GRAPH THEORY
MC-405
ASSIGNMENT-1

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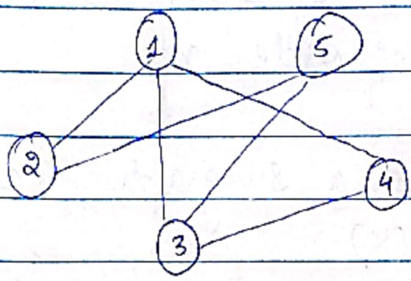
1. Graph G:



(a) Yes G is a simple graph as there are no self loops or multi edges.

(b) Degree sequence : 3, 2, 3, 2, 2

(c) G has 6 edges.

2. $G(n, m)$

To prove : $\delta \leq \frac{2m}{n} \leq \Delta$

Proof:

Let $d_1, d_2, d_3 \ldots d_n$ be the degree of vertices.

We know that

$$\Sigma d_i = 2m$$

Also, $d_1 + d_2 + \ldots + d_n \geq \delta + \delta + \cdots \delta$

$$\Sigma d_i \geq n\delta$$

$$2m \geq n\delta$$

$$\delta \leq 2m/n$$

and $\quad \sum d \leq \Delta + \Delta + \dots + \Delta$

$$2m \leq n\Delta$$

$$\frac{2m}{n} \leq \Delta$$

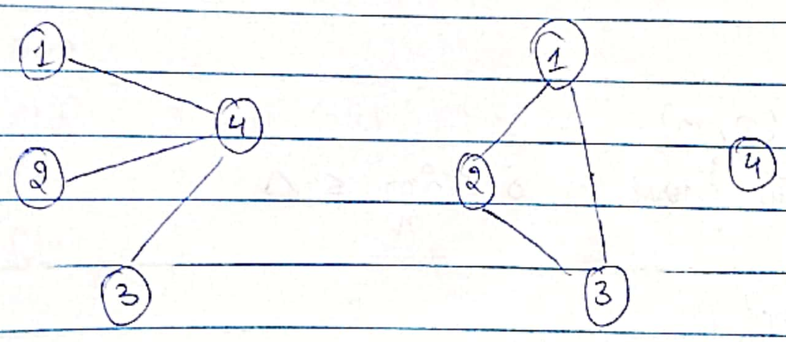
$$\therefore \quad \delta \leq \frac{2m}{n} \leq \Delta$$

(3) Let G be a graph of order n and K be a complete graph of order n.

Let us construct a subgraph $H \subset K$ where $V(H) = V(K)$ & an edge e_i with end vertices $(u_i, u_j) \in E(H)$ iff an edge $(v_i, v_j) \in E(G)$

This subgraph is isomorphic to G.

(4)



(a) (b)

(a) is a bipartite graph and (b) is its compliment (b) is not a bipartite graph.

Hence, not every complement of a bipartite graph is a bipartite.

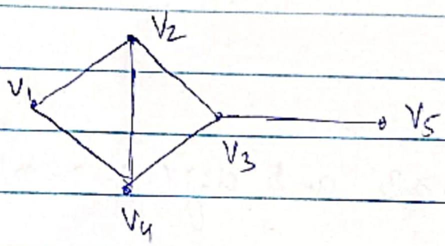
(5.) Let us assume that G has a path of maximum length $m \leq \delta(G)$ ie maximal path.

Now let $v \in V(G)$ be the terminal vertex of the path. We know that v has atleast $\delta(G)$ adjacent vertices. All of these vertices cannot be in the path as the length of the path is less than $\delta(G)$

Hence we can extend the path by atleast 1. This contradicts the assumption that maximal path has length less than $\delta(G)$.

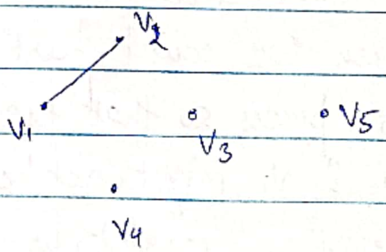
So we can always have a path of atleast $\delta(G)$ length.

(6.) G:

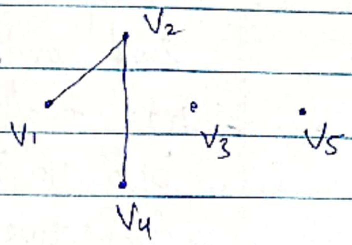


Non isomorphic spanning subgraphs:

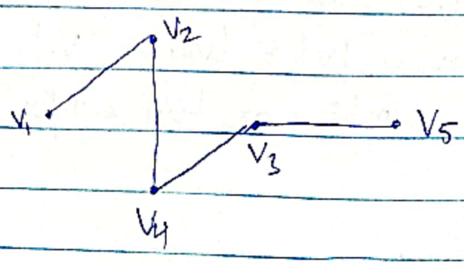
G_1:



G_2:



G_3:



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(7.) As the radius is the minimum eccentricity of any vertex and diameter is the maximum.

$$rad(G) \leq diam(G)$$

Let u & v be vertices such that $d(u,v) = diam(G)$

Let w be a central vertex such that $e(w) = rad(G)$

This means no vertex is at greater distance than $rad(G)$ from w. In particular $d(u,w)$ & $d(v,w)$ are both less than or equal to $rad(G)$

Therefore,

$$d(u,w) + d(v,w) \leq 2rad(G)$$

By triangle inequality,

$$rad(G) \leq diam(G) \leq 2rad(G)$$

Hence Proved.

(8.) $G = (V,E)$ $n \geq 3$ and $deg(v) \geq n/2$ $\forall$ $v \in V$

First we show that the graph is connected.

Suppose G is not connected so that G has atleast two components. Then we could partition $V = V_0 \cup V_1$ into two non-empty pieces so that theres no edge b/w V_0 & V_1. (V_0 & V_1 might not be components themselves, because there might be more than two components) Instead V_0 & V_1 are unions of components. Since $n = |V| = |V_0| + |V_1|$ we must have either

$$|V_0| \leq n/2 \text{ or } |V_1| \leq n/2$$

Say V_0 has size $\leq n/2$ and pick an $v \in V_0$. Then $\deg(v) \geq n/2$, but every neighbour of v is contained in V_0 & is not in v. So $\deg < n/2$.

This is a contradiction

Hence G is connected.

Now we prove that there is a hamiltonian circuit by induction.

Let p_m be the statement 'As long as $m+1 \leq n$, there is a path visiting $m+1$ distinct vertices with no repitition'.

p_0 is trivial - it is a single vertex.

p_m is true, we have a path

$$V_0 - V_1 - V_2 - \cdots - V_m$$

We need to show p_{m+1} if V_m is adjacent to a vertex not already in path we have p_{m+1}.

Let us assume all neighbours of V_0 & V_m are somewhere in path.

We want to turn our path into a cycle. If V_0 is adjacent to V_m we have it already. Suppose not we must find the following arrangement:

$$V_0 - V_1 - \cdots V_{t-1} - V_t - \cdots - V_m$$

Then we can break the link b/w V_{t-1} & V_t and have our circuit.

We know V_0 has $n/2$ neighbours, all of them in the path & none are V_m.
Let A be the vertices adjacent to V_0 so $|A| \geq n/2$
Let B be all vertices adjacent to V_m so $|B| \geq n/2$
Every vertex in B belongs on the path, so we can ask about some vertex in B immediately after it.
Let C be the set of vertices which are immediately after some vertex in B in the path.
Then $|C| = |B| \geq n/2$ if $A \cap C = \phi$, then $|A \cup C| \geq n/2 + n/2 \geq n$ so $A \cup C$ would have to include all the vertices. But V_0 is neither A nor C, so $A \cup C$ is all vertices, so there is some vertice $V_t \in A \cap C$, so $V_t \in A$ while $V_{t-1} \in B$
$\therefore$ we have a hamiltonian circuit.

9. If G is hamiltonian G+uv is hamiltonian. To prove the converse, assume G+uv is hamiltonian but G is not. Then there is a hamiltonian path b/w u & v. Let us consider this undirected path as a path from u to v. For each vertex adjacent to vertex v, the vertex in the path immediately preceding that vertex cannot be adjacent to v, there will be a Hamiltonian cycle in G. So the degree sum of u & v cannot exceed $(n-1)$.

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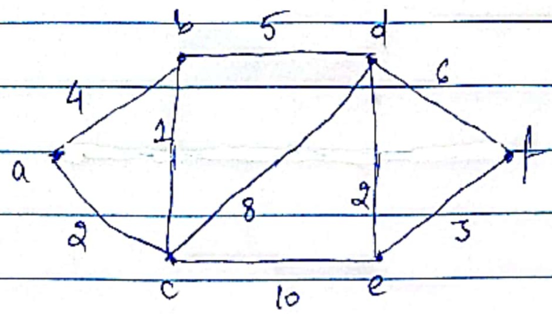
In other words the degree sum of these two non adjacent vertices is less than n, which is a contradiction.

16.



Node	a	b	c	d	e	f
	∞	∞	∞	∞	∞	∞
a	0	4	2	∞	∞	∞
c		3	2	10	12	∞
b	–	–	–	8	12	∞
d	–	–	–	–	10	14
e	–	–	–	–	–	13

We have the path:

a – c – b – d – e – f

Cost :- 13