Intro Fellowship Application: Math Challenge Solutions

AI Safety at UCLA

Fall 2024

Here are the solutions to the math challenge portion of the AI Safety Intro Fellowship application. Note that these solutions are especially verbose for clarity, but we do not neccesarily expect that your solutions be as long.

Linear Algebra Challenge Solution

First, let us consider the case where $n = 2$. In this case, let A and B be any two matrices in $M_{2\times 2}(\mathbb{R})$ (the set of 2 by 2 matrices over $\mathbb{R}$):

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

Since W consists solely of matrices of the form $AB - BA$, all matrices in W are of the form:

$$\begin{aligned} AB - BA &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix} - \begin{bmatrix} ae + cf & be + df \\ ag + ch & bg + dh \end{bmatrix} \\ &= \begin{bmatrix} bg - cf & af + bh - be - df \\ ce + dg - ag - ch & cf - bg \end{bmatrix} \end{aligned}$$

Note that the trace of $AB - BA$ is zero. In fact, $W = \{A \in M_{2\times 2}(\mathbb{R}) \mid \operatorname{tr}(A) = 0\}$, since none of the other terms in our equation above are linearly dependent. From this, we have a basis for W:

$$\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$$

It is trivial to show that this generates all matrices in W, so the dimension of W is 3.

Now let us consider the n by n case. Again, let $A, B \in M_{n\times n}(\mathbb{R})$, and let A_{ij} refer to the entry of A in the ith row and jth column. Then we have:

$$(AB - BA)_{ii} = \sum_{k=1}^{n} A_{ik}B_{ki} - \sum_{k=1}^{n} B_{ik}A_{ki}$$

From this, we have:

$$\begin{aligned} \operatorname{tr}(AB - BA) &= \sum_{i=1}^{n} (AB - BA)_{ii} \\ &= \sum_{i=1}^{n} \left(\sum_{k=1}^{n} A_{ik}B_{ki} - \sum_{k=1}^{n} B_{ik}A_{ki} \right) \\ &= \sum_{i=1}^{n} \sum_{k=1}^{n} A_{ik}B_{ki} - \sum_{i=1}^{n} \sum_{k=1}^{n} A_{ki}B_{ik} \\ &= 0 \end{aligned}$$

So we have shown that $W \subseteq \{A \in M_{n\times n}(\mathbb{R}) \mid \mathrm{tr}(A) = 0\}$. Next, we will show that the set of n by n matrices with trace zero has dimension $n^2 - 1$. Let E_{ij} be the matrix with zeros everywere except at the ith row and jth column, which instead is one. Note that the set of matrices E_{ij}, where $i \neq j$ and matrices $E_{ii} - E_{jj}$, where $i \neq j$ all have trace zero, and additionally, construct a basis for the set of matrices with trace zero, namely they form a set of $n^2 - 1$ linearly independent vectors, and the vector space of matrices with trace zero is at most dimension $n^2 - 1$, since if it was n^2, then it would just be $M_{n\times n}(\mathbb{R})$. Finally, we need to show that $W \supseteq \{A \in M_{n\times n} \mid \mathrm{tr}(A) = 0\}$. It suffices to show that the E_{ij}'s and $E_{ii} - E_{jj}$'s from above can be constructed as $AB - BA$. For E_{ij}'s this is trivial: use $A = E_{ii}$ and $B = E_{ij}$. For $E_{ii} - E_{jj}$, use $A = E_{ij}$ and $B = E_{ji}$. So, we are done.

Statistics Challenge Solution

First, we can determine the posterior distribution (up to a proportionality constant) by applying Bayes' Theorem:

$$p(\beta|X, y) \propto p(y|X)p(\beta|X)$$

Since X is fixed, we know that $p(y|X)p(\beta|X) = p(y|X, \beta)p(\beta)$. First, we will consider the case of normal (Gaussian) priors. The log-likelihood then becomes:

$$\begin{aligned}
\log p(\beta) &\propto \log \prod_{i=0}^{p} \left[\frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2} \left(\frac{\beta_i}{\sigma} \right)^2 \right) \right] \\
&= \log \left[\left(\frac{1}{\sigma\sqrt{2\pi}} \right)^2 \exp\left(-\frac{1}{2} \sum_{i=0}^{p} \left(\frac{\beta_i}{\sigma} \right)^2 \right) \right] \\
&\propto -\|\beta\|_2^2
\end{aligned}$$

So we see that maximizing log-likelihood with normal priors is equivalent to minimizing least-squares with L2 penalization. Now let us consider the Laplacian priors, where the log-likelihood becomes:

$$\begin{aligned}
\log p(\beta) &\propto \log \prod_{i=0}^{p} \left[\frac{1}{2b} \exp\left(-\frac{|\beta_i|}{b} \right) \right] \\
&\propto -\|\beta\|_1
\end{aligned}$$

And similarly, we see that that maximizing log-likelihood with Laplacian priors is equivalent to minimizing least-squares with L1 penalization.

Abstract Algebra Challenge

Let S be the set of n by m binary matrices, and G be the group corresponding to possible swaps as described in the problem statement (ie. permuting rows and columns). Note that $|G| = n!m!$, because there are $n!$ ways to permute rows and $m!$ ways to permute the columns. We would like to find a formula for $|S/G|$, because this is precisely the number of equivalence classes of S wrt. G, that is, the number of unique non-equivalent matrices.

Note that $G \cong S_n \times S_m$, where S_n and S_m are symmetric groups of order n and m, respectively, since symmetric groups represent permuations. Using Burnside's Lemma, we see that:

$$|S/G| = \frac{1}{|G|} \sum_{\alpha \in G} |\mathrm{fix}(\alpha)| = \frac{\sum_{\sigma\in \mathrm{S_n}} \sum_{\tau \in \mathrm{S_m}} |\mathrm{S}^{(\sigma,\tau)}|}{\mathrm{n!m!}}$$

But note that $|S^{(\sigma,\tau)}|$ is given by $a_{\mathrm{lcm}(\mathrm{o}(\sigma),\mathrm{o}(\tau))}^{\gcd(\mathrm{o}(\sigma),\mathrm{o}(\tau)}$, where $o(\sigma)$ is the order of the permutation σ. Additonally, a_i always 2, since we are dealing with binary matrices. So finally, we have:

$$|S/G| = \frac{\sum_{\sigma \in S_n} \sum_{\tau \in S_m} |2^{\gcd(\mathrm{o}(\sigma),\mathrm{o}(\tau))}}{n!m!}$$

This can be additionally simplfied, but we leave it here. See OEIS A028657 for more information!