Data Science and Machine Learning

5. Vectors, matrices and linear algebra

“aimed at getting you to kickass in AI”

How data is often represented

numbers

DATA

X **y**

examples

$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$	y
1	2	5	1	9
2	4	1	1	7
4	8	3	7	25

features

Given input **X**, predict **y**

- Each row of **X** corresponds to an example
- Each column of **X** corresponds to a feature
- **X** is composed of feature vectors, $\mathbf{x}_1$, $\mathbf{x}_2$, …
- **X** can be represented with a matrix
- **y** is the target (what is to be predicted)

Vectors and Matrices

An array of numbers

Numbers example

A single number, 8

An array of numbers, [2,4,5,8,9]

The dimensions of an array

It determines how arrays are indexed. Indexing an array involves retrieving its elements

1-dimensional(1D) array

e.g arr = [2,4,5,8,9]
Indexing
1st element, arr[0] 2
2nd element, arr[1] 4

2-dimensional(2D) array

e.g arr = [[2,4,5,8,9],[1,2,4,5,2], [3,2,1,5,7]]
Indexing
1st element, arr[0] [2,4,5,8,9]
1st element of 1st element, arr[0][0] 2
2nd element of 1st element, arr[0][1] 4

Vectors and Matrices

Vector - 1D array

Represented as

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 5 \\ 8 \\ 9 \end{bmatrix}$$

Matrix - 2D array

Represented as

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \end{bmatrix} = \begin{bmatrix} 2 & 4 & 5 & 8 & 9 \\ 1 & 2 & 4 & 5 & 2 \\ 3 & 2 & 1 & 5 & 7 \end{bmatrix}$$

A matrix can be seen as being composed of one or more vectors

How?

If a 1D array => an array of numbers [2,4,5,8,9]

and a 2D array => an array of 1D arrays [[2,4,5,8,9],[1,2,4,5,2],[3,2,1,5,7]]

then a **Matrix** is an array of **vectors**

Vectors and Matrices

vectors in a matrix

Consider the matrix, A

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \end{bmatrix} = \begin{bmatrix} 2 & 4 & 5 & 8 & 9 \\ 1 & 2 & 4 & 5 & 2 \\ 3 & 2 & 1 & 5 & 7 \end{bmatrix}$$

And the vectors

$$\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 8 \\ 5 \\ 5 \end{bmatrix}, \begin{bmatrix} 9 \\ 2 \\ 7 \end{bmatrix}$$ Column vectors

$$\begin{bmatrix} 2 \\ 4 \\ 5 \\ 8 \\ 9 \end{bmatrix}^T, \begin{bmatrix} 1 \\ 2 \\ 4 \\ 5 \\ 2 \end{bmatrix}^T, \begin{bmatrix} 3 \\ 2 \\ 1 \\ 5 \\ 7 \end{bmatrix}^T$$ Row vectors

T -> Transpose

$$\begin{bmatrix} 2 \\ 4 \\ 5 \\ 8 \\ 9 \end{bmatrix}^T = \begin{bmatrix} 2 & 4 & 5 & 8 & 9 \end{bmatrix}$$

Linear algebra

What does it entail?

Linear combination of vectors

Linear combinations involve multiplying vectors by numbers and adding vectors to vectors

A linear combination

$$a*\boldsymbol{v} + b*\boldsymbol{w} \qquad 2*\begin{bmatrix}1\\2\\4\\5\\2\end{bmatrix} + 1*\begin{bmatrix}2\\4\\5\\8\\9\end{bmatrix} = \begin{bmatrix}4\\8\\13\\18\\13\end{bmatrix}$$

Multiplying by numbers

$$a*\boldsymbol{v} = a*\begin{bmatrix}v_1\\v_2\\:\\.\\v_n\end{bmatrix}$$

example

$$2*\begin{bmatrix}2\\4\\5\\8\\9\end{bmatrix} = \begin{bmatrix}4\\8\\10\\16\\18\end{bmatrix}$$

Adding vectors

$$\boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix}v_1\\v_2\\:\\.\\v_n\end{bmatrix} + \begin{bmatrix}w_1\\w_2\\:\\.\\w_n\end{bmatrix}$$

example

$$\begin{bmatrix}1\\2\\4\\5\\2\end{bmatrix} + \begin{bmatrix}2\\4\\5\\8\\9\end{bmatrix} = \begin{bmatrix}3\\6\\9\\13\\11\end{bmatrix}$$

Linear algebra

Matrix and vector multiplication

Multiplying a matrix by a vector is done by a linear combination of the column vectors in the matrix

$$\mathbf{A} * \boldsymbol{v} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \end{bmatrix} * \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix}$$

$$v_1 \begin{bmatrix} A_{11} \\ A_{21} \\ A_{31} \end{bmatrix} + v_2 \begin{bmatrix} A_{12} \\ A_{22} \\ A_{32} \end{bmatrix} + v_3 \begin{bmatrix} A_{13} \\ A_{23} \\ A_{33} \end{bmatrix} + v_4 \begin{bmatrix} A_{14} \\ A_{24} \\ A_{34} \end{bmatrix} + v_5 \begin{bmatrix} A_{15} \\ A_{25} \\ A_{35} \end{bmatrix}$$

example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} * \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 * \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2 * \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

Linear algebra

Matrix and matrix multiplication

Multiplying a matrix by another matrix is done by a linear combination of all column vectors in the matrix first matrix with each column vector in the first matrix

$$\mathbf{A} * \boldsymbol{B} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} * \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

$$\begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix} = B_{11} * \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} + B_{21} * \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix}$$

$$\begin{bmatrix} C_{12} \\ C_{22} \end{bmatrix} = B_{12} * \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} + B_{22} * \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix}$$

example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} * \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 11 & 7 \end{bmatrix}$$

$$1 * \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2 * \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

$$1 * \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 1 * \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

Linear algebra

Matrix and matrix multiplication

Inverse and identity

If **A*I** = **A**, then I is an identity matrix

If **A*B** = **I**, then **B** is the inverse of **A**. i.e **B** = $\mathbf{A}^{-1}$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} * \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$1*\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 0*\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$0*\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 1*\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Linear algebra

Linear dependence ?

A vector is linearly dependent on other vectors if it can be obtained from their linear combination

Given ***u***,***v***, and ***w***

u is linearly dependent on v and w if
$\boldsymbol{u} = a*\boldsymbol{v} + b*\boldsymbol{w}$

u is linearly dependent on v if
$\boldsymbol{u} = a*\boldsymbol{v}$

u is linearly dependent on w if
$\boldsymbol{u} = a*\boldsymbol{w}$

$$\boldsymbol{u} = \begin{bmatrix}6\\2\\8\end{bmatrix} \quad \boldsymbol{v} = \begin{bmatrix}3\\1\\4\end{bmatrix} \quad \boldsymbol{w} = \begin{bmatrix}1\\2\\3\end{bmatrix}$$

$$\begin{bmatrix}6\\2\\8\end{bmatrix} = 2 * \begin{bmatrix}3\\1\\4\end{bmatrix} + 0 * \begin{bmatrix}1\\2\\3\end{bmatrix}$$

$$\begin{bmatrix}6\\2\\8\end{bmatrix} = 2 * \begin{bmatrix}3\\1\\4\end{bmatrix}$$

$$\begin{bmatrix}6\\2\\8\end{bmatrix} \qquad \begin{bmatrix}1\\2\\3\end{bmatrix}$$

Linear algebra

vector spaces ?

A vector space is a set of vectors in which any linear combination involving its elements would result in a vector that is an element of that set. Hence, if the result from linear combination is not within the set, then it is not a vector space.

Examples of vector spaces

R^2 Set of all real 2-dimensional vectors

$$\begin{bmatrix}3\\7\end{bmatrix} \begin{bmatrix}0\\1\end{bmatrix} \begin{bmatrix}0\\0\end{bmatrix} \begin{bmatrix}1\\1\end{bmatrix} \begin{bmatrix}2\\4\end{bmatrix} \dots$$

R^3 Set of all real 3-dimensional vectors

$$\begin{bmatrix}5\\4\\1\end{bmatrix} \begin{bmatrix}3\\1\\4\end{bmatrix} \begin{bmatrix}1\\2\\3\end{bmatrix} \begin{bmatrix}9\\2\\7\end{bmatrix} \begin{bmatrix}4\\2\\2\end{bmatrix} \dots$$

The dimension of a vector

This is the dimension of the vector space to which the vector belongs

2-D vectors

$$\begin{bmatrix}3\\7\end{bmatrix} \begin{bmatrix}0\\1\end{bmatrix} \begin{bmatrix}0\\0\end{bmatrix} \begin{bmatrix}1\\1\end{bmatrix} \begin{bmatrix}2\\4\end{bmatrix} \dots$$

3-D vectors

$$\begin{bmatrix}5\\4\\1\end{bmatrix} \begin{bmatrix}3\\1\\4\end{bmatrix} \begin{bmatrix}1\\2\\3\end{bmatrix} \begin{bmatrix}9\\2\\7\end{bmatrix} \begin{bmatrix}4\\2\\2\end{bmatrix} \dots$$

Linear algebra

Ax = ***b***

Systems of linear equations

Given

$4x_1 + 2x_2 = 10$

$3x_1 + x_2 = 7$

Find x_1 and x_2

More equations;

$5x_1 + 5x_2 = 15$

$3x_1 + 4x_2 = 10$

$2x_1 + 3x_2 = 7$

$x_1 + 2x_2 = 4$

They must align with our belief

How would you approach this?

I see a linear combination!

obviously

x_1 = 2, x_2 = 1

$$x_1 \begin{bmatrix} 4 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 7 \end{bmatrix}$$

I see a Matrix multiplied by a vector

$$\begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 7 \end{bmatrix}$$

A ***x*** = ***b***

$\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$

Application

Machine learning

DATA

x_1	x_2		y
4	2		10
3	1		7
5	5		15

$$\underbrace{\qquad\text{X}\qquad\qquad\text{y}\qquad}_{\text{features}}$$

examples

3	2		8

$$\mathbf{X}\boldsymbol{\theta} = \boldsymbol{y}$$

$$\begin{bmatrix} 4 & 2 \\ 3 & 1 \\ 5 & 5 \end{bmatrix} * \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 7 \\ 15 \end{bmatrix}$$

$$\theta_1 \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} + \theta_2 \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 10 \\ 7 \\ 15 \end{bmatrix}$$

$\theta_1 = 2, \quad \theta_2 = 1$

Library

numpy

Example using numpy to solve Ax=b

Thank You