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An efficient method of renewing object-induced three-way concept lattices involving decreasing attribute-granularity levels

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ABSTRACT

In three-way concept analysis, changing (decreasing or increasing) attribute-granularity levels is needed to seek desirable information. Reconstructing three-way concept lattices often requires huge computation and long elapsed time when attribute-granularity levels are changed. To avoid this problem, a good strategy is indirectly renewing three-way concept lattices. Our paper studies how to renew object-induced three-way concept lattices involving decreasing attribute-granularity levels. Firstly, we analyze changes of object-induced three-way concept lattices when attribute-granularity levels are decreased. To classify changes of object-induced three-way concepts, we classify these concepts into six categories, derive sufficient and necessary conditions of identifying these categories, and investigate their properties. To explore changes of covering relations among object-induced three-way concepts, we classify covering relations into three categories, and identify them by finding which are the destructors of deleted object-induced three-way concepts before the decrease, and analyzing which are children concepts of object-induced three-way concepts as destructors after the decrease. Secondly, by using the above analysis results, we put forward a novel algorithm called OEL-Collapse to renew object-induced three-way concept lattices when attribute-granularity levels are decreased. Finally, experiments are conducted to illustrate the efficiency of the OEL-Collapse algorithm.

1. Introduction

FCA, whose full name is formal concept analysis, was proposed by Wille [1], and used as an effective mathematical tool for conceptual knowledge presentation and discovery. There are two basic notions in FCA, namely, formal contexts and concept lattices [2]. A formal context clearly describes that which attributes are possessed by an object and which attributes are not possessed by an object. A formal concept contains extent and intent. The interacting relation of the two parts shows that the extent is a set of whole objects that own all attributes in the intent, and the intent is a set of whole attributes that are owned by all objects in the extent. For a given formal context, all formal concepts with hierarchy relationship form a concept lattice. To handle various data, incomplete context [3,4], decision context [5], multi-scale context [6,7], triadic context [8] and fuzzy context [9–12] were proposed and studied. In recent years, concept lattice construction and update [13–15], rule acquisition [16,17], knowledge reduction [18,19],

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concept learning [20–22] and three-way FCA [23–28] have been studied for application of FCA. Now FCA has become a powerful analysis tool in many research directions including data mining [29], machine learning [30,31] and so on.

In classic FCA, two-way (2-way) decision (accepting and rejecting) was adopted. That is to say, classic FCA supports that accepting means non-rejection, and rejecting means non-acceptance. However, in the real world, this is not always correct, and three-way (3-way) decision [32,33] is more appropriate. For example, interview results can be “pass”, “fail” or “pending”. At present, 3-way decision has received increasing concerns from many scholars, and rich achievements have been made in many fields, such as 3-way clustering [34,35], 3-way classification [36,37], 3-way conflict analysis [38,39], 3-way social network analysis [40,41], and 3-way approximation [42,43]. To overcome the limitation of FCA, 3-way concept analysis (3WCA) [23] was created by integrating FCA with 3-way decision, and has two basic notions of 3-way concepts and 3-way concept lattices. Notably, 3-way concepts can come in two varieties: object-induced three-way (OE3W) concept and attribute-induced three-way (AE3W) concept. Furthermore, 3WCA has been extended to fuzzy 3WCA [25,44], L-fuzzy 3WCA [45,46], partially-known 3WCA [47–49], 3-values 3WCA [50] and dual 3WCA [51]. Because 3WCA has advantages, 3WCA has been applied to rules acquisition [17], concept learning [52], conflict analysis [53], and medical diagnose [54].

Since constructing or updating 3-way concept lattices is an important issue in 3WCA, some scholars have made efforts on the issue. Qian et al. [55] defined two new kinds of formal contexts based on the given formal context, transformed the issue of constructing 3-way concept lattices into that of constructing the concept lattices of these two new kinds of formal contexts, and proposed a novel method of obtaining 3-way concepts. Wang et al. [56] gave the CBO3C algorithm to extract 3-way concepts, where partial closure canonicity, inherited failure tests and reduction condition were employed to improve the efficiency. Yang et al. [57] proposed a new algorithm of extracting 3-way concepts by combining concepts from the original formal context with those from its complement context, and demonstrated that their method outperformed the methods [55,56]. Long et al. [58] developed a new method of updating 3-way granular concepts from dynamic formal contexts, where attributes or objects were deleted. Hao et al. [40] proposed a new method of incrementally updating 3-way concepts from dynamic formal contexts of social networks, where attributes or objects were added. Hu et al. [59,60] considered how to update 3-way concepts from dynamic formal contexts, where attributes or objects were added or deleted, or granularity levels of attributes were changed. It is remarkable that: (i) the above methods only discussed how to extract or update 3-way concepts from static or dynamic formal contexts, but ignored how to establish or maintain covering relations among 3-way concepts; (ii) Hao et al. [40] and Hu et al. [59,60] did not deeply discuss the intrinsic theoretical relationships between 3-way concepts before and after changing formal contexts, reducing the efficiency of updating 3-way concept lattices.

In the classic contexts of FCA, either an object possesses some attributes, or it does not possess some attributes. Such contexts can be perceived as a binary table which only contains 0s and 1s. But some attributes in real data sets can be many-valued, such as categorical attributes, ordinal attributes and nominal attributes. For these data tables, classic contexts can also be obtained by using the scaling approach [2]. Namely, the appropriate attribute values at a specific attribute-granularity level can be used to replace those of each many-valued attribute. According to different requirements, we could choose the most appropriate attribute-granularity level. More specifically, if we want more precise concepts, it is required to increase attribute-granularity levels. Conversely, a wise idea is to decrease attribute-granularity levels. Therefore, changing attribute-granularity levels occurs frequently for discovering the required conceptual knowledge.

Whenever attribute-granularity levels undergo changes, if we directly reconstruct a new 2-way (or 3-way) concept lattice, required computation will be huge and elapsed time will be long, so indirectly renewing 2-way (or 3-way) concept lattices is a good strategy. Many studies [14,15] investigated the issue of renewing 2-way concept lattices when attribute-granularity levels are changed. However, few scholars have conducted studies on the issue of renewing 3-way concept lattices when attribute-granularity levels are changed, which is becoming a huge challenge due to the unique characteristics of 3-way concept lattices. In this study, we focus on renewing OE3W concept lattices involving decreasing attribute-granularity levels since OE3W concept lattices are more useful in knowledge discovery than AE3W concept lattices. What is more, changes of OE3W concept lattices involving the decrease are also investigated to facilitate the study.

The rest of our paper is organized as follows. Firstly, we recall basic notions and propositions of 3WCA in Section 2. Secondly, we analyze changes of OE3W concept lattices involving decreasing attribute-granularity levels in Section 3. Thirdly, we put forward a novel algorithm named OEL-Collapse to renew OE3W concept lattices involving the decrease in Section 4. Fourthly, we illustrate the efficiency of the OEL-Collapse algorithm through experiments in the second-to-last section. Finally, a brief conclusion is presented in the last section.

2. Preliminaries

In this section, to serve our work, some basic notions and propositions in FCA and 3WCA are introduced. For full details, please see the literatures [1,14,23].

Let $\mathcal{PS}(\cdot)$ be the power set of a set, and $\mathcal{CP}(\cdot)$ be the Cartesian product $\mathcal{PS}(\cdot) \times \mathcal{PS}(\cdot)$. For $(J, K), (L, M) \in \mathcal{CP}(G)$ (G is a set), the inclusion relation and join and intersection operations are defined as follows:

$$(J, K) \subseteq (L, M) \iff J \subseteq L, K \subseteq W$$

$$(J, K) \cap (L, M) = (J \cap L, K \cap W)$$

$$(J, K) \cup (L, M) = (J \cup L, K \cup W).$$

A formal context, which is the most elementary notion in FCA, could be understood as a triple (G, A, I), where every element $g \in G$ is an object, every element $a \in A$ is an attribute, and I represents the binary relation between G and A. Specifically, if object g owns attribute a, we have $(g, a) \in I$ (or $(g, a) \notin I^c$); otherwise, we have $(g, a) \notin I$ (or $(g, a) \in I^c$), where $I^c = (G \times A) - I$.

The positive forming operator $*$, the negative forming operator $\bar{*}$, and the 3-way forming operators $\lessdot$ and $\gtrdot$ in (G, A, I) are listed as follows. For $P, R \subseteq G$ and $L, K \subseteq A$,

$$* : \mathcal{PS}(G) \to \mathcal{PS}(A), P^* = \{a \in A | \forall g \in P, (g, a) \in I\},$$

$$* : \mathcal{PS}(A) \to \mathcal{PS}(G), K^* = \{g \in G | \forall a \in K, (g, a) \in I\},$$

$$\bar{*} : \mathcal{PS}(G) \to \mathcal{PS}(A), P^{\bar{*}} = \{a \in A | \forall g \in P, (g, a) \in I^c\},$$

$$\bar{*} : \mathcal{PS}(A) \to \mathcal{PS}(G), K^{\bar{*}} = \{g \in G | \forall a \in K, (g, a) \in I^c\},$$

$$\lessdot : \mathcal{PS}(G) \to \mathcal{CP}(A), P^{\lessdot} = (P^*, P^{\bar{*}}),$$

$$\lessdot : \mathcal{PS}(A) \to \mathcal{CP}(G), L^{\lessdot} = (L^*, L^{\bar{*}}),$$

$$\gtrdot : \mathcal{CP}(G) \to \mathcal{PS}(A), (P, R)^{\gtrdot} = \{a \in A | a \in P^*, a \in R^{\bar{*}}\} = P^* \cap R^{\bar{*}},$$

$$\gtrdot : \mathcal{CP}(A) \to \mathcal{PS}(G), (L, K)^{\gtrdot} = \{g \in G | g \in L^*, g \in K^{\bar{*}}\} = L^* \cap K^{\bar{*}}.$$

Applying these operators, formal concepts, N-concepts, OE3W concepts and AE3W concepts are given below. For $P, R \subseteq G$ and $L, K \subseteq A$,

(1) (P, K) is a formal concept iff $P^* = K$ and $K^* = P$;

(2) (P, K) is an N-concept (i.e., negative concept) iff $P^{\bar{*}} = K$ and $K^{\bar{*}} = P$;

(3) $(P, (L, K))$ is an OE3W concept iff $P^{\lessdot} = (L, K)$ and $(L, K)^{\gtrdot} = P$;

(4) $((P, R), L)$ is an AE3W concept iff $L^{\lessdot} = (P, R)$ and $(P, R)^{\gtrdot} = L$.

It is worth mentioning that these operators have some useful properties. Take the forming operators of OE3W concepts for example. For $P, P_1, P_2 \subseteq G$ and $L, L_1, L_2, K, K_1, K_2 \subseteq A$, the following properties hold:

(1) $P_1 \subseteq P_2 \Rightarrow P_2^{\lessdot} \subseteq P_1^{\lessdot}$ and $(L_1, K_1) \subseteq (L_2, K_2) \Rightarrow (L_2, K_2)^{\gtrdot} \subseteq (L_1, K_1)^{\gtrdot}$;

(2) $P \subseteq P^{\lessdot\gtrdot}$ and $(L, K) \subseteq (L, K)^{\gtrdot\lessdot}$;

(3) $P^{\lessdot} = P^{\lessdot\gtrdot\lessdot}$ and $(L, K)^{\gtrdot} = (L, K)^{\gtrdot\lessdot\gtrdot}$;

(4) $P \subseteq (L, K)^{\gtrdot} \Leftrightarrow (L, K) \subseteq P^{\lessdot}$;

(5) $(P_1 \cup P_2)^{\lessdot} = P_1^{\lessdot} \cap P_2^{\lessdot}$ and $((L_1, K_1) \cup (L_2, K_2))^{\gtrdot} = (L_1, K_1)^{\gtrdot} \cap (L_2, K_2)^{\gtrdot}$;

(6) $(P_1 \cap P_2)^{\lessdot} \supseteq P_1^{\lessdot} \cup P_2^{\lessdot}$ and $((L_1, K_1) \cap (L_2, K_2))^{\gtrdot} \supseteq (L_1, K_1)^{\gtrdot} \cup (L_2, K_2)^{\gtrdot}$.

Organizing all formal concepts, N-concepts, AE3W concepts and OE3W concepts, respectively, formal concept lattice, N-concept lattice, AE3W concept lattice and OE3W concept lattice are formed with the partial order $\leqslant$ as follows. Take OE3W concepts as an example. For two OE3W concepts $(P_1, (J_1, K_1))$ and $(P_2, (J_2, K_2))$, if $P_1 \subseteq P_2$ or $(J_2, K_2) \subseteq (J_1, K_1)$, then $(P_1, (J_1, K_1))$ is an OE3W sub-concept of $(P_2, (J_2, K_2))$, or $(P_2, (J_2, K_2))$ is an OE3W super-concept of $(P_1, (J_1, K_1))$, denoted by $(P_1, (J_1, K_1)) \leqslant (P_2, (J_2, K_2))$.

Based on the partial order $\leqslant$, a covering relation $\prec$ can be defined. Take OE3W concepts as an example. For two OE3W concepts $(P_1, (J_1, K_1))$ and $(P_2, (J_2, K_2))$, if $(P_1, (J_1, K_1)) \leqslant (P_2, (J_2, K_2))$ and no another OE3W concept $(P_3, (J_3, K_3))$ with $(P_1, (J_1, K_1)) \leqslant (P_3, (J_3, K_3)) \leqslant (P_2, (J_2, K_2))$ exists, then $(P_1, (J_1, K_1))$ is a child concept of $(P_2, (J_2, K_2))$, or $(P_2, (J_2, K_2))$ is a parent concept of $(P_1, (J_1, K_1))$, denoted by $(P_1, (J_1, K_1)) \prec (P_2, (J_2, K_2))$.

To employ attribute-granularity in FCA conveniently, three notions of attribute-granularity trees, cuts and decreasing attribute-granularity levels are introduced:

(1) for attribute a, an attribute-granularity tree (abbreviated as g-tree) is such a tree whose nodes are marked as the specific attribute-values, and for any node γ, the objects having any children node γ_i ($i \in \{1, \cdots, n\}$) jointly form a partition of the set γ^* of the objects having the parent node γ;

(2) for attribute a, a set of nodes $\Gamma = \{\gamma_1, \cdots, \gamma_n\}$ of the g-tree is a cut if the objects having any node of Γ jointly form a partition of the objects having the root node γ, i.e. $\gamma_1^*, \cdots, \gamma_n^*$ form a partition of γ^*; for two cuts Γ^1 and Γ^2 of the g-tree, if for any $\gamma_i \in \Gamma^1$, there exists $\gamma_j' \in \Gamma^2$ such that $\gamma_i^* \subseteq \gamma_j'^*$, Γ^1 is said to be finer than Γ^2, or Γ^2 is coarser than Γ^1, denoted by $\Gamma^1 \leqslant \Gamma^2$;

(3) for attribute a, decreasing attribute-granularity levels is to replace the existing cut Γ^1 with another coarser cut Γ^2, i.e. $\Gamma^1 \leqslant \Gamma^2$.

Example 1. A context $DT_1 = (G, A)$ is described in Table 1, where $1, 2, \ldots, 5$ are five objects in G, $a, b, \ldots, f$ are six attributes in A, and the first five attributes are one-valued attributes, while the sixth attribute f is a many-valued attribute. Fig. 1 shows the g-tree of attribute f by which we can obtain the contexts DT_2 and DT_3. $\{l\}$ and $\{l_1', l_2'\}$ are two cuts of the g-tree due to $l^* = \{1, 2, 3, 4, 5\}$ and $l_1'^* \cup l_2'^* = \{1, 2\} \cup \{3, 4, 5\} = \{1, 2, 3, 4, 5\}$. Similarly, $\{l_1', l_3, l_4\}$, $\{l_1, l_2, l_2'\}$ and $\{l_1, l_2, l_3, l_4\}$ are another three cuts of the g-tree. Note that $l_1'^*$ and $l_2'^*$ jointly form a partition of l^*, yielding $\{l_1', l_2'\} \leqslant \{l\}$. In addition, we can obtain the following statements.

(1) For attribute f, the relationships among its five cuts are as follows: $\{l_1, l_2, l_3, l_4\} \leqslant \{l_1', l_2'\} \leqslant \{l\}$, $\{l_1, l_2, l_3, l_4\} \leqslant \{l_1', l_3, l_4\} \leqslant \{l_1', l_2'\}$, and $\{l_1, l_2, l_3, l_4\} \leqslant \{l_1, l_2, l_2'\} \leqslant \{l_1', l_2'\}$.

(2) Replacing the cut $\{l_1, l_2, l_3, l_4\}$ with the cut $\{l_1', l_2'\}$ is to decrease the attribute-granularity level of attribute f, and the context DT_2 in Table 2 is obtained.

(3) Replacing the cut $\{l_1', l_2'\}$ with the cut $\{l\}$ is further to decrease the attribute-granularity level of attribute f, and the context DT_3 in Table 3 is obtained.

Table 1
The context DT_1.

G	a	b	c	d	e	f
$g_1 = 1$	1	1	0	0	0	l_1
$g_2 = 2$	1	1	0	0	0	l_2
$g_3 = 3$	0	0	1	1	0	l_3
$g_4 = 4$	1	0	1	1	1	l_4
$g_5 = 5$	0	1	0	0	1	l_3

Table 2
The context DT_2.

G	a	b	c	d	e	f
$g_1 = 1$	1	1	0	0	0	l'_1
$g_2 = 2$	1	1	0	0	0	l'_1
$g_3 = 3$	0	0	1	1	0	l'_2
$g_4 = 4$	1	0	1	1	1	l'_2
$g_5 = 5$	0	1	0	0	1	l'_2

Table 3
The context DT_3.

G	a	b	c	d	e	f
$g_1 = 1$	1	1	0	0	0	l
$g_2 = 2$	1	1	0	0	0	l
$g_3 = 3$	0	0	1	1	0	l
$g_4 = 4$	1	0	1	1	1	l
$g_5 = 5$	0	1	0	0	1	l

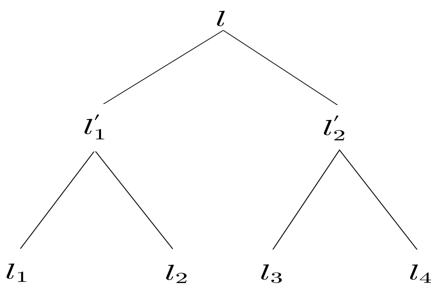

Fig. 1. The g-tree of attribute f.

3. Changes of OE3W concept lattices involving attribute-granularity levels decrease

In the previous section, we have discussed decreasing attribute-granularity levels by replacing the current cut with a coarser one. In what follows, we further divide it into two categories.

Definition 1. For attribute a, slightly decreasing attribute-granularity level is to replace the existing cut $\Gamma^{\downarrow} = \{\gamma_i | i = 1, \cdots, n^f\}$ with another cut $\Gamma^{\Downarrow} = \{\gamma'_j | j = 1, \cdots, n^c\}$, where $\Gamma^{\downarrow}$ and $\Gamma^{\Downarrow}$ are two cuts of a, $|\Gamma^{\downarrow}| \geqslant 2$, $|\Gamma^{\Downarrow}| \geqslant 2$, and $\Gamma^{\downarrow} \leqslant \Gamma^{\Downarrow}$.

In the remainder of our paper, when $|\Gamma^{\Downarrow}| = 1$ in Definition 1, it is called sharply decreasing attribute-granularity levels to distinguish this case from slightly decreasing attribute-granularity levels. In other words, decreasing attribute-granularity levels includes two cases: sharply decreasing attribute-granularity levels and slightly decreasing attribute-granularity levels. In this paper, we focus on the latter since the latter is much more common than the former in the real world and the latter can be deduced to the former.

Example 2. Continued from Example 1. Obviously, we can obtain the following statements:
(1) replacing the cut $\{l_1, l_2, l_3, l_4\}$ with the cut $\{l'_1, l'_2\}$ is slightly decreasing attribute-granularity level of attribute f;
(2) replacing the cut $\{l'_1, l'_2\}$ with the cut l is sharply decreasing the attribute-granularity level of attribute f.

In this section, we are concerned with this issue: for a formal context (G, A, I), if we slightly decrease attribute-granularity levels of attribute $a \in A$, what changes do OE3W concept lattices undergo?

Hereafter, to avoid confusion, before and after the slight decrease, positive forming operators, negative forming operators, 3-way forming operators are denoted by $*^{\downarrow}$, $\overline{*}^{\downarrow}$, $\lessdot^{\downarrow}$, $\gtrdot^{\downarrow}$ and $*^{\Downarrow}$, $\overline{*}^{\Downarrow}$, $\lessdot^{\Downarrow}$, $\gtrdot^{\Downarrow}$, respectively. OE3W concept lattices are denoted by $OEL^{\downarrow}$ and $OEL^{\Downarrow}$, respectively, covering relations and subconcept-superconcept relations in OE3W concept lattices are denoted by $\prec_{\downarrow}$, $\leqslant_{\downarrow}$ and $\prec_{\Downarrow}$, $\leqslant_{\Downarrow}$, respectively.

Notably, the attribute-value conversion function $\varphi : \Gamma^{\downarrow} \to \Gamma^{\Downarrow}$ exists, where $\varphi(\gamma_i) = \gamma'_j$. Namely, for attribute a, if object g owns attribute-value γ_i before the slight decrease, then object g owns attribute-value $\varphi(\gamma_i)$ after the slight decrease. To facilitate subsequent analysis, based on the function φ, we need to define five other mappings as follows:

$\phi : \mathcal{PS}(\Gamma^{\downarrow}) \to \mathcal{P}(\Gamma^{\Downarrow})$ satisfying $\phi(J) = \{\varphi(\gamma) \in \Gamma^{\Downarrow} | \gamma \in J\}$,

$\psi : \Gamma^{\Downarrow} \to \mathcal{PS}(\Gamma^{\downarrow})$ satisfying $\psi(\gamma'_j) = \{\gamma \in \Gamma^{\downarrow} | \varphi(\gamma) = \gamma'_j\}$,

$\rho : \Gamma^{\downarrow} \to \mathcal{PS}(\Gamma^{\downarrow})$ satisfying $\rho(\gamma) = \psi(\varphi(\gamma))$,

$\omega : \mathcal{PS}(\Gamma^{\downarrow}) \to \mathcal{PS}(\Gamma^{\downarrow})$ satisfying $\omega(J) = \{\gamma \in \Gamma^{\downarrow} | \rho(\gamma) \subseteq J, \gamma \in J\}$,

$\tau : \mathcal{PS}(\Gamma^{\downarrow}) \to \mathcal{PS}(\Gamma^{\Downarrow})$ satisfying $\tau(J) = \{\rho(\gamma) \in \Gamma^{\Downarrow} | \rho(\gamma) \subseteq J, \gamma \in J\}$.

Next, some basic and useful propositions are listed in Propositions 1 and 2, which will be used in later discussion.

Proposition 1. *For $\gamma \in \Gamma^{\downarrow}$, $\gamma' \in \Gamma^{\Downarrow}$, $\emptyset \subset J \subset \Gamma^{\downarrow}$ and $\emptyset \subset K \subset \Gamma^{\Downarrow}$, we have*

(1) $\gamma'^{*^{\Downarrow}} = \bigcup_{\varphi(\gamma)=\gamma'} \gamma^{*^{\downarrow}}$;

(2) $\gamma'^{\overline{*}^{\Downarrow}} = \bigcap_{\varphi(\gamma)=\gamma'} \gamma^{\overline{*}^{\downarrow}}$;

(3) $\gamma^{*^{\downarrow}} \subseteq \varphi(\gamma)^{*^{\Downarrow}}$;

(4) $\gamma^{\overline{*}^{\downarrow}} \supseteq \varphi(\gamma)^{\overline{*}^{\Downarrow}}$;

(5) $\gamma'^{*^{\Downarrow}} = (\Gamma^{\Downarrow} - \{\gamma'\})^{\overline{*}^{\Downarrow}}$;

(6) $\gamma^{*^{\downarrow}} = (\Gamma^{\downarrow} - \{\gamma\})^{\overline{*}^{\downarrow}}$;

(7) $\bigcup_{\gamma \in J} \gamma^{*^{\downarrow}} = \bigcap_{\chi \in \Gamma^{\downarrow} - J} \chi^{\overline{*}^{\downarrow}}$;

(8) $\bigcup_{\gamma' \in K} \gamma'^{*^{\Downarrow}} = \bigcap_{\chi' \in \Gamma^{\Downarrow} - K} \chi'^{\overline{*}^{\Downarrow}}$.

Proof. They are immediate. □

Proposition 2. *For $(P, (J, K)) \in OEL^{\downarrow}$ with $P \neq \emptyset$ and $(Q, (L, M)) \in OEL^{\Downarrow}$ with $Q \neq \emptyset$, we have*

(1) if $J \cap \Gamma^{\downarrow} \neq \emptyset$, then $|J \cap \Gamma^{\downarrow}| = 1$ and $K \cap \Gamma^{\downarrow} = \Gamma^{\downarrow} - J \cap \Gamma^{\downarrow}$;

(2) if $L \cap \Gamma^{\Downarrow} \neq \emptyset$, then $|L \cap \Gamma^{\Downarrow}| = 1$ and $M \cap \Gamma^{\Downarrow} = \Gamma^{\Downarrow} - L \cap \Gamma^{\Downarrow}$.

Proof. (1) First, proof by contradiction is applied to verify $|J \cap \Gamma^{\downarrow}| = 1$. If $|J \cap \Gamma^{\downarrow}| > 1$, then $(J \cap \Gamma^{\downarrow})^{*^{\downarrow}} = \emptyset$ stands. So, we have $P = ((J - \Gamma^{\downarrow}) \cup (J \cap \Gamma^{\downarrow}))^{*^{\downarrow}} \cap K^{\overline{*}^{\downarrow}} = \emptyset \cap K^{\overline{*}^{\downarrow}} = \emptyset$, which contradicts with $P \neq \emptyset$. Thus, $|J \cap \Gamma^{\downarrow}| = 1$ holds.

Next, we prove $K \cap \Gamma^{\downarrow} = \Gamma^{\Downarrow} - J \cap \Gamma^{\downarrow}$. Since $|J \cap \Gamma^{\downarrow}| = 1$, we denote the unique element in $J \cap \Gamma^{\downarrow}$ by γ_0. From Proposition 1, $\gamma_0{}^{*^{\downarrow}} = (\Gamma^{\downarrow} - \{\gamma_0\})^{\overline{*}^{\downarrow}}$, which implies $P^{\lessdot^{\downarrow}} \cap (\Gamma^{\downarrow}, \Gamma^{\downarrow}) = (\{\gamma_0\}, \Gamma^{\downarrow} - \{\gamma_0\})$. Equivalently speaking, $K \cap \Gamma^{\downarrow} = \Gamma^{\downarrow} - J \cap \Gamma^{\downarrow}$

(2) The proof is similar to that of (1). □

3.1. Changes of OE3W concepts involving slight attribute-granularity decrease

For OE3W concept lattices, before and after slightly decreasing attribute-granularity levels, some OE3W concepts remain constant, but some OE3W concepts need to be deleted, other OE3W concepts need to be locally amended, and the rest of OE3W concepts need to be newly generated. We describe changes of OE3W concepts involving the slight decrease in the following definition.

Definition 2. For $(P, (J, K)) \in OEL^{\downarrow}$ and $(Q, (L, M)) \in OEL^{\Downarrow}$,

(1) $(P, (J, K))$ is an old OE3W concept, written as $class(P, (J, K)) =$ "old", if $(P, (J, K)) \in OEL^{\Downarrow}$;

(2) $(Q, (L, M))$ is a new OE3W concept, written as $class(Q, (L, M)) =$ "new", if Q is not the extent of any OE3W concept in $OEL^{\downarrow}$;

(3) $(P, (J, K))$ is a deleted OE3W concept, written as $class(P, (J, K)) =$ "deleted", if P is not the extent of any OE3W concept in $OEL^{\Downarrow}$;

(4) $(P, (J, K))$ is an extended-condensed OE3W concept, written as $class(P, (J, K)) =$ "extended-condensed", if $J \cap \Gamma^{\downarrow} = \emptyset$, $K \cap \Gamma^{\downarrow} \neq \emptyset$, and $(P, (J \cup \{\gamma'_k\}, (K - \Gamma^{\downarrow}) \cup \tau(K \cap \Gamma^{\downarrow}))) \in OEL^{\Downarrow}$, where $\gamma'_k \in \Gamma^{\Downarrow}$;

(5) $(P, (J, K))$ is a modified-condensed OE3W concept, written as $class(P, (J, K)) =$ "modified-condensed", if $J \cap \Gamma^{\downarrow} \neq \emptyset$, $K \cap \Gamma^{\downarrow} \neq \emptyset$, and $(P, ((J - \Gamma^{\downarrow}) \cup \phi(J \cap \Gamma^{\downarrow}), (K - \Gamma^{\downarrow}) \cup \tau(K \cap \Gamma^{\downarrow}))) \in OEL^{\Downarrow}$;

(6) $(P, (J, K))$ is an unchanged-condensed OE3W concept, written as $class(P, (J, K)) =$ "unchanged-condensed", if $J \cap \Gamma^{\downarrow} = \emptyset$, $K \cap \Gamma^{\downarrow} \neq \emptyset$, and $(P, (J, (K - \Gamma^{\downarrow}) \cup \tau(K \cap \Gamma^{\downarrow}))) \in OEL^{\Downarrow}$.

Now we discuss sufficient and necessary conditions of these categories.

Proposition 3. *For $(P,(J,K)) \in OEL^{\Downarrow}$, P is still an extent in $OEL^{\downarrow}$.*

Proof. There are four situations for any $(P,(J,K)) \in OEL^{\Downarrow}$.

(i) If $J \cap \Gamma^{\Downarrow} = \emptyset$ and $K \cap \Gamma^{\Downarrow} = \emptyset$, we have $(J,K)^{\gtrdot\Downarrow} = J^{*\Downarrow} \cap K^{\bar{*}\Downarrow} = J^{*\downarrow} \cap K^{\bar{*}\downarrow} = (J,K)^{\gtrdot\downarrow}$. It is straightforward that P is an extent in $OEL^{\downarrow}$.

(ii) If $J \cap \Gamma^{\Downarrow} = \emptyset$ and $K \cap \Gamma^{\Downarrow} \neq \emptyset$, according to Proposition 1, we can get $(K \cap \Gamma^{\Downarrow})^{\bar{*}\Downarrow} = \Lambda^{\bar{*}\downarrow}$, where $\Lambda = \bigcup_{\gamma' \in K \cap \Gamma^{\Downarrow}} \psi(\gamma')$. Then, $(J,K)^{\gtrdot\Downarrow} = J^{*\Downarrow} \cap K^{\bar{*}\Downarrow} = J^{*\downarrow} \cap ((K - \Gamma^{\Downarrow}) \cup \Lambda)^{\bar{*}\downarrow} = (J,((K - \Gamma^{\Downarrow}) \cup \Lambda))^{\gtrdot\downarrow}$. Therefore, P is an extent in $OEL^{\downarrow}$.

(iii) If $J \cap \Gamma^{\Downarrow} \neq \emptyset$ with $|J \cap \Gamma^{\Downarrow}| = 1$ and $K \cap \Gamma^{\Downarrow} \neq \emptyset$, then the unique element in $J \cap \Gamma^{\Downarrow}$ is denoted by γ_0'. By Proposition 2, we obtain $K \cap \Gamma^{\Downarrow} = \Gamma^{\Downarrow} - \{\gamma_0'\}$. By Proposition 1, we have ${\gamma_0'}^{*\Downarrow} = (\Gamma^{\Downarrow} - \{\gamma_0'\})^{\bar{*}\Downarrow} = \Lambda^{\bar{*}\downarrow}$, where $\Lambda = \bigcup_{\gamma' \in \Gamma^{\Downarrow} - \{\gamma_0'\}} \psi(\gamma')$. Then, $(J,K)^{\gtrdot\Downarrow} = J^{*\Downarrow} \cap K^{\bar{*}\Downarrow} = ((J - \Gamma^{\Downarrow}) \cup \{\gamma_0'\})^{*\Downarrow} \cap ((K - \Gamma^{\Downarrow}) \cup (\Gamma^{\Downarrow} - \{\gamma_0'\}))^{\bar{*}\Downarrow} = (J - \Gamma^{\Downarrow})^{*\Downarrow} \cap ((K - \Gamma^{\Downarrow}) \cup (\Gamma^{\Downarrow} - \{\gamma_0'\}))^{\bar{*}\Downarrow} = (J - \Gamma^{\Downarrow})^{*\downarrow} \cap ((K - \Gamma^{\Downarrow}) \cup \Lambda)^{\bar{*}\downarrow} = ((J - \Gamma^{\Downarrow}),(K - \Gamma^{\Downarrow}) \cup \Lambda))^{\gtrdot\downarrow}$. Thus, P is an extent in $OEL^{\downarrow}$.

(iv) If $J \cap \Gamma^{\Downarrow} \neq \emptyset$ with $|J \cap \Gamma^{\Downarrow}| > 1$ and $K \cap \Gamma^{\Downarrow} \neq \emptyset$, then we have $P = \emptyset$. Since $\emptyset$ is an extent in $OEL^{\downarrow}$, P is an extent in $OEL^{\downarrow}$.

Finally, for any $(P,(J,K)) \in OEL^{\Downarrow}$, P is still an extent in $OEL^{\downarrow}$. $\square$

From Proposition 3, we can derive the following theorem.

Theorem 1. *No new OE3W concepts exist in $OEL^{\Downarrow}$.*

Proof. It is immediate from Proposition 3. $\square$

Theorem 2. *For $(P,(J,K)) \in OEL^{\downarrow}$ with $P \neq \emptyset$, $(P,(J,K))$ is a deleted OE3W concept iff the following conditions stand:*

(1) $K \cap \Gamma^{\downarrow} \neq \emptyset$;

(2) at least one OE3W concept $(Q,(L,M))$ with $L \cap \Gamma^{\downarrow} = \emptyset$ exists among parents of $(P,(J,K))$ in $OEL^{\downarrow}$, where $(Q,(L,M))$ meets requirements: $L = J - \Gamma^{\downarrow}$, $M - \Gamma^{\downarrow} = K - \Gamma^{\downarrow}$ and $\omega(M \cap \Gamma^{\downarrow}) = \omega(K \cap \Gamma^{\downarrow})$.

Proof. ($\Rightarrow$) Firstly, proof by contradiction is applied to show $K \cap \Gamma^{\downarrow} \neq \emptyset$. Assume $K \cap \Gamma^{\downarrow} = \emptyset$. It is obvious that $J \cap \Gamma^{\downarrow} = \emptyset$ from Proposition 2. Nothing that the objects in the context remain unchanged before and after the decrease, we have $P = (J,K)^{\gtrdot\downarrow} = J^{*\downarrow} \cap K^{\bar{*}\downarrow} = J^{*\Downarrow} \cap K^{\bar{*}\Downarrow}$. Therefore, we have that $P = J^{*\Downarrow} \cap K^{\bar{*}\Downarrow}$ is an extent in $OEL^{\Downarrow}$, which contradicts with $class(P,(J,K)) =$ "deleted". Thus, $K \cap \Gamma^{\downarrow} \neq \emptyset$.

Secondly, we prove the remainder. Two cases for $(P,(J,K))$ are considered.

(i) If $J \cap \Gamma^{\downarrow} = \emptyset$, combined with $K \cap \Gamma^{\downarrow} \neq \emptyset$, we easily obtain $P \subseteq (K \cap \Gamma^{\downarrow})^{\bar{*}\downarrow} \subseteq \omega(K \cap \Gamma^{\downarrow})^{\bar{*}\downarrow}$, and $P \nsubseteq (\Gamma^{\downarrow} - K \cap \Gamma^{\downarrow})^{\bar{*}\downarrow}$. Then, according to Proposition 1, for any $\gamma' \in \Gamma^{\Downarrow}$, we know that if $\gamma' \in \tau(K \cap \Gamma^{\downarrow})$, $P \subseteq {\gamma'}^{\bar{*}\Downarrow}$ holds; otherwise, $P \nsubseteq {\gamma'}^{\bar{*}\Downarrow}$ holds. Since $P^{\lessdot\downarrow} = (J,(K - \Gamma^{\downarrow}) \cup (K \cap \Gamma^{\downarrow}))$ holds, we have $P^{\lessdot\Downarrow} = (J,(K - \Gamma^{\downarrow}) \cup \tau(K \cap \Gamma^{\downarrow}))$. Therefore, $P^{\lessdot\Downarrow\gtrdot\Downarrow} = J^{*\Downarrow} \cap (K - \Gamma^{\downarrow})^{\bar{*}\Downarrow} \cap \tau(K \cap \Gamma^{\downarrow})^{\bar{*}\Downarrow} = J^{*\downarrow} \cap (K - \Gamma^{\downarrow})^{\bar{*}\downarrow} \cap \omega(K \cap \Gamma^{\downarrow})^{\bar{*}\downarrow} = (J,((K - \Gamma^{\downarrow}) \cup \omega(K \cap \Gamma^{\downarrow})))^{\gtrdot\downarrow}$. Combining the fact that $(P,(J,K))$ is a deleted OE3W concept, which leads to $P \subset P^{\lessdot\Downarrow\gtrdot\Downarrow}$, we can obtain that $P \subset (J,((K - \Gamma^{\downarrow}) \cup \omega(K \cap \Gamma^{\downarrow})))^{\gtrdot\downarrow}$ and $P \neq (J,((K - \Gamma^{\downarrow}) \cup \omega(K \cap \Gamma^{\downarrow})))^{\gtrdot\downarrow}$. Therefore, we know that there is an OE3W concept $(R,(D,E))$ in $OEL^{\downarrow}$, where $(R,(D,E)) = ((J,((K - \Gamma^{\downarrow}) \cup \omega(K \cap \Gamma^{\downarrow})))^{\gtrdot\downarrow},(J,((K - \Gamma^{\downarrow}) \cup \omega(K \cap \Gamma^{\downarrow})))^{\gtrdot\downarrow\lessdot\downarrow})$, and $(P,(J,K)) <_{\downarrow} (R,(D,E))$.

Next, we prove that $(R,(D,E))$ meets the following requirements. From $(D,E) = (J,((K - \Gamma^{\downarrow}) \cup \omega(K \cap \Gamma^{\downarrow})))^{\gtrdot\downarrow\lessdot\downarrow}$, we get $(J,(K - \Gamma^{\downarrow}) \cup \omega(K \cap \Gamma^{\downarrow})) \subseteq (D,(E - \Gamma^{\downarrow}) \cup (E \cap \Gamma^{\downarrow}))$, which demonstrates $\omega(K \cap \Gamma^{\downarrow}) \subseteq \omega(E \cap \Gamma^{\downarrow})$. Furthermore, since $(D,(E - \Gamma^{\downarrow}) \cup (E \cap \Gamma^{\downarrow})) \subseteq (J,(K - \Gamma^{\downarrow}) \cup (K \cap \Gamma^{\downarrow}))$ holds by $(P,(J,K)) \leqslant_{\downarrow} (R,(D,E))$, we have $\omega(E \cap \Gamma^{\downarrow}) \subseteq \omega(K \cap \Gamma^{\downarrow})$. Thus, we obtain that $J \subseteq D \subseteq J$, $K - \Gamma^{\downarrow} \subseteq E - \Gamma^{\downarrow} \subseteq K - \Gamma^{\downarrow}$ and $\omega(K \cap \Gamma^{\downarrow}) \subseteq \omega(E \cap \Gamma^{\downarrow}) \subseteq \omega(K \cap \Gamma^{\downarrow})$. Equivalently saying, it follows that $D = J$, $E - \Gamma^{\downarrow} = K - \Gamma^{\downarrow}$ and $\omega(E \cap \Gamma^{\downarrow}) = \omega(K \cap \Gamma^{\downarrow})$.

Consequently, we can get that $(R,(D,E))$ is either a parent of $(P,(J,K))$ or not. If $(R,(D,E))$ is not a parent of $(P,(J,K))$, then at least one OE3W concept $(Q,(L,M))$ exists among the parents of $(P,(J,K))$ in $OEL^{\downarrow}$, where $(Q,(L,M))$ meets requirements: $L = J$, $M - \Gamma^{\downarrow} = K - \Gamma^{\downarrow}$ and $\omega(M \cap \Gamma^{\downarrow}) = \omega(K \cap \Gamma^{\downarrow})$.

(ii) If $J \cap \Gamma^{\downarrow} \neq \emptyset$, according to Proposition 2, we easily obtain that $K \cap \Gamma^{\downarrow} = \Gamma^{\downarrow} - J$, $|K \cap \Gamma^{\downarrow}| = |\Gamma^{\downarrow}| - 1$ and $|J \cap \Gamma^{\downarrow}| = 1$. Then, it follows $P \subseteq (J \cap \Gamma^{\downarrow})^{*\downarrow} \subseteq \phi(J \cap \Gamma^{\downarrow})^{*\Downarrow}$ and $P \subseteq (K \cap \Gamma^{\downarrow})^{\bar{*}\downarrow} \subseteq \omega(K \cap \Gamma^{\downarrow})^{\bar{*}\downarrow} = \tau(K \cap \Gamma^{\downarrow})^{\bar{*}\Downarrow}$. Obviously, $|\phi(J \cap \Gamma^{\downarrow})| = 1$ and $|\tau(K \cap \Gamma^{\downarrow})| = |\Gamma^{\Downarrow}| - 1$. Since $P^{\lessdot\downarrow} = (J,(K - \Gamma^{\downarrow}) \cup (K \cap \Gamma^{\downarrow}))$ holds, we have $P^{\lessdot\Downarrow} = ((J - \Gamma^{\downarrow}) \cup \phi(J \cap \Gamma^{\downarrow}),(K - \Gamma^{\downarrow}) \cup \tau(K \cap \Gamma^{\downarrow}))$. Therefore, $P^{\lessdot\Downarrow\gtrdot\Downarrow} = (J - \Gamma^{\downarrow})^{*\Downarrow} \cap \phi(J \cap \Gamma^{\downarrow})^{*\Downarrow} \cap (K - \Gamma^{\downarrow})^{\bar{*}\Downarrow} \cap \tau(K \cap \Gamma^{\downarrow})^{\bar{*}\Downarrow} = (J - \Gamma^{\downarrow})^{*\downarrow} \cap (K - \Gamma^{\downarrow})^{\bar{*}\downarrow} \cap \tau(K \cap \Gamma^{\downarrow})^{\bar{*}\Downarrow} = (J - \Gamma^{\downarrow})^{*\downarrow} \cap (K - \Gamma^{\downarrow})^{\bar{*}\downarrow} \cap \omega(K \cap \Gamma^{\downarrow})^{\bar{*}\downarrow} = (J - \Gamma^{\downarrow},(K - \Gamma^{\downarrow}) \cup \omega(K \cap \Gamma^{\downarrow}))^{\gtrdot\downarrow}$. Combining $(P,(J,K))$ being a deleted OE3W concept, which leads to $P \subset P^{\lessdot\Downarrow\gtrdot\Downarrow}$, we can obtain that $P \subset (J - \Gamma^{\downarrow},(K - \Gamma^{\downarrow}) \cup \omega(K \cap \Gamma^{\downarrow}))^{\gtrdot\downarrow}$ and $P \neq (J - \Gamma^{\downarrow},(K - \Gamma^{\downarrow}) \cup \omega(K \cap \Gamma^{\downarrow}))^{\gtrdot\downarrow}$. Therefore, we know that there is an OE3W concept $(R,(D,E))$ in $OEL^{\downarrow}$, where $(R,(D,E)) = ((J - \Gamma^{\downarrow},(K - \Gamma^{\downarrow}) \cup \omega(K \cap \Gamma^{\downarrow}))^{\gtrdot\downarrow},(J - \Gamma^{\downarrow},(K - \Gamma^{\downarrow}) \cup \omega(K \cap \Gamma^{\downarrow}))^{\gtrdot\downarrow\lessdot\downarrow})$, and $(P,(J,K)) <_{\downarrow} (R,(D,E))$.

Next, we prove that $(R,(D,E))$ meets the following requirements. From $(D,E) = (J - \Gamma^{\downarrow},(K - \Gamma^{\downarrow}) \cup \omega(K \cap \Gamma^{\downarrow}))^{\gtrdot\downarrow\lessdot\downarrow}$, we get $(J - \Gamma^{\downarrow},(K - \Gamma^{\downarrow}) \cup \omega(K \cap \Gamma^{\downarrow})) \subseteq (D,(E - \Gamma^{\downarrow}) \cup (E \cap \Gamma^{\downarrow}))$, which demonstrates $\omega(K \cap \Gamma^{\downarrow}) \subseteq \omega(E \cap \Gamma^{\downarrow})$. Furthermore, since $(D,(E - \Gamma^{\downarrow}) \cup (E \cap \Gamma^{\downarrow})) \subseteq (J,(K - \Gamma^{\downarrow}) \cup (K \cap \Gamma^{\downarrow}))$ holds by $(P,(J,K)) \leqslant_{\downarrow} (R,(D,E))$, we have $\omega(E \cap \Gamma^{\downarrow}) \subseteq \omega(K \cap \Gamma^{\downarrow})$. Thus, we obtain that $J - \Gamma^{\downarrow} \subseteq D \subseteq J$, $K - \Gamma^{\downarrow} \subseteq E - \Gamma^{\downarrow} \subseteq K - \Gamma^{\downarrow}$ and $\omega(K \cap \Gamma^{\downarrow}) \subseteq \omega(E \cap \Gamma^{\downarrow}) \subseteq \omega(K \cap \Gamma^{\downarrow})$. Equivalently saying, it follows that $E - \Gamma^{\downarrow} = K - \Gamma^{\downarrow}$, $\omega(E \cap \Gamma^{\downarrow}) = \omega(K \cap \Gamma^{\downarrow})$, and D equals J or $J - \Gamma^{\downarrow}$. Then, proof by contradiction is applied to show $D = J - \Gamma^{\downarrow}$. Suppose $D = J$. It is obvious that $D \cap \Gamma^{\downarrow} = J \cap \Gamma^{\downarrow}$ and $|D \cap \Gamma^{\downarrow}| = 1$, which support $E \cap \Gamma^{\downarrow} = \Gamma^{\downarrow} - D \cap \Gamma^{\downarrow} = \Gamma^{\downarrow} - J \cap \Gamma^{\downarrow} = K \cap \Gamma^{\downarrow}$. Equivalently saying, we have $(D,E) = (J,K)$, which contradicts with $(P,(J,K)) <_{\downarrow} (R,(D,E))$. Thus, we get $D = J - \Gamma^{\downarrow}$.

Consequently, we can get that $(R,(D,E))$ is either a parent of $(P,(J,K))$ or not. If $(R,(D,E))$ is not a parent of $(P,(J,K))$, then at least one OE3W concept $(Q,(L,M))$ exists among the parents of $(P,(J,K))$ in $OEL^{\downarrow}$, where $(Q,(L,M))$ meets requirements: $L=J-\Gamma^{\downarrow}$, $M-\Gamma^{\downarrow}=K-\Gamma^{\downarrow}$ and $\omega(M\cap\Gamma^{\downarrow})=\omega(K\cap\Gamma^{\downarrow})$.

($\Leftarrow$) Assume that $K\cap\Gamma^{\downarrow}\neq\emptyset$, and $(Q,(L,M))\in OEL^{\downarrow}$ with $(P,(J,K))\prec_{\downarrow}(Q,(L,M))$ and $L\cap\Gamma^{\downarrow}=\emptyset$ meets requirements: $L=J-\Gamma^{\downarrow}$, $M-\Gamma^{\downarrow}=K-\Gamma^{\downarrow}$ and $\omega(M\cap\Gamma^{\downarrow})=\omega(K\cap\Gamma^{\downarrow})$. Then, $Q\subseteq(J-\Gamma^{\downarrow})^{*^{\downarrow}}\cap(K-\Gamma^{\downarrow})^{\overline{*}^{\downarrow}}\cap\omega(K\cap\Gamma^{\downarrow})^{\overline{*}^{\downarrow}}$. Two cases for $(P,(J,K))$ are considered.

(i) If $J\cap\Gamma^{\downarrow}=\emptyset$, according to the analysis in ($\Rightarrow$), we have $P^{\lessdot^{\Downarrow}\gtrdot^{\Downarrow}}=(J,((K-\Gamma^{\downarrow})\cup\omega(K\cap\Gamma^{\downarrow})))^{\gtrdot^{\downarrow}}$. Since $(Q,(L,M))$ is a parent of $(P,(J,K))$, $J\cap\Gamma^{\downarrow}=\emptyset$ and $Q\subseteq(J-\Gamma^{\downarrow})^{*^{\downarrow}}\cap(K-\Gamma^{\downarrow})^{\overline{*}^{\downarrow}}\cap\omega(K\cap\Gamma^{\downarrow})^{\overline{*}^{\downarrow}}$ hold, it follows $P\subset Q\subseteq J^{*^{\downarrow}}\cap(K-\Gamma^{\downarrow})^{\overline{*}^{\downarrow}}\cap\omega(K\cap\Gamma^{\downarrow})^{\overline{*}^{\downarrow}}=P^{\lessdot^{\Downarrow}\gtrdot^{\Downarrow}}$. Namely, $P\subset P^{\lessdot^{\Downarrow}\gtrdot^{\Downarrow}}$, which means that $(P,(J,K))$ is a deleted OE3W concept.

(ii) If $J\cap\Gamma^{\downarrow}\neq\emptyset$, according to the analysis in ($\Rightarrow$), we have $P^{\lessdot^{\Downarrow}\gtrdot^{\Downarrow}}=(J-\Gamma^{\downarrow},((K-\Gamma^{\downarrow})\cup\omega(K\cap\Gamma^{\downarrow})))^{\gtrdot^{\downarrow}}$. Since $(Q,(L,M))$ is a parent of $(P,(J,K))$ and $Q\subseteq(J-\Gamma^{\downarrow})^{*^{\downarrow}}\cap(K-\Gamma^{\downarrow})^{\overline{*}^{\downarrow}}\cap\omega(K\cap\Gamma^{\downarrow})^{\overline{*}^{\downarrow}}$ holds, it follows $P\subset Q\subseteq(J-\Gamma^{\downarrow})^{*^{\downarrow}}\cap(K-\Gamma^{\downarrow})^{\overline{*}^{\downarrow}}\cap\omega(K\cap\Gamma^{\downarrow})^{\overline{*}^{\downarrow}}=P^{\lessdot^{\Downarrow}\gtrdot^{\Downarrow}}$. Namely, $P\subset P^{\lessdot^{\Downarrow}\gtrdot^{\Downarrow}}$, which means that $(P,(J,K))$ is a deleted OE3W concept. □

Analogously, we can get sufficient and necessary conditions of modified-condensed OE3W concepts, extended-condensed OE3W concepts and unchanged-condensed OE3W concepts.

Theorem 3. *For $(P,(J,K))\in OEL^{\downarrow}$ with $P\neq\emptyset$, $(P,(J,K))$ is a modified-condensed OE3W concept iff the following conditions stand:*

(1) $J\cap\Gamma^{\downarrow}\neq\emptyset$ and $K\cap\Gamma^{\downarrow}\neq\emptyset$;

(2) no such an OE3W concept $(Q,(L,M))$ with $L\cap\Gamma^{\downarrow}=\emptyset$ exists among parents of $(P,(J,K))$ in $OEL^{\downarrow}$, where $(Q,(L,M))$ meets requirements: $L=J-\Gamma^{\downarrow}$, $M-\Gamma^{\downarrow}=K-\Gamma^{\downarrow}$ and $\omega(M\cap\Gamma^{\downarrow})=\omega(K\cap\Gamma^{\downarrow})$.

Proof. They are immediate from Theorem 2 and Definition 2. □

Theorem 4. *For $(P,(J,K))\in OEL^{\downarrow}$ with $P\neq\emptyset$, $(P,(J,K))$ is an extended-condensed OE3W concept iff the following conditions stand:*

(1) $J\cap\Gamma^{\downarrow}=\emptyset$, $K\cap\Gamma^{\downarrow}\neq\emptyset$ and $|\tau(K\cap\Gamma^{\downarrow})|=|\Gamma^{\Downarrow}|-1$;

(2) no such an OE3W concept $(Q,(L,M))$ with $L\cap\Gamma^{\downarrow}=\emptyset$ exists among parents of $(P,(J,K))$ in $OEL^{\downarrow}$, where $(Q,(L,M))$ meets requirements: $L=J-\Gamma^{\downarrow}$, $M-\Gamma^{\downarrow}=K-\Gamma^{\downarrow}$ and $\omega(M\cap\Gamma^{\downarrow})=\omega(K\cap\Gamma^{\downarrow})$.

Proof. They are immediate from Theorem 2 and Definition 2. □

Theorem 5. *For $(P,(J,K))\in OEL^{\downarrow}$ with $P\neq\emptyset$, $(P,(J,K))$ is an unchanged-condensed OE3W concept iff the following conditions stand:*

(1) $J\cap\Gamma^{\downarrow}=\emptyset$, $K\cap\Gamma^{\downarrow}\neq\emptyset$ and $|\tau(K\cap\Gamma^{\downarrow})|<|\Gamma^{\Downarrow}|-1$;

(2) no such an OE3W concept $(Q,(L,M))$ with $L\cap\Gamma^{\downarrow}=\emptyset$ exists among parents of $(P,(J,K))$ in $OEL^{\downarrow}$, where $(Q,(L,M))$ meets requirements: $L=J-\Gamma^{\downarrow}$, $M-\Gamma^{\downarrow}=K-\Gamma^{\downarrow}$ and $\omega(M\cap\Gamma^{\downarrow})=\omega(K\cap\Gamma^{\downarrow})$.

Proof. They are immediate from Theorem 2 and Definition 2. □

Since Theorems 2-5 exclude the case when $(P,(J,K))=(\emptyset,\emptyset^{\lessdot^{\downarrow}})$, we give the following remark to identify the category of $(\emptyset,\emptyset^{\lessdot^{\downarrow}})$ based on $(\Gamma^{\downarrow},\Gamma^{\downarrow})^{\gtrdot^{\downarrow}}=\emptyset$ and $(\Gamma^{\Downarrow},\Gamma^{\Downarrow})^{\gtrdot^{\Downarrow}}=\emptyset$.

Remark 1. For the bottom concept $(\emptyset,\emptyset^{\lessdot^{\downarrow}})$ in $OEL^{\downarrow}$, $(\emptyset,\emptyset^{\lessdot^{\downarrow}})$ is a modified-condensed OE3W concept.

As addressed in Theorem 2, we give the following definition.

Definition 3. For a deleted OE3W concept $(P,(J,K))$,

(1) if $(Q,(L,M))\in OEL^{\downarrow}$ meets requirements: $L\cap\Gamma^{\downarrow}=\emptyset$, $L=J-\Gamma^{\downarrow}$, $M-\Gamma^{\downarrow}=K-\Gamma^{\downarrow}$ and $\omega(M\cap\Gamma^{\downarrow})=\omega(K\cap\Gamma^{\downarrow})$, $(Q,(L,M))$ is a drag of $(P,(J,K))$;

(2) if $(Q,(L,M))$ is a drag of $(P,(J,K))$, $(P,(J,K))$ is a casualty of $(Q,(L,M))$;

(3) if $(R,(D,E))\in OEL^{\downarrow}$ meets requirements: $R=(J-\Gamma^{\downarrow},(K-\Gamma^{\downarrow})\cup\omega(K\cap\Gamma^{\downarrow}))^{\gtrdot^{\downarrow}}$ and $(D,E)=(J-\Gamma^{\downarrow},(K-\Gamma^{\downarrow})\cup\omega(K\cap\Gamma^{\downarrow}))^{\gtrdot^{\downarrow}\lessdot^{\downarrow}}$, $(R,(D,E))$ is a destructor of $(P,(J,K))$.

From Theorem 2, we obtain the following remarks.

Remark 2. For a deleted OE3W concept $(P,(J,K))$,

(1) there is at least one drag;

(2) there is only one destructor;

(3) the maximum OE3W concept among all the drags is the only one destructor;

(4) if a drag is not a deleted OE3W concept, it is the destructor.

Remark 3. For a drag $(Q,(L,M))$, there is at least one casualty.

According to Definition 3 and Theorems 2-5, we have Propositions 4-9.

Proposition 4. *Let $(P_1,(J_1,K_1)) \in OEL^{\downarrow}$ be a modified-condensed OE3W concept with $P_1 \neq \emptyset$. For $(P_2,(J_2,K_2)) \in OEL^{\downarrow}$ with $(P_2,(J_2,K_2)) <_{\downarrow} (P_1,(J_1,K_1))$ and $P_2 \neq \emptyset$, we have that $(P_2,(J_2,K_2))$ is also a modified-condensed OE3W concept.*

Proof. Let $\{\gamma_0\} = J_1 \cap \Gamma^{\downarrow}$, $\gamma_0' = \varphi(\gamma_0)$. According to Proposition 2, it follows $K_1 \cap \Gamma^{\downarrow} = \Gamma^{\downarrow} - \{\gamma_0\}$. Since $(P_1,(J_1,K_1))$ is a modified-condensed concept, it follows that $P_1 = (J_1 - \{\gamma_0\})^{*\downarrow} \cap (K_1 - (\Gamma^{\downarrow} - \{\gamma_0\}))^{\overline{*}\downarrow} \cap {\gamma_0}^{*\downarrow} \cap (\Gamma^{\downarrow} - \{\gamma_0\})^{\overline{*}\downarrow} = (J_1 - \{\gamma_0\})^{*\downarrow} \cap (K_1 - (\Gamma^{\downarrow} - \{\gamma_0\}))^{\overline{*}\downarrow} \cap {\gamma_0'}^{*\Downarrow} \cap (\Gamma^{\Downarrow} - \{\gamma_0'\})^{\overline{*}\Downarrow}$. By Proposition 1, we obtain $P_1 = (J_1 - \{\gamma_0\})^{*\downarrow} \cap (K_1 - (\Gamma^{\downarrow} - \{\gamma_0\}))^{\overline{*}\downarrow} \cap {\gamma_0}^{*\downarrow} = (J_1 - \{\gamma_0\})^{*\downarrow} \cap (K_1 - (\Gamma^{\downarrow} - \{\gamma_0\}))^{\overline{*}\downarrow} \cap {\gamma_0'}^{*\Downarrow}$, which means $((J_1 - \{\gamma_0\})^{*\downarrow} \cap (K_1 - (\Gamma^{\downarrow} - \{\gamma_0\}))^{\overline{*}\downarrow} - P_1) \cap {\gamma_0'}^{*\Downarrow} = \emptyset$. By $(P_2,(J_2,K_2)) <_{\downarrow} (P_1,(J_1,K_1))$ and $P_2 \neq \emptyset$, it is straightforward that $P_2 \subset P_1$, $J_2 \cap \Gamma^{\downarrow} = \{\gamma_0\}$, $K_2 \cap \Gamma^{\downarrow} = \Gamma^{\downarrow} - \{\gamma_0\}$, $J_1 - \{\gamma_0\} \subset J_2 - \{\gamma_0\}$ and $K_1 - (\Gamma^{\downarrow} - \{\gamma_0\}) \subset K_2 - (\Gamma^{\downarrow} - \{\gamma_0\})$, which support that $(J_1 - \{\gamma_0\})^{*\downarrow} \cap (K_1 - (\Gamma^{\downarrow} - \{\gamma_0\}))^{\overline{*}\downarrow} - P_1 = (J_1 - \{\gamma_0\})^{*\downarrow} \cap (K_1 - (\Gamma^{\downarrow} - \{\gamma_0\}))^{\overline{*}\downarrow} - \{\gamma_0\}^{*\downarrow} \supseteq (J_2 - \{\gamma_0\})^{*\downarrow} \cap (K_2 - (\Gamma^{\downarrow} - \{\gamma_0\}))^{\overline{*}\downarrow} - P_2 = (J_2 - \{\gamma_0\})^{*\downarrow} \cap (K_2 - (\Gamma^{\downarrow} - \{\gamma_0\}))^{\overline{*}\downarrow} - \{\gamma_0\}^{*\downarrow}$. Therefore, $((J_2 - \{\gamma_0\})^{*\downarrow} \cap (K_2 - (\Gamma^{\downarrow} - \{\gamma_0\}))^{\overline{*}\downarrow} - P_2) \cap {\gamma_0'}^{*\Downarrow} = \emptyset$, which means $P_2 = (J_2 - \{\gamma_0\})^{*\downarrow} \cap (K_2 - (\Gamma^{\downarrow} - \{\gamma_0\}))^{\overline{*}\downarrow} \cap {\gamma_0}^{*\downarrow} = ((J_2 - \{\gamma_0\})^{*\downarrow} \cap (K_2 - (\Gamma^{\downarrow} - \{\gamma_0\}))^{\overline{*}\downarrow} \cap {\gamma_0'}^{*\Downarrow}$. Thus, P_2 is an extent in $OEL^{\Downarrow}$. Finally, from Definition 2, it follows that $(P_2,(J_2,K_2))$ is a modified-condensed OE3W concept. □

Proposition 5. *For a deleted OE3W concept $(P,(J,K))$, if $(Q,(L,M))$ is the destructor of $(P,(J,K))$, then $(Q,(L,M))$ can only be an unchanged-condensed OE3W concept or an extended-condensed OE3W concept.*

Proof. According to the definition of destructors, we have that $(Q,(L,M))$ is not a deleted OE3W concept and $L \cap \Gamma^{\downarrow} = \emptyset$. Additionally, from $L \cap \Gamma^{\downarrow} = \emptyset$, we have that $(Q,(L,M))$ is not a modified-condensed OE3W concept. Thus, $(Q,(L,M))$ is either an unchanged-condensed OE3W concept or an extended-condensed OE3W concept. □

Proposition 6. *Let $(P_1,(J_1,K_1)) \in OEL^{\downarrow}$ be an extended-condensed OE3W concept (denote $P_1^{\lessdot\Downarrow} \cap (\Gamma^{\Downarrow},\Gamma^{\Downarrow})$ by $(\{\gamma'\},\Gamma^{\Downarrow} - \{\gamma'\})$). For $(P_2,(J_2,K_2)) \in OEL^{\downarrow}$ with $(P_2,(J_2,K_2)) <_{\downarrow} (P_1,(J_1,K_1))$ and $P_2 \neq \emptyset$, we have*
(1) if $J_2 \cap \Gamma^{\downarrow} = \emptyset$, then $(P_2,(J_2,K_2))$ is either an extended-condensed OE3W concept or a deleted OE3W concept;
(2) if $J_2 \cap \Gamma^{\downarrow} \neq \emptyset$ and $P_2 \neq \emptyset$, then $(P_2,(J_2,K_2))$ is either a modified-condensed OE3W concept or a deleted OE3W concept and $\varphi(J_2 \cap \Gamma^{\downarrow}) = \{\gamma'\}$.

Proof. Since $(P_1,(J_1,K_1))$ is an extended-condensed OE3W concept, it follows that $J_1 \cap \Gamma^{\downarrow} = \emptyset$, $\tau(K_1 \cap \Gamma^{\downarrow}) = \Gamma^{\Downarrow} - \{\gamma'\}$ and $P_1 = J_1^{*\downarrow} \cap (K_1 - \Gamma^{\downarrow})^{\overline{*}\downarrow} \cap (K_1 \cap \Gamma^{\downarrow})^{\overline{*}\downarrow} = J_1^{*\downarrow} \cap (K_1 - \Gamma^{\downarrow})^{\overline{*}\downarrow} \cap {\gamma'}^{*\Downarrow} \cap (\Gamma^{\Downarrow} - \{\gamma'\})^{\overline{*}\Downarrow} = J_1^{*\downarrow} \cap (K_1 - \Gamma^{\downarrow})^{\overline{*}\downarrow} \cap {\gamma'}^{*\Downarrow}$, which means $P_1 \subseteq {\gamma'}^{*\Downarrow}$. It is obvious that $P_2 \subset P_1$, $J_1 \subseteq J_2$, $K_1 - \Gamma^{\downarrow} \subseteq K_2 - \Gamma^{\downarrow}$, $K_1 \cap \Gamma^{\downarrow} \subseteq K_2 \cap \Gamma^{\downarrow}$, $K_2 \cap \Gamma^{\downarrow} \supseteq \omega(K_1 \cap \Gamma^{\downarrow}) = \omega(K_2 \cap \Gamma^{\downarrow}) = \Gamma^{\downarrow} - \psi(\gamma')$ and $\tau(K_2 \cap \Gamma^{\downarrow}) = \Gamma^{\Downarrow} - \{\gamma'\}$ from $(P_2,(J_2,K_2)) <_{\downarrow} (P_1,(J_1,K_1))$ and $P_2 \neq \emptyset$. Since $P_2 \subset P_1 \subseteq {\gamma'}^{*\Downarrow}$ and $P_2^{\lessdot\downarrow} = (J_2,(K_2 - \Gamma^{\downarrow}) \cup (K_2 \cap \Gamma^{\downarrow}))$, we have $P_2^{\lessdot\Downarrow} = (J_2 \cup \{\gamma'\},(K_2 - \Gamma^{\downarrow}) \cup (\Gamma^{\Downarrow} - \{\gamma'\}))$.

(1) Combining $J_2 \cap \Gamma^{\downarrow} = \emptyset$, $K_2 \cap \Gamma^{\downarrow} \supseteq \omega(K_1 \cap \Gamma^{\downarrow}) = \omega(K_2 \cap \Gamma^{\downarrow}) = \Gamma^{\downarrow} - \psi(\gamma')$, $\tau(K_2 \cap \Gamma^{\downarrow}) = \Gamma^{\Downarrow} - \{\gamma'\}$ with Proposition 1, we can obtain that $P_2^{\lessdot\Downarrow\gtrdot\Downarrow} = {J_2}^{*\downarrow} \cap {\gamma'}^{*\Downarrow} \cap (K_2 - \Gamma^{\downarrow})^{*\downarrow} \cap (\Gamma^{\Downarrow} - \{\gamma'\})^{*\Downarrow} = {J_2}^{*\downarrow} \cap (K_2 - \Gamma^{\downarrow})^{*\downarrow} \cap \omega(K_2 \cap \Gamma^{\downarrow})^{*\downarrow} \supseteq {J_2}^{*\downarrow} \cap (K_2 - \Gamma^{\downarrow})^{*\downarrow} \cap (K_2 \cap \Gamma^{\downarrow})^{*\downarrow} = P_2$. If $P_2^{\lessdot\Downarrow\gtrdot\Downarrow} \supset P_2$, then $(P_2,(J_2,K_2))$ is a deleted OE3W concept. If $P_2^{\lessdot\Downarrow\gtrdot\Downarrow} = P_2$, then $(P_2,(J_2,K_2))$ is an extended-condensed OE3W concept.

(2) Note that $J_2 \cap \Gamma^{\downarrow} \neq \emptyset$ and $P_2 \neq \emptyset$, which implies $|J_2 \cap \Gamma^{\downarrow}| = 1$ by Proposition 2. Suppose that the only element in $J_2 \cap \Gamma^{\downarrow}$ is γ_0. Thus, $J_2 \cap \Gamma^{\downarrow} = \{\gamma_0\}$ and $K_2 \cap \Gamma^{\downarrow} = \Gamma^{\downarrow} - \{\gamma_0\}$. Since $P_2 \subseteq {\gamma_0}^{*\downarrow}$ and $P_2 \subseteq {\gamma'}^{*\Downarrow}$, we can conclude $\phi(J_2 \cap \Gamma^{\downarrow}) = \{\gamma'\}$. Combining $K_2 \cap \Gamma^{\downarrow} \supseteq \omega(K_1 \cap \Gamma^{\downarrow}) = \omega(K_2 \cap \Gamma^{\downarrow}) = \Gamma^{\downarrow} - \psi(\gamma')$, $\tau(K_2 \cap \Gamma^{\downarrow}) = \Gamma^{\Downarrow} - \{\gamma'\}$ with Proposition 1, we can obtain that $P_2^{\lessdot\Downarrow\gtrdot\Downarrow} = (J_2 - \Gamma^{\downarrow})^{*\downarrow} \cap {\gamma'}^{*\Downarrow} \cap (K_2 - \Gamma^{\downarrow})^{*\downarrow} \cap (\Gamma^{\Downarrow} - \{\gamma'\})^{*\Downarrow} = (J_2 - \Gamma^{\downarrow})^{*\downarrow} \cap (K_2 - \Gamma^{\downarrow})^{*\downarrow} \cap \omega(K_2 \cap \Gamma^{\downarrow})^{*\downarrow} \supseteq (J_2 - \Gamma^{\downarrow})^{*\downarrow} \cap (K_2 - \Gamma^{\downarrow})^{*\downarrow} \cap (K_2 \cap \Gamma^{\downarrow})^{*\downarrow} = P_2$. If $P_2^{\lessdot\Downarrow\gtrdot\Downarrow} \supset P_2$, then $(P_2,(J_2,K_2))$ is a deleted OE3W concept. If $P_2^{\lessdot\Downarrow\gtrdot\Downarrow} = P_2$, then $(P_2,(J_2,K_2))$ is a modified-condensed OE3W concept. □

Proposition 7. *Let $(P_1,(J_1,K_1))$ be a deleted OE3W concept with $P_1 \neq \emptyset$. For any parent $(P_2,(J_2,K_2))$ of $(P_1,(J_1,K_1))$ in $OEL^{\downarrow}$, if $(P_2,(J_2,K_2))$ is not a drag of $(P_1,(J_1,K_1))$, then $(P_2,(J_2,K_2))$ is a deleted OE3W concept.*

Proof. Because $(P_1,(J_1,K_1))$ is a deleted OE3W concept and $(P_2,(J_2,K_2))$ is not a drag of $(P_1,(J_1,K_1))$, at least one OE3W concept $(P_3,(J_3,K_3))$ with $(P_3,(J_3,K_3)) \neq (P_2,(J_2,K_2))$ exists among the parents of $(P_1,(J_1,K_1))$ in $OEL^{\downarrow}$, where $(P_3,(J_3,K_3))$ is the drag of $(P_1,(J_1,K_1))$. According to Theorem 2, $(P_3,(J_3,K_3))$ meets the four conditions: $J_3 \cap \Gamma^{\downarrow} = \emptyset$, $J_3 = J_1 - \Gamma^{\downarrow}$, $K_3 - \Gamma^{\downarrow} = K_1 - \Gamma^{\downarrow}$ and $\omega(K_3 \cap \Gamma^{\downarrow}) = \omega(K_1 \cap \Gamma^{\downarrow})$. Note that $(P_2,(J_2,K_2))$ is a parent of $(P_1,(J_1,K_1))$ with $(P_2,(J_2,K_2)) \neq (P_3,(J_3,K_3))$, which leads to the fact that $P_2 \neq P_3$, $J_2 - \Gamma^{\downarrow} \subseteq J_1 - \Gamma^{\downarrow} = J_3$, $K_2 - \Gamma^{\downarrow} \subseteq K_1 - \Gamma^{\downarrow} = K_3 - \Gamma^{\downarrow}$, $K_2 \cap \Gamma^{\downarrow} \subseteq K_1 \cap \Gamma^{\downarrow}$ and $\omega(K_2 \cap \Gamma^{\downarrow}) \subseteq \omega(K_1 \cap \Gamma^{\downarrow}) = \omega(K_3 \cap \Gamma^{\downarrow})$. Thus, $(P_2 \cup P_3)^{\lessdot\downarrow} = (((J_2 - \Gamma^{\downarrow}) \cap (J_3 - \Gamma^{\downarrow})) \cup ((J_2 \cap \Gamma^{\downarrow}) \cap (J_3 \cap \Gamma^{\downarrow})), ((K_2 - \Gamma^{\downarrow}) \cap (K_3 - \Gamma^{\downarrow})) \cup ((K_2 \cap \Gamma^{\downarrow}) \cap (K_3 \cap \Gamma^{\downarrow}))) = (J_2 - \Gamma^{\downarrow}, (K_2 - \Gamma^{\downarrow}) \cup ((K_2 \cap \Gamma^{\downarrow}) \cap (K_3 \cap \Gamma^{\downarrow})))$, in which $\omega((K_2 \cap \Gamma^{\downarrow}) \cap (K_3 \cap \Gamma^{\downarrow})) = \omega(K_2 \cap \Gamma^{\downarrow}))$. Obviously, we have $P_2 \subset (P_2 \cup P_3) \subseteq (P_2 \cup P_3)^{\lessdot\downarrow\gtrdot\downarrow}$ and $P_3 \subset (P_2 \cup P_3) \subseteq (P_2 \cup P_3)^{\lessdot\downarrow\gtrdot\downarrow}$, which implies $(P_2,(J_2,K_2)) <_{\downarrow} ((P_2 \cup P_3)^{\lessdot\downarrow\gtrdot\downarrow}, (P_2 \cup P_3)^{\lessdot\downarrow})$ and $(P_3,(J_3,K_3)) <_{\downarrow} ((P_2 \cup P_3)^{\lessdot\downarrow\gtrdot\downarrow}, (P_2 \cup P_3)^{\lessdot\downarrow})$. According to Theorem 2, we can conclude that $(P_2,(J_2,K_2))$ is a deleted OE3W concept and $((P_2 \cup P_3)^{\lessdot\downarrow\gtrdot\downarrow}, (P_2 \cup P_3)^{\lessdot\downarrow})$ is a drag of $(P_2,(J_2,K_2))$. □

Proposition 8. *Let $(P_1,(J_1,K_1))$ be a deleted OE3W concept with $P_1 \neq \emptyset$. For any parent $(P_2,(J_2,K_2))$ of $(P_1,(J_1,K_1))$ in $OEL^{\downarrow}$, if $(P_2,(J_2,K_2))$ is not a deleted OE3W concept, $(P_2,(J_2,K_2))$ is the destructor of $(P_1,(J_1,K_1))$.*

Proof. If $(P_1,(J_1,K_1))$ is a deleted OE3W concept, $(P_2,(J_2,K_2))$ is a parent of $(P_1,(J_1,K_1))$, and $(P_2,(J_2,K_2))$ is not a deleted OE3W concept, by Proposition 7, $(P_2,(J_2,K_2))$ is a drag of $(P_1,(J_1,K_1))$. Since $(P_2,(J_2,K_2))$ is a drag of $(P_1,(J_1,K_1))$ and $(P_2,(J_2,K_2))$ is not a deleted OE3W concept, $(P_2,(J_2,K_2))$ is the destructor of $(P_1,(J_1,K_1))$. □

Proposition 9. *Let $(P_1,(J_1,K_1))$ and $(P_2,(J_2,K_2))$ with $(P_1,(J_1,K_1)) <_{\downarrow} (P_2,(J_2,\ K_2))$ be two deleted OE3W concepts in $OEL^{\downarrow}$, $(Q_1,(L_1,M_1))$ be the destructor of $(P_1,(J_1,K_1))$, and $(Q_2,(L_2,M_2))$ be the destructor of $(P_2,(J_2,K_2))$. If $(P_2,(J_2,K_2))$ is not a drag of $(P_1,(J_1,K_1))$, then $(Q_1,(L_1,M_1)) <_{\downarrow} (Q_2,(L_2,M_2))$.*

Proof. Note that $(P_1,(J_1,K_1))$ and $(P_2,(J_2,K_2))$ are both deleted OE3W concepts with $(P_1,(J_1,K_1)) <_{\downarrow} (P_2,(J_2,K_2))$, $(Q_1,(L_1,M_1))$ is the destructor of $(P_1,(J_1,K_1))$, $(Q_2,(L_2,M_2))$ is the destructor of $(P_2,(J_2,K_2))$, and $(P_2,(J_2,K_2))$ is not a drag of $(P_1,(J_1,K_1))$. By Definition 3, we can obtain $(Q_1,(L_1,M_1)) \leqslant_{\downarrow} (Q_2,(L_2,M_2))$. Next, we prove $(Q_1,(L_1,M_1)) \neq (Q_2,(L_2,M_2))$. If not, assume $(Q_1,(L_1,M_1)) = (Q_2,(L_2,M_2))$, which implies that $J_2 - \Gamma^{\downarrow} = J_1 - \Gamma^{\downarrow}$, $K_2 - \Gamma^{\downarrow} = K_1 - \Gamma^{\downarrow}$ and $\omega(K_2 \cap \Gamma^{\downarrow}) = \omega(K_1 \cap \Gamma^{\downarrow})$. Thus, $(P_2,(J_2,K_2))$ is either a drag of $(P_1,(J_1,K_1))$ or equal to $(P_1,(J_1,K_1))$, which contradicts the fact that $(P_2,(J_2,K_2))$ is not a drag of $(P_1,(J_1,K_1))$ and $(P_1,(J_1,K_1)) <_{\downarrow} (P_2,(J_2,K_2))$. Finally, we have $(Q_1,(L_1,M_1)) <_{\downarrow} (Q_2,(L_2,M_2))$. □

Based on the above analysis, now we can obtain changes of OE3W concepts before and after slight attribute-granularity levels decrease.

For $(P,(J,K)) \in OEL^{\downarrow}$ and $(Q,(L,M)) \in OEL^{\downarrow}$ with $class(Q,(L,M)) \neq$ "deleted", we define $\eta(P,(J,K)) = (P^{\lessdot^{\Downarrow}\gtrdot^{\Downarrow}}, P^{\lessdot^{\Downarrow}})$, $\bar{\eta}(Q,(L,M)) = (Q, Q^{\lessdot^{\Downarrow}})$. Obviously, we have the following remark.

Remark 4. For $(P,(J,K)) \in OEL^{\downarrow}$ and $(Q,(L,M)) \in OEL^{\downarrow}$ with $class(Q,\ (L,M)) \neq$ "deleted",

(1) $\eta(P,(J,K)) \in OEL^{\Downarrow}$ and $\bar{\eta}(Q,(L,M)) \in OEL^{\Downarrow}$;

(2) if $class(P,(J,K)) =$ "old", $\eta(P,(J,K)) = \bar{\eta}(P,(J,K)) = (P,(J,K))$;

(3) if $class(P,(J,K)) =$ "extended-condensed", $\eta(P,(J,K)) = \bar{\eta}(P,(J,K)) = (P,(J \cup (\Gamma^{\Downarrow} - \tau(K \cap \Gamma^{\downarrow})),(K - \Gamma^{\downarrow}) \cup \tau(K \cap \Gamma^{\downarrow})))$;

(4) if $class(P,(J,K)) =$ "modified-condensed", $\eta(P,(J,K)) = \bar{\eta}(P,(J,K)) = (P,((J - \Gamma^{\downarrow}) \cup \phi(J \cap \Gamma^{\downarrow}),(K - \Gamma^{\downarrow}) \cup \tau(K \cap \Gamma^{\downarrow})))$;

(5) if $class(P,(J,K)) =$ "unchanged-condensed", $\eta(P,(J,K)) = \bar{\eta}(P,(J,K)) = (P,(J,(K - \Gamma^{\downarrow}) \cup \tau(K \cap \Gamma^{\downarrow})))$;

(6) if $class(P,(J,K)) =$ "deleted", $\eta(P,(J,K)) = \eta(Q,(L,M))$, where $(Q,(L,M))$ is the destructor of $(P,(J,K))$;

(7) η is a mapping (not a bijection) between $OEL^{\downarrow}$ and $OEL^{\Downarrow}$, and $\bar{\eta}$ is a bijection between $OEL^{\downarrow} - \{(P,(J,K)) \in OEL^{\downarrow} | class(P,(J,K)) = \text{"deleted"}\}$ and $OEL^{\Downarrow}$.

3.2. Changes of covering relations involving slightly decreasing attribute-granularity levels

For OE3W concept lattices before and after slightly decreasing attribute-granularity levels, some covering relations remain constant, but other covering relations should be deleted, and the rest of covering relations should be newly built. We describe changes of the covering relations involving the slight decrease in the following definition.

Definition 4. For $(P,(J,K)) \in OEL^{\downarrow}$ and $(Q,(L,M)) \in OEL^{\downarrow}$,

(1) $(P,(J,K)) \prec_{\downarrow} (Q,(L,M))$ (or $\bar{\eta}(P,(J,K)) \prec_{\Downarrow} \bar{\eta}(Q,(L,M))$) is called an old covering relation, if $(P,(J,K)) \prec_{\downarrow} (Q,(L,M))$, and $\bar{\eta}(P,(J,K)) \prec_{\Downarrow} \bar{\eta}(Q,(L,\ M))$;

(2) $\eta(P,(J,K)) \prec_{\Downarrow} \eta(Q,(L,M))$ is called a new covering relation, if $(P,(J,K)) \prec_{\Downarrow} (Q,(L,M))$, and $\bar{\eta}^{-1}(P,(J,K)) \nprec_{\downarrow} \bar{\eta}^{-1}(Q,(L,M))$;

(3) $(P,(J,K)) \prec_{\downarrow} (Q,(L,M))$ is called a deleted covering relation, if $(P,(J,K)) \prec_{\downarrow} (Q,(L,M))$, and $(P,(J,K))$ or $(Q,(L,M))$ is a deleted OE3W concept.

To facilitate subsequent analysis, for three OE3W concepts $(P_1,(J_1,K_1))$, $(P_2,(J_2,K_2))$ and $(P_3,(J_3,K_3))$ in $OEL^{\downarrow}$, where $class(P_1,(J_1,K_1))) =$ "deleted", $class(P_2,(J_2,K_2))) \neq$ "deleted", $class(P_3,(J_3,K_3)) \neq$ "deleted", $(P_3,(J_3,K_3)) \prec_{\downarrow} (P_1,(J_1,K_1)) <_{\downarrow} (P_2,(J_2,K_2))$, and $(P_2,(J_2,K_2))$ is the destructor of $(P_1,(J_1,K_1))$, five notations are given as follows:

$CC'(P_2,(J_2,K_2)) = \{(P,(J,K)) \in OEL^{\downarrow} | (P,(J,K)) <_{\downarrow} (P_2,(J_2,K_2)), class(P,\ (J,K))) \neq \text{"deleted"}\}$,

$CC^{\triangle}(P_2,(J_2,K_2)) = \{(P,(J,K)) \in OEL^{\downarrow} | (P,(J,\ K)) \prec_{\downarrow} (P_2,(J_2,K_2)), class(P,(J,K))) \neq \text{"deleted"}\}$,

$CC^{\triangledown}(P_2,(J_2,K_2)) = \{(P,(J,K)) \in OEL^{\downarrow} | (P_2,(J_2,K_2))$ is the destructor of $(P',(J',K')), (P,(J,K)) \prec_{\downarrow} (P',(J',K')), class(P,\ (J,K))) \neq$ "deleted"$\}$,

$CC(P_2,(J_2,K_2)) = CC^{\triangle}(P_2,(J_2,K_2)) \cup CC^{\triangledown}(P_2,(J_2,K_2))$,

$CC^{\square}((P_2,(J_2,\ K_2)),(P_3,(J_3,K_3))) = \{(P,(J,K)) \in CC(P_2,(J_2,K_2)) | (P_3,(J_3,K_3)) <_{\downarrow} (P,(J,K))\}$.

Next, we discuss the following issues: which covering relations are deleted covering relations; which covering relations are old covering relations; which covering relations are new covering relations.

Theorem 6. *Let $(P_2,(J_2,K_2))$ be an OE3W concept in $OEL^{\downarrow}$, $class(P_2,(J_2,\ K_2)) =$ "modified-condensed" and $(P_1,(J_1,K_1))$ be a child concept of $(P_2,(J_2,K_2))$ in $OEL^{\downarrow}$. Then $\bar{\eta}(P_1,(J_1,K_1)) \prec_{\Downarrow} \bar{\eta}(P_2,(J_2,K_2))$ is an old covering relation.*

Proof. Since $(P_2,(J_2,K_2))$ is a modified-condensed OE3W concept in $OEL^{\downarrow}$, according to Proposition 4, $(P_1,(J_1,K_1))$ is also a modified-condensed OE3W concept, that is, $(P_1,(J_1,K_1))$ is not a deleted OE3W concept. Obviously, we can get that any child concept of $(P_2,(J_2,K_2))$ in $OEL^{\downarrow}$ is not a deleted OE3W concept. For any child concept $(P_3,(J_3,K_3))$ of $(P_2,(J_2,K_2))$ with $(P_3,(J_3,K_3)) \neq (P_1,(J_1,K_1))$, we have $P_1^{\lessdot\Downarrow\gtrdot\Downarrow} = P_1 \not\subset P_3 = P_3^{\lessdot\Downarrow\gtrdot\Downarrow}$. Thus, $\bar{\eta}(P_1,(J_1,K_1)) \prec_{\Downarrow} \bar{\eta}(P_2,(J_2,K_2))$ holds, or equivalently, $\bar{\eta}(P_1,(J_1,K_1)) \prec_{\Downarrow} \bar{\eta}(P_2,(J_2,K_2))$ is an old covering relation. □

Theorem 7. *Let $(P_2,(J_2,K_2))$ be an OE3W concept in $OEL^{\downarrow}$, $class(P_2,(J_2,K_2))$ = "old", $(P_2,(J_2,K_2))$ be not the destructor of any OE3W concept, and $(P_1,(J_1,K_1))$ be a child concept of $(P_2,(J_2,K_2))$ in $OEL^{\downarrow}$. Then $\bar{\eta}(P_1,(J_1,K_1)) \prec_{\Downarrow} \bar{\eta}(P_2,(J_2,K_2))$ is an old covering relation.*

Proof. Since $(P_2,(J_2,K_2))$ is an old OE3W concept in $OEL^{\downarrow}$ and $(P_2,(J_2,K_2))$ is not the destructor of any OE3W concept, according to Propositions 7 and 8, $(P_1,(J_1,K_1))$ is not a deleted OE3W concept. So we can obtain that any child concept of $(P_2,(J_2,K_2))$ in $OEL^{\downarrow}$ is not a deleted OE3W concept. For any child concept $(P_3,(J_3,K_3))$ of $(P_2,(J_2,K_2))$ with $(P_3,(J_3,K_3)) \neq (P_1,(J_1,K_1))$, we have $P_1^{\lessdot\Downarrow\gtrdot\Downarrow} = P_1 \not\subset P_3 = P_3^{\lessdot\Downarrow\gtrdot\Downarrow}$. Thus, $\bar{\eta}(P_1,(J_1,K_1)) \prec_{\Downarrow} \bar{\eta}(P_2,(J_2,K_2))$ holds, or equivalently, $\bar{\eta}(P_1,(J_1,K_1)) \prec_{\Downarrow} \bar{\eta}(P_2,(J_2,K_2))$ is an old covering relation. □

Theorem 8. *Let $(P_1,(J_1,K_1))$, $(P_2,(J_2,K_2))$, $(P_3,(J_3,K_3))$ and $(P_4,(J_4,K_4))$ be four OE3W concepts in $OEL^{\downarrow}$, where $class(P_1,(J_1,K_1)))$ = "deleted", $class(P_2,(J_2,K_2))) \neq$ "deleted", $class(P_3,(J_3,K_3))) \neq$ "deleted", $class(P_4,(J_4,K_4)) \neq$ "deleted", $(P_3,(J_3,K_3)) \prec_{\downarrow} (P_1,(J_1,K_1)) <_{\downarrow} (P_2,(J_2,K_2))$, $(P_4,(J_4,K_4)) \prec_{\downarrow} (P_2,(J_2,K_2))$, and $(P_2,(J_2,K_2))$ is the destructor of $(P_1,(J_1,K_1))$. We have the following statements:*

(1) $\{(R,(D,E)) \in OEL^{\Downarrow} | (R,(D,E)) \prec_{\Downarrow} \bar{\eta}(P_2,(J_2,K_2))\} \subseteq \{\bar{\eta}(P,(J,K)) | \ (P,(J,K)) \in CC(P_2,(J_2,K_2))\}$;

(2) $\bar{\eta}(P_4,(J_4,K_4)) \prec_{\Downarrow} \bar{\eta}(P_2,(J_2,K_2))$ is an old covering relation;

(3) $\bar{\eta}(P_3,(J_3,K_3)) \prec_{\Downarrow} \bar{\eta}(P_2,(J_2,K_2))$ is a new covering relation, if $CC^{\square}\ ((P_2,(J_2,K_2)),(P_3,(J_3,K_3))) = \emptyset$.

Proof. For any two non-deleted OE3W concepts $(P'',(J'',K''))$ and $(P',(J',K'))$ in $OEL^{\downarrow}$ with $(P'',(J'',K'')) <_{\downarrow} (P',(J',K'))$, we have $P''^{\lessdot\Downarrow\gtrdot\Downarrow} = P'' \subset P' = P'^{\lessdot\Downarrow\gtrdot\Downarrow}$. Thus, it follows $\bar{\eta}(P'',(J'',K'')) <_{\Downarrow} \bar{\eta}(P',(J',K'))$.

(1) It is straightforward that $\{(P,(J,K)) \in OEL^{\Downarrow} | (P,(J,K)) \prec_{\Downarrow} \bar{\eta}(P_2,\ (J_2,K_2)\} \subseteq \{\bar{\eta}((P,(J,K)) | (P,(J,K)) \in CC'(P_2,\ (J_2,K_2))\}$.

Given $(P,(J,K)) \in CC(P_2,(J_2,K_2))$, let $(P',(J',K')) <_{\downarrow} (P,(J,K))$ with $class(P',(J',K')) \neq$ "deleted". It is obvious that $\bar{\eta}(P',(J',K')) <_{\Downarrow} \bar{\eta}(P,(J,K))$.

Given $(Q,(L,M)) \in \{(Q,(L,M)) \in OEL^{\downarrow} | (Q,(L,M)) <_{\downarrow} (Q'',(L'',M'')) <_{\downarrow} (P_2,(J_2,K_2)), class(Q'',(L'',M''))$ = "deleted", $class(Q,(L,M))$ = "deleted", $(P_2,(J_2,K_2))$ is the destructor of $(Q'',(L'',M''))$, and $(P_2,(J_2,K_2))$ is not the destructor of $(Q,(L,M))\}$, let $(Q',(L',M')) <_{\downarrow} (Q,(L,M))$ with $class(Q',(L',M')) \neq$ "deleted". It is obvious that $(Q',(L',M')) <_{\downarrow} (Q,(L,M)) <_{\downarrow} (R,(D,E))$, where $(R,(D,E))$ is the destructor of $(Q,(L,M))$. Because $(Q,(L,M))$ and $(Q'',(L'',M''))$ are both deleted OE3W concepts with $(Q,(L,M)) <_{\downarrow} (Q'',\ (L'',M''))$ and $(Q'',(L'',M''))$ is not a drag of $(Q,(L,M))$, based on Proposition 9, we have $(R,(D,E)) <_{\downarrow} (R'',(D'',E'')) = (P_2,(J_2,K_2))$, where $(R'',(D'',E''))$ is the destructor of $(Q'',(L'',M''))$. There are two possibilities: $(R,(D,E)) \prec_{\downarrow} (P_2,(J_2,K_2))$ and $(R,(D,E)) <_{\downarrow} (P_2,(J_2,K_2))$ but $(R,(D,E)) \nprec_{\downarrow} (P_2,(J_2,K_2))$. If $(R,(D,E)) \prec_{\downarrow} (P_2,(J_2,K_2))$, then $(R,(D,E)) \in CC^{\triangle}(P_2,(J_2,K_2))$ holds, which yields $(Q',(L',M')) <_{\downarrow} (R,(D,E)) \prec_{\downarrow} (P_2,(J_2,K_2))$. If $(R,(D,E)) <_{\downarrow} (P_2,(J_2,K_2))$ but $(R,(D,E)) \nprec_{\downarrow} (P_2,(J_2,K_2))$, then there exists $(P,(J,K)) \in CC^{\triangle}(P_2,(J_2,K_2))$ such that $(R,(D,E)) <_{\downarrow} (P,(J,K))$, which yields $(Q',(L',M')) <_{\downarrow} (P,(J,K)) \prec_{\downarrow} (P_2,(J_2,K_2))$. Thus, there must exist some OE3W concept $(P,(J,K)) \in CC^{\triangle}(P_2,\ (J_2,K_2))$ such that $(Q',(L',M')) <_{\downarrow} (P,(J,K)) \prec_{\downarrow} (P_2,(J_2,K_2))$, that is, $\bar{\eta}(Q',(L',M')) <_{\Downarrow} \bar{\eta}(P,(J,K)) <_{\Downarrow} \bar{\eta}(P_2,(J_2,K_2))$.

Consequently, $\{(R,(D,E)) \in OEL^{\Downarrow} | (R,(D,E)) \prec_{\Downarrow} \bar{\eta}(P_2,(J_2,K_2))\} \subseteq \{\bar{\eta}(P,(J,K)) | (P,(J,K)) \in CC(P_2,(J_2,K_2))\}$.

(2) Given $(P,(J,K)) \in CC^{\triangle}(P_2,(J_2,K_2))$ with $(P,(J,K)) \neq (P_4,(J_4,\ K_4))$, since $(P_4,(J_4,K_4))$ with $class(P_4,(J_4,\ K_4)) \neq$ "deleted" and $(P,(J,K))$ with $class(P,(J,K)) \neq$ "deleted" are both children concepts of $(P_2,(J_2,K_2))$, it is obvious that $P_4^{\lessdot\Downarrow\gtrdot\Downarrow} = P_4 \not\subseteq P = P^{\lessdot\Downarrow\gtrdot\Downarrow}$. Thus, $\bar{\eta}(P_4,(J_4,K_4)) \nless_{\Downarrow} \bar{\eta}(P,(J,K))$.

Given $(Q,(L,M)) \in CC^{\triangledown}(P_2,(J_2,K_2))$, there exists an OE3W concept $(P_0,(J_0,\ K_0))$ in $OEL^{\downarrow}$ such that $(Q,(L,M)) <_{\downarrow} (P_0,(J_0,K_0)) \prec_{\downarrow} (P_2,(J_2,K_2))$, $class(P_0,(J_0,K_0))$ = "deleted", and $(P_2,(J_2,K_2))$ is the destructor of $(P_0,(J_0,K_0))$. It is obvious that $(P_0,(J_0,K_0)) \neq (P_4,(J_4,K_4))$. Since $(P_0,(J_0,K_0))$ with $class(P_0,(J_0,K_0))$ = "deleted" and $(P_4,(J_4,K_4))$ with $class(P_4,(J_4,K_4)) \neq$ "deleted" are both children of $(P_2,(J_2,K_2))$, it is obvious that $P_4^{\lessdot\Downarrow\gtrdot\Downarrow} = P_4 \not\subset P_0$. Because $Q^{\lessdot\Downarrow\gtrdot\Downarrow} = Q \subset P_0$, we can get $P_4^{\lessdot\Downarrow\gtrdot\Downarrow} = P_4 \not\subset Q = Q^{\lessdot\Downarrow\gtrdot\Downarrow}$. Thus, $\bar{\eta}(P_4,(J_4,K_4)) \nless_{\Downarrow} \bar{\eta}(Q,(L,M))$.

Consequently, according to Item (1) in Theorem 8, $\bar{\eta}(P_4,(J_4,K_4)) \prec_{\Downarrow} \bar{\eta}(P_2,(J_2,K_2))$ holds, or equivalently, $\bar{\eta}(P_4,(J_4,\ K_4)) \prec_{\Downarrow} \bar{\eta}(P_2,(J_2,K_2))$ is an old covering relation.

(3) For any OE3W concept $(P',(J',K')) \in CC(P_2,(J_2,K_2))$ with $(P_3,(J_3,\ K_3)) <_{\downarrow} (P',(J',K'))$, we have $\bar{\eta}(P_3,(J_3,\ K_3)) <_{\Downarrow} \bar{\eta}(P',(J',K')) <_{\Downarrow} \bar{\eta}(P_2,(J_2,K_2))$. As long as one OE3W concept $(P',(J',K'))$ with $(P_3,(J_3,K_3)) <_{\downarrow} (P',(J',K'))$ exists in $CC(P_2,(J_2,K_2))$, we can obtain $\bar{\eta}(P_3,(J_3,K_3)) \nprec_{\Downarrow} \bar{\eta}(P_2,(J_2,K_2))$. Consequently, according to Item (1) in Theorem 8, if $CC^{\square}((P_2,(J_2,K_2)),(P_3,(J_3,K_3))) = \emptyset$, we have $\bar{\eta}(P_3,(J_3,K_3)) \prec_{\Downarrow} \bar{\eta}(P_2,(J_2,K_2))$, or equivalently, $\bar{\eta}(P_3,(J_3,K_3)) \prec_{\Downarrow} \bar{\eta}(P_2,(J_2,K_2))$ is a new covering relation. □

Example 3. Continued from Example 2. DT_1 and DT_2 can be converted to formal contexts FT_1 (Table 4) and FT_2 (Table 5). Fig. 2 and Fig. 3 show the OE3W concept lattices $OEL^{\downarrow}(FT_1)$ and $OEL^{\Downarrow}(FT_2)$ before and after the slight decrease, respectively. The red, white, green, gray and blue nodes, and the red, black and purple lines in Fig. 2 and Fig. 3 represent deleted, old, unchanged-condensed, modified-condensed and extended-condensed OE3W concepts, and the deleted, old and new covering relations, respectively. The values of key variables required to find children of OE3W concepts in $OEL(FT_2)$ are shown in Table 6, where $(P,(J,K))$ is an OE3W concept of $OEL^{\downarrow}(FT_1)$, $Casualties(P,(J,K))$ denotes the set of all the casualties of $(P,(J,K))$, and "/" means that "it is unnecessary to calculate the value".

Table 4
The formal context FT_1.

G	a	b	c	d	e	l_1	l_2	l_3	l_4
$g_1 = 1$	1	1	0	0	0	1	0	0	0
$g_2 = 2$	1	1	0	0	0	0	1	0	0
$g_3 = 3$	0	0	1	1	0	0	0	1	0
$g_4 = 4$	1	0	1	1	1	0	0	0	1
$g_5 = 5$	0	1	0	0	1	0	0	1	0

Table 5
The formal context FT_2.

G	a	b	c	d	e	l'_1	l'_2
$g_1 = 1$	1	1	0	0	0	1	0
$g_2 = 2$	1	1	0	0	0	1	0
$g_3 = 3$	0	0	1	1	0	0	1
$g_4 = 4$	1	0	1	1	1	0	1
$g_5 = 5$	0	1	0	0	1	0	1

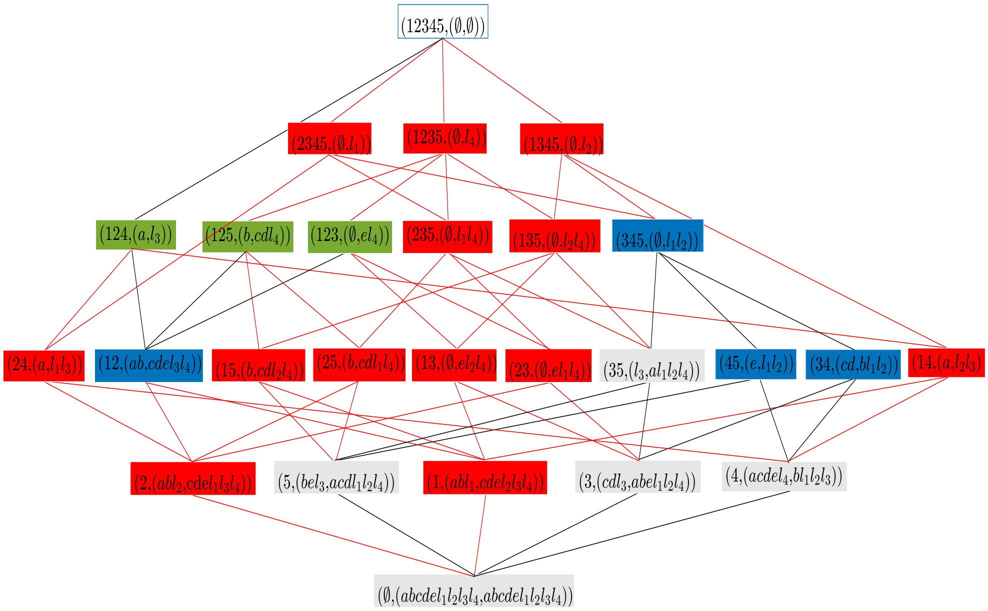

Fig. 2. The OE3W concept lattice $OEL^{\downarrow}(FT_1)$.

4. The OEL-Collapse algorithm

Now we put forward the OEL-Collapse algorithm of renewing OE3W concept lattices involving slightly decreasing attribute-granularity levels by making full use of the changes of OE3W concept lattices. The proposed algorithm includes two key tasks: (i) classifying OE3W concepts and altering intents of unchanged-condensed OE3W concepts, modified-condensed OE3W concepts and extended-condensed OE3W concepts; (ii) fixing covering relations and removing deleted OE3W concepts.

4.1. Handling

In order to facilitate classifying OE3W concepts and fixing covering relations, we need to handle all OE3W concepts except the bottom OE3W concept recursively. Such a simple idea is performed in Algorithm 1.

Algorithm 1 receives four arguments: an OE3W concept $(P,(J,K))$ with $P \neq \emptyset$, $\Gamma^{\downarrow}$, $\Gamma^{\Downarrow}$ and OEL. In fact, Algorithm 1 handles $(P,(J,K))$ only after OE3W sub-concepts of $(P,(J,K))$ are handled. Namely, all OE3W concepts except the bottom OE3W concept are handled in the bottom-up process.

If a child concept $(Q,(L,M))$ of $(P,(J,K))$ is not handled, the procedure calls Algorithm 1 using $(Q,(L,M))$, $\Gamma^{\downarrow}$, $\Gamma^{\Downarrow}$ and OEL as arguments (Lines 1-3). Then, the algorithm calls Algorithms 2 and 3 to classify $(P,(J,K))$ and fix the covering relations associated with $(P,(J,K))$ and OEL (Lines 4-5). At last, $(P,(J,K))$ is marked as handled (Line 6). Next, let us elucidate Algorithms 2-3 one by one.

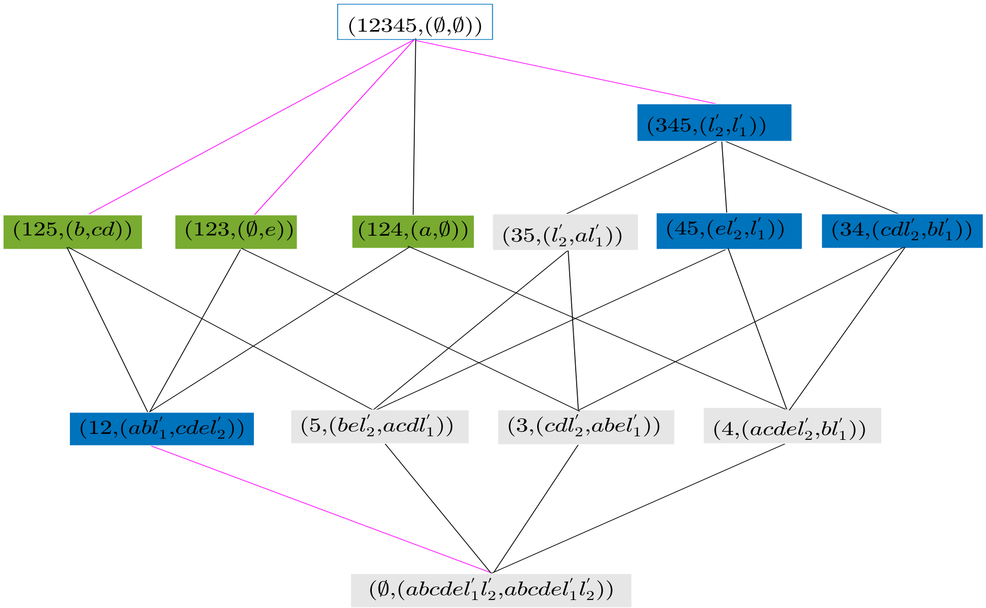

Fig. 3. The OE3W concept lattice $OEL^{\Downarrow}(FT_2)$.

Table 6
The values of key variables.

No.	$(P,(J,K))$	$\phi(J\cap\Gamma^{\downarrow})$	$\omega(K\cap\Gamma^{\downarrow})$	$\tau(K\cap\Gamma^{\downarrow})$	$Casualties(P,(J,K))$	$CC^{\triangle}$ $(P,(J,K))$	$CC^{\triangledown}$ $(P,(J,K))$
1	$(\emptyset,(\emptyset,\emptyset^{\lessdot\downarrow}))$	$l'_1l'_2$	$l_1l_2l_3l_4$	$l'_1l'_2$	∅	/	/
2	$(1,(abl_1,cdel_2l_3l_4))$	∅	l_3l_4	l'_2	{2,4}	/	/
3	$(12,(ab,cdel_3l_4))$	l'_1	l_3l_4	l'_2	{2,4}	∅	{1}
4	$(2,(abl_2,cdel_1l_3l_4))$	l'_1	l_3l_4	l'_2	∅	/	/
5	$(123,(\emptyset,el_4))$	∅	∅	∅	{6,7}	{3}	{8}
6	$(13,(\emptyset,el_2l_4))$	∅	∅	∅	∅	/	/
7	$(23,(\emptyset,el_1l_4))$	∅	∅	∅	∅	/	/
8	$(3,(cdl_3,abel_1l_2l_4))$	l'_2	l_1l_2	l'_1	∅	/	/
9	$(12345,(\emptyset,\emptyset))$	∅	∅	∅	{2, 11, 13, 17, 19}	{10}	{5, 18, 23, 24}
10	$(124,(a,l_3))$	∅	∅	∅	{12, 14}	{3}	{16}
11	$(1345,(\emptyset,l_2))$	∅	∅	∅	{19}	/	/
12	$(14,(a,l_2l_3))$	∅	∅	∅	/	/	
13	$(2345,(\emptyset,l_1))$	∅	∅	∅	{21}	/	/
14	$(24,(a,l_1l_3))$	∅	∅	∅	∅	/	/
15	$(34,(cd,bl_1l_2))$	∅	l_1l_2	l'_1	∅	/	/
16	$(4,(acdel_4,bl_1l_2l_3))$	l'_2	l_1l_2	l'_1	∅	/	/
17	$(1235,(\emptyset,l_4))$	∅	∅	∅	{19, 21}	/	/
18	$(125,(b,cdl_4))$	∅	∅	∅	{20, 22}	{3}	{26}
19	$(135,(\emptyset,l_2l_4))$	∅	∅	∅	∅	/	/
20	$(15,(b,cdl_2l_4))$	∅	∅	∅	∅	/	/
21	$(235,(\emptyset,l_1l_4))$	∅	∅	∅	∅	/	/
22	$(25,(b,cdl_1l_4))$	∅	∅	∅	∅	/	/
23	$(345,(\emptyset,l_1l_2))$	∅	l_1l_2	l'_1	∅	/	/
24	$(35,(y_3,al_1l_2l_4))$	∅	l_1l_2	l'_1	∅	/	/
25	$(45,(e,l_1l_2))$	∅	l_1l_2	l'_1	∅	/	/
26	$(5,(bel_3,acdl_1l_2l_4))$	l'_2	l_1l_2	l'_1	∅	/	/

4.1.1. *Classification of OE3W concepts*

It is important to classify OE3W concepts and amend intents of non-deleted OE3W concepts. In order to achieve this task, Algorithm 2 is developed with theoretical support of Theorems 1-5.

Algorithm 2 receives four arguments: an OE3W concept $(P,(J,K))$ with $P\neq\emptyset$, $\Gamma^{\downarrow}$, $\Gamma^{\Downarrow}$ and OEL, and it computes $class(P,(J,K))$ and renews the set of all casualties of $(Q,(L,M))$ for each parent $(Q,(L,M))$ of $(P,(J,K))$. Meanwhile, Algorithm 2 sets two flags for $(P,(J,K))$ to indicate whether $(P,(J,K))$ is a drag and whether $(P,(J,K))$ is a destructor, respectively.

In order to classify $(P,(J,K))$ with $P\neq\emptyset$, for each parent $(Q,(L,M))$ of $(P,(J,K))$, the algorithm tests if $(Q,(L,M))$ meets requirements: $L=J-\Gamma^{\downarrow}$, $M-\Gamma^{\downarrow}=K-\Gamma^{\downarrow}$ and $\omega(M\cap\Gamma^{\downarrow})=\omega(K\cap\Gamma^{\downarrow})$. For any parent $(Q,(L,M))$ of $(P,(J,K))$, if $(Q,(L,M))$ passes the test, $(Q,(L,M))$ is marked as a drag, and $(P,(J,K))$ and all casualties of $(P,(J,K))$ are added to the set of all casualties of

Algorithm 1: An algorithm for classifying OE3W concepts and fixing covering relations.

Input: $(P,(J,K)),\Gamma^{\downarrow},\Gamma^{\Downarrow},OEL$
Output: renewed OEL

for *each child* $(Q,(L,M))$ *of* $(P,(J,K))$ **do**
  if $(Q,(L,M))$ *has not been handled* **then**
    Handle $(Q,(L,M))$ by Algorithm 1
Classify $(P,(J,K))$ by Algorithm 2
Fix the covering relations afterwards associated with $(P,(J,K))$ by Algorithm 3
Mark $(P,(J,K))$ as handled
return *renewed* OEL

Algorithm 2: An algorithm for classifying OE3W concepts.

Input: $(P,(J,K)),\Gamma^{\downarrow},\Gamma^{\Downarrow},OEL$
Output: renewed OEL

1 **if** $K\cap\Gamma^{\downarrow}\neq\emptyset$ **then**
2   $flag=False$
3   **for** *each parent* $(Q,(L,M))$ *of* $(P,(J,K))$ **do**
4   **if** $L\cap\Gamma^{\downarrow}=\emptyset,\ L-\Gamma^{\downarrow}=J-\Gamma^{\downarrow},\ M-\Gamma^{\downarrow}=K-\Gamma^{\downarrow}$ *and* $\omega(M\cap\Gamma^{\downarrow})=\omega(K\cap\Gamma^{\downarrow})$ **then**
5   Add $(P,(J,K))$ and all casualties into the set of all casualties of $(Q,(L,M))$
6   Mark $(Q,(L,M))$ as a drag
7   **if** $flag==False$ **then**
8         $flag=True$
9   **if** $flag==False$ **then**
10   **if** $|\tau(K\cap\Gamma^{\downarrow})|==|\Gamma^{\Downarrow}|-1$ **then**
11   **if** $J\cap\Gamma^{\downarrow}\neq\emptyset$ **then**
12         $class(P,(J,K))=$ "modified-condensed", $J=(J-\Gamma^{\downarrow})\cup\phi(J\cap\Gamma^{\downarrow})$, $K=(K-\Gamma^{\downarrow})\cup\tau(K\cap\Gamma^{\downarrow})$
13   **else**
14         $class(P,(J,K))=$ "extended-condensed", $J=J\cup(\Gamma^{\Downarrow}-\tau(K\cap\Gamma^{\downarrow}))$, $K=(K-\Gamma^{\downarrow})\cup\tau(K\cap\Gamma^{\downarrow})$
15   **else**
16       $class(P,(J,K))=$ "unchanged-condensed", $K=(K-\Gamma^{\downarrow})\cup\tau(K\cap\Gamma^{\downarrow})$
17   **else**
18     $class(P,(J,K))=$ "deleted"
19 **else**
20   $class(P,(J,K))=$ "old"
21 **if** $(P,(J,K))$ *is not an deleted OE3W concept, and* $(P,(J,K))$ *is a drag* **then**
22   Mark $(P,(J,K))$ as a destructor
23 **return** *renewed* OEL

$(Q,(L,M))$ to facilitate later fixing the covering relations (Lines 4-6). If there exists one or more OE3W concepts which pass the test among all the parents of $(P,(J,K))$, $flag$ is assigned to be "True" (Line 8).

In what follows, the category of $(P,(J,K))$ is determined: (i) if $(P,(J,K))$ with $P\neq\emptyset$ meets conditions in Theorem 3, the algorithm assigns "modified-condensed" to $class(P,(J,K))$, amends $J\cap\Gamma^{\downarrow}$ to $\phi(J\cap\Gamma^{\downarrow})$ in J, and replaces $K\cap\Gamma^{\downarrow}$ with $\tau(K\cap\Gamma^{\downarrow})$ in K (Line 12); (ii) if $(P,(J,K))$ with $P\neq\emptyset$ meets conditions in Theorem 4, the algorithm assigns "extended-condensed" to $class(P,(J,K))$, adds the only element in $\Gamma^{\Downarrow}-\tau(K\cap\Gamma^{\downarrow})$ to J, and replaces $K\cap\Gamma^{\downarrow}$ with $\tau(K\cap\Gamma^{\downarrow})$ in K (Line 14); (iii) if $(P,(J,K))$ with $P\neq\emptyset$ meets conditions in Theorem 5, the algorithm assigns "unchanged-condensed" to $class(P,(J,K))$, and meanwhile amends $K\cap\Gamma^{\downarrow}$ to $\tau(K\cap\Gamma^{\downarrow})$ in K (Line 16); (iv) if $(P,(J,K))$ with $P\neq\emptyset$ meets conditions in Theorem 2, the algorithm assigns "deleted" to $class(P,(J,K))$ (Line 18); (v) if $K\cap\Gamma^{\downarrow}=\emptyset$, the algorithm assigns "old" to $class(P,(J,K))$ (Line 20).

At last, the algorithm recognizes all the destructors: if $(P,(J,K))$ is not a deleted OE3W concept, and $(P,(J,K))$ is a drag, $(P,(J,K))$ is marked as a destructor (Lines 21-22).

4.1.2. Fixing covering relations

For OE3W concepts, fixing the lattice order relations afterwards is essentially to find their children. In order to achieve this task, Algorithm 3 is developed with theoretical support of Theorems 6-8.

Algorithm 3 receives two arguments: an OE3W concept $(P,(J,K))$ and OEL, and it aims at finding children concepts of $(P,(J,K))$ if $(P,(J,K))$ is a destructor.

At first, the algorithm obtains three useful sets S_1, S_2 and S_3 required for later steps (Lines 2-6), where $S_1=CC^{\triangle}(P,(J,K))$, $S_2=CC^{\triangledown}(P,(J,K))$, and $S_3=CC(P,(J,K))$. Then, for every deleted child concept $(Q,(L,M))$ of $(P,(J,K))$, links between $(P,(J,K))$ and $(Q,(L,M))$ should be broken (Line 7). Furthermore, for each concept $(Q,(L,M))$ in S_2 (if $|S_2|>0$), the algorithm performs the following operations: links between $(Q,(L,M))$ and every parent of $(Q,(L,M))$ with $class(parent)=$ "deleted" should be broken (Line 9); if $(R,(D,E))\in S_3$ with $(Q,(L,M))<_{\downarrow}(R,(D,E))$ does not exist, links between $(P,(J,K))$ and $(Q,(L,M))$ should be established (Line 11) according to Theorem 8. At last, the algorithm deletes all casualties of $(P,(J,K))$ from OEL (Line 12).

Algorithm 3: An algorithm for fixing the covering relations.

Input: $(P,(J,K))$ and OEL
Output: renewed OEL
if $(P,(J,K))$ *is a destructor* **then**
  $S_1 = \{(Q,(L,M)) | (Q,(L,M)) \prec_{\downarrow} (P,(J,K)), class(Q,(L,M)) \neq$ "deleted"$\}$
  $S_2 = \emptyset$
  for *each casualty* $(Q,(L,M))$ *of* $(P,(J,K))$ **do**
    $S_2 = S_2 \cup \{(R,(D,E)) | (R,(D,E)) \prec_{\downarrow} (Q,(L,M)), class(R,(D,E)) \neq$ "deleted"$\}$
  $S_3 = S_1 \cup S_2$
  Remove links between $(P,(J,K))$ and every child of $(Q,(L,M))$ with $class(child) =$ "deleted"
  for *each OE3W concept* $(Q,(L,M))$ *in* S_2 **do**
    Remove links between $(Q,(L,M))$ and every parent of $(Q,(L,M))$ with $class(parent) =$ "deleted"
    if *an OE3W concept* $(R,(D,E)) \in S_3$ *with* $(Q,(L,M)) \prec_{\downarrow} (R,(D,E))$ *does not exist* **then**
      Set links between $(P,(J,K))$ and $(Q,(L,M))$
  Delete all casualties of $(P,(J,K))$ from OEL
return *renewed* OEL

4.2. The overall procedure

Since the argument $(P,(J,K))$ in Algorithms 1-3 is required to satisfy $P \neq \emptyset$, we need to handle $(\emptyset, \emptyset^{\lessdot\downarrow})$ alone before invoking these algorithms. Such a simple idea is performed in Algorithm 4.

Algorithm 4: An algorithm of renewing OE3W concept lattices involving slightly decreasing attribute-granularity levels (OEL-Collapse).

Input: $OEL, \Gamma^{\downarrow}, \Gamma^{\Downarrow}$
Output: renewed OEL
1 Find the top OE3W concept $(P_t,(J_t,K_t))$ and the bottom OE3W concept $(P_b,(J_b,K_b))$ of OEL
2 **for** *each concept* $(P,(J,K))$ *in* OEL **do**
3   Compute $\phi(J \cap \Gamma^{\downarrow})$, $\omega(K \cap \Gamma^{\downarrow})$ and $\tau(K \cap \Gamma^{\downarrow})$
4 $class(P_b,(J_b,K_b)) =$ "modified-condensed", $J_b = (J_b - \Gamma^{\downarrow}) \cup \Gamma^{\Downarrow}$, $K_b = (K_b - \Gamma^{\downarrow}) \cup \Gamma^{\Downarrow}$
5 Mark the bottom OE3W concept $(P_b,(J_b,K_b))$ in OEL as handled
6 Handle $(P_t,(J_t,K_t))$ by Algorithm 1
7 **return** *renewed* OEL

Algorithm 4 receives three arguments: the OE3W concept lattice (OEL) before slightly decreasing attribute-granularity levels, and attribute-granularity levels ($\Gamma^{\downarrow}$ and $\Gamma^{\Downarrow}$) before and after the slight decrease.

Firstly, the procedure searches the top OE3W concept $(P_t,(J_t,K_t))$ and the bottom OE3W concept $(P_b, (J_b,K_b))$ of OEL (Line 1). Secondly, compute $\phi(J \cap \Gamma^{\downarrow})$, $\omega(K \cap \Gamma^{\downarrow})$ and $\tau(K \cap \Gamma^{\downarrow})$ for each concept $(P,(J,K))$ of OEL (Lines 2-3). Thirdly, the procedure handles $(P_b,(J_b,K_b)) = (\emptyset, \emptyset^{\lessdot\downarrow})$. Specifically, $(P_b, (J_b,K_b))$ is identified as a modified-condensed OE3W concept, amends $J_b \cap \Gamma^{\downarrow}$ to $\Gamma^{\Downarrow}$ in J_b, and replaces $K_b \cap \Gamma^{\downarrow}$ with $\Gamma^{\Downarrow}$ in K_b (Line 4). Since $(P_b, (J_b,K_b))$ is a modified-condensed OE3W concept and the bottom OE3W concept in OEL, there is no need to find its children. In the end, we mark $(P_b,(J_b,K_b))$ as handled (Line 5), invoke Algorithm 1 to handle every OE3W concept $(P,(J,K))$ with $P \neq \emptyset$ in OEL (Line 6), and return renewed OEL (Line 7).

Next, we analyze the time complexities of Algorithms 1-4. Firstly, the time complexities of Algorithms 2 and 3 are $O(\kappa_1\kappa_2|M|)$ and $O(\kappa_3\kappa_4|M|)$, respectively, where κ_1 is the maximum number of parents of OE3W concepts in OEL, κ_2 is the maximum number of casualties of OE3W concepts in OEL, $\kappa_3 = \max\limits_{(P,(J,K))\in OEL}\{|CC^{\nabla}(P,(J,K))|\}$, $\kappa_4 = \max\limits_{(P,(J,K))\in OEL}\{|CC(P,(J,K))|\}$ and $|M|$ is the number of attributes. Secondly, the time complexity of Algorithm 1 is $|OEL|$ times the sum of the time complexities of Algorithms 2 and 3, i.e. $O(|OEL||M|(\kappa_1\kappa_2 + \kappa_3\kappa_4))$. Finally, the time complexity of Algorithm 4 is $O(|OEL||M|(\kappa_1\kappa_2 + \kappa_3\kappa_4))$.

5. Experimental evaluation

To evaluate our OEL-Collapsee algorithm, some experiments are conducted in this section. For this purpose, we compare the OEL-Collapse algorithm with the conventional approach, where the OE3W concept latices generated by the conventional approach were directly constructed by using FastAddIntent [13]. All the experiments were implemented in Matlab R2018b and on a computer with an Intel(R) Xeon(R) Platinum 8251C CPU @ 3.80 GHz processor, 256 GB RAM and a 64-bit operating system.

In the experiments, we selected 13 data sets from the UCI machine learning repository [61] and the KEEL-dataset repository [62]. The details of these chosen data sets are shown in Table 7. For our purpose, the objects having missing values were all removed from the ninth data set, e.g. Mammographic mass. To transform these chosen data sets into formal contexts, we used the following pretreatment method. Specifically, firstly, for all continuous attributes in the chosen data sets, we employed the discretization method of "six-section", which means "classify the values of each continuous attribute into six pairwise disjoint intervals with equal width". Secondly, we further applied the scaling approach [2] to these discretized data sets. Then, we obtained thirteen formal contexts, denoted by $T_1 - T_{13}$. It should be pointed out that for the first continuous attribute in these data sets, the operations of decreasing

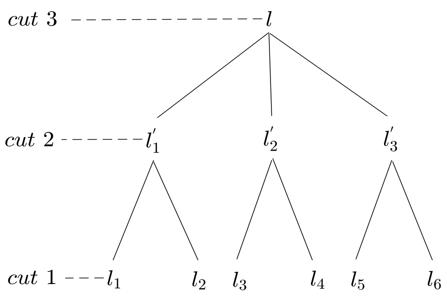

Fig. 4. The g-tree for the first continuous attribute.

Table 7
The details about the thirteen chosen data sets.

No.	Data sets	Objects	Attributes
1	Challenger USA space shuttle O-Ring	23	3 (discrete but not Boolean), 2 (continuous)
2	Caesarian section classification	80	2 (Boolean), 3 (discrete but not Boolean), 1 (continuous)
3	Cryotherapy	90	1 (Boolean), 2 (discrete but not Boolean), 4 (continuous)
4	Hayes-roth	132	5 (discrete but not Boolean), 1 (continuous)
5	Servo	167	4 (discrete but not Boolean), 1 (continuous)
6	Teaching assistant evaluation	151	3 (discrete but not Boolean), 3 (continuous)
7	Haberman's survival	306	3 (discrete but not Boolean), 1 (continuous)
8	Blood transfusion service center	748	1 (Boolean), 4 (continuous)
9	Mammographic mass	961	1 (Boolean), 4 (discrete but not Boolean), 1 (continuous)
10	Banknote authentication	1372	1 (Boolean), 4 (continuous)
11	Stulong	1417	5 (continuous)
12	Plastic	1650	1 (discrete but not Boolean), 2 (continuous)
13	Quake	2178	4 (continuous)

attribute-granularity levels were performed by using the g-tree shown in Fig. 4, where $l_1, \cdots, l_6$ are six intervals under six-section, cut 1, cut 2 and cut 3 are $\{l_1, \cdots, l_6\}$, $\{l_1', l_2', l_3'\}$ and $\{l\}$, respectively, and $\{l_1, \cdots, l_6\} \leqslant \{l_1', l_2', l_3'\} \leqslant \{l\}$. Note that, although the principles for decreasing the attribute-granularity levels should be determined according to domain knowledge, we designed a unified mechanism of decreasing attribute-granularity levels without considering domain knowledge to simplify the experiments. Specifically, for the first continuous attribute in these chosen data sets, the operations of decreasing the attribute-granularity levels are performed by replacing $\{l_1, \cdots, l_6\}$ with $\{l_1', l_2', l_3'\}$ and $\{l_1', l_2', l_3'\}$ with $\{l\}$, and we denote them by $\downarrow_{cut2}^{cut1}$ and $\downarrow_{cut3}^{cut2}$, respectively. For the formal contexts $T_1 - T_{13}$, after applying $\downarrow_{cut2}^{cut1}$, we obtained thirteen formal contexts, denoted by $\mathbb{T}_1 - \mathbb{T}_{13}$; for the formal contexts $\mathbb{T}_1 - \mathbb{T}_{13}$, after applying $\downarrow_{cut3}^{cut2}$, we obtained thirteen formal contexts, denoted by $\mathcal{T}_1 - \mathcal{T}_{13}$.

Now, it is ready to evaluate our OEL-Collapse algorithm. Fig. 5 shows the running time of OEL-Collapse for updating the OE3W concept lattices $OEL(T_1) - OEL(T_{13})$ and the conventional method for constructing the OE3W concept lattices of the formal contexts $\mathbb{T}_1 - \mathbb{T}_{13}$, and the number of OE3W concepts in $OEL(T_1) - OEL(T_{13})$. Fig. 6 reports the running time of OEL-Collapse for updating the OE3W concept lattices $OEL(\mathbb{T}_1) - OEL(\mathbb{T}_{13})$ and the conventional approach for constructing the OE3W concept lattices of the formal contexts $\mathcal{T}_1 - \mathcal{T}_{13}$, and the numbers of OE3W concepts in $OEL(\mathbb{T}_1) - OEL(\mathbb{T}_{13})$. From Fig. 5 and Fig. 6, we can see that our OEL-Collapse algorithm outperforms the traditional method considerably, especially when the sizes of OE3W concept lattices are big.

6. Conclusions

Now we describe our major work, our main contribution and the future study.

6.1. Our major work

Firstly, we have examined changes of OE3W concepts lattices involving decreasing attribute-granularity levels. Specifically, we have classified OE3W concepts and covering relations into six types and three types, respectively, given methods of quickly identifying the categories of OE3W concepts and covering relations, and discussed properties about these kinds of OE3W concepts. Namely, we have investigated the intrinsic theoretical relationships of OE3W concepts lattices before and after decreasing attribute-granularity levels. Secondly, on the basis of the above results, we have proposed a novel algorithm (named OEL-Collapse) to renew OE3W concept lattices involving decreasing attribute-granularity levels. Finally, we have conducted experiments to show the computational efficiency of the OEL-Collapse algorithm.

6.2. Our main contribution

We emphasize our contribution by comparing the differences between our method and the existing work.

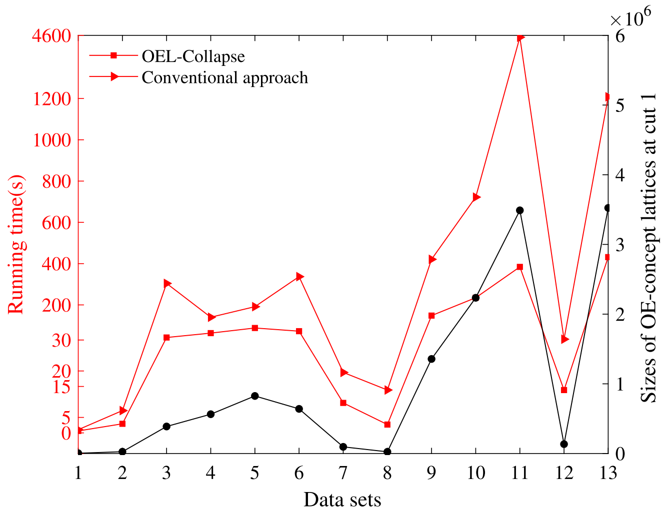

Fig. 5. The running time of OEL-Collapse and conventional approach and the sizes of OE3W concept lattices when applying $\downarrow^{cut1}_{cut2}$.

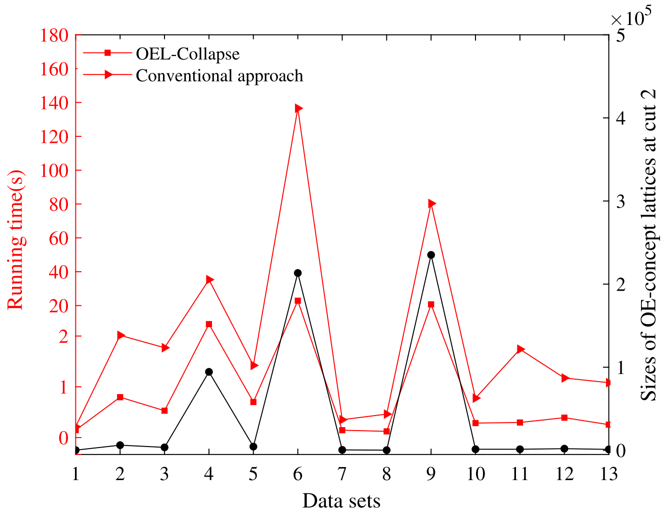

Fig. 6. The running time of OEL-Collapse and conventional approach and the sizes of OE3W concept lattices when applying $\downarrow^{cut2}_{cut3}$.

- The differences between our method and the existing work on OE3W concepts updating with decreasing attribute-granularity levels
(i) Although the existing work [60] has discussed the issue of updating OE3W concepts with decreasing attribute-granularity levels, how to update the covering relation among the OE3W concepts with decreasing attribute-granularity levels has not been studied yet. In our work, the transformation from the covering relation among the OE3W concepts under a fine attribute-granularity level to that among the OE3W concepts under a coarse attribute-granularity level, has been established to efficiently update the covering relation among the obtained OE3W concepts.
(ii) Although the existing work [60] has obtained the OE3W concepts under a coarse attribute-granularity level by using the operators $\lessdot^{\Downarrow}$ and $\gtrdot^{\Downarrow}$ on the OE3W concepts under a fine attribute-granularity level, many repetitive elements existed in the obtained OE3W concepts. To avoid this case, it requires enormous amounts of pairwise comparison between the obtained OE3W concepts. However, we have clarified the intrinsic theoretical relationships of the OE3W concept lattices before and after decreasing attribute-granularity levels, which can be used to identify the deleted OE3W concepts under a fine attribute-granularity level and further transform the non-deleted OE3W concepts into the OE3W concepts under a coarse attribute-granularity level without involving the operators $\lessdot^{\Downarrow}$ and $\gtrdot^{\Downarrow}$. In other words, our method can improve the computational efficiency of updating OE3W concepts when decreasing attribute-granularity levels.
- The differences between our work and the method of directly constructing OE3W concept latices
Our algorithm has used the intrinsic theoretical relationships of OE3W concept lattices before and after decreasing attribute-granularity levels to update OE3W concept lattices, which can make full use of the information of the obtained OE3W concepts to generate desired OE3W concepts, but directly constructing OE3W concept latices ignored the information of the obtained OE3W

concepts, let alone utilizing the old conceptual knowledge. Thus, our algorithm outperforms the method of directly constructing OE3W concept latices in the aspect of elapsed time.

6.3. Our future study

As the other situation about changing attribute-granularity levels is increasing attribute-granularity levels, how to efficiently renew OE3W concepts lattices in this situation? Additionally, based on these dynamic updating methods, how to quickly select an appropriate attribute-granularity level for a specific real problem? These issues will be discussed in our future study.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

The public data sets employed in the experiment were downloaded from https://archiveics.uci.edu/datasets and https://sci2s.ugr.es/keel/datasets.php.

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