly positive and risk free. This is not possible due to no arbitrage principle.

□

Example 2.4.1 *Let at the beginning of the year, a stock be sold for Rs 45 and risk free interest rate be 6%. Consider a forward contract on this stock with delivery date as one year. Find its forward price. Also find its value after 9 months if it is given that the stock price at that time turns out to be Rs 49.*

Solution From (2.2), the initial forward price $F(0, 1)$ is given by

$$F(0, 1) = S(0)\, e^{rT} = 45\, e^{0.06} = Rs\ 47.78.$$

Also it is given that $S(9/12) = 49$ and hence

$$F(9/12, 1) = 49\ \exp((\cdot 06)(3/12)) = Rs\ 49 \cdot 74.$$

Therefore by Theorem 2.4.1, the value of the forward contract after 9 months is

$$\begin{aligned} f(9/12) &= [F(9/12, 1) - F(0, 1)]\ \exp((-0 \cdot 06)(1 - (9/12))) \\ &= 1 \cdot 93. \end{aligned}$$

□

Example 2.4.2 *Consider the data of Example 2.4.1 and assume that a dividend of Rs 2 is being paid after 6 months. Find the forward price of the contract and also its value after 9 months.*

Solution By (2.7), the initial forward price $F(0, 1)$ is given by

$$\begin{aligned} F(0, 1) &= [S(0) - (div)e^{-rt}]\ e^{rT} \\ &= \left[45 - 2e^{-0.06(1/2)}\right]\ e^{\cdot 06} \\ &= Rs\ 45 \cdot 72. \end{aligned}$$

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Also by (2.3)

$$F(9/12, 1) = S(9/12)\, e^{(0\cdot06)(1-9/12)} = Rs\ 49 \cdot 74.$$

Hence by Theorem 2.4.1,

$$f(9/12) = Rs\ 3 \cdot 96.$$

□

2.5 Futures Contract

It is obvious that when two parties enter into a forward contract, one of them is certainly going to lose money. Therefore a forward contract is always exposed to a risk of default by the party suffering a loss. To eliminate this risk of default, futures contracts have been introduced which are managed by an organized exchange. Thus even though futures contracts are very much in the spirit of forward contracts, their working is entirely different. Here the role of an organized exchange becomes very important because individual contracts are made with the exchange, which itself becomes counter party to both long and short trades. Therefore individuals themselves do not need to search for an appropriate counterparty because this job is taken care off by the exchange itself. The exchange also takes appropriate measures so as to eliminate the risk of counterparty default. This is done by a process called *marking to market*. Before we explain the working of the process, we list the basic features of a futures contract.

(i) Similar to a forward contact, a futures contract also has an underlying asset (e.g. stock) and a delivery date, say time $t = T$.

(ii) In addition to the (underlying) asset prices, the market also dictates the futures price $f(t, T)$ at discrete time steps, say $f(n, T)$ for $n = 0, 1, 2, \ldots$ with $nt \leq T$. In practice these discrete time steps may refer to day1, day2 etc. $f(n, T)$ for $n = 1, 2, \ldots$ with $nt \leq T$ are unknown and

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3
Basic Theory of Option Pricing-I

3.1 Introduction

We have already seen some examples of European options in Chapter 1. Though these examples were specific to single step discrete time scenario, they were general enough to guide the basic principle of pricing methodology. There the strategy had been to replicate the given option in terms of stock and bond so that at the end of expiry, the pay-off of the option matches with the value of the replicating portfolio. Then, by no arbitrage principle, the initial value of the portfolio became the price of the given option.

The aim of this chapter is to formalize the above pricing methodology and introduce single and multi-period binomial lattice models for pricing of European and American call/put options. This discussion is continued in Chapter 4 as well, where first the Cox-Ross-Rubinstein (C.R.R.) model is presented, and then the celebrated Black-Scholes formula of option pricing is introduced. The Black-Scholes formula will be re-visited in Chapter 10 after readers have acquired the necessary background in stochastic calculus.

3.2 Basic Definitions and Preliminaries

In this section we define European and American call/put options and discuss some of their simple properties.

Definition 3.2.1 (European Call Option) *A European call option is a contract giving the holder the right to buy an asset, called the underlying (e.g. stock), for a price K fixed in advance, called the strike price or exercise price, at a specified future time T, called the exercise time or the expiry time.*

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Here the term 'underlying' has a general meaning. It could be stock, commodity, foreign currency, stock index or even interest rate. However, unless it is otherwise stated, we shall always mean stock option, i.e. the underlying asset is being taken as stock. Thus a European (stock) call option is a *derivative security* whose underlying security is stock. It gives the holder the right, to buy stock under specified terms as prescribed in Definition 3.2.1. There is absolutely no obligation for the holder to buy stock, but he/she has got the right to buy if he/she so wishes.

A *European put option* is exactly similar to a European call option, except that the word 'buy' changes to 'sell'. Therefore a European put option is a contract giving the holder the right to sell the underlying asset for the strike price K at the exercise time T.

Along with the European call/put options another term, namely *American call/put options*, is also very commonly used. The basic distinction between a European option and an American option is that an American option allows exercise at any time before and including the expiry; whereas a European option can be exercised only at the expiry. Here we may note that the word *European* and *American* refer to two different conventions of exercise rather than having any geographical significance. Thus the words *European* and *American* have become two standards in option market, referring to two different structures, no matter where they are issued.

The above discussion suggests that an option can be described by describing its four basic features. These are as listed below.

(i) The description of the underlying asset.
(ii) The nature of the option - whether a call or a put; a European or an American.
(iii) The strike or exercise price.
(iv) The exercise or expiry date.

Next we try to understand the meaning of the term *option pricing*. By definition, the holder of an option gets the right to buy or sell the asset (depending upon the nature of the option) but has no obligation, so some amount has to be paid at the time of contract to get this right. This amount is termed as the *premium* or *price* of the option, and the problem of option pricing revolves around the methodologies to find this price in a *fair* way.

An option has two sides or parties. The party that grants the option is said to *write* the option and the party that obtains the option is said to *purchase* it. The party that purchases the option becomes its holder, and it has no risk of loss other than the original premium paid because it has the right to exercise the option and has no obligation attached to it. Whereas the party that writes the

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option has major risk. This is because if the option, say a call, is exercised then the writer has to arrange for the asset. If the writer does not already own the asset then it might have to be acquired from the market at a price higher than the agreed strike price. Similarly in the case of a put option, the writer may have to accept the asset at a much lower price (namely the strike price) than what is prevailing in the market. The problem of option pricing is to be *fair* to both the parties and determine the *fair* price of the option under consideration. There is another terminology used for the writer and the purchaser of the option. The buyer (purchaser) of the option is said to have taken the *long position* and the seller (writer) is said to have taken the *short position*. Thus we have terms like *long call*, *short call*, *long put* and *short put* etc.

The next important concept to understand is the *pay-off* or the *value of the option at expiry*. To fix our ideas, let us consider the case of a European call option with strike price K and expiry as T. Let $S(T)$ denote the price of the underlying asset (stock) at the expiry. Then, the holder will not exercise the option at the expiry if $S(T) \leq K$ but will certainly exercise the option if $S(T) > K$. Therefore the pay-off to the holder is zero for $S(T) \leq K$ and $S(T) - K$ for $S(T) > K$. If we now introduce the notation x^+ as

$$x^+ = \begin{cases} x, & x > 0 \\ 0, & \text{otherwise,} \end{cases}$$

then the pay-off of the given call option can be written as $(S(T) - K)^+$.

Definition 3.2.2 (Pay-off of a European Call Option) *Let C be a European call option with specifications as prescribed in Definition 3.2.1. Then*

$$C(T, K, S) = (S(T) - K)^+ = Max\,(S(T) - K,\ 0),$$

is called the value or the pay-off of the call option C. Here $S(T)$ denotes the price of the underlying at the exercise time T.

It is simple to define the pay-off or the value of a European put option P. Obviously the holder of a European put option will exercise the option only when $K > S(T)$ and get the profit as $K - S(T)$. Therefore the pay-off or the value of the given European put option is defined as

$$P(T, K, S) = (K - S(T))^+ = \text{Max}(K - S(T),\ 0).$$

The left hand side diagram of Fig. 3.1 depicts the pay-off of a European call option whereas the pay-off of a European put option is depicted in the right hand side diagram of Fig. 3.1.

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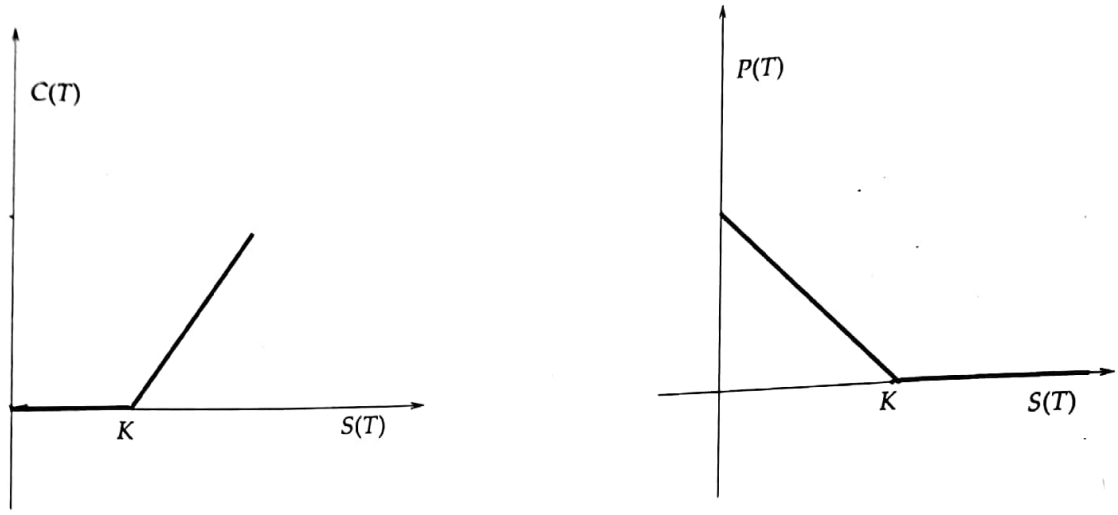

Fig. 3.1. Pay-off curves of call and put options

We say that a European call option is *in the money*, *at the money* or *out of money* depending upon $S(T) > K$, $S(T) = K$ or $S(T) < K$. A European put option is *in the money*, *at the money* or *out of money* depending upon $S(T) < K$, $S(T) = K$ or $S(T) > K$.

We also define the *gain* of an option. The gain of an option buyer is the pay-off modified by the premium $C(0)$ or $P(0)$ paid for the option. Thus for the call option C, the gain is $(S(T) - K)^+ - C(0)e^{rT}$. Similarly the gain of the buyer of the put option P is $(K - S(T))^+ - P(0)e^{rT}$.

We shall discuss the pay-off corresponding to an American option at a later place. Also sometimes to make the context specific, we shall write C^E or C^A (respectively P^E or P^A) to identify whether the option is European or American. However, if only C or P is used, it will always mean a European option.

Though there are many results connecting C^E, P^E and C^A, P^A, the following lemma, called *put-call parity* is interesting and useful.

Lemma 3.2.1. ***(Put-Call Parity)*** *Let C^E and P^E be the prices of a European call and a European put defined over the same stock with price S. Let the given call and the given put have the same strike price K and the same expiry date T. Further, let the underlying stock pay no dividend. Then*

$$C^E(0) - P^E(0) = S(0) - Ke^{-rT}$$

where r is the constant risk-free interest rate under continuous compounding.

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Though a formal proof could be given to the above lemma, we present here only an intuitive argument. Suppose we construct a portfolio by writing and selling one put and buying one call option, both with the same strike price K and expiry date T. Now if $S(T) \geq K$, then the call will pay $S(T) - K$ and the put will be worthless. If $S(T) < K$ then the call will be worthless and the writer of the put will need to pay $K - S(T)$. In either case, the value of the portfolio will be $S(T) - K$ at expiry. But $S(T) - K$ is also the pay-off of a long forward contract with forward price K and delivery time T. Therefore by no arbitrage principle, the current value of the constructed portfolio of options should be that of the forward contract, i.e. $C^E(0) - P^E(0)$ should be $S(0) - Ke^{-rT}$. Thus $C^E(0) - P^E(0) = S(0) - Ke^{-rT}$. Fig. 3.2. depicts this argument.

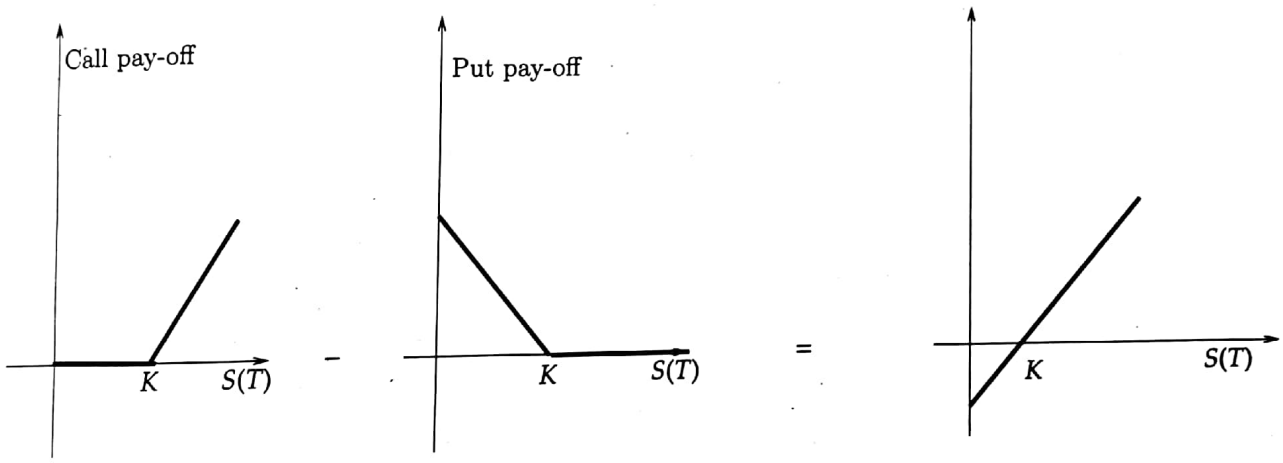

Fig. 3.2. Put-call parity

It is also quite obvious that $C^E(0) \leq C^A(0)$ and $P^E(0) \leq P^A(0)$. This is because an American option gives us more freedom to exercise the right and therefore we have to pay more premium in comparison to a European option. Here of course, it is being assumed that both American and European options have the same parameters, T, K and S.

The below given lemma states that for a non-dividend paying stock, the American call option will never be exercised prior to expiry. This being so, it should be equivalent to the European call option, i.e. $C^A(0) = C^E(0)$.

Lemma 3.2.2. *Let C^E and C^A be respectively the prices of a European call and an American call defined over the same stock with price S. Let both these calls have the same strike price and the same expiry date. Further, let the stock be non-dividend paying. Then $C^A(0) = C^E(0)$.*

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Again rather than giving a formal proof, let us give an intuitive argument for the above lemma. We have already seen that $C^A(0) \geq C^E(0)$ because in comparison to a European call option, an American option gives more right with regard to the time of exercise of the given option. Also, by using the put-call parity, $C^E(0) \geq S(0) - Ke^{-rT}$, as $P^E(0) \geq 0$. Therefore $C^A(0) \geq C^E(0) \geq S(0) - Ke^{-rT}$. But $r > 0$ and hence this last inequality gives $C^A(0) > S(0) - K$.

But $C^A(0) > (S(0) - K)$ implies that the price of the given American call option is greater than the pay-off. Therefore the option should be sold sooner than exercised at $t = 0$. Here the choice of $t = 0$ is for the purpose of reference time and is totally arbitrary. Hence taking any $t < T$ instead of $t = 0$, the above arguments show that the American call option will not be exercised at time t, i.e. the American call option will not be exercised prior to expiry, and then it is equivalent to the European call option. It is important to note here that the assumption of stock being non-dividend paying is crucial for the above assertion. The situation would be different for dividend paying stock. Moreover, this result does not hold for American put, i.e. $P^A(0)$ is different from $P^E(0)$ even when the stock is non-dividend paying (see Example 3.8.1).

Though the above lemma convinces that for a non dividend paying stock, $C^A(0) = C^E(0)$, we attempt to give an independent formal proof of this equality. This proof does not make use of the result that a non-dividend paying stock, an American call option will always be exercised at the expiry.

Lemma 3.2.3. *Let the stock be non-dividend paying. Then for the same K, T and r we have $C^A(0) = C^E(0)$.*

Proof. We know that $C^A(0) \geq C^E(0)$. Therefore we assume that $C^A(0) > C^E(0)$ and then arrive at a contradiction.

At time $t = 0$, let the investor write and sell an American call so as to get the amount $C^A(0)$. Further let a European call be bought for the amount $C^E(0)$ and the balance $C^A(0) - C^E(0)$ be invested risk free at the interest rate r.

If the American call is exercised at time $t < T$, then the investor borrows a share of stock and sells the same for K to the buyer so as to settle his/her obligation as writer of the American call option. Further the investor invests this amount K risk free up to time T. Now at time $t = T$, the investor exercises his/her European call option to buy a share of stock for K and closes the short position on stock. This will result in an arbitrage profit of $\left(C^A(0) - C^E(0)\right)e^{rT} + Ke^{r(T-t)} - K > 0$. If the American option is not exercised at all, then the investor will end up with the European call and even an arbitrage profit $\left(C^A(0) - C^E(0)\right)e^{rT} > 0$. Therefore $C^A(0) = C^E(0)$. □

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An obvious question at this stage is as follows : Do we have put call parity for American option as well? Unfortunately there is no exact put call parity for American option. But we have certain estimates for $C^A(0) - P^A(0)$ as detailed in the below given lemma.

Lemma 3.2.4. ***(Put-Call parity Estimate for American Option)*** *Let the stock be non dividend paying. Then for the same K, T and S, we have*

$$S(0) - Ke^{-rT} \geq C^A(0) - P^A(0) \geq S(0) - K.$$

Proof. We shall first prove that $S(0) - Ke^{-rT} \geq C^A(0) - P^A(0)$. For this, we assume that $C^A(0) - P^A(0) - S(0) + Ke^{-rT} > 0$, and arrive at some contradiction. Now at $t = 0$, we write and sell an American call, buy an American put, buy a share of stock, and finance the transactions in the money market.

If the holder of the American call chooses to exercise it at time $t \leq T$, then we shall receive K for the share of stock and settle the money market position, ending up with the put and a positive amount

$$\begin{aligned} K + \left(C^A(0) - P^A(0) - S(0)\right)e^{rt} &= \left(Ke^{-rt} + C^A(0) - P^A(0) - S(0)\right)e^{rt} \\ &\geq \left(Ke^{-rT} + C^A(0) - P^A(0) - S(0)\right)e^{rt} \\ &> 0. \end{aligned}$$

This argument presumes that $C^A(0) - P^A(0) - S(0) > 0$. In case $C^A(0) - P^A(0) - S(0) < 0$, then this much money is borrowed from the bank at $t = 0$. Then at t, we get K because of the sold American call. But under our assumption $K > \left(C^A(0) - P^A(0) - S(0)\right)e^{rt}$ and therefore we close the position in the money market and still having a positive amount. This violates no arbitrage principle.

Next suppose that $C^A(0) - P^A(0) - S(0) + K < 0$. In this case at $t = 0$, we write and sell a put, buy a call, sell short one share of stock and finance the transactions in the money market. If the American put is exercised at $t \leq T$, then we can withdraw K from the money market to buy a share of stock and close the short sale. We shall be left with the call option and a positive amount $\left(-C^A(0) + P^A(0) + S(0)\right)e^{rt} - K > Ke^{rt} - K \geq 0$. If the put is not exercised at all, then we can buy a share of stock for K by exercising the call at time T and close the short position on stock. On closing the money market position, we shall also end up with a positive amount. This again contradicts the no arbitrage principle. □

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3.3 Behavior of Option Prices With Respect to Variables

The price of an option depends on a number of variables. These are

(i) the strike price K
(ii) the expiry time T
(iii) the current stock price $S(0)$
(iv) the dividend rate r_{div}, and
(v) the risk free rate r.

Both from theoretical as well as from applications point of view, it should be useful to analyze the behavior of option price as function of one of the variables, keeping the remaining variables constant. We now present a sample of some results and refer to the text by Capinski and Zastawniak [25] for further details. The notation $C^E(K)$ denotes the price of European call options over the same underlying and with the same exercise time, but with different values of the strike price K. Similar interpretation is given for other notations, namely $C^E(S)$ and $C^E(T)$ etc.

Lemma 3.3.1. *Let $K_1 < K_2$. Then*

(i) $C^E(K_1) \geq C^E(K_2)$,
(ii) $P^E(K_1) \leq P^E(K_2)$.

Thus $C^E(K)$ is a non-increasing and $P^E(K)$ is a non-decreasing function of K. These assertions are obvious because to have the right to buy at a lower price, we have to pay more premium than to have the right to buy at a higher price.

Lemma 3.3.2. *Let $K_1 < K_2$. Then*

(i) $C^E(K_1) - C^E(K_2) \leq e^{-rT}(K_2 - K_1)$,
(ii) $P^E(K_2) - P^E(K_1) \leq e^{-rT}(K_2 - K_1)$.

Proof. From put-call parity, we have

$$C^E(K_1) - P^E(K_1) = S(0) - K_1 e^{-rt},$$

and

$$C^E(K_2) - P^E(K_2) = S(0) - K_2 e^{-rt}.$$

If we now subtract the above two equations, we get

$$C^E(K_1) - C^E(K_2) + P^E(K_2) - P^E(K_1) = (K_2 - K_1)e^{-rt}.$$

But then an application of Lemma 3.3.1 gives the result. □

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Lemma 3.3.3. *Let $K_1 < K_2$ and $0 \le \alpha \le 1$. Then $C^E(K)$ and $P^E(K)$ satisfy*

(i) $C^E(\alpha K_1 + (1-\alpha)K_2) \le \alpha C^E(K_1) + (1-\alpha)C^E(K_2)$
(ii) $P^E(\alpha K_1 + (1-\alpha)K_2) \le \alpha P^E(K_1) + (1-\alpha)P^E(K_2)$.

This means that $C^E(K)$ and $P^E(K)$ are convex functions of K.

Proof. Let us denote $\widehat{K} = \alpha\ K_1 + (1-\alpha)\ K_2$. Suppose that $C^E(\widehat{K}) > \alpha\ C^E(K_1) + (1-\alpha)\ C^E(K_2)$.
Now at $t = 0$, let the investor write and sell the option with strike $\widehat{K}$ and get the amount (premium) $C^E(\widehat{K})$. Then let the investor purchase α option with strike price K_1 and $(1-\alpha)$ options with strike price K_2. This will involve a payment (premium) of $\alpha\ C^E(K_1) + (1-\alpha)\ C^E(K_2)$. But then under the assumed inequality the investor is still left with the amount $C^E\ (\widehat{K}) - (\alpha\ C^E(K_1) + (1-\alpha)\ C^E(K_2)) > 0$ which he/she invests risk free.

At the expiry, if the option with the strike price $\widehat{K}$ is exercised, then the investor shall have to pay the amount $\left(S(T) - \widehat{K}\right)^+ = \text{Max}\left(S(T) - \widehat{K}, 0\right)$. But he/she can raise the amount $\alpha\ (S(T) - K_1)^+ + (1-\alpha)\ (S(T) - K_2)^+$ by exercising α calls with strike price K_1 and $(1-\alpha)$ calls with strike price K_2. But it is known that

$$\left(S(T) - \widehat{K}\right)^+ \le \alpha\ (S(T) - K_1)^+ + (1-\alpha)\ (S(T) - K_2)^+,$$

because the function $(S(T) - K)^+$ is a convex function of K. Therefore the investor will realize an arbitrage profit of $\left[C^E(\widehat{K}) - (\alpha\ C^E(K_1) + (1-\alpha)\ C^E(K_2))\right]\ e^{rT}$ at expiry. Therefore our supposition is wrong and we get the result.

□

The below given Lemmas can also be proved on the similar lines.

Lemma 3.3.4. *Let $T_1 < T_2$. Then*

(i) $C^E(T_1) \le C^E(T_2)$
(ii) $P^E(T_1) \le P^E(T_2)$.

Proof. If possible let $C^E(T_1) > C^E(T_2)$. We write and sell one option expiring at time T_1 and buy one with the same strike price but expiry as T_2, investing the balance without risk. If the written option is exercised at T_1, we can exercised the option immediately to cover our liability. The balance $(C^E(T_1) - C^E(T_2)) > 0$ invested without risk will be our arbitrage profit.

The inequality *(ii)* can be proved on similar lines. □

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Lemma 3.3.5. *Let $0 < x_1 < x_2$. Further let at time $t = 0$, $\overline{S} = x_1 S(0)$ and $\hat{S} = x_2 S(0)$. Then*

(i) $C^E(\overline{S}) \leq C^E(\hat{S})$
(ii) $P^E(\overline{S}) \geq P^E(\hat{S})$.

Proof. We shall prove the first inequality. If possible let $C^E(\overline{S}) > C^E(\hat{S})$. We can write and sell a call on a portfolio with x_1 shares and buy a call on a portfolio with x_2 shares having the same strike price K and exercise time T. Also we invest the amount $C^E(\overline{S}) - C^E(\hat{S})$ risk-free. As $x_1 < x_2$, we have $(x_1 S(T) - K)^+ \leq (x_2 S(T) - K)^+$. If the sold option is exercised at time T, we can exercised other option to cover our liability. The balance $C^E(\overline{S}) - C^E(\hat{S}) > 0$ invested risk-free will be our arbitrage profit.

□

Lemma 3.3.6. *Let $\overline{S}$ and $\hat{S}$ be as defined in Lemma 3.3.5. Then*

(i) $C^E(\hat{S}) - C^E(\overline{S}) \leq (\hat{S} - \overline{S})$
(ii) $P^E(\overline{S}) - P^E(\hat{S}) \leq (\hat{S} - \overline{S})$.

Proof. Using put-call parity we have

$$C^E(\hat{S}) - P^E(\hat{S}) = \hat{S} - Ke^{-rT},$$

and

$$C^E(\overline{S}) - P^E(\overline{S}) = \overline{S} - Ke^{-rT}.$$

On subtraction, we get

$$\left(C^E(\hat{S}) - C^E(\overline{S})\right) + \left(P^E(\overline{S}) - P^E(\hat{S})\right) = (\hat{S} - \overline{S}),$$

which gives inequalities (i) and (ii) because both terms on the right hand side are non-negative.

□

Results similar to above Lemmas also hold for American options. For these details we may refer to Capinski and Zastawniak [25].

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obvious manner by utilizing the definition of gain of an option.

Example 3.4.1 *Let the sale and purchase of options over the same stock and having the same expiry be given by the expression*

$$-P_{100} + P_{120} + 2C_{150} - C_{180}.$$

Sketch the spread of the given option combination.

Solution To draw the spread, we draw the individual pay-off curves and then combine them. The details in Fig. 3.6 are self explanatory.

□

3.5 Single Period Binomial Lattice Model for Option Pricing

The simple example of pricing a European option, presented in Chapter 1, is in fact an example of a single period binomial lattice model. Here we formally develop the theory for the single period case and extend the same for multi-period case in the next section. For this model we assume the following

(i) The initial value of the stock is $S(0)$, i.e. the stock price at $t = 0$ is $S(0)$.
(ii) At the end of the period, the price is either $u\,S(0)$ with probability p, or $d\,S(0)$ with probability $(1-p)$, $0 < p < 1$.
(iii) $u > d > 0$.
(iv) At every period, it is possible to borrow or lend at a common risk free interest rate r. Let $R = (1+r)$, and $u > R > d$. This assumption is needed to avoid arbitrage.

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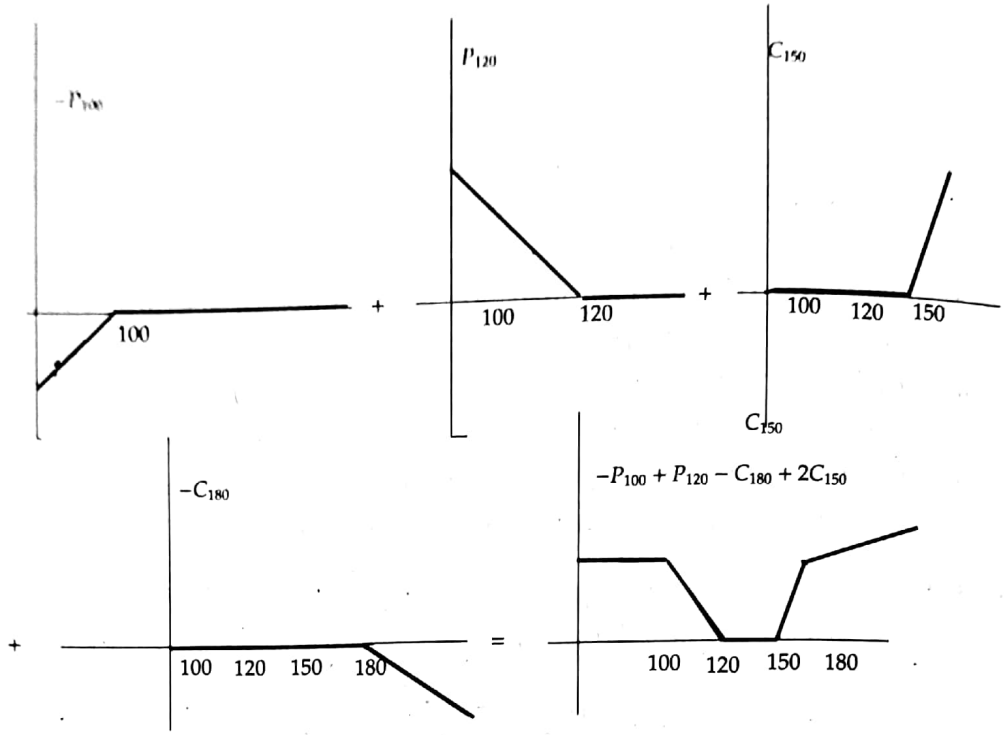

Fig. 3.6. Spread of Example 3.4.1

Let C be the given European call option on the stock with strike price K and expiry at the end of period (i.e. $t = 1$). Our aim is to find $C(0)$. Since at $t = 1$, the stock price could be either $u\,S(0)$ (with probability p) or $d\,S(0)$ (with probability $(1-p)$). We have

$$C(1) = \begin{cases} C_u = \text{Max}(u\,S(0) - K,\ 0), & \text{with probability } p \\ C_d = \text{Max}(d\,S(0) - K,\ 0), & \text{with probability } (1-p). \end{cases}$$

Though the value of the risk free asset (say bond) is deterministic, we can take it as a (degenerate) derivative of the stock-degenerate in the sense that the same value R is assigned at the end of each arc. Thus we have the following figures

$$S(0) \xrightarrow{p} u\,S(0), \qquad S(0) \xrightarrow{1-p} d\,S(0)$$

$$1 \xrightarrow{p} R, \qquad 1 \xrightarrow{1-p} R$$

$$C(0) \xrightarrow{p} C_u = \text{Max}(u\,S(0) - K,\ 0), \qquad C(0) \xrightarrow{1-p} C_d = \text{Max}(d\,S(0) - K,\ 0)$$

Before we actually derive the pricing formula for the given scenario, let us try to justify the assumption $u > R > d$. For this we have below given lemma.

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Lemma 3.5.1. *If $u > R > d$ does not hold then "no arbitrage principle" is violated.*

Proof. We first consider the case $R \geq u > d$. Now construct the portfolio P : $\left(x = -\frac{1}{S(0)},\ y = \frac{1}{B(0)}\right)$, where $B(0)$ denotes the price of the bond at $t = 0$. Let $V_P(0)$ and $V_P(1)$ respectively denote the value of the portfolio P at $t = 0$ and $t = 1$. Then

$$V_P(0) = x\,S(0) + y\,B(0) = \left(-\frac{1}{S(0)}\right)S(0) + \left(\frac{1}{B(0)}\right)B(0) = 0,$$

and

$$\begin{aligned} V_P(1) &= x\,S(1) + y\,B(1) \\ &= \begin{cases} -\frac{1}{S(0)}u\,S(0) + \frac{1}{B(0)}R\,B(0), & \text{with probability } p \\ -\frac{1}{S(0)}d\,S(0) + \frac{1}{B(0)}R\,B(0), & \text{with probability } (1-p) \end{cases} \\ &= \begin{cases} R - u, & \text{with probability } p \\ R - d, & \text{with probability } (1-p). \end{cases} \end{aligned}$$

But under the assumption $R \geq u > d$, $(R - u) \geq 0$ and $(R - d) > 0$. As $0 < p < 1$, $V_P(1) \geq 0$ with probability 1 and it can take positive value with positive probability even though $V_P(0) = 0$. This clearly violates the "no arbitrage principle".

We next consider the case $u > d \geq R$, and construct the portfolio Q : $\left(x = \frac{1}{S(0)}, y = -\frac{1}{B(0)}\right)$ and note that $V_Q(0) = 0$. Further $V_Q(1) \geq 0$ with probability 1 and it can take positive value $(u - R)$ with positive probability, which again violates the "no arbitrage principle". Therefore to avoid the arbitrage opportunity, we need to assume that $u > R > d$.

□

Now to find the price $C(0)$ of the call, we need to replicate it in terms of stock and bond so that at $t = 1$, the value of this replicating portfolio equals the pay-off of the call at $t = 1$. Let the replicating portfolio be $RP : (x = a, y = b)$, where a is the number of shares of stock and b is the units of bond. Then we need to find a and b such that $V_{RP}(1) = C(1)$. But

$$\begin{aligned} V_{RP}(1) &= a\,S(1) + b\,B(1) \\ &= \begin{cases} a\,u\,S(0) + b\,R\,B(0), & \text{with probability } p \\ a\,d\,S(0) + b\,R\,B(0), & \text{with probability } (1-p), \end{cases} \end{aligned}$$

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and

$$C(1) = \begin{cases} C_u, & \text{with probability } p \\ C_d, & \text{with probability } (1-p). \end{cases}$$

Hence $V_{RP}(1) = C(1)$ gives

$$a\,u\,S(0) + b\,R\,B(0) = C_u \tag{3.1}$$
$$a\,d\,S(0) + b\,R\,B(0) = C_d. \tag{3.2}$$

Solving the above system we obtain

$$a = \frac{C_u - C_d}{S(0)(u-d)}, \tag{3.3}$$

and

$$b = \frac{u\,C_d - d\,C_u}{R\,(u-d)B(0)}. \tag{3.4}$$

Also

$$V_{RP}(0) = a\,S(0) + b\,B(0). \tag{3.5}$$

Therefore using (3.3), (3.4) and (3.5) we have

$$\begin{aligned} V_{RP}(0) &= a\,S(0) + b\,B(0) \\ &= \left(\frac{C_u - C_d}{S(0)(u-d)}\right)S(0) + \left(\frac{u\,C_d - d\,C_u}{R\,(u-d)A(0)}\right)B(0) \\ &= \left(\frac{C_u - C_d}{(u-d)}\right) + \left(\frac{u\,C_d - d\,C_u}{R\,(u-d)}\right) \\ &= \frac{1}{R}\left(\left(\frac{R-d}{u-d}\right)C_u + \left(\frac{u-R}{u-d}\right)C_d\right) \\ &= \frac{1}{R}\left(\hat{p}\,C_u + (1-\hat{p})\,C_d\right), \end{aligned} \tag{3.6}$$

where $\hat{p} = \left(\frac{R-d}{u-d}\right)$ and $(1-\hat{p}) = \left(\frac{u-R}{u-d}\right)$.

Since at $t = 1$, the value $V_{RP}(1)$ of the replicating portfolio and the pay-off $C(1)$ of the call are equal, by no arbitrage principle, we should have $C(0) = V_{RP}(0)$. Hence from (3.6)

$$C(0) = \frac{1}{R}\left(\hat{p}\,C_u + (1-\hat{p})\,C_d\right). \tag{3.7}$$

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In (3.7) we should note that $0 < \hat{p} < 1$ because $u > R > d$. Hence $\hat{p}$ can be considered a probability. This probability $\hat{p}$ is called the *risk neutral probability*. Before we discuss its interpretation and nomenclature, we observe from (3.7) that $C(0) = \frac{1}{R}E_{\hat{p}}(C(1))$, where $E_{\hat{p}}$ denotes the expectation under risk neutral probability $\hat{p}$. The formula

$$C(0) = \frac{1}{R}E_{\hat{p}}(C(1)) \tag{3.8}$$

is very general and is valid for any derivative security with "appropriate" modifications.

In the above derivation, there is no role of $B(0)$. In fact we may take $B(0) = 1$, giving $B(1) = R$. Therefore we can think of bond as cash, as Rs 1 gives $Rs\ (1+r) = Rs\ R$ after one time period. In this light, formula (3.8) tells that to find the price $C(0)$ of the call, we should first take the expectation of the pay-off at expiry with respect to the risk neutral probability, and then discount the same according to the risk free rate.

Here we should note that risk neutral probability $\hat{p}$ is different from p. The actual probability p describes the (stochastic) price movement but the risk neutral probability $\hat{p}$ has a totally different interpretation. In fact the actual probability p enters nowhere in the derivation of $C(0)$. However it gives a motivation to introduce risk neutral probability. This is because one would invest in stock only if the expected growth rate of stock is higher than that of the rate at money market (bank/bond/cash), i.e.

$$p\,(u\ S(0)) + (1-p)\,(d\ S(0)) > R\ S(0),$$

i.e.

$$S(0) < \frac{1}{R}\left[p\,(u\ S(0)) + (1-p)\,(d\ S(0))\right]. \tag{3.9}$$

But

$$\hat{p}\,(u\ S(0)) + (1-\hat{p})\,(d\ S(0)) = \left(\frac{R-d}{u-d}\right)(u\ S(0)) + \left(\frac{u-R}{u-d}\right)(d\ S(0)),$$

which on simplification gives

$$\frac{1}{R}\left[\hat{p}\,(u\ S(0)) + (1-\hat{p})\,(d\ S(0))\right] = S(0). \tag{3.10}$$

Equation (3.10) tells that under risk neutral probability $\hat{p}$, the mean rate of growth of stock equals the rate of growth in the money market account which is risk free. Now if this is the case then the investor must be neutral about the risk

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and hence the name *risk neutral probability*. The formula (3.7) is therefore referred as the *risk neutral pricing formula* for European call option.

The role of risk neutral probability is very important because it provides *fair* opportunity to the writer of the option to *hedge* his/her risk and thereby provides a *fair* price of the call. The hedging problem in the present context is to find an equivalent replicating portfolio, i.e. a portfolio of stock and bond at $t = 0$ such that the value of this portfolio at $t = 1$ matches with the pay-off of the call. This process of replicating the option is called the *hedging problem*, and the replicating portfolio is called the *hedge of the given option*.

Remark 3.5.1 *Though the actual probabilities p and $(1-p)$ have not entered in the pricing formula (3.7), they have played an important role indirectly. Thus the investor will like to buy a call option provided he/she feels that there is high probability of the stock price going up at $t = 1$. Also the agreed strike price K depends to some extent on these probabilities. This is because if there is a feeling that at $t = 1$, the stock price will go up with high value of p then he/she may agree for a higher value of K.*

The above procedure of finding the call price is valid provided we guarantee that the risk neutral probability measure (RNPM) always exists and is unique. The below given result tells that under no arbitrage principle RNPM always exists. Further if in addition, the market is *complete*, then the RNPM is unique.

Lemma 3.5.2. *A risk neutral probability measure exists if and only if no arbitrage principle holds. Further if the market is complete then, the RNPM is unique.*

We shall partly prove this Lemma in Section 3.7 by making use of the concept of duality in linear programming. Specifically we shall prove that RNPM exists if and only if, no arbitrage principle holds. The discussion about the uniqueness of RNPM will be postponed till the concept of completeness of the market is introduced.

Example 3.5.1 *Find the price of a European call option with the given data as $B(0) = 100$, $B(1) = 110$, $S(0) = 100$, $K = 100$, $T = 1$ and*

$$S(1) = \begin{cases} 120, & \text{with probability } 0\cdot 6 \\ 80, & \text{with probability } 0\cdot 4. \end{cases}$$

Will the price change if the probabilities p and $(1-p)$ are taken as 0.3 and 0.7 respectively?

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Solution As $B(1) = 110$, $B(0) = 100$, we get $r = 10\%$, i.e. $R = 1 + r = 1.1$. Further $S(0) = 100$, $u\ S(0) = 120$ and $d\ S(0) = 80$ yield $u = 1.2$ and $d = 0.8$. Here the risk neutral probability $\hat{p}$ is $\frac{3}{4}$ as shown below.

$$\hat{p} = \frac{R-d}{u-d} = \frac{1.1-0.8}{1.2-0.8} = \frac{3}{4}$$

i.e. $1 - \hat{p} = 1 - \frac{3}{4} = \frac{1}{4}$.

Also

$$C_u = \text{Max}(u\ S(0) - K,\ 0) = \text{Max}(120 - 100, 0) = 20$$
$$C_d = \text{Max}(d\ S(0) - K,\ 0) = \text{Max}(80 - 100, 0) = 0.$$

Therefore

$$\begin{aligned} C(0) &= \frac{1}{R}\left[\hat{p}\ C_u + (1-\hat{p})\ C_d\right] \\ &= \frac{1}{1.1}\left[\frac{3}{4}(20) + \frac{1}{4}(0)\right] \\ &= 13.63. \end{aligned}$$

We exhibit the above details in the form of following tables

t	0	1
$S(t)$		
	100 $\xrightarrow{p}$	120
	100 $\xrightarrow{(1-p)}$	80

t	0	1
$C(t)$		
	C(0) (?) $\xrightarrow{\hat{p}=\frac{3}{4}}$	$C_u(= 20)$
	C(0) (?) $\xrightarrow{(1-\hat{p})=\frac{1}{4}}$	$C_d(= 0)$

This gives $C(0) = Rs\ 13.64$. As call price does not depend on p, it will not change if 0.6 and 0.4 are changed to 0.7 and 0.3 respectively.

□

To find the price of a European put option we can follow exactly the same derivation as for the European call. We need to use pay-off of the put at expiry instead of the pay-off of the call to get the following pricing formula

$$P(0) = \frac{1}{R}\left[\hat{p}\ P_u + (1-\hat{p})\ P_d\right], \tag{3.11}$$

where $P_u = \text{Max}\,(K - u\ S(0),\ 0)$ and $P_d = \text{Max}\,(K - d\ S(0),\ 0)$.

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Example 3.5.2 *For the data given in Example 3.5.1, find the price of the corresponding European put option.*

Solution We have already obtained $\hat{p} = \frac{3}{4}$, $(1-\hat{p}) = \frac{1}{4}$ and $R = 1.1$. Further $P_u = \text{Max}(K - u\,S(0),\ 0) = 0$ and $P_d = \text{Max}(K - d\,S(0),\ 0) = 20$. Therefore

$$\begin{aligned} P(0) &= \frac{1}{R}\left[\hat{p}\,P_u + (1-\hat{p})\,P_d\right] \\ &= \frac{1}{1.1}\left[\left(\frac{3}{4}\times 0\right) + \left(\frac{1}{4}\times 20\right)\right] \\ &= 4.54, \end{aligned}$$

and the corresponding table is

t	0	1
$P(t)$		$P_u(=0)$ ($\hat{p}$)
	$P(0)$	
		$P_d(=20)$ ($(1-\hat{p})$)

At this stage we can also verify the put-call parity. For the given data we have already obtained $C(0)$ =Rs 13.63 and $P(0)$ =Rs 4.54. Hence the value of the expression $C(0) - P(0) + d(0,1)K$ comes out to be $13.64 - 4.54 + (1.1)^{-1} \times 100 = 100$, which is same as the value $S(0)$. Here $d(0,1) = \frac{1}{R}$ is the discount factor between $t = 0$ and $t = 1$. □

3.6 Multi Period Binomial Lattice Model for Option Pricing

We shall now extend the pricing formula (3.7) to multi period binomial lattice model. For this let us first take the two period case. Our notations are same as those in Section 3.5. Thus $S(0)$ is the price of the stock at $t = 0$ (initial time), u and d are up and down factors of the stock price movement, K is the strike price, $t = 2$ is expiry and $R = (1+r)$. The following table illustrates the price movement of the stock along the binomial lattice

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t	0	1	2
$S(t)$			$u^2S(0)$
		$u\ S(0)$	
	$S(0)$		$u\ d\ S(0)$
		$d\ S(0)$	
			$d^2S(0)$

Along the line of the single period case, the table for the call price should be

t	0	1	2
$C(t)$			C_{uu}
		C_u	
	$C(0)$ $=?$		C_{ud}
		C_d	
			C_{dd}

Here C_{uu}, C_{ud} and C_{dd} are pay-offs of the call at the expiry (i.e. at $t = 2$). As per our definition, it is obvious that

$$C_{uu} = \text{Max}(u^2S(0) - K,\ 0)$$
$$C_{ud} = \text{Max}(u\ d\ S(0) - K,\ 0)$$
$$C_{dd} = \text{Max}(d^2S(0) - K,\ 0).$$

But what are the values of C_u and C_d? For this let us re-look the binomial lattice for the price of the stock. We can visualize this bigger lattice as a combination of three binomial lattices of the following type

$u\ S(0)$	$u^2S(0)$	$u\ d\ S(0)$
$S(0)$	$u\ S(0)$	$d\ S(0)$
$d\ S(0)$	$u\ d\ S(0)$	$d^2S(0)$

Let above lattices be referred as $L(1)$, $L(2)$ and $L(3)$ respectively. Treating lattice $L(3)$ as the single period binomial lattice and using formula (3.7), we obtain

$$C_d = \frac{1}{R}\left[\hat{p}\ C_{ud} + (1 - \hat{p})\ C_{dd}\right], \tag{3.12}$$

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where as before $\hat{p}$ is the risk neutral probability given by $\hat{p} = \dfrac{R-d}{u-d}$. Similarly using lattice $L(2)$ we obtain

$$C_u = \frac{1}{R}\left[\hat{p}\, C_{uu} + (1-\hat{p})\, C_{ud}\right]. \tag{3.13}$$

Finally using lattice $L(1)$ we obtain

$$C_0 = \frac{1}{R}\left[\hat{p}\, C_u + (1-\hat{p})\, C_d\right]. \tag{3.14}$$

Here the same risk neutral probability $\hat{p}$ has been used for all three lattices $L(1)$, $L(2)$ and $L(3)$. The reason being that for these single period lattices the up-tick probability p, the down-tick probability $1-p$ and the factors u and d do not change - they remain same for $t = 0$ as well as for $t = 1$. Now substituting for C_d and C_u from (3.12) and (3.13), equation (3.14) gives

$$C_0 = \frac{1}{R^2}\left[\hat{p}^2 C_{uu} + 2\hat{p}\,(1-\hat{p})\, C_{ud} + (1-\hat{p})^2 C_{dd}\right]. \tag{3.15}$$

Formula (3.15) can also be written as

$$C_0 = \frac{1}{R^2}\left[\sum_{j=0}^{2} \frac{2!}{j!(2-j)!}(\hat{p})^j(1-\hat{p})^{2-j}\left(u^j d^{2-j} S(0) - K\right)^+\right], \tag{3.16}$$

where $\left(u^j d^{2-j} S(0) - K\right)^+ = \text{Max}\left(u^j d^{2-j} S(0) - K,\ 0\right)$.

Remark 3.6.1 *Here, unlike the single period case,* C_u *and* C_d *are NOT the payoffs - Max*$(u\ S(0) - K,\ 0)$ *and Max*$(d\ S(0) - K,\ 0)$ *at* $t = 1$, *because the call is being exercised at* $t = 2$ *only.*

In deriving formula (3.15), we have not made use of the concept of replicating portfolio directly. This has been possible because we have visualized the given 2-period binomial lattice as a combination of three 1-period binomial lattices which have already been studied. But it will be interesting to derive the same formula by employing the replicating portfolio strategy as well.

Though we are not giving the complete details here, we are giving enough hint to get the complete solution. In order to have replication at maturity $t = 2$ we start replicating backwards, from the end of lattice. For this at $t = 1$, we consider the upper node of the lattice L_2 and determine the scalars a_1 and b_1 such that

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$$a_1\left(u^2S(0)\right) + b_1R = C_{uu},$$

and

$$a_1\left(udS(0)\right) + b_1R = C_{ud}.$$

This system can be solved to get

$$a_1 = \frac{C_{uu} - C_{ud}}{uS(0)(u-d)},$$

and

$$b_1 = \frac{C_{uu} - a_1(u^2S(0))}{R}.$$

Then $C_u = a_1\left(uS(0)\right) + b_1$ becomes the value of the replicating portfolio in the upper node at $t = 1$. We can similarly find the value of C_d. Then we need to replicate these two values in the first period. Thus we need to find a_0 and b_0 such that

$$a_0\left(uS(0)\right) + b_0 = C_u,$$

and

$$a_0\left(dS(0)\right) + b_0 = C_d.$$

The above system can be solved as before to get the values of a_0 and b_0. Now a little manipulation and the expression $C(0) = a_0S(0) + b_0R$ will give the formula (3.15).

Remark 3.6.2 *It is to be noted that in this model, the number of shares in the replicating portfolio is always equal to the ratio of ΔC and ΔS, where ΔC is the change in the future values of the call and ΔS is the change in the future values of the stock. This ratio is called the* delta *of the call, and it is different at different nodes of the lattice. Delta is one of the* Greeks *to be studied later for a derivative security defined over a given undertaking. Greeks have been used extensively in derivative pricing for hedging purposes.*

Example 3.6.1 *Find the price of a European call option with the given data as $S(0) = 100$, $K = 100$, $u = 1.2$, $d = 0.8$, $r = 10\%$ per year and time to expiry $T = 2$ years.*

Solution We have

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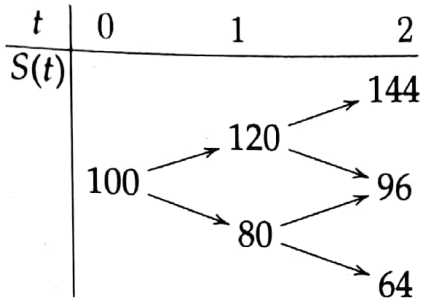

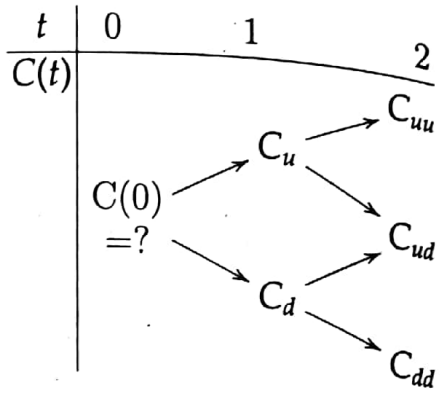

From the given data $R = 1 + r = 1.1$ and $\hat{p} = \dfrac{R-d}{u-d} = \dfrac{1.1-0.8}{1.2-0.8} = \dfrac{3}{4}$. Also

$$C_{uu} = \text{Max}(u^2S(0) - K,\ 0) = \text{Max}(144 - 100, 0) = 44$$
$$C_{ud} = \text{Max}(u\ d\ S(0) - K,\ 0) = \text{Max}(96 - 100, 0) = 0$$
$$C_{dd} = \text{Max}(d^2S(0) - K,\ 0) = \text{Max}(64 - 100, 0) = 0.$$

Therefore

$$C_u = \frac{1}{R}\left[\hat{p}\ C_{uu} + (1-\hat{p})\ C_{ud}\right] = \frac{1}{1.1}\left[\left(\frac{3}{4}\times 44\right) + \left(\frac{1}{4}\times 0\right)\right] = 30$$
$$C_d = \frac{1}{1.1}\left[\left(\frac{3}{4}\times 0\right) + \left(\frac{1}{4}\times 0\right)\right] = 0,$$

which gives

$$\begin{aligned} C(0) &= \frac{1}{R}\left[\hat{p}\ C_u + (1-\hat{p})\ C_d\right] \\ &= \frac{1}{1.1}\left[\left(\frac{3}{4}\times 30\right) + \left(\frac{1}{4}\times 0\right)\right] \\ &= 20.45. \end{aligned}$$

Alternatively, we can use the formula (3.15) to get

$$\begin{aligned} C(0) &= \frac{1}{R^2}\left[\hat{p}^2C_{uu} + 2\hat{p}\ (1-\hat{p})\ C_{ud} + (1-\hat{p})^2C_{dd}\right] \\ &= \frac{1}{1.1^2}\left[\left(\frac{9}{64}\times 44\right)\right] \\ &= 20.45. \end{aligned}$$

Example 3.6.2 *For the data given in Example 3.6.1, find the price of the corresponding European put option.*

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Solution We have already obtained $\hat{p} = \frac{3}{4}$, $(1-\hat{p}) = \frac{1}{4}$ and $R = 1.1$. Further

$$P_{uu} = \text{Max}(K - u^2 S(0),\ 0) = \text{Max}(-44, 0) = 0$$
$$P_{ud} = \text{Max}(K - u\ d\ S(0),\ 0) = \text{Max}(4, 0) = 4$$
$$P_{dd} = \text{Max}(K - d^2 S(0),\ 0) = \text{Max}(36, 0) = 36.$$

Therefore

$$P_u = \frac{1}{1.1}\left[\left(\frac{3}{4}\times 0\right) + \left(\frac{1}{4}\times 4\right)\right] = \frac{1}{1.1}$$
$$P_d = \frac{1}{1.1}\left[\left(\frac{3}{4}\times 4\right) + \left(\frac{1}{4}\times 36\right)\right] = \frac{12}{1.1}.$$

This gives

$$\begin{aligned} P(0) &= \frac{1}{R}\left[\hat{p}\ P_u + (1-\hat{p})\ P_d\right] \\ &= \frac{1}{1.1}\left[\left(\frac{3}{4}\times\frac{1}{1.1}\right) + \left(\frac{1}{4}\times\frac{12}{1.1}\right)\right] \\ &= 3.10. \end{aligned}$$

Alternatively, we can get

$$\begin{aligned} P(0) &= \frac{1}{R^2}\left[\hat{p}^2 P_{uu} + 2\hat{p}\ (1-\hat{p})\ P_{ud} + (1-\hat{p})^2 P_{dd}\right] \\ &= \frac{1}{1.1^2}\left[\left(\left(\frac{3}{4}\right)^2\times 0\right) + \left(2\times\frac{3}{4}\times\frac{1}{4}\times 4\right) + \left(\left(\frac{1}{4}\right)^2\times 36\right)\right] \\ &= 3.10. \end{aligned}$$

We can also use put-call parity to find $P(0)$, once $C(0)$ is known. Specifically

$$C(0) - P(0) + d(0,2)K = S(0),$$

gives

$$P(0) = C(0) + d(0,2)K - S(0) = 20.45 + d(0,2)100 - 100.$$

We can take $d(0,2) = (1+2r)^{-1}$ or e^{-2r} depending upon the nature of the compounding of the interest rate r; but it has to be same for the entire calculations. □

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The development of the pricing methodology for the case of multi period binomial lattice model is similar to the one discussed for the two period lattice model. The single period risk free discounting is carried out at every node of the lattice as has been done for the two period case. Obviously we need to start from the final period ($t = N$) and then proceed backward till we reach the initial time period ($t = 0$). For the case of European call option, this process will result in the following formula

$$C(0) = \frac{1}{R^N}\left[\sum_{j=0}^{N} \frac{N!}{j!(N-j)!}(\hat{p})^j(1-\hat{p})^{N-j}\left(u^j d^{N-j}S(0) - K\right)^+\right], \tag{3.17}$$

which reduces to (3.16) for $N = 2$.

3.7 Existence of Risk Neutral Probability Measure

We have already seen in Section 3.5 that for the single period binomial pricing model, risk neutral probability measure (RNPM) plays the most fundamental role. We have also exhibited the RNPM $\hat{p}$ and used the same for deriving the option pricing formula (3.7). Here it may be noted that in this scenario the state space Ω consists of only two points say $\Omega = \{\omega_1, \omega_2\}$ where ω_1 is the up-tick movement of the stock price and ω_2 is the down-tick movement of the stock price. But realistically Ω may consist of m points $\Omega = \{\omega_1, \omega_2, \ldots, \omega_m\}$ or Ω may be an interval $[a, b]$. Can we still guarantee the existence of RNPM? To answer this question is important if we wish to discuss derivative pricing for more general scenarios.

Let us consider a more general model where $\Omega = \{\omega_1, \omega_2, \ldots, \omega_m\}$, ω_j being the jth state of economy. Thus Ω represents the finite set of possible values; e.g. these could be m possible values of the stock price $S(1)$ (at $t = 1$), namely $S_1(\omega_1)$, $S_1(\omega_2)$, ..., $S_1(\omega_m)$. For the single period binomial lattice, $\Omega = \{\omega_1, \omega_2\}$ with $S_1(\omega_1) = uS(0)$ and $S_1(\omega_2) = dS(0)$.

Also we could have more than two securities, say $S^{(k)}$ $(k = 0, 1, 2, \ldots, n)$. Let $S_1^{(k)}(\omega_j)$, for $(k = 0, 1, 2, \ldots, n)$ and $(j = 1, 2, \ldots, m)$ denote the price of the kth security at $t = 1$ when the state of the economy is ω_j. We may think of k securities as bond, stock, option, forward contracts, etc having different defining variables but over the same underlying.

Let $S_0^{(k)}$ $(k = 0, 1, 2, \ldots, n)$ denote the current $(t = 0)$ price of the kth security. Here we may note that $S_0^{(k)}$ is deterministic, but $S_1^{(k)}$ is a random variable taking values $\left\{S_1^{(k)}(\omega_j), (j = 1, 2, \ldots, m)\right\}$. Since all derivative securities are defined on

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the same underlying (stock), they all will be random variables taking m possible values.

As a convention, we take $k = 0$ for bond, $k = 1$ for the underlying (stock) and $(k = 2, \ldots, n)$ for other (derivative) securities. Let r be the risk free interest rate for the period $t = 0$ to $t = 1$. It is convenient to assume that $S_0^{(0)} = 1$ and $S_1^{(0)}(\omega_j) = R = (1 + r)$, $(j = 1, 2, \ldots, m)$.

Definition 3.7.1 (Risk Neutral Probability Measure) *A risk neutral probability measure (RNPM) is a vector $\hat{p} = (\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_m)^T$ such that*

(i) $\hat{p}_j > 0$ $(j = 1, 2, \ldots, m)$,

(ii) $\sum_{j=1}^{m} \hat{p}_j = 1$,

and for every security k $(k = 0, 1, 2, \ldots, n)$, we have

$$S_0^{(k)} = \frac{1}{R}\left(\sum_{j=1}^{m} \hat{p}_j S_1^{(k)}(\omega_j)\right). \tag{3.18}$$

If we denote by $E_{\hat{p}}\left(S_1^{(k)}\right)$ the expected value of $S_1^{(k)}$ with respect to RNPM $\hat{p}$, then (3.18) can be written as

$$S_0^{(k)} = \frac{1}{R} E_{\hat{p}}\left(S_1^{(k)}\right). \tag{3.19}$$

Here $S_1^{(k)}$ denotes the value of the kth security at $t = 1$ and $S_0^{(k)}$ is the price of the same security at $t = 0$.

Regarding the existence and uniqueness of RNPM we have the following two main theorems

Theorem 3.7.1 (First Fundamental Theorem of Asset Pricing) *A risk neutral probability measure $\hat{p}$ exists if and only if no arbitrage principle holds.*

Theorem 3.7.2 (Second Fundamental Theorem of Asset Pricing) *The RNPM is unique if and only if the market is complete.*

Thus for an arbitrage free market, there is unique RNPM $\hat{p}$ if and only if the market is *complete*. We have not yet discussed the meaning of *market completeness* but that we postpone for the time being.

We shall now give a linear programming based proof of Theorem 3.7.1. Let us recall the following primal-dual pair of linear programming problem

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$$(LP) \qquad \begin{array}{ll} \text{Min} & c^T x \\ \text{subject to} & \\ & Ax \geq b \\ & x \geq 0, \end{array}$$

and

$$(LD) \qquad \begin{array}{ll} \text{Max} & b^T y \\ \text{subject to} & \\ & A^T y \leq c \\ & y \geq 0. \end{array}$$

It is well known in duality theory that if (LP) and (LD) both are feasible then both have optimal solutions. The below given theorem, called *strict complementarity theorem*, gives some additional information as well.

Theorem 3.7.3 (Goldman-Tucker Theorem) *Let (LP) and (LD) both be feasible. Then they both have optimal solutions x^* (for(LP)) and y^* (for(LD)) satisfying*

$$x^* + (c - A^T y^*) > 0. \tag{3.20}$$

The condition (3.20) is called the strict complementarity condition. Here it may be noted that Theorem 3.7.3 does not tell that (3.20) holds for every pair $(\overline{x}, \overline{y})$ of optimal solution of (LP) and (LD). But rather it guarantees the existence of a pair (x^*, y^*) of optimal solution of (LP) and (LD) for which (3.20) holds. We can refer to Goldman and Tucker [52] for the proof of Theorem 3.7.3.

We now proceed to prove Theorem 3.7.1. For this let us consider a portfolio $P : (x_0, x_1, \ldots, x_n)$. Then the value of this portfolio at $t = 0$ is

$$V_P(0) = \sum_{k=0}^{n} x_k \, S_0^{(k)}.$$

Further its value at $t = 1$ will be one of the m values, namely $V_{P,1}(\omega_j) = \sum_{k=0}^{n} x_k \, S_1^{(k)}(\omega_j)$, $(j = 1, 2, \ldots, m)$, depending upon the state of economy ω_j where $\Omega = \{\omega_1, \omega_2, \ldots, \omega_m\}$. We now consider the following linear programming problem (LPP)

$$\begin{array}{ll} \text{Min} & V_P(0) \\ \text{subject to} & \\ & V_{P,1}(\omega_j) \geq 0 \qquad (j = 1, 2, \ldots, m), \end{array}$$

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i.e.

$$\text{Min} \quad \sum_{k=0}^{n} x_k \, S_0^{(k)}$$

subject to

$$\sum_{k=0}^{n} x_k \, S_1^{(k)}(\omega_j) \geq 0 \qquad (j = 1, 2, \ldots, m). \tag{3.21}$$

The dual of problem (3.21) is

$$\text{Max} \quad \sum_{j=1}^{m} 0.p_j$$

subject to

$$\begin{aligned} \sum_{j=1}^{m} S_1^{(k)}(\omega_j) \, p_j &= S_0^{(k)} \qquad (k = 0, 1, 2, \ldots, n) \\ p_j &\geq 0 \qquad (j = 1, 2, \ldots, m). \end{aligned} \tag{3.22}$$

Next we note that LPP (3.21) is feasible because $\{x_k = 0, \ (k = 0, 1, 2, \ldots, n)\}$ satisfies all its constraints. Also this is optimal because by definition to meet the assumption of solvency, only admissible portfolios are to be considered, i.e. $V_P(0) \geq 0$, i.e. $\sum_{k=0}^{n} x_k \, S_0^{(k)} \geq 0$. But (3.21) is a minimization problem and therefore $\{x_k = 0, \ (k = 0, 1, 2, \ldots, n)\}$ is optimal to (3.21) and the optimal value is zero. Therefore its dual LPP (3.22) also has an optimal solution with optimal value as zero.

We now write the primal-dual pair (3.21) - (3.22) in the form of the pair (LP)-(LD) and then apply the Goldman-Tucker theorem (Theorem 3.7.3) to this pair. This implies that there exists $\overline{x}$ optimal to (3.21) and $\overline{p}$ optimal to (3.22) such that

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$$-\sum_{k=0}^{n} S_1^{(k)}(\omega_j)\bar{x}_k \le 0 \qquad (j = 1,2,\ldots,m), \tag{3.23}$$

$$\sum_{j=1}^{m} S_1^{(k)}(\omega_j)\,\bar{p}_j = S_0^{(k)} \qquad (k = 0,1,2,\ldots,n), \tag{3.24}$$

$$\bar{p}_j \ge 0 \qquad (j = 1,2,\ldots,m), \tag{3.25}$$

$$\sum_{j=1}^{m} 0.\bar{p}_j = \sum_{k=0}^{n} S_0^{(k)}\,\bar{x}_k = 0, \tag{3.26}$$

and

$$\bar{p}_j + \left(0 - \left(-\sum_{k=0}^{n} S_1^{(k)}(\omega_j)\,\bar{x}_k\right)\right) > 0 \qquad (j = 1,2,\ldots,m). \tag{3.27}$$

But from (3.26), $V_{\bar{p}}(0) = 0$ for the portfolio $(\bar{x}_k,\ (k = 0,1,2,\ldots,n))$. Hence by *no arbitrage principle*, $V_{\bar{p}}(1)$ should be zero with probability 1, i.e. $\sum_{k=0}^{n} S_1^{(k)}(\omega_j)\,\bar{x}_k = 0$ $(j = 1,2,\ldots,m)$. Therefore (3.27) gives $\bar{p}_j > 0$ $(j = 1,2,\ldots,m)$.

Now we recall that $k = 0$ refers to the bond. Taking $S_0^{(0)} = 1$ we get $S_1^{(0)}(\omega_j) = 1 + r = R$ $(j = 1,2,\ldots,m)$. Then (3.24) gives

$$\sum_{j=1}^{m} R\,\bar{p}_j = 1,$$

i.e.

$$\sum_{j=1}^{m} \bar{p}_j = \frac{1}{R}. \tag{3.28}$$

If we now define $\hat{p}$ such that $\hat{p}_j = R\,\bar{p}_j$ $(j = 1,2,\ldots,m)$, then (3.24), (3.25) and (3.28) give $\hat{p}_j > 0$ $(j = 1,2,\ldots,m)$, $\sum_{j=1}^{m} \hat{p}_j = 1$ and

$$\frac{1}{R}\left(\sum_{j=1}^{m} S_1^{(k)}(\omega_j)\,\hat{p}_j\right) = S_0^{(k)} \qquad (k = 0,1,2,\ldots,n). \tag{3.29}$$

But then this shows that $\hat{p} = (\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_j, \ldots, \hat{p}_m)$ is a risk neutral probability measure.

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We can also write (3.29) in a more familiar notation as

$$S_0^{(k)} = \frac{1}{R} E_{\hat{p}}\left(S_1^{(k)}\right) \qquad (k = 0, 1, 2, \ldots, n), \tag{3.30}$$

where $E_{\hat{p}}$ denotes the expectation under RNPM $\hat{p}$. This proves that if no arbitrage principle holds then RNPM exists. The converse can be proved on similar lines. □

Remark 3.7.1 *The formula (3.30) is very general. Apart from the fact that it holds for the stock and bond, it should hold for other securities ($k = 2, 3, \ldots, n$) as well, be it European call, put, forward contract etc. What we simply need to do is to find the expectation of the pay-off of the given security at expiry under RNPM and discount the same for $t = 0$.*

Remark 3.7.2 *For the case of single period binomial lattice model the linear programming problem to find RNPM is*

$$\begin{aligned} &Max \qquad 0\,\hat{p}_1 + 0\,\hat{p}_2 \\ &subject\ to \\ &\qquad (u\ S(0))\,\hat{p}_1 + (d\ S(0))\,\hat{p}_2 = RS(0) \\ &\qquad\qquad \hat{p}_1 + \hat{p}_2 = 1 \\ &\qquad\qquad \hat{p}_1 \geq 0, \hat{p}_2 \geq 0, \end{aligned}$$

such that at optimality $\hat{p}_1 > 0$, $\hat{p}_2 > 0$. It is not difficult to see that the unique optimal solution is $\left(\hat{p}_1 = \frac{R-d}{u-d},\ \hat{p}_2 = \frac{R-u}{u-d}\right)$, which is the same as obtained earlier by replicating portfolio arguments.

Also if $k = 2$ and $k = 3$ respectively refer to European call and European put options, then formula (3.30) gives

$$C(0) = \frac{1}{R}\left[\hat{p}_1\ C_u + \hat{p}_2\ C_d\right],$$

and

$$P(0) = \frac{1}{R}\left[\hat{p}_1\ P_u + \hat{p}_2\ P_d\right].$$

Example 3.7.1 *Consider a forward contract with the given data as $S(0) = 100$, $u = 1.2$, $d = 0.8$, $T = 1$ year and $r = 10\%$ per year. Determine the forward price F.*

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Solution We know that $F = RS(0) = (1.1) \times 100 = Rs\ 110$. But here we have to determine F by utilizing the formula (3.30). For this we note that the pay-off of the forward contract is $(S - F)$ if $S > F$ and $-(F - S)$ if $S \leq F$. Also from the given data ($\hat{p}_1 = \frac{3}{4}$, $\hat{p}_2 = \frac{1}{4}$). Therefore formula (3.30) gives

$$F(0) = \frac{1}{1.1}\left[\frac{3}{4}(120 - F) + \frac{1}{4}(80 - F)\right]. \tag{3.31}$$

But in the case of forward contract $F(0) = 0$, and then (3.31) gives $F = 110$.

□

Let us again look at formula (3.29) and concentrate for the case $k = 1$. We recall that $k = 1$ refers to the stock and therefore if we write $S_0^1 = S(0)$ and $S_1^1 = S(1)$ then

$$\begin{aligned} S(0) &= \frac{1}{R}E_{\hat{p}}(S(1)) \\ &= E_{\hat{p}}\left(\frac{S(1)}{R}\right) \\ &= E_{\hat{p}}\left(\frac{S(1)}{B(1)}\right) \\ &= E_{\hat{p}}(\widetilde{S}(1)). \end{aligned} \tag{3.32}$$

Here we have taken $B(0) = 1$, $\widetilde{S}(1) = (S(1)/B(1))$ and written $E_{\hat{p}}$ to emphasize that expectation has been taken with respect to RNPM $\hat{p}$. Therefore (3.32) gives

$$\widetilde{S}(0) = E_{\hat{p}}(\widetilde{S}(1)) \tag{3.33}$$

The expectation in (3.32) is in fact conditional because it is computed once the stock price $S(0)$ becomes known at $t = 0$. Therefore (3.33) is actually

$$\widetilde{S}(0) = E_{\hat{p}}\left(\widetilde{S}(1)/\widetilde{S}(0)\right),$$

which, in general, is expressed as

$$\widetilde{S}(l) = E_{\hat{p}}\left((\widetilde{S}(l+1))/\widetilde{S}(l)\right). \tag{3.34}$$

The relationship (3.34) is expressed as: *the discounted stock prices* $\widetilde{S}(0), \widetilde{S}(1), \widetilde{S}(2), \ldots$ *form a martingale with respect to RNPM* $\hat{p}$. This holds for other securities as well. i.e.

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$$\widetilde{S}^{(k)}(l) = E_{\hat{p}}\left(\widetilde{S}^{(k)}(l+1)/\widetilde{S}^{(k)}(l)\right) \quad (k = 0, 1, 2, \ldots, n).$$

Therefore the problem of asset pricing gets translated into the problem of finding a unique RNPM $\hat{p}$ or to be precise $\hat{p}$-*martingale* on the set of scenarios Ω. Moreover $E_{\hat{p}}$ is called the *risk neutral* or *martingle expectation* with respect to $\hat{p}$. These concepts will probably get much clearer once we are familiar with basics of stochastic process and stochastic calculus.

Since there are no goods in the model other than money, it is convenient to pick one of the security as reference and normalize others with respect to it. The security so chosen for normalization purpose is called *numeraire*. In our context, the usual choice of numeraire is the bond price and that is what exactly we have done in defining $\widetilde{S}(0) = S(0)/B(0)$, $\widetilde{S}(1) = S(1)/B(1)$ etc.

Example 3.7.2 *Consider the data:* $S(0) = 100$, $u = 1.2$, $d = 0.9$, $r = 10\%$ *per year and* $T = 2$ *years. Determine RNPM* $\hat{p}$ *and show that discounted stock prices form a* $\hat{p}$*-martingale.*

Solution We have $\hat{p} = \frac{2}{3}$ and $(1-\hat{p}) = \frac{1}{3}$. We have the dynamics of the stock price as

t	t=0	t=1	t=2
$S(t)$			144
		120	
	100		108
		90	
			81

We can check the following

$$\frac{1}{1.1}\left(\frac{2}{3}(144) + \frac{1}{3}(108)\right) = 120$$
$$\frac{1}{1.1}\left(\frac{2}{3}(108) + \frac{1}{3}(81)\right) = 90$$
$$\frac{1}{1.1}\left(\frac{2}{3}(120) + \frac{1}{3}(90)\right) = 100.$$

These give

$$E_{\hat{p}}\left[\left(\frac{S(2)}{(1.1)^2}\right)\Big/\left(\frac{S(1)}{1.1}\right) = \frac{120}{1.1}\right] = \frac{120}{1.1}$$
$$E_{\hat{p}}\left[\left(\frac{S(2)}{(1.1)^2}\right)\Big/\left(\frac{S(1)}{1.1}\right) = \frac{90}{1.1}\right] = \frac{90}{1.1}$$

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etc. Thus

$$E_{\hat{p}}\left(\widetilde{S}(2)/\widetilde{S}(1)\right) = \widetilde{S}(1).$$

The below given theorem is very general and is valid for any European derivative security.

Theorem 3.7.4 *Let D be a European derivative security whose pay-off in the N-period binomial model is $f(S(N))$. Then*

$$D(0) = \frac{1}{R^N} E_{\hat{p}}\left(f\left(S(N)\right)\right). \tag{3.35}$$

The above theorem essentially tells that the price of a European derivative security D with pay-off $f(S(N))$ in the N-period binomial model is the expectation of the discounted pay-off under the risk neutral probably measure.

We know that for a European call, $f(S(N)) = (S(N) - K)^+$; for a European put $f(S(N)) = (K - S(N))^+$ and for a forward contract $f(S(N)) = (S(N) - F)$. Therefore the same formula (3.35) can be used to price each of these derivative securities.

Example 3.7.3 *Use Theorem 3.7.1 to justify the pricing formula (3.15) for the European call option.*

Solution Here $N = 2$ and the three possible value for $f(S(2))$ are

$$\begin{aligned} C_{uu} &= \text{Max}\left(u^2 S(0) - K,\ 0\right) \\ C_{ud} &= \text{Max}\left(u\ d\ S(0) - K,\ 0\right) \\ C_{dd} &= \text{Max}\left(d^2 S(0) - K,\ 0\right). \end{aligned}$$

Also $\hat{p} = \dfrac{(R-d)}{(u-d)}$. Therefore if we define $\hat{p}_1 = (\hat{p})^2$, $\hat{p}_2 = 2\hat{p}(1-\hat{p})$ and $\hat{p}_3 = (1-\hat{p})^2$ then

(i) $\hat{p}_i > 0 \quad (i = 1, 2, 3)$

(ii) $\sum_{i=1}^{3} \hat{p}_i = 1$ and

(iii) $E_{\hat{p}}(S(2)) = R^2 S(0)$.

The third assertion can be verified as follows

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$$
\begin{aligned}
E_{\hat{p}}(S(2)) &= \hat{p}_1\left(u^2S(0)\right) + \hat{p}_2\left(u\ d\ S(0)\right) + \hat{p}_3\left(d^2S(0)\right) \\
&= S(0)\left[u^2(\hat{p})^2 + 2\hat{p}\ (1-\hat{p})\ u\ d\ S(0) + (1-\hat{p})^2d^2S(0)\right] \\
&= S(0)\left[u\ \hat{p} + (1-\hat{p})\ d\right]^2 \\
&= S(0)\left[\frac{u(R-d)+d(u-R)}{(u-d)}\right]^2 \\
&= S(0)\left[\frac{R(u-d)}{u-d}\right]^2 = R^2S(0),
\end{aligned}
$$

i.e.

$$S(0) = \frac{1}{R^2}\left(E_{\hat{p}}(S(2))\right).$$

Therefore $\hat{p}_i$ $(i = 1, 2, 3)$ as defined above is in fact RNPM. Hence

$$
\begin{aligned}
C(0) &= \frac{1}{R^2}\left(E_{\hat{p}}\left(f\left(S(2)\right)\right)\right) \\
&= \frac{1}{R^2}\left[(\hat{p})^2C_{uu} + 2\hat{p}\ (1-\hat{p})\ C_{ud} + (1-\hat{p})^2C_{dd}\right],
\end{aligned}
$$

as obtained by formula (3.15).

□

Remark 3.7.3 *Apparently we have employed two distinct approaches to price a given derivative. These are the replicating portfolio approach and the RNPM approach. Various illustrative examples presented above and also the derivation of single period binomial lattice model suggest that these two approaches are related. In fact for a complete market, the two approaches are equivalent. This is because the replicating portfolio approach uses the law of one price which is essentially a consequence of no arbitrage principle. This is because then the unique RNPM exists which can be used to price any contingent claim. For the existence of RNPM, the market has to be arbitrage free. Therefore if no arbitrage principle does not hold then we cannot price the given derivative even if the corresponding unique replicating portfolio exists. The below given example illustrates the point. We shall further discuss this aspect in Section 3.10.*

Example 3.7.4 *Let* $B(0) = 100, B(1) = 120,\ S(0) = 100$ *and* $S(1)$ *take values* 120 *and* 80 *with probabilities* 0.8 *and* 0.2 *respectively. Let* C *be a European call with* $K = 100$ *and* $T = 1$ *year. Find the replicating portfolio* (x, y) *for the call* C*. Are we justifying in taking* $C(0) = x\ S(0) + y\ B(0)$*? Determine RNPM if it exists.*

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Solution The pay-off of the call C is

$$C(1) = \begin{cases} 20, & \text{with probability } 0.8 \\ 0, & \text{with probability } 0.2. \end{cases}$$

Therefore if (x, y) is the replicating portfolio, then we have $x\,S(1) + y\,B(1) = C(1)$. This gives the following two equations

$$120x + 120y = 20$$

$$80x + 120y = 0,$$

having solution as $(x = 1/2, y = -1/3)$. Here we cannot take $C(0) = 1/2S(0) - 1/3B(0)$ because this utilizes the law of one price which is valid only under no arbitrage principle. But as $u = 1.2$ and $R = 1 + r = 1.2$ we do not have the required condition $u > R > d$ for no arbitrage principle to hold.

To determine the RNPM we need to solve the system

$$\hat{p}_1 + \hat{p}_2 = 1$$

$$\frac{1}{1.2}(120\hat{p}_1 + 80\hat{p}_2) = 100$$

$$\hat{p}_1 > 0,\ \hat{p}_2 > 0.$$

The above system does not have a solution because the first two equations give $\hat{p}_1 = 1,\ \hat{p}_2 = 0$. Therefore the RNPM does not exist. This is again because $u = 1.2 = R$ which violates the condition $u > R > d$ and therefore no arbitrage principle does not hold.

□

3.8 Pricing American Options: A Binomial Lattice Model

Following earlier notations, let D denote the derivative security under consideration and stock be its underlying. Let $D^A(n)$ denote the price of an American option at the time period n, $0 \le n \le N$. To be specific, let $N = 2$ and D denote either an American call or an American put. We now wish to analyze the given American option expiring after two time periods.

If the option has not already been exercised then at the expiry it will have the value $D^A(2) = f(S(2))$, where f is the pay-off function of the derivative D.

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ould the arbitrage
ne.

Here we may note that for the two period binomial lattice model, $f(S(2))$ will have three possible values, depending upon the three possible values of $S(2)$. If D denote an American put then these values are denoted by P^A_{uu}, P^A_{ud} and P^A_{dd}. For the case of an American call these values are denoted by C^A_{uu}, C^A_{ud} and C^A_{dd}. Since D is a American derivative and the binomial lattice is of two periods, the holder can exercise his/her option at $t = 0$, $t = 1$ or $t = 2$.

At $t = 1$, the option holder has a choice. He/She can exercise immediately or wait until $t = 2$. If he/she exercises his/her option at $t = 1$, the pay-off is $f(S(1))$, otherwise it is $f(S(2))$. But is it worth waiting until $t = 2$ to exercise the option? This question can be answered by computing the *value of waiting*. For this we may treat $f(S(2))$ as a one step European option to be priced at $t = 1$. Let as before $\hat{p} = \dfrac{(R-d)}{(u-d)}$ denote one step RNPM. Then the value of the given derivative at $t = 1$ is given by $\frac{1}{R}E_{\hat{p}}(f(S(2)))$, where

$$\frac{1}{R}E_{\hat{p}}(f(S(2))) = \frac{1}{R}\left(\hat{p}\, f(u\, S(1)) + (1-\hat{p})\, f(d\, S(1))\right). \tag{3.36}$$

Therefore the option holder has to choose the higher of two values $f(S(1))$ and $\frac{1}{R}E_{\hat{p}}(f(S(2)))$ as given at (3.36). Thus at $t = 1$, the given American option is worth the price $D^A(1)$, where

$$D^A(1) = \text{Max}\left[f(S(1)), \frac{1}{R}(\hat{p}\, f(u\, S(1)) + (1-\hat{p})\, f(d\, S(1)))\right].$$

In a similar manner we can argue to get

$$D^A(0) = \text{Max}\left[f(S(0)), \frac{1}{R}(\hat{p}\, f_1(u\, S(0)) + (1-\hat{p})\, f_1(d\, S(0))\right], \tag{3.37}$$

where

$$f_1(x) = \text{Max}\left[f(x), \frac{1}{R}(\hat{p}\, f(u\, x) + (1-\hat{p})\, f(d\, x))\right]. \tag{3.38}$$

Remark 3.8.1 *As we have assumed that the stock is non-dividend paying, an American call is the same as a European call and $C^A(0) = C^E(0)$. Therefore the formula (3.37) is essentially for the case of an American put, though it can be applied for an American call as well if one so wishes. For a dividend paying stock, an American call will behave differently from a European call.*

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Remark 3.8.2 *Using backward induction, formula (3.37) can be extended to an N-period binomial lattice model in an obvious manner. Unlike the closed form formula (3.35) for the case of European scenario, we do not have any closed form formula for pricing of an American derivative. We do not have any choice except to resort to the backward induction as explained above.*

Example 3.8.1 *Find the price of an American put option with the given data as* $S(0) = 80,\ K = 80,\ u = 1.8,\ d = 0.95,\ r = 5\%$ *per year and time to expiry* $T = 2$ *years.*

Solution We detail the solution in below given steps

Step 1. The table for the stock price at different periods is

n	0	1	2
$S(n)$			96.80
		88	
	80		83.60
		76	
			72.20

Also $\hat{p} = \dfrac{R-d}{u-d} = \dfrac{1.05-0.95}{1.8-0.95} = \dfrac{2}{3}$, $(1-\hat{p}) = \dfrac{1}{3}$.

Step 2. Let $P^A(n)$, $(n = 0,1,2)$, denote the price of given American put. To determine $P^A(n)$, we first need to compute

$$P^f_{uu} = \text{Max}\left(K - u^2S(0),\ 0\right) = \text{Max}(80 - 96.80, 0) = 0$$
$$P^f_{ud} = \text{Max}\left(K - u\ d\ S(0),\ 0\right) = \text{Max}\left(80 - 83.60, 0\right) = 0$$
$$P^f_{dd} = \text{Max}\left(K - d^2S(0),\ 0\right) = \text{Max}(80 - 72.2, 0) = 7.80.$$

Here P^f_{uu} etc are same as for the European put option. We have used the suffix 'f' to denote the forward value. The notation P^b_u etc will be used to denote the backward value to be computed in step 3. Thus $f(S(2))$ is a random variable taking values as P^f_{uu}, P^f_{ud} and P^f_{dd}.

Step 3. Compute

$$P^b_u = \frac{1}{R}\left(\hat{p}\ P^f_{uu} + (1-\hat{p})\ P^f_{ud}\right) = \frac{1}{1.05}\left[(\frac{2}{3}\times 0) + (\frac{1}{3}\times 0)\right] = 0$$
$$P^b_d = \frac{1}{R}\left(\hat{p}\ P^f_{ud} + (1-\hat{p})\ P^f_{dd}\right) = \frac{1}{1.05}\left[(\frac{2}{3}\times 0) + (\frac{1}{3}\times 7.80)\right] = 2.476.$$

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Step 4. Evaluate

$$P_u^f = \text{Max}\,(K - u\ S(0),\ 0) = \text{Max}(80 - 88, 0) = 0$$
$$P_d^f = \text{Max}\,(K - d\ S(0),\ 0) = \text{Max}(80 - 76, 0) = 4.$$

Step 5. Evaluate

$$P^A(1) = \begin{cases} \text{Max}(P_u^f, P_u^b) = \text{Max}(0,0) = 0 = P_u^{Max} \\ \text{Max}(P_d^f, P_d^b) = \text{Max}(4, 2.476) = 4 = P_d^{\text{Max}}. \end{cases}$$

Step 6. Compute

$$\begin{aligned} P^A(0) &= \text{Max}\left(K - S(0),\ \frac{1}{R}\left[\hat{p}\ P_u^{\text{Max}} + (1-\hat{p})\ P_d^{\text{Max}}\right]\right) \\ &= \text{Max}\left((80 - 80), \frac{1}{1.05}\left[(\frac{2}{3} \times 0) + (\frac{1}{3} \times 4)\right]\right) \\ &= 1.27. \end{aligned}$$

The above calculation are depicted in the following tables which are now self explanatory

n	0	1	2
$P^A(n)$			$(P_{uu}^f = 0)$
		(?)	
	(?)		$(P_{ud}^f = 0)$
		(?)	
			$(P_{dd}^f = 7.80)$

n	0	1	2
$P^A(n)$			0.0
		$P_u^{\text{Max}} = [\underline{0}, 0]$	
	(?)		0.0
		$P_d^{\text{Max}} = [\underline{4}, 2.47]$	
			7.80

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n	0	1	2
$P^A(n)$			0.0
		$P_u^{\text{Max}} = [\underline{0}, 0]$	
	$[80-80, \underline{1.27}]$		0.0
		$P_d^{\text{Max}} = [\underline{4}, 2.47]$	
			7.80

Here the notation $[a, \underline{b}]$ is used to denote that b is the maximum of a and b, i.e. $b = \text{Max}(a, b)$. Therefore $P^A(0) = 1.27$, and the given American put should be exercised early in the down state at $t = 1$. This will give a pay-off of *Rs* 4.00 which is more than the value of holding it to the expiry for the value of *Rs* 2.47. We may also compute $P^E(0)$ and get $P^E(0) = 0.79$. Clearly $P^E(0) < P^A(0)$. Hence these are different even for a non-dividend paying stock.

□

Example 3.8.2 *Find the price of an American call and a European call for the data as* $S(0) = 120$, $K = 120$, $u = 1.2$, $d = 0.9$, $r = 10\%$ *per year and time to expiry* $T = 2$ *years.*

Solution As the stock is non-dividend paying, $C^A(0) = C^E(0)$. Computing $C^E(0)$ by formula (3.15) or utilizing the below given tables we get $C^E(0) = 22.92$. Here $\hat{p} = \frac{2}{3}$.

n	0	1	2
$S(n)$			172.80
		144	
	12		129.60
		96	
			97.20

n	0	1	2
$C^E(n)$			52.80
		34.91	
	22.92		9.60
		5.82	
			0.0

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It may also be interesting to compute $C^A(0)$ as per the procedure outlined in this section and verify that $C^A(0)$ also comes out to be *Rs* 22.92. This calculation is shown in the below given tables

n	0	1	2
$C^A(n)$			52.80
		(?)	
	(?)		9.60
		(?)	
			0.0

n	0	1	2
$C^A(n)$			52.80
		[24, $\underline{34.9}$]	
	(?)		9.60
		[0, $\underline{5.8}$]	
			0.00

$$
\begin{aligned}
C^A(0) &= \text{Max}\left((120-120), \frac{1}{1.1}\left(\frac{2}{3}(34.9)+\frac{1}{3}(5.80)\right)\right) \\
&= \text{Max}(0, 22.92) \\
&= 22.92.
\end{aligned}
$$

At time $T = 2$, both C^E and C^A have the same pay-off. At $t = 1$, The given American option will not be exercised in the up state because it will bring only *Rs* 24 which is less than the value of holding it up to expiry. In the down state, the American call will be out of money and will not be exercised either.

□

3.9 Options on Dividend Paying Stock

It has been remarked in the last section that for a non-dividend paying stock, $C^A(0)$ equals $C^E(0)$. But for a dividend paying stock, in general, $C^A(0)$ and $C^E(0)$ will have different value.

Let the stock pay a dividend of amount Q at time τ, $0 \le \tau \le T$. We wish to determine the price of a European call option on this stock, using the multi period

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binomial model. For this we divide the time interval $[0, T]$ into N time periods. Assuming that the dividend date τ occurs in the kth period, we form a lattice in the usual way, but subtract the amount Q from the nodes at the kth period. But then the nodes at time period k do not recombine and therefore we get a binomial tree model rather than a binomial lattice model. Except for this change and some obvious modifications, the procedure for finding $P^E(0)$, $C^E(0)$, $P^A(0)$ and $C^A(0)$ remains same as discussed in Section 3.7 and Section 3.8. We illustrate the procedure in the below given exercise.

Example 3.9.1 *Compute the price of the European put with the given data as* $S(0) = 12$, $K = 14$, $u = 1.1$, $d = 0.95$, $r = 2\%$ *per year and time to expiry in two years. Assume that a dividend of Rs 2.00 is paid at time* $t = 1$.

Solution The ex-dividend price are as given in the below given table

n	0	1	2
$S(n)$ (ex-div)			12.32
		11.20	
			10.64
	12.00		10.34
		9.40	
			8.93

Then following the methodology discussed in Section 3.8 we get $\hat{p} = \frac{7}{15}$ and

n	0	1	2
P^E			1.68
		2.53	
			3.36
	(?)		3.66
		4.33	
			5.07

Here

$$2.53 = \frac{1}{1.02}\left[\frac{7}{15}(1.68) + \frac{8}{15}(3.36)\right]$$

$$4.33 = \frac{1}{1.02}\left[\frac{7}{15}(3.66) + \frac{8}{15}(5.07)\right].$$

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Therefore

$$P^E(0) = \frac{1}{1.02}\left[\frac{7}{15}(2.53) + \frac{8}{15}(4.33)\right] = 3.42.$$

□

Example 3.9.2 *For the data given in Example 3.9.1, evaluate $P^A(0)$.*

Solution Continuing with the data of Example 3.9.1, we have

n	0	1	2
P^A			
			1.68
		[2.80, 2.53]	
			3.36
	(?)		3.66
		[4.60, 4.33]	
			5.07

Therefore

$$\begin{aligned} P^A(0) &= \text{Max}\left(K - S(0), \frac{1}{1.02}\left(\frac{7}{15}(2.80) + \frac{8}{15}(4.60)\right)\right) \\ &= \frac{1}{1.02}\left[\frac{7}{15}(2.80) + \frac{8}{15}(4.60)\right] \\ &= 3.69. \end{aligned}$$

□

Example 3.9.3 *Find the price $C^E(0)$ and $C^A(0)$ for the data, $S(0) = 120$, $K = 120$, $u = 1.2$, $d = 0.90$, $r = 10\%$ per year and time to expiry is $T = 2$ periods. What are the value of $C^E(0)$ and $C^A(0)$ if it is additionally given that a dividend of Rs 14 is paid at $T = 2$?*

Solution If there is no dividend on the stock then $C^A(0) = C^E(0)$. The value of $C^E(0)$ can be easily determined as *Rs* 22.92. Here $\hat{p} = \frac{2}{3}$.

We next consider the case when there is a dividend of *Rs* 14.00 at time $t = 2$. This gives the following table for the stock price at different periods.

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n	0	1	2
$S(n)$ (ex-div)			$(1.2)^2(120) - 14 = 158.80$
		144	$(1.2)(0.9)(120) - 14 = 115.60$
	120		
		108	$(0.9)^2(120) - 14 = 83.20$

n	0	1	2
C^A			38.80
		(?)	0.0
	(?)		
		(?)	0.0

n	0	1	2
C^A			38.80
		[24, 21.5]	0.0
	(?)		
		[0, 0]	0.0

Therefore

$$C^A(0) = \frac{1}{1.1}\left[\frac{2}{3}(24) + \frac{1}{3}(0)\right]$$
$$= 14.54.$$

To compute $C^E(0)$ we follow the usual methodology to get the following table

n	0	1	2
C^E			38.80
		23.5	0.0
	(?)		
		0.0	0.0

This gives

$$C^E(0) = \frac{1}{1.1}\left[\frac{2}{3}(23.5) + \frac{1}{3}(0)\right] = 14.24.$$

As expected, here $C^E(0)$ and $C^A(0)$ are not equal because the stock is dividend paying. □

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3.10 Notion of Complete Markets

The concept of completeness property for a financial market is very important. This is because, if the market is complete then the unique RNPM exists and therefore every financial contract has unique fair price. In view of Theorems 3.7.1 and 3.7.2 a complete market has certainly to be arbitrage free.

In order to introduce the notion of completeness, we first introduce the notion of *contingent claim*. A *contingent claim* is a financial contract with random pay-off that can be positive or negative. The term 'contingent' reflects the situation that because the pay-off is random, it is contingent (i.e. dependent) on which state of nature is realized at the time of pay-off. All the securities that we have described earlier are contingent claims. Thus stock, bond, call, put, forward contract, futures etc are all contingent claims.

We say that a portfolio *replicates* a contingent claim when the pay-off of the portfolio matches the pay-off the claim in all possible states. Here 'matching' has to be understood in the sense that 'matches with probability one'. Such a portfolio is called a *replicating portfolio*. Here it must be added that we require replicating portfolio strategy to be *self financing*, i.e. the investor can not use funds other than the initial wealth to finance the position in the market and he/she is not allowed to spend money outside market. We say that the market is complete when it is arbitrage free and we are able to uniquely replicate any contingent claim with the existing securities.

Some readers must have realized that single and multi-period binomial lattice models are complete market models as they are capable of pricing any contingent claim. But what about a trinomial model? Here we have three possible outcomes for the price of stock in the next period which are: it goes up, it does not change or it goes down. Thus if $S(0)$ is the price of the stock at $t = 0$ then $S(1)$ takes three values $uS(0)$, $S(0)$ and $dS(0)$ with probabilities p_1, p_2 and p_3 respectively. The trinomial model is certainly arbitrage free because we have again assumed that $u > R > d$. But what about the existence of replicating portfolio? Let C denote European call with strike price K and expiry $t = 1$. Then the portfolio (x, y) will replicate the pay-off of the call if

$$\begin{aligned} xuS(0) + yB(1) &= C_u \\ xS(0) + yB(1) &= C_n \\ xdS(0) + yB(1) &= C_d, \end{aligned}$$

where $C_u = \text{Max}(uS(0) - K, 0)$, $C_n = \text{Max}(S(0) - K, 0)$ and $C_d = \text{Max}(dS(0) - K, 0)$.

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We see that the above linear system only permits a solution for a select subset of contingent claims (e.g. European calls here) with pay-offs within a 2-dimensional subspace. Some readers may like to verify that the linear system under consideration is consistent only when

$$(1-d)C_u + (u-1)C_d - (u-d)C_n = 0.$$

Thus not all contigent claims can be priced by the trinomial model. This means that the trinomial model is not complete.

In fact if we try to determine the unique RNPM for the trinomial model we shall again face the problem. Here we need to find $\hat{p}_1 > 0$, $\hat{p}_2 > 0$, and $\hat{p}_3 > 0$ such that

$$\hat{p}_1 + \hat{p}_2 + \hat{p}_3 = 1,$$

and

$$\hat{p}_1\,(uS(0)) + \hat{p}_2 S(0) + \hat{p}_3\,(dS(0)) = RS(0).$$

Obviously this system is consistent but it cannot have unique solution. This again shows that the trinomial model is not complete.

To tackle such situation we enlarge the set of basic securities. For example, we may take bond, stock and (say) an option whose price $C(0)$ is known. Now we wish to obtain $\hat{p}_1$, $\hat{p}_2$, $\hat{p}_3$ $(\hat{p}_i > 0,\ i = 1,2,3)$ such that

$$\begin{aligned} \hat{p}_1 + \hat{p}_2 + \hat{p}_3 &= 1 \\ \hat{p}_1\,(uS(0)) + \hat{p}_2\,(S(0)) + \hat{p}_3\,(dS(0)) &= RS(0) \\ \hat{p}_1 C_u + \hat{p}_2 C_n + \hat{p}_3 C_d &= RC(0). \end{aligned}$$

The above system has unique solution giving the unique RNPM. This unique RNPM can now be used to price other contingent claims. In this way we have artificially made the trinomial model as a complete market model.

Example 3.10.1 *Let $S(0) = 100$ and $S(1)$ takes three values 120, 100 and 90 with probabilities 0.4, 0.2 and 0.4 respectively. The price of the European call on this stock with the strike price 105 is 5 and the price of the European call with strike price 95 is 10, with both option expiring at $t = 1$. Replicate the risk-free security that pays Rs 1 at $t = 1$ regardless of what happens.*

Solution Let x denote the number of shares of stock, z_1 the number of calls with strike price 100 and z_2 the number of calls with strike price 95. Therefore in order to replicate the risk-free security we need to solve

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$$120x + 15z_1 + 25z_2 = 1$$
$$100x + 0z_1 + 5z_2 = 1$$
$$90x + 0z_1 + 0z_2 = 1.$$

The solution of the above system is $x = 1/90$, $z_1 = 2/135$ and $z_3 = -1/45$. Further the cost of this portfolio is $(1/90)100 + (2/135)5 - (1/45)10$, i.e. 0.963.

Therefore the price of risk-free security is 0.963. Since its relative return is $(1/0.963) - 1 = 3.85\%$, the risk-free rate in this model is 3.85%.

□

3.11 Summary and Additional Notes

- Certain basic definitions and preliminaries with regard to European call/put options are presented in Section 3.2. An interesting and useful result of this section is the put- call parity. Here it is also mentioned that there is no such exact put-call parity for American options, but only certain estimates are provided for $C^A(0) - P^A(0)$.
- Section 3.3 is devoted to the analysis of the behavior of option prices with respect to defining variables K, S and T. Pay-off curves/gain curves of option combinations is the topic of discussion in Section 3.4. In particular, we have introduced bull spread, bear spread, and butterfly spread; and also a general procedure to draw spread of a given option combination.
- The core of this chapter has been discussion on single and multi period binomial lattice models for option pricing. The single period case is discussed in Section 3.5, while multi period case is presented in Section 3.6. These are very important topics on option pricing because they form the basis of the elaborated Black-Scholes formula to be discussed in the next chapter.
- An important question about the existence of RNPM is answered in Section 3.7. Here the readers are able to see the application of the linear programming duality in proving that under no arbitrage principle RNPM certainly exists. Another important aspect of this section is to motivate the readers to study martingales so as to be ready for detailed study in a later chapter.
- Section 3.8 discusses the applicability of binomial lattice approach for pricing American options. Here, unlike the case of European option, we cannot have any closed form pricing formula and backward induction is the only recourse.
- The case of dividend paying stock is considered in Section 3.9 and results of Section 3.5 and 3.6 are restated in this new scenario.

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4
Basic Theory of Option Pricing-II

4.1 Introduction

The world of options underwent a revolutionary change in 1973 when Fisher Black and Maryon Scholes [15] and Robert C. Merton [94] published their seminal papers on theory of option pricing. The basic idea behind their studies is that the price of an option is determined implicitly by the price of the underlying stock and certain other parameters whose values, except one of them, are easily observable. Black-Scholes option pricing model constructed a continuously hedging strategy to protect the writer's short position in option. Also Merton designed a self-financing and dynamically replicating hedged position containing the options, the underlying risky stock and a riskless asset. Both approaches led to a partial differential equation, now famously known as the Black-Scholes partial differential equation or simply Black-Scholes equation, with appropriate boundary conditions defined according to the contractual specifications of the option. These models will be presented and discussed later in Chapter 10. The price of a particular derivative security or option is obtained by solving the Black-Scholes equation subject to boundary conditions. In recognition to their pioneering contributions, Scholes and Merton were awarded 1997 Nobel prize in Economics. Black's contribution was also recorded by the Swedish Academy though he was found ineligible for prize as he died in 1995.

Despite its richness and simplicity, the Black-Scholes model has one major limitation. It cannot be used to accurately price options with an American-style exercise options as it only calculates the option price at expiration. It does not consider the steps along the way where there could be the possibility of early exercise of an option. To this end, came the alternative proposal of using binomial option pricing model. Both the Black-Scholes model and the binomial model are based on the same theoretical foundations and assumptions but there are also some

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important differences between the two. The mathematical tools Black, Scholes and Merton employed are quite advanced, like solving a partial differential equation, and this fact does not make the underlying economics more clear. In contrast to the Black-Scholes model the binomial option-pricing model is simple and can easily be understood with undergraduate knowledge. Partly the same has already been experienced in previous chapter wherein the readers have been provided with the glimpse of multi period binomial lattice models for option pricing through simple two period examples.

As already exhibited in the previous chapter, the guiding principle of the binomial model is to break down the time to expiration into a large number of time intervals or steps. Thereafter, at each step, it is assumed that the stock price will move up or down by certain amounts. This produces a binomial lattice of underlying stock prices. Subsequently the option prices at each node of the lattice are calculated, working backward from expiration to the present. For almost all practical purposes, in practice the discrete model of option pricing is preferred.

In this chapter, we aim to present the binomial lattice model proposed by John C. Cox, Stephen A. Ross and Mark Rubinstein [33] in 1979 for option pricing. This model is now famously called the *CRR model* for option pricing. Quoting Cox, Ross and Rubinstein, "Our formulation, by its very construction, leads to an alternative numerical procedure which is both simpler, and for many purposes, computationally more efficient". In fact the Black-Scholes formula turns out to be a particular limiting case of the discrete binomial CRR model. In Section 4.2, we describe the construction of the CRR model. In Section 4.3, we present the case when the CRR model can be matched with the multi-period binomial model. This matching finally leads to the Black-Scholes formula in the limiting case. Section 4.4 presents an extension of the Black-Scholes formula for options on dividend paying underlying stocks. Section 4.5 briefly describes the parameters, called the Greeks, which are vital tools in risk management.

4.2 The CRR Model

We illustrate how the knowledge gained in previous chapter on a single period binomial model and the strategy of replication can be used to extend the option pricing theory to multi-period scenario. The assumptions on the financial market made for single period binomial model are carried forward here too. For the sake of completeness, let us recall them as follows.

(i) The underlying stock on which option is written is perfectly divisible.

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(ii) The underlying stock pays no dividend.
(iii) There is no transaction costs in buying or selling the option, and no taxes.
(iv) Short selling is allowed.
(v) The riskless interest rate is known and constant till time to expiration of option.
(vi) No arbitrage principle holds.

Let us now divide the time horizon interval $[0, T]$ into n subintervals each of length $\Delta t = T/n$, and assume that in each subinterval the stock price changes like a one period binomial case. Thus in each period the underlying stock price either moves up by a constant factor u or moves down by a constant factor d with positive probability. For $k = 1, \ldots, n$, we also define

$$E_k = \begin{cases} u, & \text{with probability } p \\ d, & \text{with probability } 1-p, \end{cases}$$

with $0 < p < 1$. Here we note that E_k is a Bernoulli random variable.

At time to expiration T, the stock price $S(T)$ is given by

$$\begin{aligned} S(T) &= S(0)E_1E_2\cdots E_n \\ &= S(0)\,e^H, \qquad \text{(say)} \end{aligned}$$

where $e^H = E_1E_2\cdots E_n$. Thus,

$$ln(S(T)) = ln(S(0)) + H; \qquad H = \sum_{k=1}^{n} ln(E_k).$$

Here, H represents the logarithmic growth of the stock price.

The above expression helps us to visualize it as a discrete approximation to the continuous price process. The same is shown in Figure 4.1.

We recollect that if a random variable X, defined as

$$X = \begin{cases} a, & \text{with probability } p \\ b, & \text{with probability } 1-p, \end{cases}$$

the expectation of X is $E(X) = ap + b(1-p)$ and the variance of X is $Var(X) = E(X^2) - (E(X))^2 = p(1-p)(a-b)^2$. Using this we can easily derive that

$$E(ln(E_k)) = pln(u) + (1-p)ln(d), \tag{4.1}$$

$$Var(ln(E_k)) = p(1-p)(ln(u) - ln(d))^2. \tag{4.2}$$

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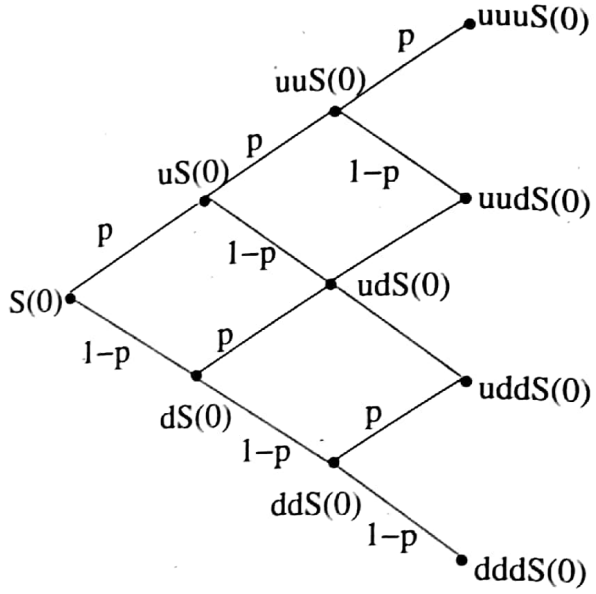

Fig. 4.1. The underline binomial lattice model

We now introduce two new parameters μ and σ^2 as follows

$$\mu\Delta t = E(ln(E_k)) \quad \text{and} \quad \sigma^2\Delta t = Var(ln(E_k)). \tag{4.3}$$

The parameter μ is called the *drift* and the parameter σ is called the *volatility*. It may appear a little strange as to how and why we suddenly bring in two new parameters μ and σ^2. As of now we urge to take them on face value as we shall be providing their physical interpretation later in this chapter. Now for each $k = 1, \ldots, n$, let us introduce another random variable

$$X_k = \frac{ln(E_k) - E(ln(E_k))}{\sqrt{Var(ln(E_k))}}.$$

Then, by using (4.1), (4.2) and definition of E_k, we can easily work out that

$$X_k = \begin{cases} \dfrac{1-p}{\sqrt{p(1-p)}}, & \text{with probability } p \\ \dfrac{-p}{\sqrt{p(1-p)}}, & \text{with probability } 1-p. \end{cases}$$

Clearly, for each $(k = 1, \ldots, n)$, $E(X_k) = 0$ and $Var(X_k) = 1$. Now,

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$$
\begin{aligned}
H &= \sum_{k=1}^{n} ln(E_k) \\
&= \sum_{k=1}^{n} (\mu \Delta t + \sigma \sqrt{\Delta t} X_k) \\
&= \mu T + \sigma \sqrt{\Delta t} Y,
\end{aligned}
$$

where

$$
Y = \sum_{k=1}^{n} X_k, \tag{4.4}
$$

is a simple random walk. Though the notion of simple random walk will be explained in details in Chapter 7, here we just wish to get familiar with it for clarity.

Definition 4.2.1 (Random Walk) *A random walk is a stochastic process* $\{S_n, n = 0, 1, \ldots\}$, *with* $S_0 = 0$, *defined by*

$$
S_n = \sum_{k=1}^{n} X_k,
$$

where $\{X_k\}$ *are independent and identically distributed random variables. The random walk is simple if for each* $k = 1, \ldots, n$, X_k *takes value from* $\{a, b\}$ *only,* a *and* b *are real constants, with* $P(X_k = a) = p$ *and* $P(X_k = b) = 1 - p$, $p \in [0, 1]$.

It is important to observe that the constants a, b, p are independent of k. Generally a and b are taken as 1 and -1 respectively. A simple random walk can be visualized as a game involving two persons, which consists of a sequence of independent identically distributed moves, and the sum S_n represents the score of the first person, say, after n moves, with the assumption that the score of the second person is $-S_n$.

In view of the above discussion we have the following lemma.

Lemma 4.2.1 *For a CRR model with probability of up tick* u *equals* p *and probability of down tick* d *equals* $1 - p$, *life time* T *and time increment* $\Delta t = T/n$, *the stock price is given by*

$$
S(T) = S(0) \exp\left(\mu T + \sigma \sqrt{\Delta T}\, Y\right),
$$

where μ *is the drift and* σ *is the volatility described by (4.3) and* Y *is a simple random walk given by (4.4).*

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We calculate expectation and variance of the random variable H.

$$\begin{aligned} E\Big(ln\Big(\frac{S(T)}{S(0)}\Big)\Big) &= E(H) \\ &= E(\mu T + \sigma\sqrt{\Delta t}Y) \\ &= \mu T + \sigma\sqrt{\Delta t}\sum_{k=1}^{n} E(X_k) \\ &= \mu T \qquad (\text{as } E(X_k) = 0,\ \forall\, k), \end{aligned}$$

and

$$\begin{aligned} Var\Big(ln\Big(\frac{S(T)}{S(0)}\Big)\Big) &= Var(H) \\ &= Var(\mu T + \sigma\sqrt{\Delta t}Y) \\ &= \sigma^2\Delta t\sum_{k=1}^{n} Var(X_k) \\ &= \sigma^2 T \qquad (\text{as } Var(X_k) = 1,\ \forall\, k). \end{aligned}$$

The above two relations made us ready to give interpretations of the two parameters - the drift and the volatility. Suppose we have n quotes of stock prices per year. A typical example is the daily closing stock prices for $T = 1$ year (or 252 trading days per year), so in this case $n = 252$ and $\Delta t = 1$ day. We consider the random stock prices, initially $S(0) = S_0$ and on subsequent days $S_1, S_2, \ldots, S_{251}$ and finally at the end of one year $S_{252} = S(T) = S(1)$. As an investor we are interested in the yearly return $\frac{S(1)}{S(0)}$, or its logarithm $ln\Big(\frac{S(1)}{S(0)}\Big)$. Note that the logarithmic returns of the daily observed stock prices are $\{ln(S_1) - ln(S_0), ln(S_2) - ln(S_1), \ldots, ln(S_{252}) - ln(S_{251})\}$. The sum of these daily returns yield $(ln(S_{252}) - ln(S_0))$, which is the logarithmic yearly return. This simple characteristic is not available in decimal returns $\{\frac{S_1 - S_0}{S_0}, \frac{S_2 - S_1}{S_1}, \ldots, \frac{S_n - S_{n-1}}{S_{n-1}}\}$. By using the logarithmic prices we can convert an exponential problem to a linear problem; the logarithmic returns can simply be added. For this simple reason the logarithmic yearly returns are often used. What we observe is that the drift μ is the expectation of the logarithmic return and the volatility σ is the standard deviation of the logarithmic return.

The volatility is the most critical of all option trading concepts. It provides investors with an estimate of how much movement a stock can be expected to make over a given time frame. Two types of volatilities are generally talked about - historical and implied. At present we shall not be talking about the implied volatility

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as it requires other conceptual knowledge which we have not yet acquired. (Please refer to the third point in the notes, Section 4.6).

Also, there are many sophisticated models in financial econometric for computing historical volatility, like ARCH (autoregressive conditional heteroskedasticity), GARCH (generalized ARCH), risk metrics EWMA (equally weighted moving average), threshold autoregressive model, and so on so forth. Again we are not in a position to elaborate on them. However, for the sake of understanding, we briefly explain a very basic model for computing the historical volatility. First calculate the natural log of the ratio of a stock price from the current day t to the previous day $t-1$, that is, $R_t = ln(S_t/S_{t-1})$. Then find the average of all these for the given period n (for example, $n = 20$ days), say, $\overline{R} = (\sum_{t=1}^{n} R_t)/n$. The historical volatility is the standard deviation of R_t, $(t = 1, \ldots, n)$, from the mean $\overline{R}$. If a stock price follows a Brownian motion or a geometric Brownian motion then the above result is multiplied by the square root of the average number of trading days in a year to quote the volatility on an annual basis. The reason for the last step lies in the fact that the volatility increases with the square root of the unit of time. Though we agree that one may find this discussion a bit incomplete, but the details have been avoided just to skip the complications involved with volatility. However, we encourage the readers to search the web and find sufficient material/examples on this concept or refer to an excellent text [138].

We shall next attempt to approximate the CRR model by a multi-period binomial model. It is important to point out here that not every CRR model is a multi-period binomial model.

4.3 Matching of CRR Model with a Multi-Period Binomial Model

The primary question is to find the values of u, d, and p for given values of μ and σ^2 from (4.1), (4.2), (4.3). It is worth to mention that, in the conclusive analysis, the value of μ will interestingly not be required. However, it is the actual value of σ which is rather hard to capture. Nevertheless, we assume that σ is known from some past data of the stock (i.e. say historical volatility).

Now, in lieu of (4.3) we have two relations (4.1) and (4.2) to determine three parameters u, d, p. In order to describe these parameters uniquely we need a third condition consistent with the above two relations. To meet this need, let us denote by $U = ln(u)$, $D = ln(d)$, and assume that $D = -U$. In other words, we shall be matching the CRR model with a particular multi-period binomial model

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in which $d = \frac{1}{u}$. Now, (4.1) and (4.2), on using (4.3), can be re-expressed as follows

$$\begin{aligned}(2p-1)U &= \mu\Delta t \\ 4p(1-p)U^2 &= \sigma^2\Delta t.\end{aligned} \tag{4.5}$$

Simplifying the above equations, we get

$$(2p-1)^2U^2 + 4p(1-p)U^2 = \mu^2(\Delta t)^2 + \sigma^2\Delta t,$$

that is,

$$U = \sqrt{\mu^2(\Delta t)^2 + \sigma^2\Delta t}, \text{ and } D = -U.$$

Furthermore, (4.5) yields

$$p = \frac{1}{2} + \frac{\mu\Delta t}{2\sqrt{\mu^2(\Delta t)^2 + \sigma^2\Delta t}}.$$

Let us take n sufficiently large (equivalently, the time steps between successive trading instances approach zero) so that $(\Delta t)^2$ terms can be neglected. We get

$$U = ln(u) = \sigma\sqrt{\Delta t} \quad \text{and} \quad D = -U = -\sigma\sqrt{\Delta t}.$$

Consequently, we have

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}} \quad \text{and} \quad p = \frac{1}{2} + \frac{\mu}{2\sigma}\sqrt{\Delta t}. \tag{4.6}$$

We take a small break here for some examples to illustrate the choice of the parameters.

Example 4.3.1 *A non-dividend paying stock is currently selling at Rs* 100 *with annual volatility* 20%. *Assume that the continuously compounded risk-free interest rate is* 5%. *Using a two-period CRR binomial option pricing model, find the price of one European call option on this stock with a strike price of Rs* 80 *and time to expiration 4 years.*

Solution We are given $S(0) = Rs\ 100$, $K = Rs\ 80$, $T = 4$, $\Delta t = 2$, $r = 0.05$, $\sigma = 0.2$. Applying the CRR model, the up-factor and the down-factor are respectively given by

$$u = e^{\sigma\sqrt{\Delta t}} = 1.3269, \quad d = \frac{1}{u} = 0.7536.$$

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Now,

$$S_{uu} = u^2 S(0) = 176.0664,\ S_{ud} = udS(0) = 100.00,\ S_{dd} = 56.7913.$$

Thus,

$$\begin{aligned} C_{uu} &= \text{Max}\{S_{uu} - K, 0\} = 96.0664, \\ C_{ud} &= \text{Max}\{S_{ud} - K, 0\} = 20.00, \\ C_{dd} &= \text{Max}\{S_{dd} - K, 0\} = 0. \end{aligned}$$

Since the annual rate of risk-free interest is continuously compounded (so instead of $1 + r$ or R we shall use $e^{r\Delta t}$), the risk neutral probability is

$$\hat{p} = \frac{e^{r\Delta t} - d}{u - d} = 0.6132.$$

Hence for a two period binomial model,

$$\begin{aligned} C(0) &= e^{-r\Delta t}((\hat{p})^2 C_{uu} + 2\hat{p}(1 - \hat{p})C_{ud} + (1 - \hat{p})^2 C_{dd}) \\ &= e^{-0.1}((0.3760)(96.0664) + (0.4744)(20.00) + 0) \\ &= \text{Rs } 41.27. \end{aligned}$$

□

For computational purpose, we take the annualized risk-free interest rate and volatility. Although we take 365 days for counting a year but the exchanges across the world remain closed and no trading take place on Saturday and Sunday. Moreover, there are some other holidays observed by exchanges. Consequently the number of trading days are less. In practice options are priced taking into account the number of trading days. The following example throw light on this aspect.

Example 4.3.2 *A non-dividend paying stock is selling at Rs* 1,500 *on March 1, 2010, with annual volatility of* 22%. *Assume that the continuously compounded risk-free interest rate is* 3%. *Compute the price of a European call option written on this stock struck at Rs* 1470 *expiring in April 2010 using a single-period CRR binomial model.*

Solution It is given that $S(0) = Rs\ 1,500$, $K = Rs\ 1,470$, $r = 0.03$, $\sigma = 0.22$. The option is written on March 1 and it expires in the business day immediately preceding the last business day of the contract month which is April 29, 2010. The total number of trading days are 44 days. The number of trading days in 2010 is 252. Thus, $T = 44/252 = 0.1746$. For a single-period model, $\Delta t = T = 0.1746$. Then, $u = e^{\sigma\sqrt{\Delta t}} = 1.0963$, $d = u^{-1} = 0.9124$. The risk neutral probability measure is $\hat{p} = \dfrac{e^{r\Delta t} - d}{u - d} = 0.505$. Thus,

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$$C(0) = e^{-rT}(\hat{p}C_u + (1-\hat{p})C_d) = (0.9948)((0.505)174.45 + (0.495)0) = Rs\ 87.64.$$

□

However, for text purpose we continue to take 365 days or 12 months in a year. This may lead to calculation differences between the theoretical value, when applied on a real data, and the actual financial quotation (available in newspapers or www) for the option. The readers are cautioned.

Continuing with our previous discussion, we next define a counter on up tick and down tick movements on stock price at time $k\Delta t$, $(k = 1, \ldots, n)$, as a Bernoulli random variable

$$Y_k = \begin{cases} 1, & \text{with probability } p \text{ if stock goes up} \\ 0, & \text{with probability } 1-p \text{ if stock goes down.} \end{cases}$$

Then,

$$\begin{aligned} S(n\Delta t) &= S(T) \\ &= S(0)u^{\sum_{k=1}^{n} Y_k} d^{(n-\sum_{k=1}^{n} Y_k)} \\ &= d^n S(0)\left(\frac{u}{d}\right)^{\sum_{k=1}^{n} Y_k}, \end{aligned}$$

giving

$$\frac{S(T)}{S(0)} = d^{\frac{T}{\Delta t}}\left(\frac{u}{d}\right)^{\sum_{k=1}^{T/\Delta t} Y_k}.$$

Observe that $Y = \sum_{k=1}^{T/\Delta t} Y_k$ is a simple random walk with $E(Y) = p(T/\Delta t)$ and $Var(Y) = p(1-p)(T/\Delta t)$.

Using the value of $ln(d)$ from (4.6), we have

$$ln\left(\frac{S(T)}{S(0)}\right) = \frac{-T\sigma}{\sqrt{\Delta t}} + 2\sigma\sqrt{\Delta t}\sum_{k=1}^{T/\Delta t} Y_k.$$

Hence,

$$\begin{aligned} E\left(ln\left(\frac{S(T)}{S(0)}\right)\right) &= \frac{-T\sigma}{\sqrt{\Delta t}} + 2\sigma\sqrt{\Delta t}\sum_{k=1}^{T/\Delta t} E(Y_k) \\ &= \frac{-T\sigma}{\sqrt{\Delta t}} + 2\sigma\sqrt{\Delta t}\frac{T}{\Delta t}p \\ &= \mu T, \end{aligned}$$

where the last equality follows on using the value of p from (4.6). Also,

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$$\begin{aligned} Var\Big(\ln\Big(\frac{S(T)}{S(0)}\Big)\Big) &= 4\sigma^2\Delta t\sum_{k=1}^{T/\Delta t} Var(Y_k) \\ &= 4\sigma^2 Tp(1-p) \\ &\to \sigma^2 T, \text{ as } p \to \frac{1}{2} \text{ when } n \to \infty. \end{aligned}$$

Furthermore, by application of the central limit theorem, we can assume that Y_k follows a normal distribution when time steps approach zero. Summarizing the above discussion we can conclude that

$$ln\Big(\frac{S(T)}{S(0)}\Big) \sim \mathcal{N}(\mu T, \sigma^2 T).$$

Moving ahead, observe that

$$u = e^{\sigma\sqrt{T/n}} \cong 1 + \sigma\sqrt{\frac{T}{n}} + \frac{\sigma^2 T}{2n} \tag{4.7}$$

$$d = e^{-\sigma\sqrt{T/n}} \cong 1 - \sigma\sqrt{\frac{T}{n}} + \frac{\sigma^2 T}{2n}. \tag{4.8}$$

So, the risk neutral probability measure (RNPM) is given by

$$\begin{aligned} \hat{p} &= \frac{R-d}{u-d} \\ &= \frac{1+\frac{rT}{n}-d}{u-d} \\ &\cong \frac{\frac{rT}{n}+\sigma\sqrt{\frac{T}{n}}-\frac{\sigma^2 T}{2n}}{2\sigma\sqrt{\frac{T}{n}}}. \end{aligned}$$

Thus,

$$\hat{p} = \frac{1}{2} + \frac{2r-\sigma^2}{4\sigma}\sqrt{\frac{T}{n}}. \tag{4.9}$$

We have seen that the European call or put options are simple to price. We therefore concentrate on pricing the European call option using the above described scheme. As established in previous chapter, the European call option price for an n-period binomial lattice model is described as follows.

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$$\begin{aligned} C(0) &= \left(1+\frac{rT}{n}\right)^{-n} E_{\hat{p}}((S(T)-K)_+) \\ &= \left(1+\frac{rT}{n}\right)^{-n} E_{\hat{p}}((S(0)u^Y d^{n-Y}-K)_+) \\ &= \left(1+\frac{rT}{n}\right)^{-n} E_{\hat{p}}((S(0)\left(\frac{u}{d}\right)^Y d^n-K)_+). \end{aligned}$$

It follows from (4.7) and (4.8), that

$$C(0) = \left(1+\frac{rT}{n}\right)^{-n} E_{\hat{p}}((S(0)e^w-K)_+), \quad \text{where } w = 2\sigma\sqrt{\frac{T}{n}}Y - \sigma\sqrt{nT}. \tag{4.10}$$

It is to be noted here that

$$\begin{aligned} E(w) &= 2\sigma\sqrt{\frac{T}{n}}E(Y) - \sigma\sqrt{nT} \\ &= 2\sigma\sqrt{\frac{T}{n}}np - \sigma\sqrt{nT} \\ &= (2p-1)\sigma\sqrt{nT} \\ &= \mu\sqrt{\Delta t}\sqrt{nT} \\ &= \mu T, \end{aligned}$$

where the second last equality follows by virtue of first relation in (4.5) and the value of $U = \sigma\sqrt{\Delta t}$. Also, using that $Var(Y) = p(1-p)(T/\Delta t)$ and $n\Delta t = T$, we have

$$\begin{aligned} Var(w) &= 4\sigma^2\frac{T}{n}Var(Y) \\ &= 4p(1-p)\sigma^2 T \\ &\to \sigma^2 T, \end{aligned}$$

where the last relation follows in the limiting case when $n \to \infty$ (or $\Delta t \to 0$), and $p \to \frac{1}{2}$ from (4.6). Moreover it is important to note that

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$$
\begin{aligned}
E_{\hat{p}}(w) &= 2\sigma\sqrt{\frac{T}{n}}E_{\hat{p}}(Y) - \sigma\sqrt{nT} \\
&= 2\sigma\sqrt{\frac{T}{n}}np^* - \sigma\sqrt{nT} \\
&= 2\sigma\sqrt{nT}\Big(\frac{1}{2} + \frac{2r-\sigma^2}{4\sigma}\sqrt{\frac{T}{n}}\Big) - \sigma\sqrt{nT} \qquad \text{(using (4.9))} \\
&= \Big(r - \frac{\sigma^2}{2}\Big)T
\end{aligned}
$$

and

$$
\begin{aligned}
Var(w)_{\hat{p}} &= 4\sigma^2\hat{p}(1-\hat{p})T \\
&\to \sigma^2 T,
\end{aligned}
$$

(using the limiting case when $n \to \infty$ and $p \to \frac{1}{2}$ in (4.9)).

As a consequence of above relations, (4.10) gives

$$
C(0) = e^{-rT}\int_{-\infty}^{\infty}(S(0)e^w - K)_+\frac{1}{\sigma\sqrt{2\pi T}}\, e^{-\frac{1}{2}\left(\frac{w-(r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)^2} dw.
$$

Now $S(0)e^w > K$, implies $w > ln\Big(\frac{K}{S(0)}\Big) = w_1$. Therefore,

$$
C(0) = e^{-rT}\int_{w>w_1}(S(0)e^w - K)\frac{1}{\sigma\sqrt{2\pi T}}\, e^{-\frac{1}{2}\left(\frac{w-(r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)^2} dw.
$$

To evaluate this integral, substitute

$$
y = \frac{w - \Big(r - \frac{\sigma^2}{2}\Big)T}{\sigma\sqrt{T}}.
$$

Then,

$$
w = y\sigma\sqrt{T} + \Big(r - \frac{\sigma^2}{2}\Big)T, \quad \text{so } dw = \sigma\sqrt{T}\,dy.
$$

Moreover, $w > w_1$ gives $y > y_1$, where

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$$y_1 = \frac{1}{\sigma\sqrt{T}}\Big(ln\Big(\frac{K}{S(0)}\Big) - \Big(r - \frac{\sigma^2}{2}\Big)T\Big).$$

Subsequently,

$$\begin{aligned} C(0) &= \frac{e^{-rT}}{\sqrt{2\pi}}\int_{y>y_1}(S(0)e^{y\sigma\sqrt{T}+(r-\frac{\sigma^2}{2})T} - K)e^{-\frac{y^2}{2}}\,dy \\ &= \frac{e^{-rT}}{\sqrt{2\pi}}\int_{y>y_1} S(0)e^{y\sigma\sqrt{T}+(r-\frac{\sigma^2}{2})T}e^{-\frac{y^2}{2}}\,dy - \frac{1}{\sqrt{2\pi}}Ke^{-rT}\int_{y_1}^{\infty} e^{-\frac{y^2}{2}}dy \\ &\equiv I - Ke^{-rT}\Phi(-y_1), \end{aligned} \tag{4.11}$$

where, $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{z^2}{2}}dz$ is the distribution function of a standard normal random variable and

$$\begin{aligned} I &= \frac{e^{-rT}}{\sqrt{2\pi}}\int_{y>y_1} S(0)e^{y\sigma\sqrt{T}+(r-\frac{\sigma^2}{2})T}e^{-\frac{y^2}{2}}\,dy \\ &= \frac{S(0)}{\sqrt{2\pi}}\int_{y>y_1} e^{-\frac{(y\ -\ \sigma\sqrt{T})^2}{2}}\,dy. \end{aligned}$$

To evaluate I we define a new variable $s = y - \sigma\sqrt{T}$. Then for $y > y_1$, we have $s > s_1 = y_1 - \sigma\sqrt{T}$. Thus,

$$\begin{aligned} I &= \frac{S(0)}{\sqrt{2\pi}}\int_{s>s_1} e^{-\frac{s^2}{2}}\,ds \\ &= S(0)\Phi(-s_1) \\ &= S(0)\Phi\Big(\sigma\sqrt{T} - \frac{1}{\sigma\sqrt{T}}\Big(ln\Big(\frac{K}{S(0)}\Big) - \Big(r - \frac{\sigma^2}{2}\Big)T\Big)\Big) \\ &= S(0)\Phi(d_1), \end{aligned}$$

where

$$d_1 = \frac{ln\Big(\frac{S(0)}{K}\Big) + \Big(r + \frac{\sigma^2}{2}\Big)T}{\sigma\sqrt{T}}. \tag{4.12}$$

Summarizing the discussion, from (4.11), we have

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$$C(0) = S(0)\Phi(d_1) - Ke^{-rT}\Phi(d_2), \tag{4.13}$$

where $d_2 = -y_1 = d_1 - \sigma\sqrt{T}$ and d_1 is as described in (4.12). The expression in (4.13) is known as the *Black-Scholes formula* for computing the European call option price under the set of assumptions described at the beginning of Section 4.2.

Observe that the expected rate of return μ on the stock does not appear in the Black-Scholes formula. Moreover all parameters except for σ, the volatility of the underlying stock, can be observed in the market. Regarding σ, we assume to estimate it from historical market data (historical volatility) even though it may not present the true picture. We shall briefly be touching the concept of *implied volatility* at the end of this chapter.

The next example underlines the strength of the Black-Scholes formula.

Example 4.3.3 *Suppose a non-dividend paying stock is currently selling at Rs* 100 *and the stock's volatility is* 24%. *Assume that the continuously compounded risk-free interest rate is* 5%. *A European call option is offered on this stock with time to maturity 3 months and strike price Rs* 125. *Calculate the price of the block of 100 options in the Black-Scholes framework.*

Solution Here, $S(0) = Rs\ 100$, $K = Rs\ 125$, $T = 3/12 = 0.25$, $r = 0.05$, $\sigma = 0.24$.

$$d_1 = \frac{ln(100/125) + (0.05 + \frac{1}{2}(0.24)^2)\,0.25}{0.24\sqrt{0.25}} = -1.695363$$

and

$$d_2 = d_1 - \sigma\sqrt{T} = -1.815363.$$

Because d_1 and d_2 are negative, we use, $\Phi(d_1) = 1-\Phi(-d_1)$ and $\Phi(d_2) = 1-\Phi(-d_2)$. We round off d_1 to 1.70 and d_2 to 1.82 before looking up the standard normal distribution table. Thus,

$$\Phi(d_1) = 1 - 0.9554 = 0.0446, \quad \Phi(d_2) = 1 - 0.9656 = 0.0344.$$

Invoking (4.13), we get

$$C(0) = 100(0.0446) - 125e^{(-0.05)(0.25)}(0.0344) = 0.2134.$$

Therefore, the cost of the block of 100 options is Rs 21.34.

□

In case we wish to compute the European call option price at any time t, $0 \leq t < T$, then we simply need to make a small change in the Black-Scholes formula as follows.

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$$C(t) = S(t)\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)$$

$$d_1 = \frac{ln\left(\frac{S(t)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t},$$

where $S(t)$ is the spot price of the stock at time t and $T-t$ is the remaining time to maturity.

The price of the European put option with the strike price K and time to maturity $T-t$, $0 \leq t < T$, can either be calculated using the put-call parity or directly by the following Black-Scholes formula.

$$P(t) = Ke^{-r(T-t)}\Phi(-d_2) - S(t)\Phi(-d_1)$$

$$d_1 = \frac{ln\left(\frac{S(t)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

Obviously if we enter into the European put option on the day it is offered then the put price $P(0)$ can be obtained by taking $t = 0$ in the above formula. Here, we encouraged the readers to derive the Black-Scholes formula for pricing European put option using the analogous arguments as done above in the derivation of $C(0)$.

So far, we have discussed a European call or a European put options cases. It is well known that, for a non-dividend paying stock, the value of an American call option is same as that of a European call option with the same strike price and time to expiration. Hence (4.13) can be used to compute price of an American call option on a non-dividend paying underlying stock. Notably, the Black-Scholes formula for an American put option is not the same as for European put option as it may pay to exercise them early. An American put price has to be approximated using the binomial method explained in previous chapter wherein we can simulate sufficiently large binomial lattice by taking sufficiently small time steps for more accurate approximation.

4.4 Black-Scholes Formula for Dividend Paying Stock

One of the assumptions in the Black-Scholes basic model is that the underlying stock pays no dividend. We can easily relax this by adjusting the initial stock

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price by the present value of the dividend. To make us understand this, let us first explore how the option gets affected by dividends on the underlying stock.

Dividends are a way by which the companies distribute part of their profits to their investors. The question of who should be paid dividends becomes complex as the composition of shareholders changes each day. To settle this question, companies designate a date known as the record date. Dividends are paid to the list of shareholders who hold stock on the record date. In order to allow time for settling stock purchases, stock exchanges set a date generally two business days prior to the record date known as the ex-dividend date. Someone who purchases the stock on or after the ex-dividend date is not eligible to receive dividends.

Now suppose we own a call option on a dividend paying stock, but we don't own any of the actual underlying stock. Then the bad news is that we are not entitled to get any dividends because dividends are paid only to the actual shareholders. We can only receive the dividend if we own the stock before the ex-dividend date. All what we own is the right to become a shareholder and do so only when, for instance, we exercise the call option. But then all is not lost. The good news is that not getting the dividends while holding only the option is compensated on the price of option. This is due to the fact that on the ex-dividend date the underlying stock price falls by roughly the amount of the dividend. This drop causes call option price becoming cheaper relative to the amount of the dividend and put option price becoming slightly more expensive. In other words, with dividends, the premium we have to pay for call option gets reduced while the put option premium increases.

The Black-Scholes option pricing formula has to be adjusted when the underlying stock pays dividends. For adjusting the price of European options, for known discrete dividends, we merely have to subtract the present value of the dividend from the current price of the underlying asset in calculating the Black-Scholes value. That is, to take into account the drop in stock price on ex-dividend date, instead of $S(0)$ work with $S(0) - div\, e^{-r t_{div}}$ in the Black-Scholes formula, where div denotes the dividend amount in proportions to stock, and t_{div} denotes the dividend time. In case if dividends $div_1, \ldots, div_p$ are paid at discrete times $t_{div_1}, \ldots, t_{div_p}$ then take $S(0) - \sum_{j=1}^{p} div_j e^{-r t_{div_j}}$ instead of $S(0)$ in the Black-Scholes formula (4.13) and (4.12). A word of caution: be careful to see that $S(0) > \sum_{j=1}^{p} div_j e^{-r t_{div_j}}$ for we have to take its logarithm while computing d_1 in the Black-Scholes formula.

The following examples help us to appreciate the discussed scenario.

Example 4.4.1 *A stock currently trades for Rs* 100 *per share. The annual continuously compounded risk-free interest rate is* 5% *and the annual price volatility*

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relevant for the Black-Scholes formula is 30%. Call options are written on this stock with a strike price of Rs 80 and time to expiration of 5 years. The stock will pay a dividend of Rs 20 in 2 years and another dividend of Rs 30 in 3 years. Use the Black-Scholes formula to find the price of one such call option.

Solution Here. $S(0) = Rs\ 100$, $K = Rs\ 80$, $T = 5$, $r = 0.05$, $\sigma = 0.3$, $div = Rs\ 20$ for first 2 years and $Rs\ 30$ for next 3 years. We first compute the adjusted stock price by decreasing the current stock price by the present value of the dividend. The adjusted stock price is denoted by say S_a.

$$S_a = S(0) - (div)e^{-rt_{div}} = 100 - 20e^{-(0.05)2} - 30e^{-(0.05)3} = 56.082.$$

Now we use the formula

$$d_1 = \frac{ln(S_a/K) + (r + 0.5\sigma^2)T}{\sigma\sqrt{T}}$$

$$= \frac{ln(56.08/80) + (0.05 + 0.5(0.3)^2)5}{0.3\sqrt{5}} = 0.1786.$$

$$d_2 = d_1 - \sigma\sqrt{T} = -0.4922.$$

$$\Phi(d_1) = \Phi(0.18) = 0.5714, \quad \Phi(d_2) = 1 - \Phi(-d_2) = 1 - 0.6879 = 0.3121.$$

$$C(0) = S_a\Phi(d_1) - Ke^{-rT}\Phi(d_2).$$

Thus,

$$C(0) = (56.082)(0.5714) - (62.304)(0.3121) = \text{Rs } 12.60.$$

□

Remark 4.4.1 *Suppose in the above example the underlying stock pays no dividend in the first two years but thereafter pays a dividend of Rs 30 in 3 years. Then,*

$$S_a = 74.1787,\ d_1 = 0.5955,\ d_2 = -0.0753.$$

$$\Phi(d_1) = 0.7123,\ \Phi(d_2) = 1 - \Phi(-d_2) = 0.4721 \text{ and } C(0) = Rs\ 23.42.$$

Now assume the stock pays no dividend in its entire lifespan of 5 years, that is, $div = 0$. Then

$$d_1 = \frac{ln(100/80) + (0.05 + 0.5(0.3)^2)5}{0.3\sqrt{5}} = 1.0407, \quad d_2 = 0.3699.$$

$$\Phi(d_1) = 0.8508,\ \Phi(d_2) = 0.6443,$$

and hence $C(0) = Rs\ 44.94$.

Thus, the dividend paid by the underlying stock gets reflected in price reduction of the European call option.

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Remark 4.4.2 *Working with the same data as in Example 4.4.1, we assume that a European put has been offered on the stock. Although we can directly compute the put price using the Black-Scholes formula for European put option, we use the put-call parity with dividends to do so.*

(i) In case the stock pays dividends exactly in the same manner as described in the Example 4.4.1 then

$$P(0) = C(0) - (S(0) - (div)e^{-rt_{div}} - Ke^{-rT}) = Rs\ 18.82.$$

(ii) Consider the other case when the stock pays no dividends in the first 2 years and pays a dividend of Rs 30 in the next 3 years. Then

$$P(0) = C(0) - (S(0) - (div)e^{-rt_{div}} - Ke^{-rT}) = Rs\ 11.55.$$

(iii) If stock pays no dividend in 5 years then $P(0) = C(0) - (S(0) - Ke^{-rT}) = Rs\ 7.24$.

This clearly illustrates that the European put option become more costly on a dividend paying underlying stock.

It is worth to note that if the underlying stock pays continuous dividend at a rate δ_{div}, then the price of the European call option on such a stock is computed as follows.

$$C(0) = S(0)e^{-\delta_{div}T}\Phi(d_1) - Ke^{-rT}\Phi(d_2),$$

$$d_1 = \frac{ln\left(\frac{S(0)e^{-\delta_{div}T}}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}},$$

$$d_2 = d_1 - \sigma\sqrt{T}.$$

Similarly the price of a European put option is given by

$$P(0) = Ke^{-rT}\Phi(-d_2) - S(0)e^{-\delta_{div}T}\Phi(-d_1),$$

where d_1 and d_2 are described immediately above.

Example 4.4.2 *A stock currently trades for Rs 100 per share. The annual continuously compounded risk-free interest rate is 5% and the annual price volatility relevant for the Black-Scholes formula is 20%. European call option is written on this stock with a strike price of Rs 80 and time to expiration of 3 months. The stock pays dividend continuously at the rate of 2%. Use the Black-Scholes formula to find the price of one such call option.*

Solution The data is $S(0) = Rs\ 100$, $K = Rs\ 80$, $T = 3/12 = 0.25$, $r = 0.05$, $\sigma = 0.2$, and $\delta_{div} = 0.02$.

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$$d_1 = \frac{ln(S(0)e^{-\delta_{div}T}/K) + (r + 0.5\sigma^2)T}{\sigma\sqrt{T}} = 2.3564, \quad d_2 = 2.2564.$$

$$\Phi(d_1) = 0.9909, \ \Phi(d_2) = 0.9881.$$

$$C(0) = 100e^{-(0.02)0.25}(0.9909) - 80e^{-(0.05)0.25}(0.9881) = Rs\ 20.53.$$

□

4.5 The Greeks

Let us shift our attention to another noteworthy aspect of the Black-Scholes formula that we had already worked out for the equilibrium price of the European call option on a non-dividend paying stock. Recall that

$$C(t) = S(t)\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2), \tag{4.14}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}, \ \ d_1 = \frac{ln(S(t)/K) + (r + (\sigma^2/2))(T-t)}{\sigma\sqrt{T-t}}. \tag{4.15}$$

Note that, besides the two constants - time to maturity (T) and the strike price (K), quite a few parameters are involved in the Black-Scholes formula, viz., the price of the underlying asset $S(t)$; time to price t; the volatility of the underlying asset σ; and the interest rate r. Even a minor change in one or more of them will affect the price of the call. It is therefore essential to understand the effect of changes in these parameters on the call price.

In calculus, the notion of "derivative" is a pivotal tool used to measure the rate of change of a dependent variable as an independent variable is changed. A similar situation is prevalent in the aforementioned context. Here we will calculate the partial derivatives of option value formula (4.14), and they in turn will allow us to determine how sensitive the value of option is to the changes in the parameters. In this section we shall be learning about *the delta*, *the gamma*, *the theta*, *the vega*, and *the rho*. These famous fives are collectively known as *the Greeks*.

Let us first begin with some simple derivatives. The cumulative distribution function is

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}dy.$$

By the fundamental theorem of calculus, the derivative of Φ with respect to x is

$$\Phi'(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}. \tag{4.16}$$

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Also, a basic calculus working in (4.15) yield the following.

$$\frac{\partial d_1}{\partial t} = \frac{1}{2\sigma\sqrt{T-t}}\left(\frac{ln(S/K)}{T-t} - r - \frac{\sigma^2}{2}\right). \tag{4.17}$$

$$\frac{\partial d_1}{\partial S} = \frac{1}{S\sigma\sqrt{T-t}}. \tag{4.18}$$

$$\frac{\partial d_1}{\partial r} = \frac{\sqrt{T-t}}{\sigma}. \tag{4.19}$$

$$\frac{\partial d_1}{\partial \sigma} = \sqrt{T-t} - \frac{d_1}{\sigma}. \tag{4.20}$$

We encourage the readers to compute the above expressions.

Since time is the only deterministic independent variable we are certain, we first introduce the Greek related to time; also called the *Theta* denoted by Θ. The theta of the European call option is defined as $\frac{\partial C}{\partial t}$. From (4.14), we have

$$\Theta = \frac{\partial C}{\partial t} = S\Phi'(d_1)\frac{\partial d_1}{\partial t} - rKe^{-r(T-t)}\Phi(d_2) - Ke^{-r(T-t)}\Phi'(d_2)\left(\frac{\partial d_1}{\partial t} + \frac{\sigma}{2\sqrt{T-t}}\right).$$

We can plug in the values of the partial derivatives from (4.16) and (4.17) to get the final expression for Θ. Moreover, using the put-call parity, we can easily compute the Θ of the non-dividend paying European put option.

Perhaps the most significant Greek is *the delta* denoted by Δ. For the European call option, it is defined to be the partial derivative of call price C with respect to the price of the stock (underlying derivative security). From (4.14), and noting that $\partial d_2/\partial S = \partial d_1/\partial S$, we have

$$\Delta = \frac{\partial C}{\partial S} = \Phi(d_1) + (S\Phi'(d_1) - Ke^{-r(T-t)}\Phi'(d_2))\frac{\partial d_1}{\partial S}. \tag{4.21}$$

Now concentrate on the second term in the expression (4.21), we will try to simplify it. Using (4.16) and (4.18), we obtain

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$$
\begin{aligned}
(S\Phi'(d_1) - Ke^{-r(T-t)}\Phi'(d_2))\frac{\partial d_1}{\partial S} &= \frac{1}{S\sigma\sqrt{2\pi(T-t)}}(Se^{-d_1^2/2} - Ke^{-r(T-t)-d_2^2/2}) \\
&= \frac{1}{S\sigma\sqrt{2\pi(T-t)}}(Se^{-d_1^2/2} - Ke^{-r(T-t)-\frac{(d_1-\sigma\sqrt{T-t})^2}{2}}) \\
&= \frac{e^{-d_1^2/2}}{S\sigma\sqrt{2\pi(T-t)}}(S - Ke^{-(r+\sigma^2/2)(T-t)+\sigma d_1\sqrt{T-t}}) \\
&= \frac{e^{-d_1^2/2}}{S\sigma\sqrt{2\pi(T-t)}}(S - Ke^{ln(S/K)}) \\
&= 0.
\end{aligned}
$$

Therefore, from (4.21) we get $\Delta = \Phi(d_1)$ for the non-dividend paying European call. Notice that $\Delta > 0$, and it is less than 1. Using the put-call parity we can easily work out that for a non-dividend paying European put option $\Delta = \Phi(d_1) - 1$.

Example 4.5.1 *The current price of a stock is Rs 100 and its volatility is 30% per year. The risk-free interest rate is 4% per year. A portfolio is constructed consisting of one 6-month European call option with a strike price of Rs 80 and the cash obtained from shorting Δ shares of the stock. The portfolio value is non-random. What is Δ?*

Solution By assumption that the portfolio value is non-random, we have, $C - \Delta S = 0$. Thus

$$\frac{\partial C}{\partial S} = \Delta = \Phi(d_1),$$

where

$$d_1 = \frac{ln(100/80) + (0.04 + (0.3)^2/2)(0.5 - 0)}{0.3\sqrt{0.5 - 0}} = 1.2522.$$

Consequently, $\Delta = 0.894752$.

□

Before we proceed in describing the other Greeks, we would like to get more comfortable as to what roles these Greeks play in option theory. We present a situation below in this context.

A company has sold a 3-month European call option on 1,000 shares of a non-dividend paying stock at a strike price of *Rs* 100. The share is currently trading at *Rs* 95 per unit, the interest free rate of return is 5% per annum, and the market volatility is estimated at 20% per annum. Using (4.14) and (4.15) we can easily compute the value of 1,000 call options $C(0) = Rs\ 2269.30$. The company is a writer of call, so it is exposed to a risk on the call options which is the difference

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between the value of the call premium (for 1000 calls) invested at a risk free rate of interest for 3 months and the value of the call at the exercise time. Thus the risk profile of the company is $C(0)e^{rT} - N\text{Max}\{S(T) - K, 0\} = Rs\ (2297.84 - 1000\text{Max}\{S(\frac{1}{4}) - 100, 0\})$. *A natural question is that how does the company hedge this risk?* If it decides not to take any action (called the naked position) then the company is exposed to lose if the stock price rises from present Rs 95. See, if $S(\frac{1}{4}) = Rs\ 103$, then the company loses $Rs\ 2,702.16$ (the amount may appears to be meagre but the numeric data in the example can be changed to make this figure large). On the other hand, if the company decided to buy say $1,000$ shares of the underlying stock at $t = 0$ (called the covered position), then it is exposed to lose if stock decreases. See, if $S(\frac{1}{4}) = Rs\ 90$, the company will lose $Rs\ 2,702.16$. Both positions expose the company to risk! One can easily think of designing some other strategy, like wait and buy $1,000$ shares as soon as price of the stock reaches Rs 100 and sell $1,000$ shares as soon as the price of the stock declines below Rs 100. But simple hedging strategy sometime does not work well.

What we are looking for is a more concrete and workable strategy to hedge the risk positions in options created due to changes in the stock price. It is here when the delta, Δ, becomes a handy tool. In the immediate aforementioned situation, the company has $S(0) = Rs\ 95$, $C(0) = Rs\ 2,269.30$, and $\Delta = \Phi(d_1) = \Phi(-0.338) = 0.3685$. By linear approximation from the classical calculus, we have, $dC = \Delta\, dS$. So, if stock price goes up by Rs 1 (i.e. it becomes Rs 96) then the call price increases approximately by Rs 0.3685 or Rs368.50 for $1,000$ call options. Now, since the company is the writer of $1,000$ call, it must own 368.50 shares of stock so that a change of one unit in the stock price is offset by the change in the short call.

Generally speaking, a writer (short position) of any derivative security can take a hedge position against risk in the security whose price depends on the underlying stock. An important point is that the writer should frequently re-balance the hedge position by constantly monitoring the underlying stock price movements. A portfolio embedding the derivative security can be constructed such that it becomes delta-neutral where by the *delta-neutral portfolio* we mean the portfolio having delta equal to zero. Traders usually ensure that their portfolios are delta-neutral at least once a day. A trading strategy that dynamically maintains a delta-neutral portfolio is called *the delta hedging* which is based on the simple trading rule of "buy high, sell low".

Continuing with our discussion on Greeks, another important Greek is *the gamma* denoted by Γ. The gamma for the non-dividend paying European call

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option is the second order partial derivative of C with respect to S, thus, $\Gamma = \frac{\partial^2 C}{\partial S^2}$. Obviously,

$$\Gamma = \frac{\partial^2 C}{\partial S^2} = \Phi'(d_1)\frac{\partial d_1}{\partial S} = \frac{e^{-d_1^2/2}}{S\sigma\sqrt{2\pi(T-t)}}.$$

Note that since $\Gamma > 0$ for the European call option, thus C is a convex function of S, keeping other parameters constants. Again, using the put-call parity one can work out the gamma of the European put option. We left this for the readers to complete.

Just like delta hedging, there is another hedging strategy called the *delta-gamma hedging*. A delta-gamma hedge is a delta hedge that maintains zero portfolio gamma. The additional gamma neutrality is maintained by constructing a portfolio comprising of a short position in the derivative security, a long position in the underlying stock, and a long position in another hedging call such that the delta and the gamma of the portfolio become zero. Note that the delta hedge is based on the first-order approximation; and when change in S is not small enough, the second-order approximation (or gamma) helps. For this reason, a *delta-gamma neutral portfolio* generally offers better protection against the stock price changes than a simple delta neutral portfolio.

Also, we shall see later in Chapter 10 that the Black-Scholes PDE is

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2(S(t))^2\frac{\partial^2 f}{\partial S^2} + rS(t)\frac{\partial f}{\partial S} = rf.$$

Since $\frac{\partial f}{\partial t} = \Theta$, $\frac{\partial f}{\partial S} = \Delta$, and $\frac{\partial^2 f}{\partial S^2} = \Gamma$, the Black-Scholes PDE can be thought of as

$$\Theta + rS(t)\Delta + \frac{1}{2}\sigma^2 S^2\Gamma = rf. \tag{4.22}$$

Here, f in (4.22) can be price of any derivative security which is governed by the Black-Scholes formulas (4.14) and (4.15). Also, (4.22) provides a relationship between the three Greeks Θ, Δ and Γ. The readers can skip this discussion for the time being and can come back to it after going through Chapter 10. It will provide better understanding then. But we have included this text here for the sake of completeness.

The fourth Greek that we would like to get familiar with is called *the vega* which is the partial derivative with respect to volatility σ. It is denoted by $\mathcal{V}$. For the European call option it is $\frac{\partial C}{\partial \sigma}$. From (4.14), we have

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$$\mathcal{V} = \frac{\partial C}{\partial \sigma} = S\Phi'(d_1)\frac{\partial d_1}{\partial \sigma} - Ke^{-r(T-t)}\Phi'(d_2)(\frac{\partial d_1}{\partial \sigma} - \sqrt{T-t}).$$

Using (4.20), we get

$$\mathcal{V} = \frac{\sqrt{T-t}}{\sqrt{2\pi}}Se^{-d_1^2/2} + \frac{d_1}{\sigma}(Ke^{-r(T-t)}\Phi'(d_2) - S\Phi'(d_1))$$

$$= \frac{\sqrt{T-t}}{\sqrt{2\pi}}Se^{-d_1^2/2}.$$

The latter equation follows on account of $Ke^{-r(T-t)}\Phi'(d_2) - S\Phi'(d_1) = 0$. For detail working of this point, we urge the readers to take a look back at the analogous instance while computing $\Delta = \Phi(d_1)$ for the European call option. It is immediate from the put-call parity that the vega of the European put option is identical to that of the European call option.

Example 4.5.2 *Consider a 3-month European put option on a stock whose current value is Rs 100 and whose volatility is 30% per annum. The option has a strike price of Rs 95 and the risk-free interest rate is 3.25% per annum. Find the vega of the option. If the volatility of the stock increases to 31%, approximate the change in the value of the put.*

Solution We provide the hints to solve; the gaps can be filled by the readers. First compute d_1 using (4.15), thereafter compute $\mathcal{V} = \frac{\sqrt{T-t}}{\sqrt{2\pi}}Se^{-d_1^2/2}$ for the put option. Use the linear approximation $dP = \mathcal{V}d\sigma$, to find approximate change in the value of the put.

□

If volatility changes then the delta-gamma hedge may not work well. An enhancement is *the delta-gamma-vega hedge*, which also maintains vega zero portfolio. This is accomplished by bringing in one more security in the portfolio.

The final Greek to be introduced herein is *the rho*. It is denoted by ρ and is the partial derivative with respect to the risk-free interest rate r. For the European call option it is described as follows.

$$\rho = \frac{\partial C}{\partial r} = S\Phi'(d_1)\frac{\partial d_1}{\partial r} + K(T-t)e^{-r(T-t)}\Phi(d_2) - Ke^{-r(T-t)}\Phi'(d_2)\frac{\partial d_2}{\partial r}$$

$$= K(T-t)e^{-r(T-t)}\Phi(d_2).$$

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Again the last equation follows on similar argument as the one given above along with (4.19) and the fact that $\frac{\partial d_2}{\partial r} = \frac{\partial d_1}{\partial r}$. We encourage you to find the ρ of the European put.

Example 4.5.3 *Consider a 3-month European put option on a stock whose current value is Rs 100 and whose volatility is 30% per annum. The option has a strike price of Rs 90 and the risk-free interest rate is 3.25% per annum. Find the rho of the option. If the interest rate increases to 4%, then approximate the change in the value of the put option.*

Solution Given that $K = \text{Rs } 90$, $S(0) = \text{Rs } 100$, $r = 0.0325$, $\sigma = 0.3$, $T = \frac{3}{12} = 0.25$, $t = 0$. We first compute d_1 as follows

$$d_1 = \frac{ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}} = 0.83157.$$

The rho of the European put is given by

$$\begin{aligned}\rho &= -K(T - t)e^{-r(T-t)}\Phi(\sigma\sqrt{T - t} - d_1)\\ &= (-22.13733)\Phi(-0.68157) = -5.48463.\end{aligned}$$

Using the linear approximation from the classical calculus, the change in value of put option is given by

$$dP = \rho dr = (-5.48463)(0.0075) = 0.041134.$$

□

4.6 Summary and Additional Notes

- American options are much harder to deal with than European ones. Except for American call option on non-dividend paying underlying stock, there is no explicit formula as Black-Scholes for American put options (on dividend or non-dividend paying stock) or American call options on dividend paying stock. There is only one known analytical formula for American call options due to Roll-Geske-Whaley [65] on a stock that pays a known dividend. Also refer to http://www.bus.lsu.edu/academics/finance/ faculty/dchance/Instructional/TN98-01.pdf. Otherwise, the CRR binomial model can be applied to compute prices of such options. This has already been explained in previous chapter. For more accuracy in pricing these options, one

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