Introduction to Nuclear Magnetic Resonance Spectroscopy

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July 13, 2017

1.1. Introduction to Magnetic Resonance

- Magnetic resonance (MR) is a phenomenon of resonant energy absorption by a system of nuclei (and electrons).
- Nuclear magnetic resonance (NMR) results from the intrinsic magnetic moment of the nuclei of some atoms. Magnetic moments of electrons are exploited in electron spin resonance (ESR).
- Magnetic resonance (MR) generally involves placing a sample in a strong magnetic field (to generate polarisation at a fixed resonant frequency) and detecting signals produced following application of pulsed radio-frequency electromagnetic fields (RF pulses).
- MR is a very powerful method for studying the structure of materials: used in physics, chemistry, biology, medicine etc.

- NMR spectroscopy is used for chemical analysis and for molecular structure determination

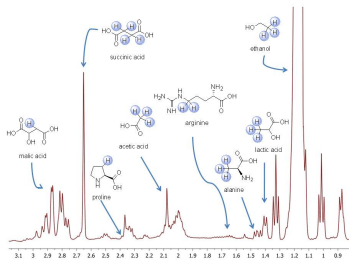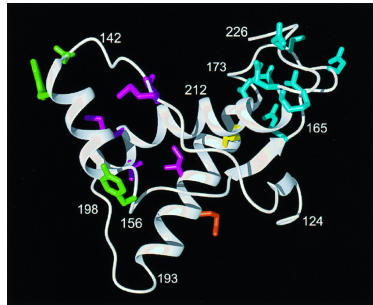Fig.1: ^{1}H NMR spectrum of a sample of Spanish wine (http://www.unirioja.es/gsoe/NMR.htm)

Fig.2: NMR-derived structure of a prion http://www.pnas.org/content/94/14/7281.full

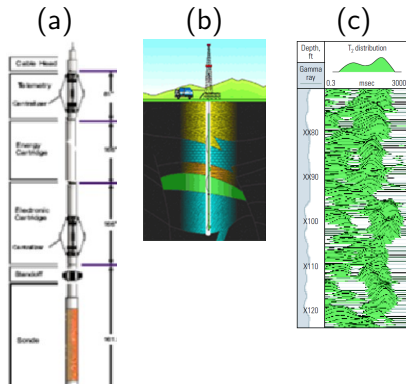- NMR relaxometry can be used to monitor molecular environment

Fig.3: (a) NMR-logging probe, (b) Schematic positioning of the probe in a well, (c) T_2-relaxation profile along the bore. Sources: 1) Allen et al. Oilfield review, Autumn 2000; 2) Coates, Xiao NMR Logging Principles and Applications, Hulliburton

- NMR forms the basis for magnetic resonance imaging (MRI)
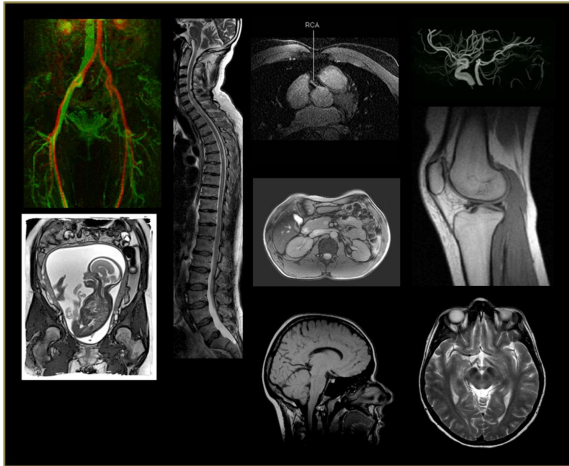

Fig.4: Example magnetic resonance images of blood vessel (in legs), fetus in utero, spine, heart, abdomen, head, blood vessels (in brain), knee, brain (courtesy of Prof. Richard Bowtell)

- Consider charges moving in a limited volume. The position of a charge e_n will be given by a vector $\mathbf{r_n}$ and its velocity by $\mathbf{v_n}$. The overall magnetic moment of such a system is defined as:

$$\mathbf{M} = \frac{1}{2} \sum_n e_n \mathbf{r_n} \times \mathbf{v_n} \tag{1}$$

- If all the charges and masses are the same, then $\mathbf{M}$ can be rewritten as:

$$\mathbf{M} = \frac{e}{2m} \sum_n m \mathbf{r_n} \times \mathbf{v_n} = \gamma \mathbf{L}, \tag{2}$$

where

$$\mathbf{L} = \sum_n \mathbf{p_n} \times \mathbf{r_n} \tag{3}$$

is the mechanical angular momentum.
- Gyromagnetic ratio (or magnetogyric):

$$\gamma = \frac{e}{2m} \tag{4}$$

- When a magnetic moment **M** is placed into an external uniform permanent magnetic field **B**, its energy is given by:

$$E = -\mathbf{M} \cdot \mathbf{B} \qquad (5)$$
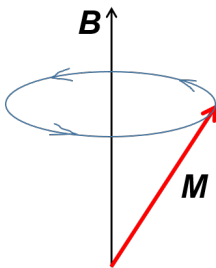
- The torque acting on the system:

$$\frac{d\mathbf{L}}{dt} = \mathbf{M} \times \mathbf{B} \qquad (6)$$

- Now using equation 2 we can obtain the equation describing the motion of vector **M**:

$$\frac{d\mathbf{M}}{dt} = \gamma \mathbf{M} \times \mathbf{B} \qquad (7)$$

- In a uniform magnetic field directed along z-axis $\mathbf{B} = (0, 0, B_0)$, the equation for individual components of $\mathbf{M}$ follow the equations:

$$\frac{dM_x}{dt} = \omega_L M_y$$

$$\frac{dM_y}{dt} = -\omega_L M_x \qquad (8)$$

$$\frac{dM_z}{dt} = 0,$$

where $\omega_L = \gamma B_0$ - Larmor frequency.

- A solution to this system of differential equations with initial values of $M_x(0)$, $M_y(0)$, $M_z(0)$ has the following form:

$$
\begin{aligned}
M_x(t) &= M_x(0)\cos(\omega_L t) + M_y(0)\sin(\omega_L t) \\
M_y(t) &= -M_y(0)\sin(\omega_L t) + M_y(0)\cos(\omega_L t) \qquad (9) \\
M_z(t) &= M_z(0)
\end{aligned}
$$

1.4. Orbital angular momentum operator

- In quantum mechanics physical quantity A ise represented by an operator $\hat{A}$. The mechanical angular momentum is replaced by its corresponding operator:

$$\mathbf{L} = \sum_n \mathbf{p_n} \times \mathbf{r_n} \longleftrightarrow \hat{\mathbf{L}} = \frac{1}{\hbar} \sum_n \hat{\mathbf{r}}_\mathbf{n} \times \hat{\mathbf{p}}_\mathbf{n} = -i \sum_n \hat{\mathbf{r}}_\mathbf{n} \times \boldsymbol{\nabla}_\mathbf{n} \quad (10)$$

- Angular momentum operator properties. Commutation:

$$[\hat{L}_y, \hat{L}_z] = i\hat{L}_x, [\hat{L}_z, \hat{L}_x] = i\hat{L}_y, [\hat{L}_x, \hat{L}_y] = i\hat{L}_z \quad (11)$$

- Angular momentum squared, and its commutation properties:

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 \quad (12)$$

$$[\hat{L}^2, \hat{L}_x] = [\hat{L}^2, \hat{L}_y] = [\hat{L}^2, \hat{L}_z] = 0 \quad (13)$$

- Eigenfunctions of both $\hat{L}^2$ and $\hat{L}_z$ operators can be characterized by integer quantum numbers l and m respectively. These eigen functions will be denoted as $|lm\rangle$. Their eigenvalues are:

$$\hat{L}_z|lm\rangle = m|lm\rangle \tag{14}$$

$$\hat{L}^2|lm\rangle = l(l+1)|lm\rangle \tag{15}$$

- Another useful operator are raising and lowering operators:

$$\hat{L}_+ = \hat{L}_x + i\hat{L}_y, \hat{L}_- = \hat{L}_x - i\hat{L}_y \tag{16}$$

$$\langle lm|\hat{L}_+|l(m-1)\rangle = \langle l(m-1)|\hat{L}_-|lm\rangle = \sqrt{(l+m)(l-m+1)} \tag{17}$$

Calculate $[\hat{L}_+, \hat{L}_x] =?$, $[\hat{L}_+, \hat{L}_-] =?$

$$[\hat{L}_y, \hat{L}_z] = i\hat{L}_x, [\hat{L}_z, \hat{L}_x] = i\hat{L}_y, [\hat{L}_x, \hat{L}_y] = i\hat{L}_z$$

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$$
$$[\hat{L}^2, \hat{L}_x] = [\hat{L}^2, \hat{L}_y] = [\hat{L}^2, \hat{L}_z] = 0$$
$$\hat{L}_+ = \hat{L}_x + i\hat{L}_y, \hat{L}_- = \hat{L}_x - i\hat{L}_y$$

- Classical magnetic moment will have its own quantum analogue, the operator of angular momentum:

$$\mathbf{M} = \gamma \mathbf{L} \longleftrightarrow \hat{\boldsymbol{\mu}} = \gamma \hbar \hat{\mathbf{L}} \tag{18}$$

- Given the electron charge $e = 1.6 \cdot 10^{-19}$C, and mass $m = 9.1 \cdot 10^{-31}$kg Bohr magneton:

$$\beta_e = \gamma \hbar = \frac{e\hbar}{2m} \approx 9.27 \cdot 10^{-24} \text{J} \cdot \text{T}^{-1} \tag{19}$$

- Similarly a nuclear magneton could be calculated for a proton (^{1}H nucleus):

$$\beta = \gamma_N \hbar = \frac{e\hbar}{2m_p} \approx 5.05 \cdot 10^{-27} \text{J} \cdot \text{T}^{-1} \tag{20}$$

However, real nuclei and electrons have spins (intrinsic magnetic moment). Their z-axis projection m takes integer and half-integer values: $m = \frac{1}{2}, 1, \frac{3}{2}, 2$ etc. Similar to the equation for the orbital angular momentum Eq.18. For nuclei spins we get its magnetic moment as:

$$\hat{\boldsymbol{\mu}}_{\mathbf{N}} = \gamma_N \hbar \hat{\mathbf{I}}, \qquad (21)$$

where $\hat{\mathbf{I}}$ stands for the nuclear spin operator. All the properties of angular momentum operators listed in Eqs.11-17 will be true for $\hat{\mathbf{I}}$.

- Many nuclei in the periodic table are magnetic, i.e. have spin $I \neq 0$. Their magnetic moments could be measured in units of β_N:

$$\hat{\boldsymbol{\mu}}_{\mathbf{N}} = \gamma_N \hbar \hat{\mathbf{I}} = g_N \beta_N \hat{\mathbf{I}}, \qquad (22)$$

where g_N - dimensionless g-factor.

Nucleus	Natural abundance %	Nuclear spin (I)	g_N, g-factor	γ_N, Gyromagnetic ratio (10^7 rad/T*s)
^{1}H	99.98	$\frac{1}{2}$	5.585	26.7519
^{2}H	$1.5*10^{-2}$	1	0.857	4.1066
^{13}C	1.108	$\frac{1}{2}$	1.405	6.7283
^{14}N	99.635	1	0.403	1.9338
^{15}N	0.365	$\frac{1}{2}$	-0.567	-2.712

- Electron magnetic moments can be measured in units of Bohr magnetons: $\hat{\boldsymbol{\mu}}_{\mathbf{S}} = -\gamma_e \hbar \hat{\mathbf{S}} = -g_e \beta_e \hat{\mathbf{S}}$, and for a free electron spin $g_e \approx 2.0023$.

- Applications of NMR: chemistry, biology, medicine, industry ...
- Magnetic moment in magnetic field: Classical description
- Recap of angular momentum operator properties: commutation properties.
- Nuclei have their own nuclear magnetic moment. Described using spin angular momentum operator.

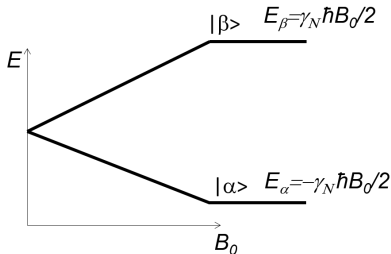**Suggested reading:**   Harris: 1.1, 1.2, 1.3, 1.4, 1.6, 2.4

- Let's quantum mechanically describe the system of spins in the magnetic field. Eq. 5 can be rewritten in a form of Hamiltonian:

$$E = -\mathbf{M} \cdot \mathbf{B} \longleftrightarrow \mathcal{H} = -\hat{\boldsymbol{\mu}}_\mathbf{N} \cdot \mathbf{B} \qquad (23)$$

- when the magnetic field is directed along z-axis $\mathbf{B} = (0, 0, B_0)$:

$$\mathcal{H} = -\hat{\boldsymbol{\mu}}_\mathbf{N} \cdot \mathbf{B} = -\gamma_N \hbar B_0 \hat{I}_z \qquad (24)$$

- For spin $I = \dfrac{1}{2}$ such Hamiltonian produces a two-level system. Its energy levels corresponding to eigenfunctions $|\alpha\rangle$ and $|\beta\rangle$:

1.6. Spin in a magnetic field

- The transition between the two states requires an energy quantum[1]:

$$h\nu_L = \gamma\hbar B_0, \omega_L = \gamma B_0, \nu_L = \frac{\gamma B_0}{2\pi} \qquad (25)$$

ω_L and ν_L is the Larmor frequency (anglular and cyclic respectively)

Nucleus	Natural abundance %	Nuclear spin (I)	Larmor frequency at 11.744T, MHz	γ_N, Gyro-magnetic ratio (10^7 rad/T*s)
^{1}H	99.98	$\frac{1}{2}$	500	26.7519
^{2}H	$1.5*10^{-2}$	1	76.753	4.1066
^{13}C	1.108	$\frac{1}{2}$	125.721	6.7283
^{14}N	99.635	1	36.118	1.9338
^{15}N	0.365	$\frac{1}{2}$	50.664	-2.712

NMR measurements are generally made on bulk samples which contain very large numbers of nuclear spins (e.g. 1 cm^3 contains $N \approx 6.7 \cdot 10^{22}$ ^{1}H atoms) The measured signals therefore result from the collective effect of a large number of magnetic moments that can be described using a bulk magnetization. At thermal equilibrium, the numbers of nuclei in the $|\alpha\rangle$ state N_α and $|\beta\rangle$ state N_β follow Boltzmann distribution:

$$\frac{N_\alpha}{N_\beta} = e^{-\frac{\gamma B_0}{kT}} \approx (1 - \frac{\gamma B_0}{kT}), \qquad (26)$$

when $\gamma B_0 \ll kT$. Overall magnetization then can be calculated as:

$$M_z = N_\alpha(-\frac{1}{2}\gamma\hbar) + N_\beta(\frac{1}{2}\gamma\hbar) = N\frac{\gamma^2\hbar^2 B_0}{4kT} \qquad (27)$$

Problem

- What is the value of $\frac{\gamma_N \hbar B_0}{kT}$ for proton nuclei (^{1}H) at 9.4 T magnetic field at 300 K?

- What is the value of $\frac{\gamma_e \hbar B_0}{kT}$ electron (^{1}H) at 9.4 T magnetic field at 4 K?

Electron charge $e = 1.602 \cdot 10^{-19}$ C

Electron mass $m_e = 9.109 \cdot 10^{-31}$ kg

Proton mass $m_p = 1.673 \cdot 10^{-27}$ kg

Plank constant $\hbar = 1.054 \cdot 10^{-34} \frac{J \cdot s}{rad}$

Proton g-factor $g_p = 5.585$

Electron g-factor $g_e = 2.0023$

Nuclear magneton $\beta_N = 5.05 \cdot 10^{-27}$ J $\cdot$ T^{-1}

Bohr magneton $\beta_e = 9.27 \cdot 10^{-24}$ J $\cdot$ T^{-1}

Proton gyromagnetic ratio $\gamma_N = 26.7519 \cdot 10^7 \frac{rad}{T \cdot s}$

Electron gyromagnetic ratio $\gamma_e = 1.76 \cdot 10^{11} \frac{rad}{T \cdot s}$

Boltzmann constant $k = 1.38 \cdot 10^{-23}$ J $\cdot$ K^{-1}

Avogadro's constant $N_A = 6.023 \cdot 10^{23}$ mol^{-1}

- Let's apply oscillating magnetic field to our system. A spin system Hamiltonian becomes time-dependent and for an oscillation along the x-axis we obtain:

$$
\begin{aligned}
\mathcal{H}(t) &= -\hat{\boldsymbol{\mu}}(\mathbf{B_0} + \mathbf{B(t)}) = \\
&= -\gamma\hbar\hat{I}_z(B_0 + B_1(t)) = \\
&= -\gamma\hbar\hat{I}_z B_0 - \gamma\hbar\hat{I}_x B_1\cos(\omega t),
\end{aligned}
\tag{28}
$$

where H_1 and ω are the amplitude and the frequency of the oscillating magnetic field.

- According to perturbation theory the transition probability between the initial state $|a\rangle$ and the final state $|b\rangle$ with a time dependent Hamiltonian $\hat{V}(t) = 2\hat{F}\cos(\omega t)$ is (Fermi's golden rule):
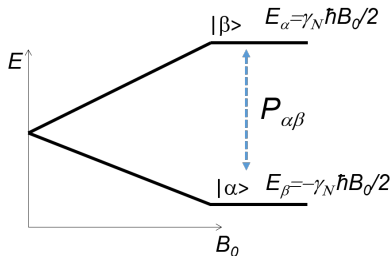
$$
P_{ab} = \frac{2\pi}{\hbar}|\langle a|\hat{F}|\rangle b|^2\delta(E_{ab} - \hbar\omega),
\tag{29}
$$

where $E_{ab} = E_a - E_b$ is an energy difference between the energies of levels a and b.

- For a two level system described before, the matrix element $\langle\alpha|\hat{I}_x|\beta\rangle = \frac{1}{2}$. The transition probability then becomes:

$$P_{\alpha\beta} = \frac{\pi}{2\hbar}(\gamma H_1)^2 \delta(E_{\alpha\beta} - \hbar\omega), \qquad (30)$$

- **The effect of resonant absorption (and emission) of electromagnetic irradiation at the frequency matching the energy difference in a nuclear system is called Nuclear Magnetic Resonance (NMR).**

1.9. Populations dynamics in two-level system.

- In a two-level system the transition will take place due to the action of external irradiation, but also due to interaction with the environment.
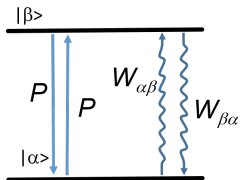
P - the rate of transitions driven by external field, $W_{\alpha\beta}$, $W_{\beta\alpha}$ - rates of spontaneous spin flips due to interaction with environment.

- In thermal equilibrium:

$$N_\alpha^0 W_{\alpha\beta} = N_\beta^0 W_{\beta\alpha}, \text{i.e.} \tag{31}$$

$$\frac{W_{\beta\alpha}}{W_{\alpha\beta}} = exp^{-\dfrac{\gamma\hbar B_0}{kT}} \approx 1 - \frac{\gamma\beta B_0}{kT} \tag{32}$$

- Equation for populations of levels:

$$
\begin{aligned}
\frac{dN_\alpha}{dt} &= -N_\alpha(P + W_{\alpha\beta}) + N_\beta(P + W_{\beta\alpha}) \\
\frac{dN_\beta}{dt} &= N_\alpha(P + W_{\alpha\beta}) - N_\beta(P + W_{\beta\alpha})
\end{aligned}
\tag{33}
$$

- If we introduce the average rate of spontaneous transitions
$W = \frac{1}{2}(W_{\alpha\beta} - W_{\beta\alpha})$, then $W_{\alpha\beta} = W(1 + \frac{\gamma\hbar B_0}{2kT})$ and
$W_{\beta\alpha} = W(1 - \frac{\gamma\hbar B_0}{2kT})$, the equations can be rewritten:

$$
\begin{aligned}
\frac{dN_\alpha}{dt} &= (N_\beta - N_\alpha)P + (N_\beta - N_\alpha)W - W\frac{\gamma\beta B_0}{2kT}N \\
\frac{dN_\beta}{dt} &= -(N_\beta - N_\alpha)P - (N_\beta - N_\alpha)W + W\frac{\gamma\beta B_0}{2kT}N
\end{aligned}
\tag{34}
$$

- Denote the population difference as $n = N_\beta - N_\alpha$ and thermal equilibrium population difference $n_0 = N_\beta^0 - N_\alpha^0 \approx N\dfrac{\gamma\hbar B_0}{2kT}$ the equations can be rewritten as:

$$\frac{dn}{dt} = -2nP - 2nW + 2Wn_0, \tag{35}$$

or

$$\frac{dn}{dt} = -2nP - \frac{(n - n_0)}{T_1}, \tag{36}$$

where $T_1 = \dfrac{1}{2W}$ is called spin-lattice relaxation time determines how quickly a spin system reaches a thermal equilibrium with environment.

- In equilibrium, when $\dfrac{dn}{dt} = 0$:

$$n = \frac{n_0}{1 + 2PT_1} \tag{37}$$

when the power is very large $PT_1 \gg 1$, $n \to 0$, i.e. the system is saturated and no signal can be observed.

- Spin $I = \dfrac{1}{2}$ in a magnetic field. Two-level system.
- System of spins in a magnetic field is capable of absorbing radiation at a resonant frequency.
- Population dynamics in a two-level system. Signal as function of radiation power and saturation.

Suggested reading: Harris 1.5, 1.7, Slichter 1.3
Harris 1.20 - CW NMR spectrometer

- Electrons in atoms in molecules interact with external magnetic field and in turn produce their own magnetic field B_0. The Larmor frequency get shifted in a chemical specific manner - this is known as the chemical shift. The spin Hamiltonian for a nucleus is:

$$\hat{H} = -\gamma_N \hbar (1 - \sigma) B_0 \hat{I}_z, \qquad (38)$$

- Classical illustration of diamagnetic chemical shift:

$$\boldsymbol{\omega} = \frac{e}{2m_e} \mathbf{B_0}$$

$$\mathbf{j} = -e[\boldsymbol{\omega} \times \mathbf{r}]\rho_e = -\frac{e^2}{2m_e}[B_0 \times \mathbf{r}]\rho_e$$

$$d\mathbf{B_i} = \frac{\mu_0}{4\pi r^3}[\mathbf{j} \times \mathbf{r}]dV$$

$$d\mathbf{B_i} = -\frac{\mu_0 e^2}{8\pi m_e r^3}[[\mathbf{B_0} \times \mathbf{r}] \times \mathbf{r}]\rho_e\, dV$$

$$B_{iz} = -B_0 \frac{\mu_0 e^2}{8\pi m_e} \int \rho_e \frac{x^2 + y^2}{r^3} dV$$

Quantum mechanical result:

$$\sigma = -\frac{\mu_0 e^2}{8\pi m_e} \langle \Psi | \frac{x^2 + y^2}{r^3} | \Psi \rangle$$
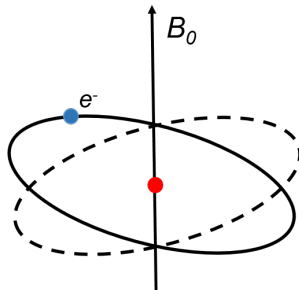
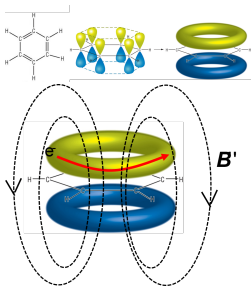Let's calculate the effect of ring current in cyclic aromatic molecules. Consider benzene molecule:

Fig.5: (top) Schematic repsresentation of electron orbitals in a benzene molecule, (bottom) local fields in a benzene molecule produced by a magnetic field.

- Larmor precession frequency of electrons $\omega_L = \frac{eB_0}{2m_e}$. Current can be calculated as charge $6e$, divided by precession period $\frac{2\pi}{\omega_L}$:

$$i = \frac{3e^2 B_0}{2\pi m_e} \qquad (39)$$

- Circular conductor creates a magnetic moment $\mu = i \cdot \pi r^2$, totalling in:

$$\mu = -\frac{3e^2 B_0 r^2}{2m_e} \qquad (40)$$

- The magnetic field created by a magnetic moment $\mathbf{B} = \frac{\mu_0}{4\pi}(\frac{\mathbf{r(mr)}}{r^5} - \frac{\mathbf{m}}{r^3})$ reduces to:

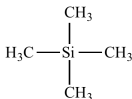$$B_i = -\frac{\mu_0}{4\pi}\frac{m}{r^3} = \frac{3\mu_0 e^2}{8\pi}\frac{r^2}{(r+d)^3}B_0 \qquad (41)$$

$$\sigma = --\frac{\mu_0}{4\pi m_e}\frac{m}{r^3} = \frac{3\mu_0 e^2}{8\pi}\frac{r^2}{(r+d)^3} \qquad (42)$$

- Given benzene molecule radius $r = 1.4\text{Å}$, CH-bond length $d = 1.1\text{Å}$ we obtain:

$$\sigma \approx -5.3 \cdot 10^{-6}, \sigma_{iso} \approx -1.8 \cdot 10^{-6} \qquad (43)$$

2.1. Chemical shifts.

- Chemical shifts are usually measured in ppm's and are referenced with respect to the signals of tetramethylsilane (TMS).

$$\delta = \frac{\nu_{sample} - \nu_{TMS}}{\nu_{TMS}} \times 10^6 \text{ppm} \qquad (44)$$

$$\delta = (\sigma_{TMS} - \sigma_{sample}) \times 10^6 \text{ppm} \qquad (45)$$

- In general the ^{1}H chemical shift is greater for nuclei to atoms/bonds that reduce the electron density at the atom.
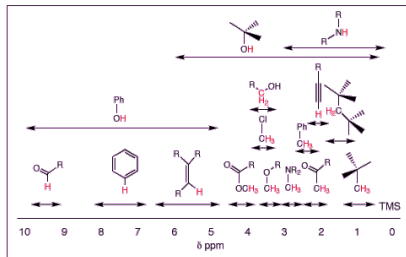

Fig.6: Chemical shifts in various organic molecules Source: http://orgchem.colorado.edu/Spectroscopy/nmrtheory/protonchemshift.html

Consider the ^{1}H-spectrum of methyl acetate.

- the TMS appears at 0 ppm
- chemical shift increases from right to left
- two resonances correpond to protons of the two methyl groups
- the presence of electronegative oxygen atom in methoxy group produces weaker shielding (σ) thus makes bigger chemical shift δ
- three nuclei of methyl groups resonate at the same frequency. Peak heights are the same.
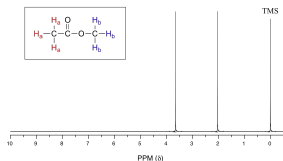
Fig.7: ^{1}H spectrum of methyl acetate. Source: http://chemwiki.ucdavies.edu/Organic_Chemistry

For other nuclei, paramagnetic shifts which arise from mixing of excited state with the ground state dues to the effect of the applicaed field, B_0 on the Hamiltonian can be important, and the range of chemical shifts is usully larger than for ^1H nuclei.
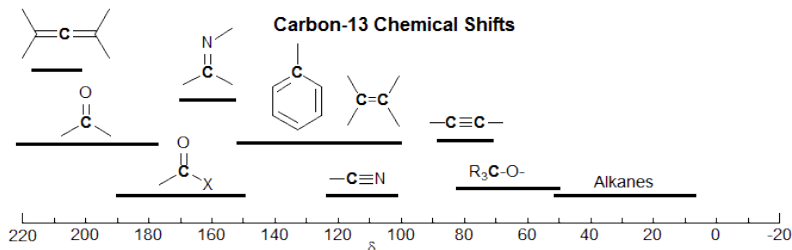

Fig.8: ^{13}C chemical shifts, https://www.chem.wisc.edu/areas/reich/nmr/

2.2. J-couplings.

Nuclear spins interact with one another. In solution, one prominent spin-spin interaction is called J-coupling.

This intra-molecular scalar coupling is caused by the combination of two effects: the Pauli principle means that the electons in the bond have opposite spin-state(spin-up and spin-down), while hyperfine couplings (specifically Fermi contact interaction) meanse that it is energetically favourable for each nuclear spin to be anti-parallel to the electron spin.
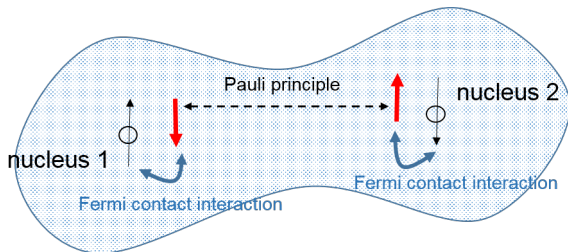

Fig.9: Origin of J-couplings. Low energy configuration in which nuclear spins are antiparallel

- Hamiltonian of J-coupling in solution (all anisotropy is averaged):

$$
\begin{aligned}
\hat{H} &= J\hat{\mathbf{I}}_1 \cdot \hat{\mathbf{I}}_2 = J(\hat{I}_{1x}\hat{I}_{2x} + \hat{I}_{1y}\hat{I}_{2y} + \hat{I}_{1z}\hat{I}_{2z}) = \\
&= J\hat{I}_{1z}\hat{I}_{2z} + \frac{J}{2}(\hat{I}_{1+}\hat{I}_{2-} + \hat{I}_{1-}\hat{I}_{2+})
\end{aligned}
\tag{46}
$$

J is usually measured in units of frequency, i.e. Hz.

- In frequency units the full spin Hamiltonian for a system of two nuclei A and B of the same king would be:

$$
\hat{H} = -\nu_0(1-\sigma_A) - \nu_0(1-\sigma_B) + J\hat{I}_{1z}\hat{I}_{2z} + \frac{J}{2}(\hat{I}_{1+}\hat{I}_{2-} + \hat{I}_{1-}\hat{I}_{2+})
\tag{47}
$$

Let's consider nuclei of the same type (i.e. both ^{1}H or both ^{13}C). When $\gamma_N \hbar \mid \omega_1 - \omega_2 \mid \ll J$,