2 sep - test

3 sep - no class

Lecture 13

(7th Sep)

① { THEOREM }

Every non-trivial ($n \geq 2$) tree has atleast two vertices of degree one. (leaves)

Proof :- Let T be a tree on n vertices ($n \geq 2$).

Assume, on the contrary, that T has less than two vertices of degree one

(I) T has exactly one vertex of degree one, say v_1 and

$$d(v_i) \geq 2 \quad \forall \; i \in \{2, \ldots, n\}$$

Also, since # of edges $= n-1$

By handshaking lemma,

$$\sum_{v \in V(G)} d(v) = 2(n-1) = 2n-2$$

Now $\displaystyle\sum_{v \in V(G)} d(v) = d(v_1) + d(v_2) + \dots + d(v_n)$

$$= 1 + d(v_2) + \dots + d(v_n)$$

$$\Rightarrow \quad 2n-2 \geq 1 + \underbrace{2 + 2 + \dots + 2}_{(n-1) \text{ times}} = 2n-1$$

which is a contradiction.

(II) T has no vertices of degree one

i.e

$$d(v_i) \geq 2 \quad \forall \; i \in \{1, \dots n\}$$

Then $\displaystyle\sum_{v \in V(G)} d(v) = 2(n-1) = 2n-2$

$$\geq 2n$$

which is a contradiction.

Hence, T must have atleast two vertices of degree one.

$\blacksquare$

② **THEOREM**

A seq $s: d_1, \ldots, d_n$ of $n \geq 2$ positive integers is the deg sequence of a tree on n-vertices iff

$$\sum_{i=1}^{n} d_i = 2n-2$$

Proof

$\Rightarrow$ let T be a tree of order n with degree sequence $d_1, \ldots, d_n$

Then $$\sum_{i=1}^{n} d(v_i) = 2(\#\text{ of edges})$$

$$= 2(n-1) = 2n-2$$

($\Leftarrow$)

Conversely, let $s: d_1, \ldots, d_n$ $(n \geq 2)$ be a sequence such that

$$\sum_{i=1}^{n} d_i = 2n-2$$

Then we claim that $\exists$ a tree with this degree sequence.

We prove this by induction.

Basis :- $n=2$ so $d_1 + d_2 = 2 \times 2 - 2 = 2$

T_2 :

Inductive step :- Suppose that the statement is true for $n=k$ for some $k \geq 2$.

Goal :- Show that the result is true for $n = k+1$

Let $s : d_1, \ldots, d_{k+1}$ be a sequence of $(k+1)$ positive integers such that

$$\sum_{i=1}^{n} d_i = 2(k+1) - 2 = 2k.$$

WLOG, we can assume, that $d_i \geq d_{i+1}$ $\forall$ $i \in \{1, 2, \ldots, k\}$

We have $k+1$ positive integers which sum to $2k$. So it follows that

→ Not all of these integers are equal to 1 because if all of them were equal to 1 then $\sum d_i = k+1$.

and for that to be equal to $2k$, k must be equal to 1, which is a contradiction as $k \geq 2$.

→ Not all of these integers can be ≥ 2 (why?)

Because then $\sum_{i=1}^{k+1} d_i \geq 2(k+1) = 2k+2$

which means $2k \geq 2k+2$ or $0 \geq 2$ a contradiction. Thus, some d_i's are less than 2.

→ At least 2 of the d_i's are equal to 1.

i.e $d_k = d_{k+1} = 1$.

Now consider the sequence

$$S_1 : d_1 - 1, d_2, \ldots, d_k$$

This is a sequence of positive integers (why?) which sum up to $2k-2$.

$\downarrow$

$\left(\because \; d_2 \geq d_3 \cdots \geq d_k = 1 \text{ and } d_1 \geq 2 \text{ so } d_1 - 1 \geq 1 \right)$

Now, $d_1 - 1 + d_2 + \dots + d_k$

$$= d_1 + \dots + d_{k+1} - 1 - d_{k+1}$$

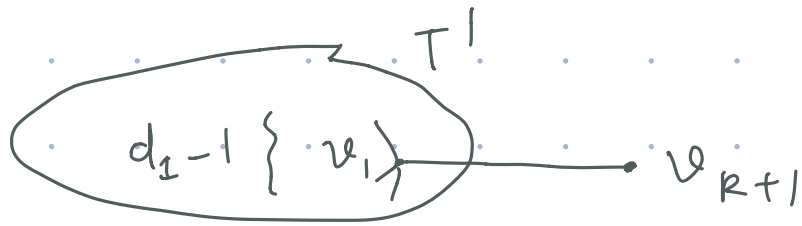
$$= 2k - 1 - 1 = 2k - 2$$

Now the inductive hypothesis says that $\exists$ a tree T' of order k with

$$V(T') = \{v_1, \dots, v_k\} \text{ such that}$$

$$\deg(v_1) = d_1 - 1$$
$$\deg(v_2) = d_2$$
$$\vdots$$
$$\deg(v_k) = d_k$$

Let T be a tree obtained from T' by adding a new vertex v_{k+1} and the edge $v_1 v_{k+1}$



So

$$\deg_T(v_1) = d_1$$

$$\deg_T(v_i) = d_i \text{ for } i = 2, \ldots k$$

$$\deg_T(v_{i+1}) = 1 = d_{k+1}$$

$\therefore$ $d_1, \ldots, d_{k+1}$ is a degree sequence of a tree T.

$\blacksquare$

Spanning Tree

① { DEFINITION }

Def:-

A spanning tree of a graph G is a spanning subgraph of G that is a tree

Observation:- A graph G can only have a spanning tree if G is connected.

② { NOTE }

Let G be a graph of order n and size m i,e

$$|V(G)| = n$$

and $$|E(G)| = m.$$

and let say G has a spanning tree

T, then $V(T) = V(G)$ ($\because T$ is sp..)

So $|V(T)| = n$

and since T is a tree

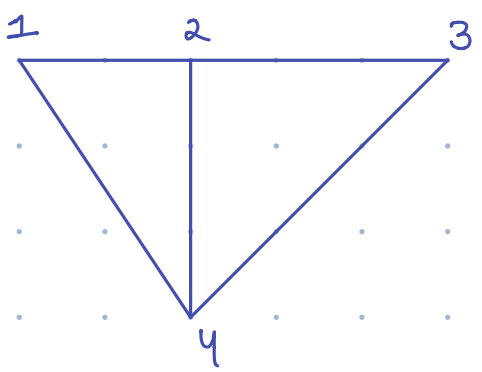
$$|E(T)| = n-1$$

Thus, to get T, we need to remove $m-(n-1)$ edges and ensure that at each step, the graph remains connected and has no cycle.

③ { Questions }

Q. Find all spanning trees of the graph G

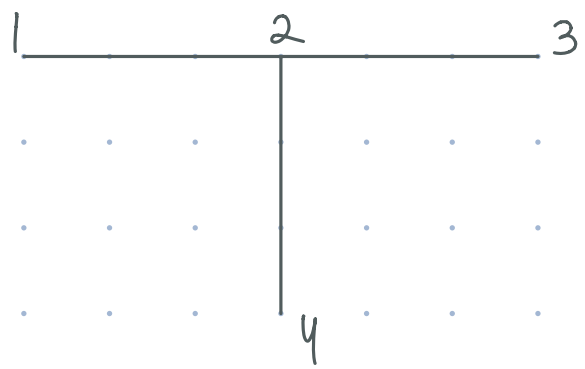
G :



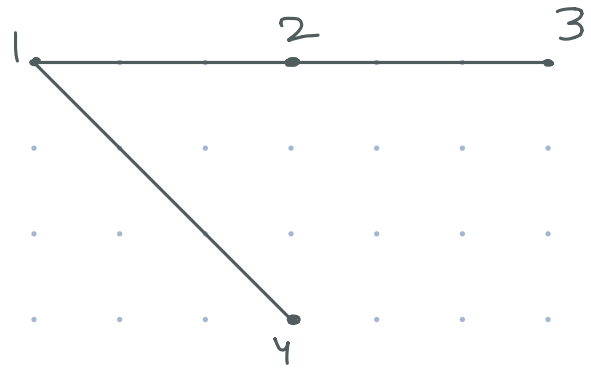
Solution :- Here $n = 4$
and $m = 5$

We need to remove $m-(n-1) = 5-3 = 2$ edges to get a spanning tree.

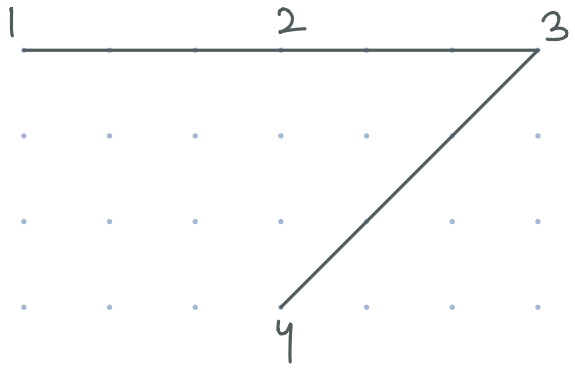
So the possible spanning trees are :-



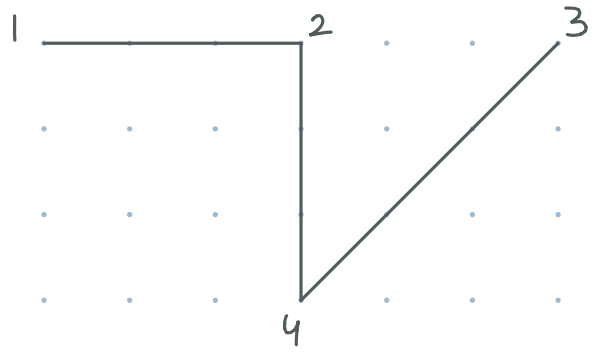
(T_1)



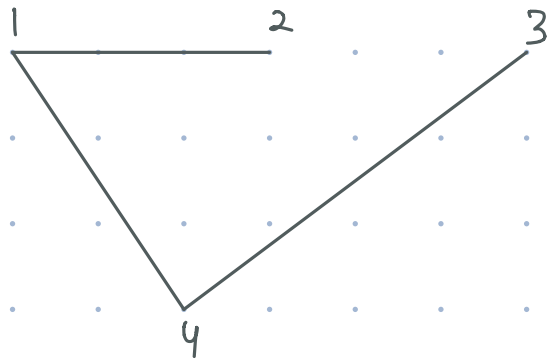
$(T_2) \cong P_4$



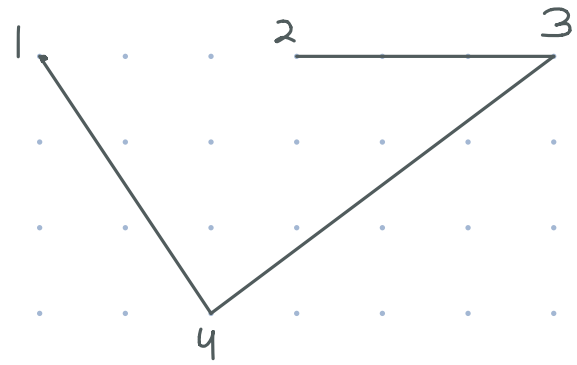
$(T_3) \cong P_4$



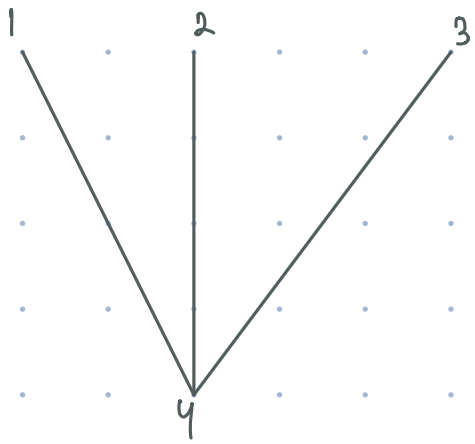
$(T_4) \cong P_4$



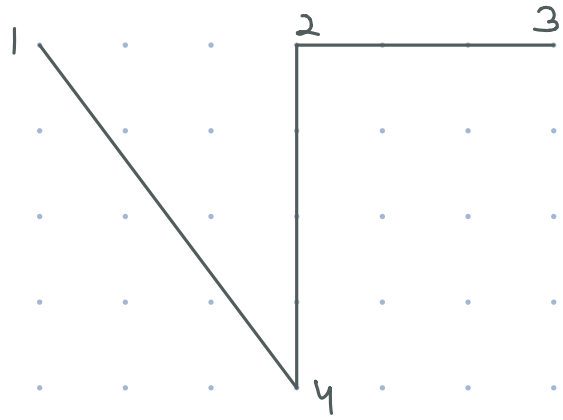
$(T_5) \cong P_4$



$(T_6) \cong P_4$



$(T_7) \cong K_{1,3}$



$(T_8) \cong P_4$

$\blacksquare$

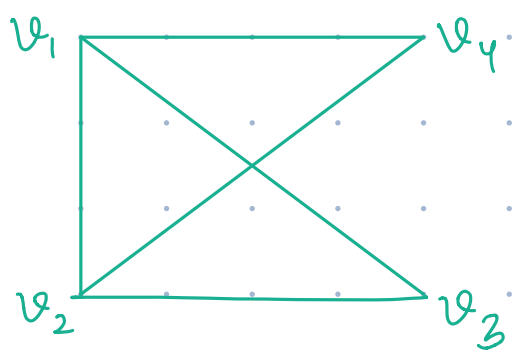
④ { # of spanning trees of a graph of order n ? }

⑤ { KIRCHOFF'S MATRIX TREE THEOREM }

Let M be the matrix obtained from the adj matrix of a connected graph G by changing all 1's to -1's and diagonal 0 to the degree of the corr vertex

Then the # of spanning trees of G is the value of any cofactor of M.

⑥ { Example of Kirchoff's theorem }



Then

$$A(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Then

$$M = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix}$$

Now $\quad \text{cof}(a_{11}) = (-1)^{1+1} \begin{vmatrix} 3 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{vmatrix}$

$$= 3(4) + 1(-2) + (-1)(2)$$

$$= 12 - 4 = 8$$

So # of spanning trees of $G = 8$

Slly, we can verify this by cofactor of any element.

So let's consider cofactor of a_{14}

$$\text{cof } a_{14} = (-1)^{1+4} \begin{vmatrix} -1 & 3 & -1 \\ -1 & -1 & 2 \\ -1 & -1 & 0 \end{vmatrix}$$

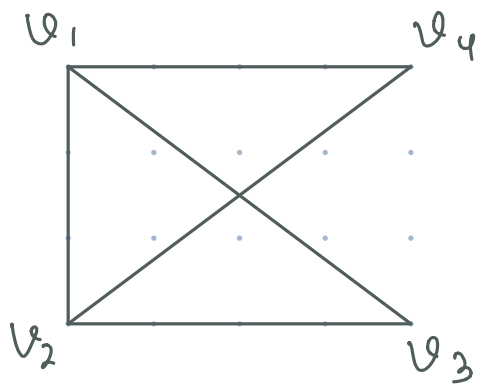
$$= (-1)\left\{-2 - 3(2) + (-1)(0)\right\}$$

$$= 8$$

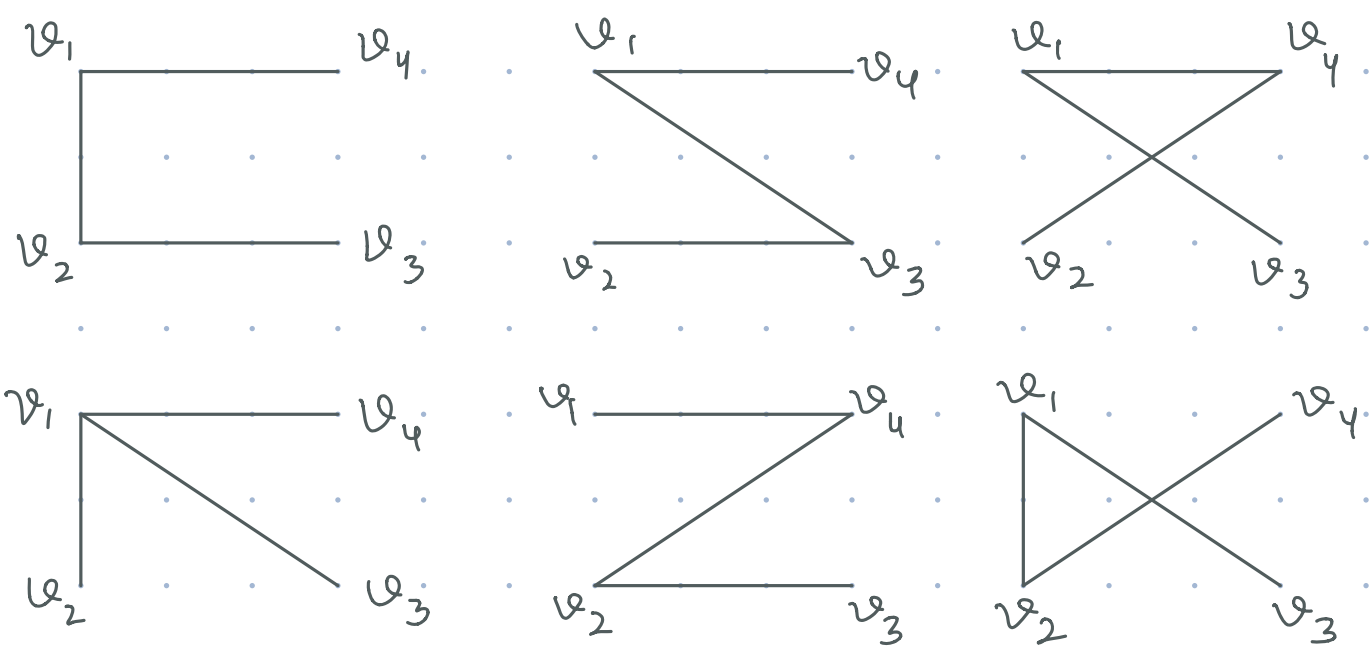
Thus, # of spanning trees of $G = 8$

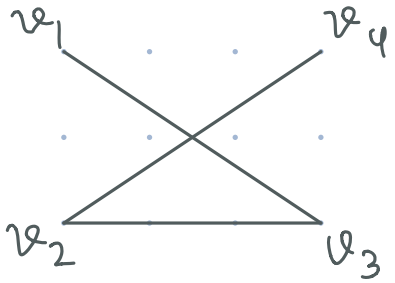
Let's make these spanning trees.

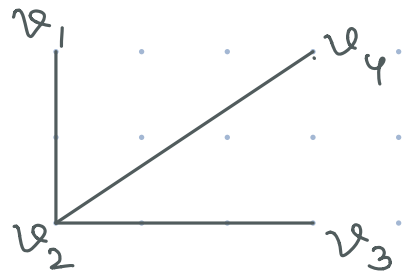
we have $n = 4$ $m = 5$ so we need to remove $m - (n-1) = 5 - (4-1) = 2$ edges.



(G)







(7) FUNDAMENTAL CYCLE

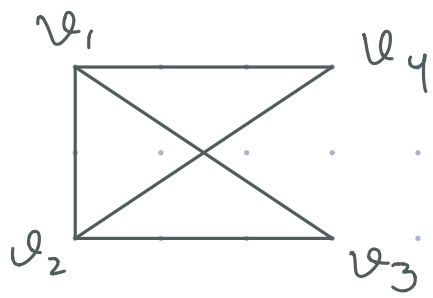
Let G be a connected graph and let T be a spanning tree of G.

Let e be an edge in G between u and v that is not in T. Now in T there is a unique path between u and v, and since e is not in T, that path does not use e. Therefore, that path together with e forms a cycle in G.

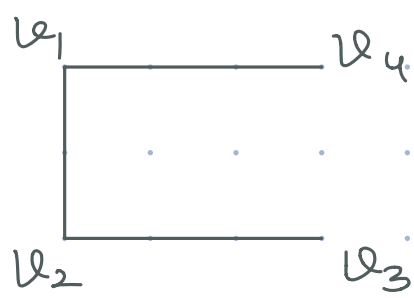
A cycle built this way is called fundame. cycle.

The set of all fundamental cycles of a graph G form a basis for cycle space of

G.

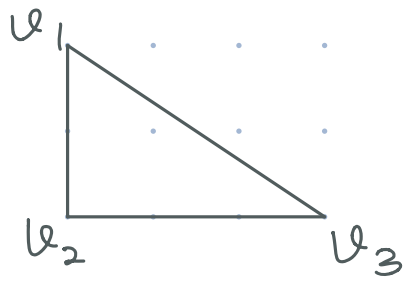


(G)



(T_1)

Then edge $v_1v_3 \in G$ but $v_1v_3 \notin T_1$
This edge v_1v_3 forms a cycle in G



(C)

Thus, C is a fundamental cycle of G.