Optimizing Low Density Parity Check Code for two Parallel Erasure links

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***Abstract*—In this paper, we solve the optimizing problem of designing Low-Density Parity Check codes for two parallel erasure links based on the known SDP approach. We show that our reformulation is suitable for this optimizing problem. Our results show that the optimal rate code design problem is a good way and worth-mentioning tool. One can extend our work to the other area.**

I. INTRODUCTION

Gallager found the Low Density Parity Check (LDPC) codes in the mid1960's in [1]. Gallager codes were generally not considered in the next 35 taking after years, as it was difficult to implement the iterative soft-decoding techniques, the guideline behind these codes. It was chiefly as a result of the absence of computational assets around then. Later on in 1984, Tanner recommended the speculation of Gallager codes using the idea of codes on graphs popularly known as Tanner graphs to describe the iterative soft decision decoding techniques. After the extensive research on iterative decoding on Tanner graphs by MacKay and Neal, it was rediscovered as LDPC codes in 1996 [2]. LDPC codes are built by outlining a sparse parity-check matrix. Subsequently, the name low-density parity-check originates from the distinguishing characteristic of their parity-check matrix which contains only a few non-zero components than zero elements.

It has been shown that, LDPC codes can approach the Shannon's limit similar to turbo codes or stunningly better than turbo codes. The low decoding complexity of LDPC codes were accomplished by utilizing Sum-Product Algorithm (SPA), a linear-time complex algorithm for iterative decoding. This made these codes perform extremely well even in various channel conditions. Henceforth, LDPC codes have found a spot as an error control code in many of the wired and wireless applications and digital storage systems where the reliability is a major concern. But, the basic assumption is that the channel state information is known 'a priori' to both the transmitter and receiver. However, it is not a promising technique for time-varying channels and time sensitive applications. Hence, there is a requirement for an alternate coding scheme which is suitable for time-varying channels.

Achieving or approaching to the capacity of the graph based codes such as LDPC code is a major research these days [3-4]. At the other hand, using the optimization methods is a very good perspective and one can reduce the complexity of these decoders as in [5].

In [6-8], the authors introduce and extend the SDP method for solving the optimal rate LDPC code design problem. This contribution can be developed in the others subjects such as in [9-10]. The SDP method has the constraint of the decoding strategy, i.e.: decoding convergence property, which is called Density Evolution (DE). Density Evolution constraint is introduced in [3] by Richardson and Urbanke and generalized at [11,12]. Density Evolution shows the convergence property of the iterative Message Passing decoder in finite and infinite mode [13-17].

In this paper, first, we review the fundamental optimizing Low Density Parity Check codes method based on the Semi Definite Programming approach in Section II. Second, we present the optimizing Low Density Parity Check codes for two parallel erasure links problem definition in Section III. Third, by using the known Semi Definite Programming transformation we transform the optimizing problem in an optimization problem in Section IV. Last but not least, by presenting some simulation results we conclude the paper in Section V and Section VI.

II. FUNDAMENTAL OPTIMIZING LDPC CODES

Degree distributions of a graph based code such as LDPC code are considered. The design rate of this graph based code is [9]:

$$\Re = \overline{\left(\frac{\int \beta}{\int \alpha}\right)} \tag{1}$$

where the not operator is for one-complement and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$ are two non-negative polynomials related to the definition of the graph based code. Maximization of the $\Re$ is desirable, which is subjected to the DE function, $\wp$:

$$\wp: \tau \to \ell\alpha\left(\overline{\beta(\overline{\tau})}\right) \quad (2)$$

$\ell$is the erasing probability in this model. It is clear that (2) is a non-linear function respect to τ. So, the non-linear optimization problem is:

$\text{Max} \int \alpha$

Subjected to:

$$C1: \wp(\tau) \le \tau \; for \; \tau \in [0,1],$$
$$C2: \wp(1) = \ell. \quad (3)$$

An approach based on SDP, for the first time, presented by the author in [5]. Eq.(3) is a semi-infinite optimization problem based on the $C1$ constraint. One can reformulate the main optimization problem, Eq.(3), by a semi-definite optimization problem and solve it in polynomial time. Related reformulation of Eq.(3) is:

$\text{Max}\; \gamma^T \alpha$

Subjected to:

$$\text{C1}: \varsigma^T \alpha = \mathfrak{P}$$
$$\text{C2}: \mathfrak{P} \succcurlyeq 0 \quad (4)$$

where the notation $\succcurlyeq$ stands for a matrix with SD positive property. The relation between SD matrix $\mathfrak{P}$ and vectors γ, ς is based on SDP reformulation method. Some examples presented in follows:

Example1: Find the solution for the setup problem $\alpha = (\alpha_1, 2,2)$, $\gamma = (1,0,0)$ and $\tau \in \mathbb{R}^+$ based on the Eq.(3) and Eq.(4). By using definitions in [5-7], the equivalent optimization problem is as follows:

$\text{Max}\; \alpha_1$

Subjected to:

$$C1: \begin{bmatrix} \alpha_1 & 1 \\ 1 & 2 \end{bmatrix} \succcurlyeq 0. \quad (5)$$

Note that if we have $\tau \in [0,1]$, the corresponding optimization reformulation is:

$\text{Max}\; \alpha_1$

Subjected to:

$$C1: \begin{bmatrix} 2 & 0 & \mathfrak{P}_{13} \\ 0 & \mathfrak{P}_{22} & 0 \\ \mathfrak{P}_{31} & 0 & 4+\alpha_1 \end{bmatrix} \succcurlyeq 0.$$
$$C2: \mathfrak{P}_{13} + \mathfrak{P}_{22} + \mathfrak{P}_{31} = 6. \quad (6)$$

Example2: Find the solution for the setup problem $\alpha = (2, \alpha_2, 2)$, $\gamma = (0,1,0)$ and $\tau \in \mathbb{R}^+$ based on the Eq.(3) and Eq.(4). By using definitions in [5-7], the equivalent optimization problem is as follows:

$\text{Max}\; \alpha_2$

Subjected to:

$$C1: \begin{bmatrix} 2 & \frac{\alpha_2}{2} \\ \frac{\alpha_2}{2} & 2 \end{bmatrix} \succcurlyeq 0. \quad (7)$$

Note that if we have $\tau \in [0,1]$, the corresponding optimization reformulation is:

$\text{Max}\; \alpha_2$

Subjected to:

$$C1: \begin{bmatrix} 2 & 0 & \mathfrak{P}_{13} \\ 0 & \mathfrak{P}_{22} & 0 \\ \mathfrak{P}_{31} & 0 & 4+\alpha_2 \end{bmatrix} \succcurlyeq 0.$$
$$C2: \mathfrak{P}_{13} + \mathfrak{P}_{22} + \mathfrak{P}_{31} = \alpha_2 + 4. \quad (8)$$

Example3: Find the solution for the setup problem $\alpha = (2,2,\alpha_3)$, $\gamma = (0,0,1)$ and $\tau \in \mathbb{R}^+$ based on the Eq.(3) and Eq.(4). By using definitions in [5-7], the equivalent optimization problem is as follows:

$\text{Max}\; \alpha_3$

Subjected to:

$$C1: \begin{bmatrix} 2 & 1 \\ 1 & \alpha_3 \end{bmatrix} \succcurlyeq 0. \quad (9)$$

Note that if we have $\tau \in [0,1]$, the corresponding optimization reformulation is:

$\text{Max}\; \alpha_2$

Subjected to:

$$C1: \begin{bmatrix} \alpha_3 & 0 & \mathfrak{P}_{13} \\ 0 & \mathfrak{P}_{22} & 0 \\ \mathfrak{P}_{31} & 0 & 4+\alpha_3 \end{bmatrix} \succcurlyeq 0.$$
$$C2: \mathfrak{P}_{13} + \mathfrak{P}_{22} + \mathfrak{P}_{31} = 2\alpha_3 + 2. \quad (10)$$

III. Problem Definition of Optimizaing LDPC

Now, we represent the optimizing method of the LDPC code for two links BEC channel. By the way, a sender and a receiver are considered. These channels are independent channels with different erasure rate between the sender and the receiver. The erasure probability of the first link is $\ell_1$and the erasure probability of the second link is ℓ_2.

In this model, all links send a copy of original data to the receiver and the used code in the sender and receiver is the same. This model can be used for the special case of the multipath fading channel.

Theorem1: There are three constraints, which forces to the LDPC code for transmitting in the two links erasure channel:

$$\ell_1\alpha\left(\overline{\beta(\overline{\tau})}\right) \le \tau \;\; \forall 0 \le \tau \le \ell_1, \quad (11)$$
$$\ell_2\alpha\left(\overline{\beta(\overline{\tau})}\right) \le \tau \;\; \forall 0 \le \tau \le \ell_2, \quad (12)$$
$$\ell_1\ell_2\alpha\left(\overline{\beta(\overline{\tau})}\right) \le \tau \;\; \forall 0 \le \tau \le \ell_1\ell_2. \quad (13)$$

Proof: Due to the behavior of two parallel links, the receiver receives two copy of the sent information from the sender. Eq.(11) certify that decoding procedure success at the receiver just by using the first link, the Eq.(12) certify that decoding procedure success at the receiver just by using the second link and Eq.(13) certify that decoding procedure success at the receiver just by using both links.

In fact, receiver node receives two copy of information with different erasing probability. First, by different looking information at the receiver, one can see two independent channels which each of them has related eraser probability and the degree distribution of the graph based code must satisfy the Eq.(11) and Eq.(12). Second, by together looking information at the receiver, one can use both received information at the receiver and generate a sequence with erasing probability equal to the product of the erasing probabilities and the degree distribution of the graph based code must satisfy the Eq.(13). ■

By using the constraints Equations (11-13), Theorem1, and by using an SDP representation formula, the optimizing code graph based code is:

Max $\int \alpha$

Subjected to:

$$C1: \ell_1 \alpha\left(\overline{\beta(\overline{\tau})}\right) \leq \tau \;\; \forall 0 \leq \tau \leq \ell_1,$$

$$C2: \ell_2 \alpha\left(\overline{\beta(\overline{\tau})}\right) \leq \tau \;\; \forall 0 \leq \tau \leq \ell_2,$$

$$C3: \ell_1 \ell_2 \alpha\left(\overline{\beta(\overline{\tau})}\right) \leq \tau \;\; \forall 0 \leq \tau \leq \ell_1 \ell_2. \quad (14)$$

Eq. (14) has three constraint which each of them has infinite constraints. A problem with infinite constraints is not solvable in polynomial time. At the other hand, each constraints has different range for τ. According to optimization problem of the Eq.(14), we want to find a graph based code that satisfy all three constraints of Eq. (14).

IV. Solution of Optimizing LDPC

In following, we present a way for solving the Eq.(14) based on the SDP representation method. Following theorem provides the goal of this section.

Theorem2: According to the definition of Eq.(2), one can consider different $\wp_i$ function. For given β-vector, one can find the solution of the following optimization problem by SDP method.

Max $\int \alpha$

Subjected to:

$$C1: \wp_1(\tau) \leq \tau \; for \; \tau \in [0,1], \wp_1(1) = \ell_1,$$

$$C2: \wp_2(\tau) \leq \tau \; for \; \tau \in [0,1], \wp_2(1) = \ell_2,$$

$$C3: \wp_3(\tau) \leq \tau \; for \; \tau \in [0,1], \wp_3(1) = \ell_3. \quad (15)$$

Proof: for the fixed polynomial α,it is clear that the cost function, which is the integration of α is equal to the linear combination of the coefficients. Now, according to Eq.(4), equal of $(C1.15)$ is:

$$\varsigma_1^T \alpha = \mathfrak{P}_1, \mathfrak{P}_1 \succcurlyeq 0. \quad (16)$$

So, the equal SDP of Eq.(15) is:

Max $\gamma^T \alpha$

Subjected to:

$$C1: \varsigma_1^T \alpha = \mathfrak{P}_1, \mathfrak{P}_1 \succcurlyeq 0,$$

$$C2: \varsigma_2^T \alpha = \mathfrak{P}_2, \mathfrak{P}_2 \succcurlyeq 0,$$

$$C3: \varsigma_3^T \alpha = \mathfrak{P}_3, \mathfrak{P}_3 \succcurlyeq 0. \quad (17)$$

Therefore, the optimization problem of Eq.(17) can be stated as follows:

Max $\gamma^T \alpha$

Subjected to:

$$C1: \varsigma^T \alpha = \mathfrak{P}, \mathfrak{P} \succcurlyeq 0. \quad (18)$$

where, $\varsigma = [\varsigma_i]_{i=1}^3$ and $\mathfrak{P} = \mathrm{diag}[\mathfrak{P}_i]_{i=1}^3$, where it completes the proof. ■

V. Simulation Results

Some simulation result for finding the solution of the Eq.(18) is presented. Our simulation results are based on the presenting some graph based code which has a small gap to the capacity.

Example4: For the single tap $\beta_4 = 1$ with 30% erasing probability, the SDP method find the solution of $\alpha = [0 \quad 0.4976 \quad 0.5024 \quad 0]$, and the gap to the capacity is 0.1. Now, this code can be used for a two parallel erasure links with constraint $\ell_1 \ell_2 = 0.3$. We find that for $\ell_1 \ell_2 = 0.3345$, α_4 will have the value near to 0.0001 and we have $\alpha = [0 \quad 0.3130 \quad 0.6869 \quad 0.0001]$ and the gap to the capacity is 0.0885. By this vision we have three regions. Figure 1 shows these regions. The horizontal axis of the Figure 1 is the value of ℓ_1 and the vertical axis of the Figure 1 is the value of ℓ_2. At first region, we observe that the value of the α_4 at R_1 and R_2 is zero and at R_3 it has the value.

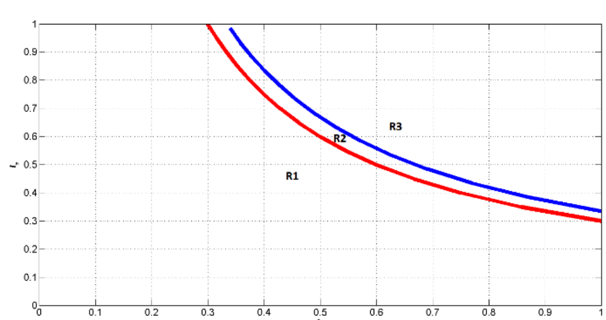

Figure 1. Different region for Example4

VI. CONCLUSION

We use effectively the optimization methods for designing the graph based codes in order to optimizing Low Density Parity Check codes for two parallel erasure links. Some numerical examples and simulation results for verifying the proposed method presented. We observed that the product of the erasing probability is very important for designing the graph based code.

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