Title

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Abstract

This paper shows the derivation of a closed-form solution to a generalized point correspondences problem.

1 Introduction

Consider the following minimization problem:

$$\min \sum_k |R(\varphi)\mathbf{p}_k + \mathbf{t} - \mathbf{q}_k|_{C_k}$$

Setting $P_k = \begin{bmatrix} \mathbf{p}_{k1} & -\mathbf{p}_{k2} \\ \mathbf{p}_{k2} & \mathbf{p}_{k1} \end{bmatrix} = \begin{bmatrix} \mathbf{p}_k & \boldsymbol{R}(\pi/2)\mathbf{p}_k \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} \mathbf{t} \\ \cos\varphi \\ \sin\varphi \end{bmatrix}$, $M_k = \begin{bmatrix} I_{2\times 2} & P_k \end{bmatrix}$, $W = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$, this can be written as

$$\min J(\mathbf{x}) = \sum_k |\left(M_k\mathbf{x} - \mathbf{q}_k\right)^T C_k \left(M_k\mathbf{x} - \mathbf{q}_k\right)$$

$$\text{s.t.} \quad \mathbf{x}^T W \mathbf{x} = 1$$

Now expand to obtain

$$\sum_k \left(M_k\mathbf{x} - \mathbf{q}_k\right)^T C_k \left(M_k\mathbf{x} - \mathbf{q}_k\right) = \sum_k \left(\mathbf{x}^T M_k^T C_k M_k \mathbf{x} + \mathbf{q}_k^T C_k \mathbf{q}_k - 2\mathbf{q}_k^T C_k M_k \mathbf{x}\right)$$

Therefore the new function to minimize is

$$\min J'(\mathbf{x}) = \mathbf{x}^T \underbrace{\left(\sum_k M_k^T C_k M_k\right)}_{M} \mathbf{x} + \underbrace{\left(\sum_k -2\mathbf{q}_k^T C_k M_k\right)}_{\boldsymbol{g}} \mathbf{x}$$

The new problem is

$$\min J'(\mathbf{x}) = \mathbf{x}^T M \mathbf{x} + \boldsymbol{g}^T \mathbf{x} \tag{1}$$

$$\text{s.t.} \quad \mathbf{x}^T W \mathbf{x} = 1 \tag{2}$$

1.1 Minimizing using Lagrange

Solve using Lagrange's multipliers. This is the Lagrangian:

$$L(\mathbf{x}) = \mathbf{x}^T M \mathbf{x} + g^T \mathbf{x} + \lambda(\mathbf{x}^T W \mathbf{x} - 1)$$

This is the condition for optimality ($\partial L / \partial \mathbf{x} = \mathbf{0}^T$):

$$2\mathbf{x}^T M + g^T + 2\lambda \mathbf{x}^T W = \mathbf{0}^T$$

that can be turned into this:

$$\mathbf{x} = (2M + 2\lambda W)^{-T} k^T \tag{3}$$

If we put this last relation into the constraint (2), we obtain

$$k\,(M + 2\lambda W)^{-1} W\,(M + 2\lambda W)^{-T} k = 1 \tag{4}$$

Even if not objoius (4) is a fourth-order polynomial in λ. The following computations will prove this assertion. Partition the matrix $(M + 2\lambda W)$ into four sub-matrixes:

$$M = \begin{bmatrix} A & B \\ B^T & D + 2\lambda I \end{bmatrix}$$

In the middle of (4) there is the matrix W, which has a sparse form, it is needed to compute only the last column of $(M + 2\lambda W)^{-1}$. Using the matrix inversion lemma, one obtains that

$$(M + 2\lambda W)^{-1} = \begin{bmatrix} A & B \\ B^T & (D + 2\lambda I) \end{bmatrix}^{-1} = \begin{bmatrix} * & -A^{-1} B Q^{-1} \\ * & Q^{-1} \end{bmatrix}$$

Where $Q = \left(D - B^T A^{-1} B + 2\lambda I\right) \triangleq (S + 2\lambda I)$. The constraint (2) now appears as

$$g^T \begin{bmatrix} A^{-1} B Q^{-1} Q^{-T} B^T A^{-T} & -A^{-1} B Q^{-1} Q^{-T} \\ * & Q^{-1} Q^{-T} \end{bmatrix} g = 1$$

Now write Q in this way:

$$Q = (S + 2\lambda I)^{-1} = \frac{S^A + 2\lambda I}{p(\lambda)}$$

where $S^A = \det(S) \cdot S^{-1}$ and $p(\lambda) = \det(S + 2\lambda I)$. Because

$$Q^{-1} Q^{-T} = \frac{\left(S^A + 2\lambda I\right)\left(S^A + 2\lambda I\right)^T}{p(\lambda)^2} = \frac{S^A S^{A^T} + 4\lambda^2 I + 4\lambda S^A}{p(\lambda)^2}$$

One finally obtains the following polynomials:

$$\begin{aligned} &\lambda^2 \cdot 4k^T \begin{bmatrix} A^{-1} B B^T A^{-T} & -A^{-1} B \\ * & I \end{bmatrix} k && + \\ &\lambda \cdot 4k^T \begin{bmatrix} A^{-1} B S^A B^T A^{-T} & -A^{-1} B S^A \\ * & S^A \end{bmatrix} k && + \\ &k^T \begin{bmatrix} A^{-1} B S^{A^T} S^A B^T A^{-T} & -A^{-1} B S^{A^T} S^A \\ * & S^{A^T} S^A \end{bmatrix} k && - (p(\lambda))^2 = 0 \end{aligned} \tag{5}$$

Because $p(\lambda)$ is a second-order polynomial, the order of (5) is 4, therefore the solution can be found in closed form. After one has found λ, one can obtain $\mathbf{x}$ using (3).