Intro Fellowship Application: Math Challenge

AI Safety at UCLA

Fall 2024

Welcome to the Math Challenge for the AI Safety Intro Fellowship application. Please choose any **one** of the three following questions to submit a solution for. These problems are tricky, and asking for help is encouraged, please feel free to reach out to the organizers with questions. Solutions prefered typeset in LaTeX, but scanned handwritten is OK too. Please show all your work. You may cite common theorems, ie. Rank-Nullity, but use your best judgement: don't trivialize the problem. If you see a term you don't recognize, Wikipedia is your friend!

Notation

This document uses standard mathematical notation as shorthand to represent various concepts. In case you are not already familiar with this shorthand, or need a refresher, here is a short glossary. Feel free to use these symbols in your solutions.

- $\exists$ is shorthand for "there exists"
- $\forall$ is shorthand for "for all"
- $\in$ is shorthand for "in" or "is in"
- $\subset$ means "subset of"
- $f : A \to B$ means f is function with domain A and range B.

Linear Algebra Challenge

Definition. A set V with operators $+ : V \times V \to V$ and $\cdot : F \times V \to V$ is a **vector space over a field** F if it satisfies the following axioms:

1. Associativity of addtion: $u + (v + w) = (u + v) + w$
2. Commutativity of addition: $u + v = v + u$
3. Additive identity: $\exists 0 \in V$ such that $\forall v \in V$, $v + 0 = v$.
4. Additive inverses: $\forall v \in V$, $\exists -v \in V$ such that $v + (-v) = 0$.
5. Compatiblity of scalar multiplication: $a(bv) = (ab)v$
6. Multiplicative identity: $1v = v$, where 1 is the multiplicative identity in F.
7. Distributivity: $a(u + v) = au + av$
8. Distributivity: $(a + b)v = av + bv$

Definition. Given V a vector space over a field F, $W \subset V$ is a subspace of V if under the operations of V, W forms a vector space over V.

Problem 1. *Let V be a vector space of n by n matrices over $\mathbb{R}$. Let W be the subspace generated by matrices of the form $AB - BA$, for $A, B \in V$. What is the dimension of W? Prove your answer.*

Hint: start by considering $n = 2$.

Statistics Challenge

For this problem, we will concern ourselves with linear models of the form:

$$y = \beta_0 + X_1\beta_1 + \cdots + X_p\beta_p + \varepsilon$$

Where X a matrix of observations on independent variables, y a vector of measurements (ie. dependent variable), β a vector of coefficents to be learned, and ε errors.

Theorem (Bayes' Theorem)**.** *Given events A and B, where $P(B) \neq 0$:*

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Moreover, given events A, B, C

$$P(A|B,C) = \frac{P(B|A,C)P(A|C)}{P(B|C)}$$

Definition. L2 regularization refers to modifying the typical least-squares regression by adding a term $\lambda \sum_i \beta_i^2$, while L1 regularization refers to adding a term $\lambda \sum_i \beta_i$, where, in both cases, λ is a (postive) tuning parameter. For this problem, L2 regression manifests as:

$$\operatorname*{argmin}_{\beta} \|y - \beta_0 + X_1\beta_1 + \cdots + X_p\beta_p\|_2^2 + \lambda\|\beta\|_2^2$$

and similarly for L1:

$$\operatorname*{argmin}_{\beta} \|y - \beta_0 + X_1\beta_1 + \cdots + X_p\beta_p\|_2^2 + \lambda\|\beta\|_1$$

Problem 2. *Given $p(\beta)$ the prior distribution for coefficent vector β, assume that $p(\beta) = \prod_{j=1}^{p} g(\beta_j)$ for some density function g. Explain why the posterior mode for β given some set of data X, y is:*

1. *given by L2 regression, assuming Gaussian priors for g: $g(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2\right)$*
2. *given by L1 regression, assuming Laplacian priors for g: $g(x) = \frac{1}{2b} \exp\left(-\frac{|x|}{b}\right)$*

Abstract Algebra Challenge

Warning: this one is fun brain teaser, but almost certainly the hardest one.

Definition. Given G a group with identity e and X a set, then a (left) group action α of G on X is a function $\alpha : G \times X \to X$ satisfying two axioms:

1. Identity: $\alpha(e, x) = x$
2. Compatiblity: $\alpha(g, \alpha(h, x)) = \alpha(gh, x)$

for all $g, h \in G$ and $x \in X$. G is said to **act on** X.

Theorem (Burnside's Lemma)**.** *Let G be a group acting on a set S. For $\alpha \in G$, let $\mathrm{fix}(\alpha)$ denote the set of fixed points of α. Then:*

$$|G||S/G| = \sum_{\alpha \in G} |\mathrm{fix}(\alpha)|$$

Definition. Any matrix A consisting of elements belonging to $\{0, 1\}$ is called a **binary** matrix.

Problem 3. *Given any two binary matrices A and B, say that A and B are equivalent if you can get from A to B by swapping rows and columns any number of times that is, by permuting the axes. Determine a formula for the number of unique, non-equivalent n by m binary matrices.*