ENM 5310: Data-driven Modeling and Probabilistic Scientific Computing

Lecture #1

Probability theory, statistics, and information theory fundamentals

Penn
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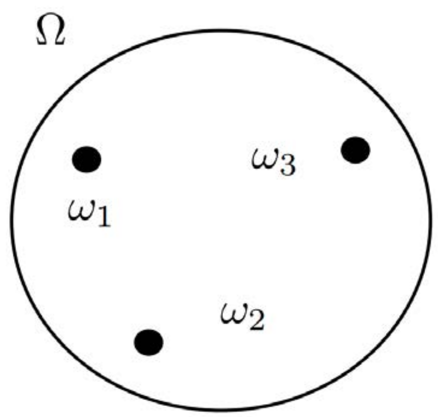
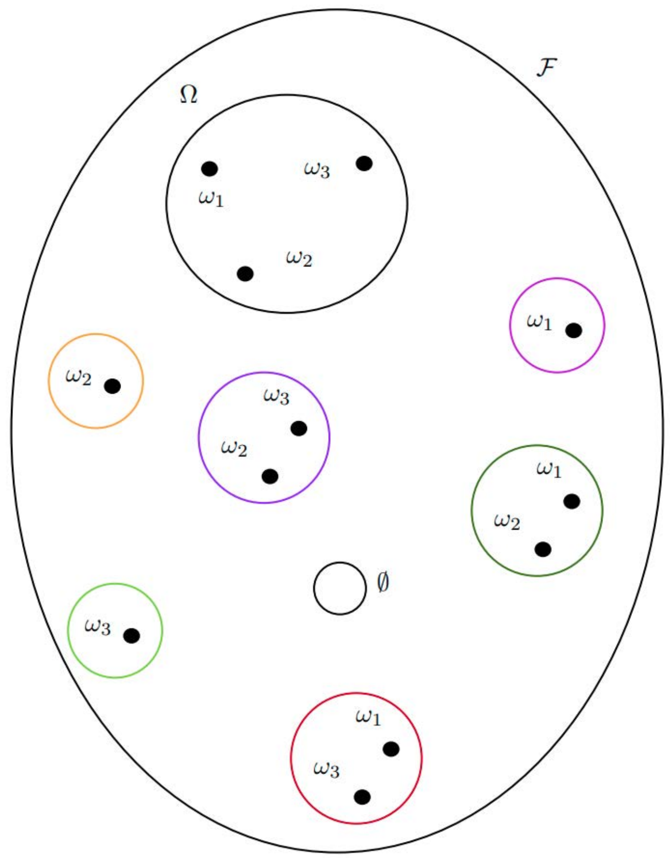
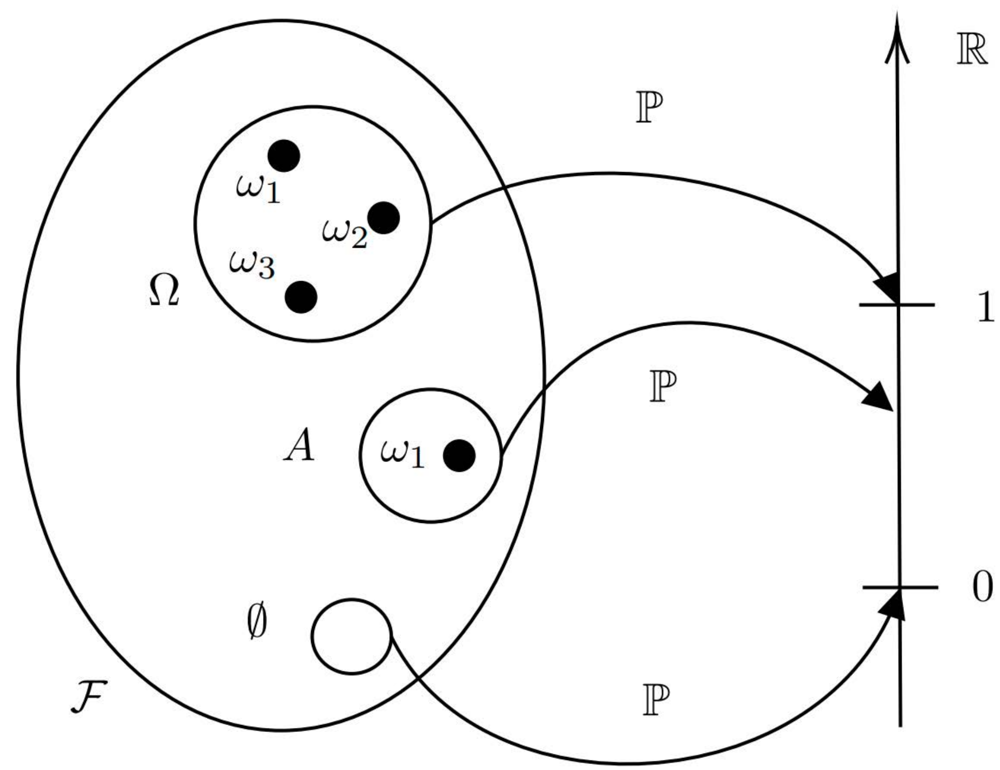
Probability spaces & random variables



Sample space



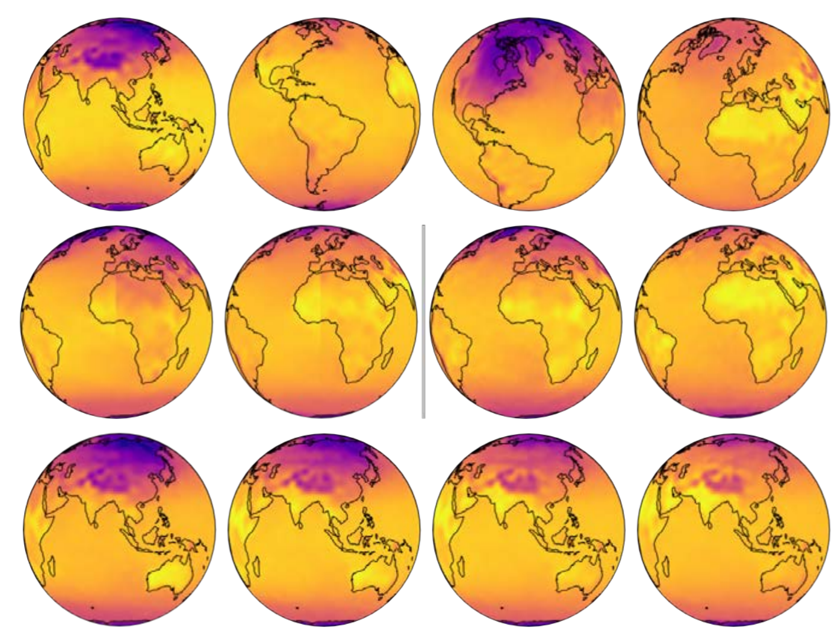
σ-algebra of events



Probability measure

vectors

matrices



functions

Discrete random variables

- A ***discrete random variable*** is one which may take on only a countable number of distinct values such as 0,1,2,3,4,........ Discrete random variables are usually (but not necessarily) counts. If a random variable can take only a finite number of distinct values, then it must be discrete. Examples of discrete random variables include the number of children in a family, the Friday night attendance at a cinema, the number of patients in a doctor's surgery, the number of defective light bulbs in a b
- The ***probability distribution*** of a discrete random variable is a list of probabilities associated with each of its possible values. It is also sometimes called the probability function or the probability mass function.

Continuous random variables

- A ***continuous random variable*** is one which takes an infinite number of possible values. Continuous random variables are usually measurements. Examples include height, weight, the amount of sugar in an orange, the time required to run a mile.
- A continuous random variable is not defined at specific values. Instead, it is defined over an *interval* of values, and is represented by the ***area under a curve*** (in advanced mathematics, this is known as an *integral*). The probability of observing any single value is equal to 0, since the number of values which may be assumed by the random variable is infinite.

- Suppose a random variable X may take all values over an interval of real numbers. Then the probability that X is in the set of outcomes A, $P(A)$, is defined to be the area above A and under a curve. The curve, which represents a function $p(x)$, must satisfy the following:
- ***1:*** *The curve has no negative values ($p(x) \geq 0$ for all x)*
- ***2:*** *The total area under the curve is equal to 1.*
- A curve meeting these requirements is known as a ***density curve***.

Basic rules of probability

Sum rule

$$p(X) = \sum_Y p(X, Y)$$

Product rule

$$p(X, Y) = p(Y|X)p(X)$$

Bayes rule

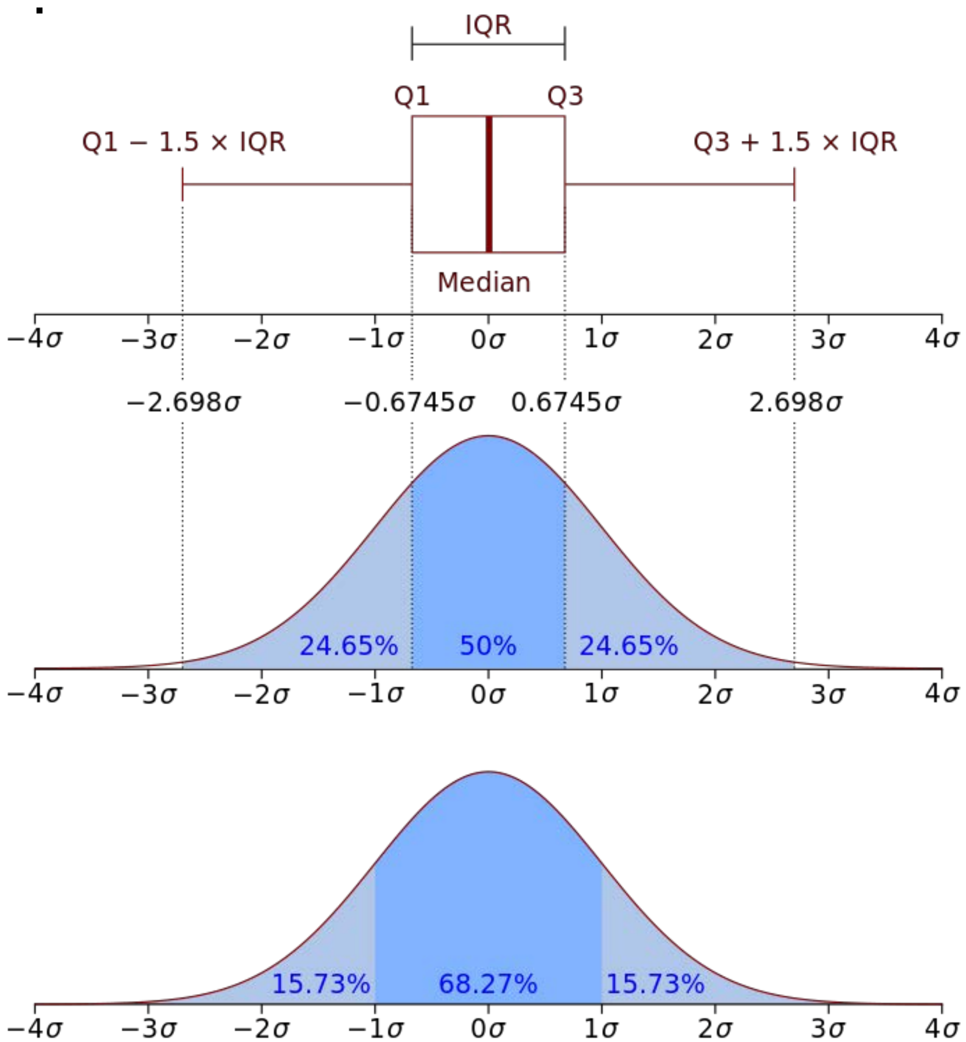
$$p(Y|X) = \frac{p(X|Y)p(Y)}{p(X)}$$

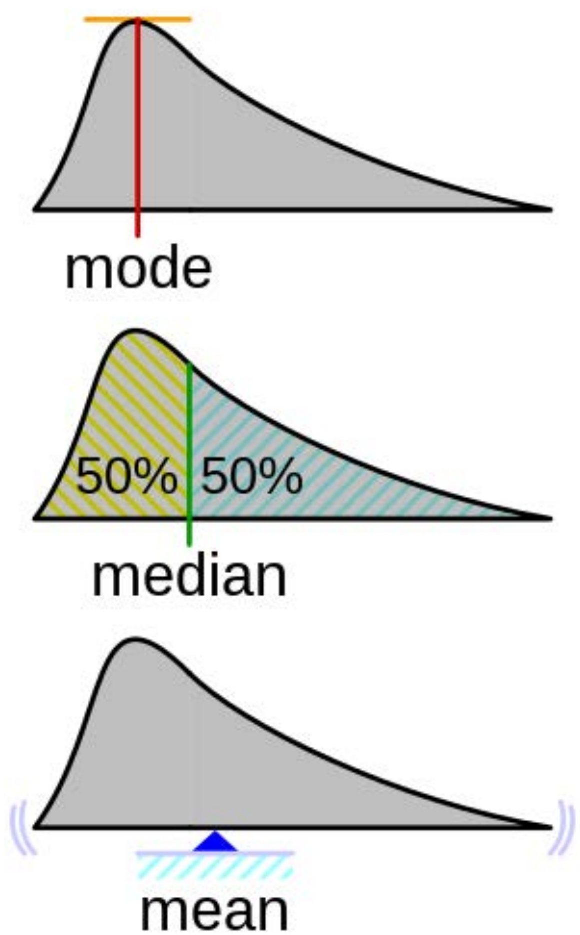
Density function

Closely related to the distribution function is the density function. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a nonnegative function, satisfying $\int_{\mathbb{R}} f d\lambda = 1$. The function f is called a density function (with respect to the Lebesgue measure) and the associated probability measure for a random variable X, defined on $(\Omega, \mathcal{F}, P)$, is

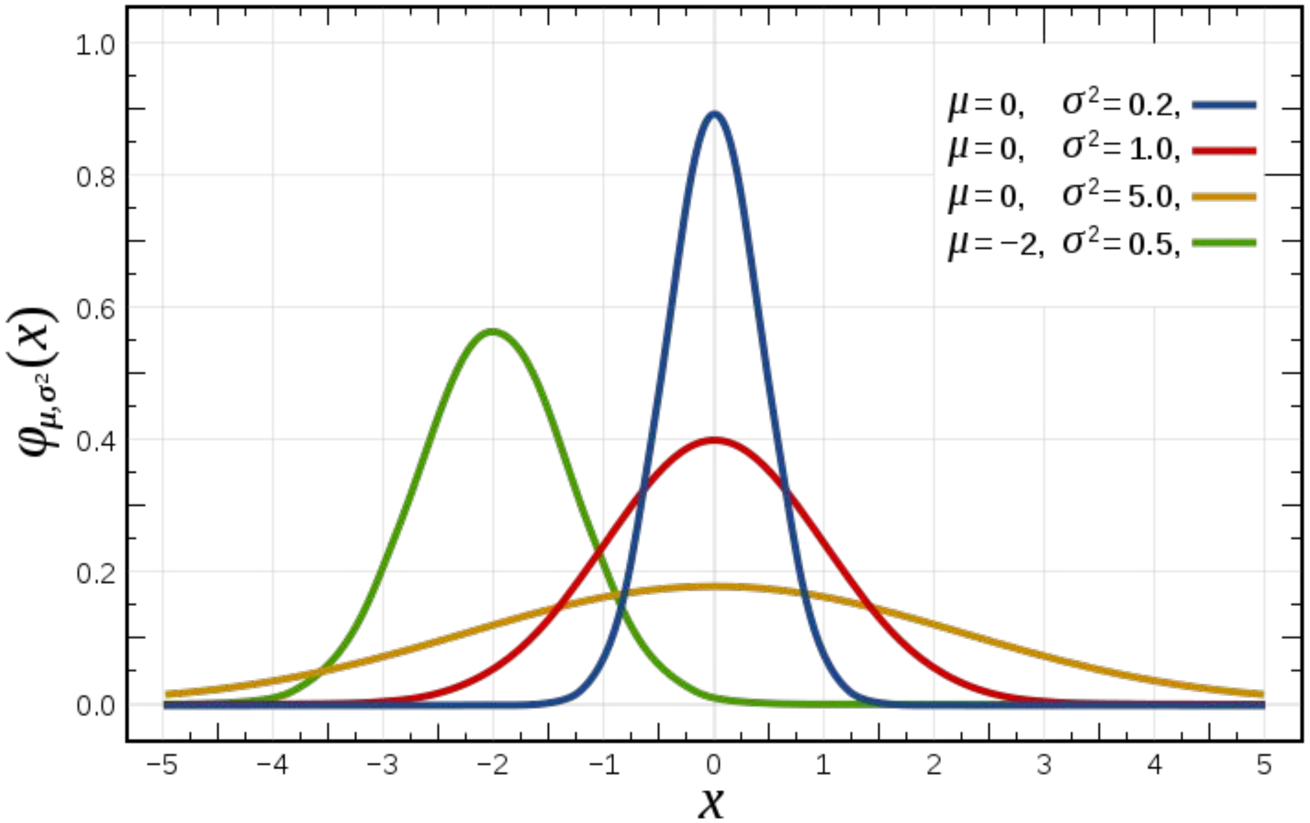
$$P(\{\omega : \omega \in A\}) = \int_A f d\lambda.$$

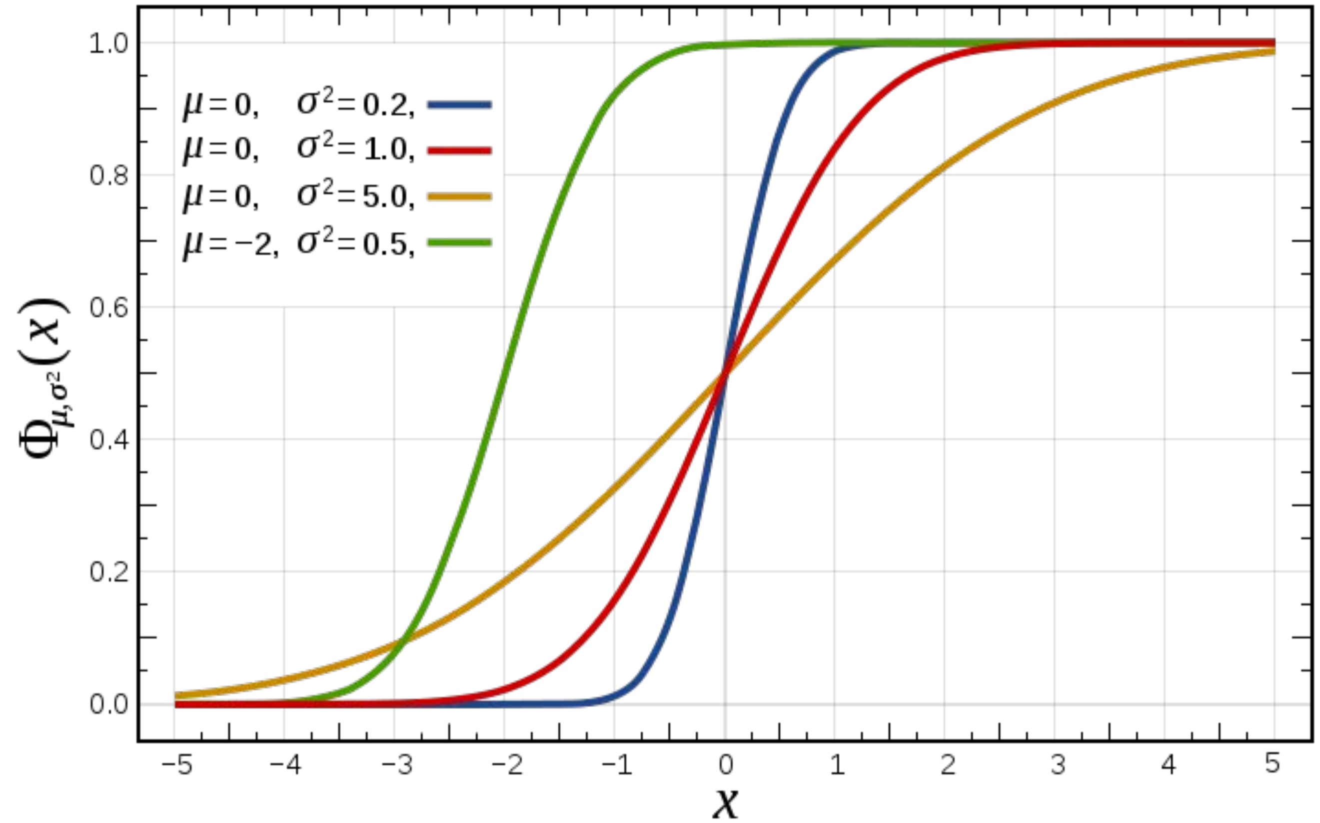
for all $A \in \mathcal{F}$.





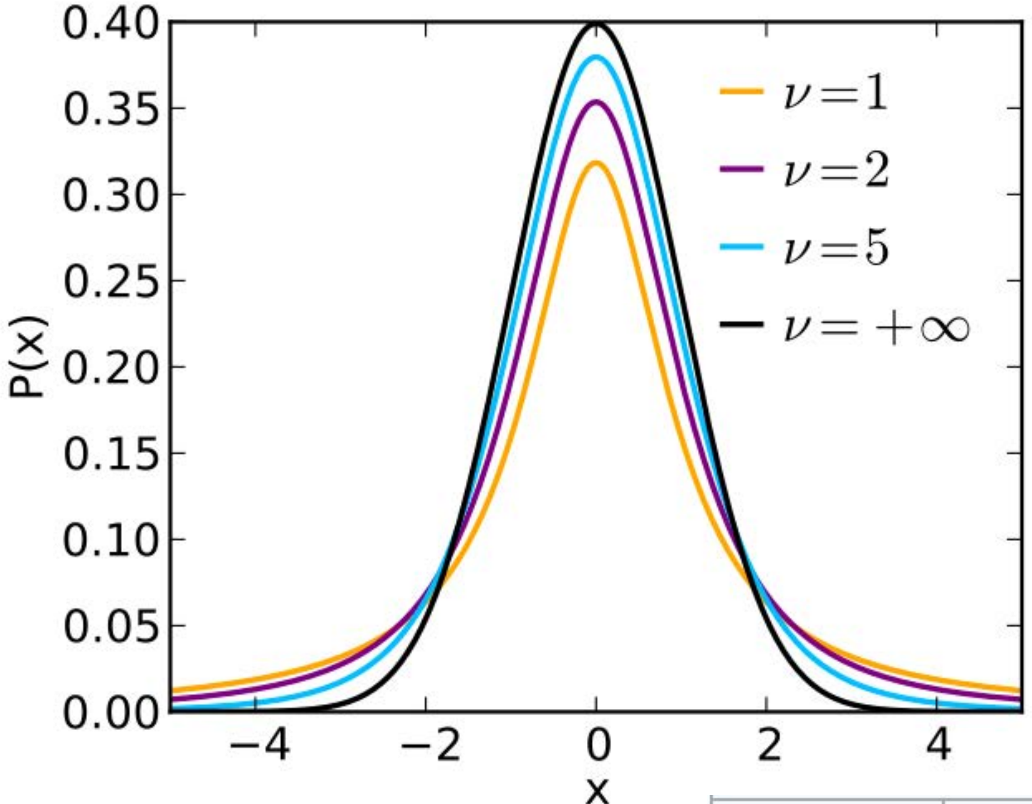
The Gaussian distribution

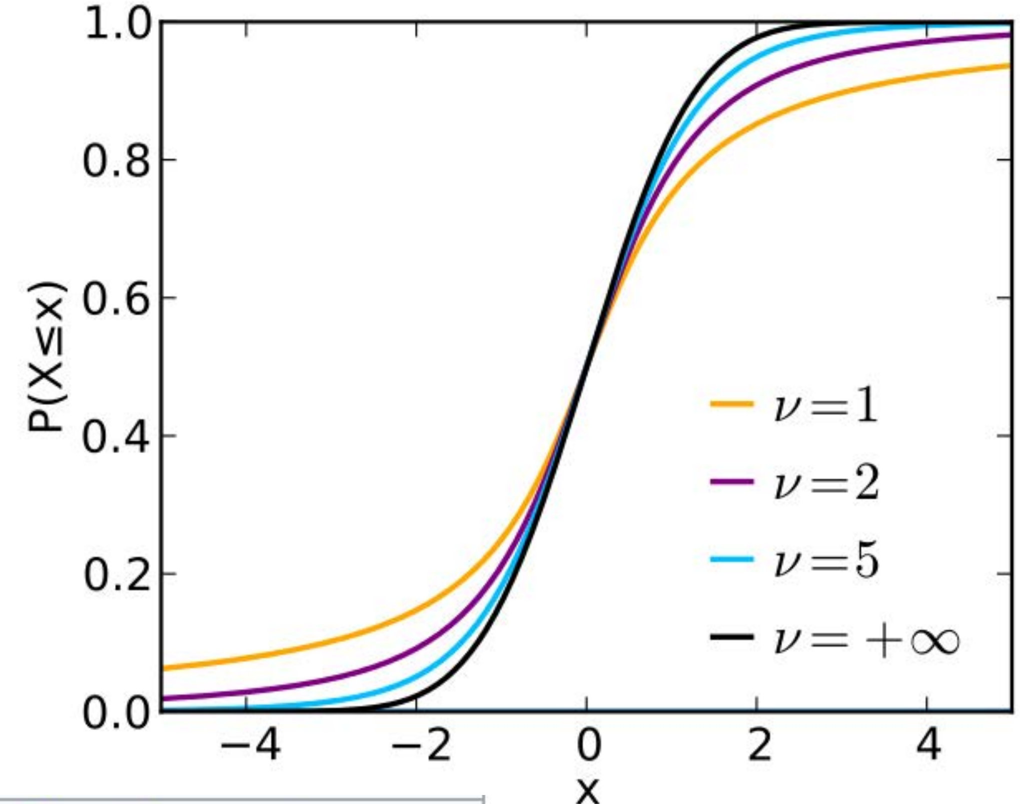




Notation	$\mathcal{N}(\mu, \sigma^2)$
Parameters	$\mu \in \mathbb{R}$ = mean (location) $\sigma^2 > 0$ = variance (squared scale)
Support	$x \in \mathbb{R}$
PDF	$\frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
CDF	$\frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]$
Quantile	$\mu + \sigma\sqrt{2}\operatorname{erf}^{-1}(2F-1)$
Mean	μ
Median	μ
Mode	μ
Variance	σ^2

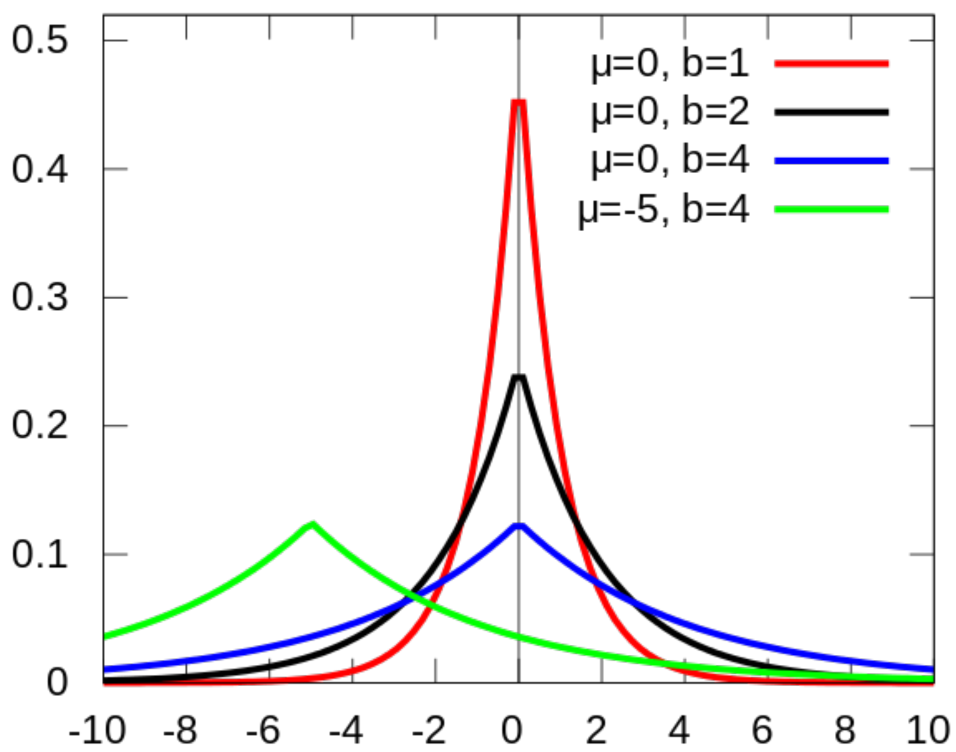
The Student-t distribution

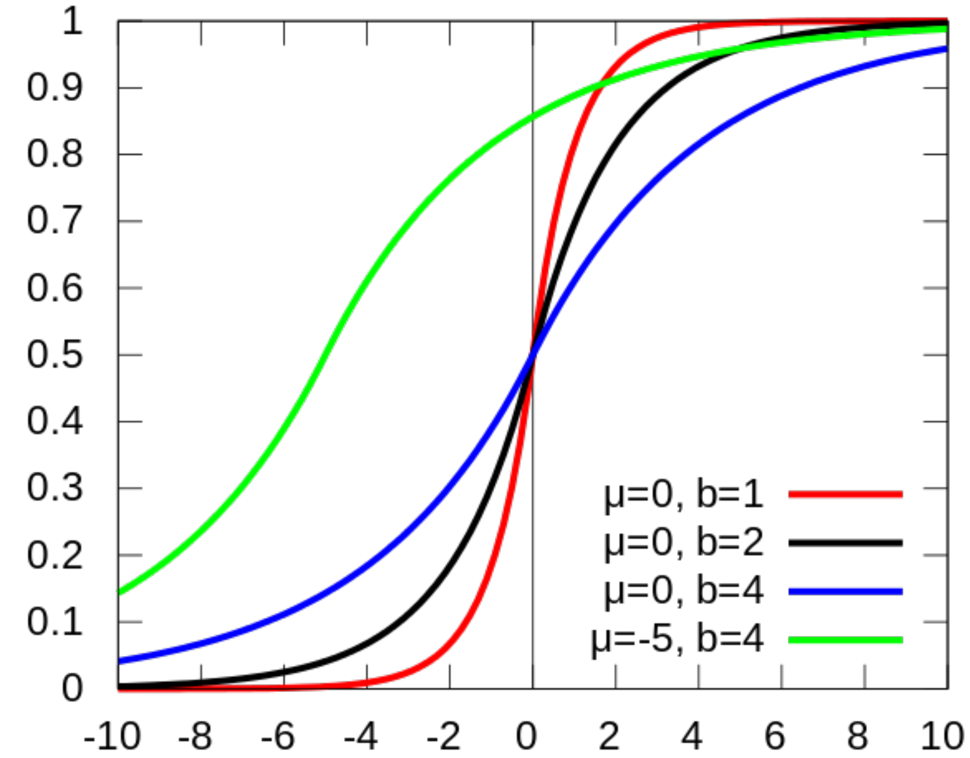




Parameters	$\nu > 0$ degrees of freedom (real)
Support	$x \in (-\infty; +\infty)$
PDF	$\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}$
CDF	$\frac{1}{2} + x\Gamma\left(\frac{\nu+1}{2}\right) \times \frac{{}_2F_1\left(\frac{1}{2},\frac{\nu+1}{2};\frac{3}{2};-\frac{x^2}{\nu}\right)}{\sqrt{\pi\nu}\,\Gamma\left(\frac{\nu}{2}\right)}$ where ${}_2F_1$ is the hypergeometric function
Mean	0 for $\nu > 1$, otherwise undefined
Median	0
Mode	0
Variance	$\frac{\nu}{\nu-2}$ for $\nu > 2$, ∞ for $1 < \nu \leq 2$, otherwise undefined

The Laplace distribution





Parameters	μ location (real) $b > 0$ scale (real)
Support	$x \in (-\infty; +\infty)$
PDF	$\frac{1}{2\,b} \exp\left(-\frac{\lvert x-\mu \rvert}{b}\right)$
CDF	$\begin{cases} \frac{1}{2} \exp\left(\frac{x-\mu}{b}\right) & \text{if } x < \mu \\ 1 - \frac{1}{2} \exp\left(-\frac{x-\mu}{b}\right) & \text{if } x \geq \mu \end{cases}$
Mean	μ
Median	μ
Mode	μ
Variance	$2b^2$

Gaussian vs Student-t vs Laplace

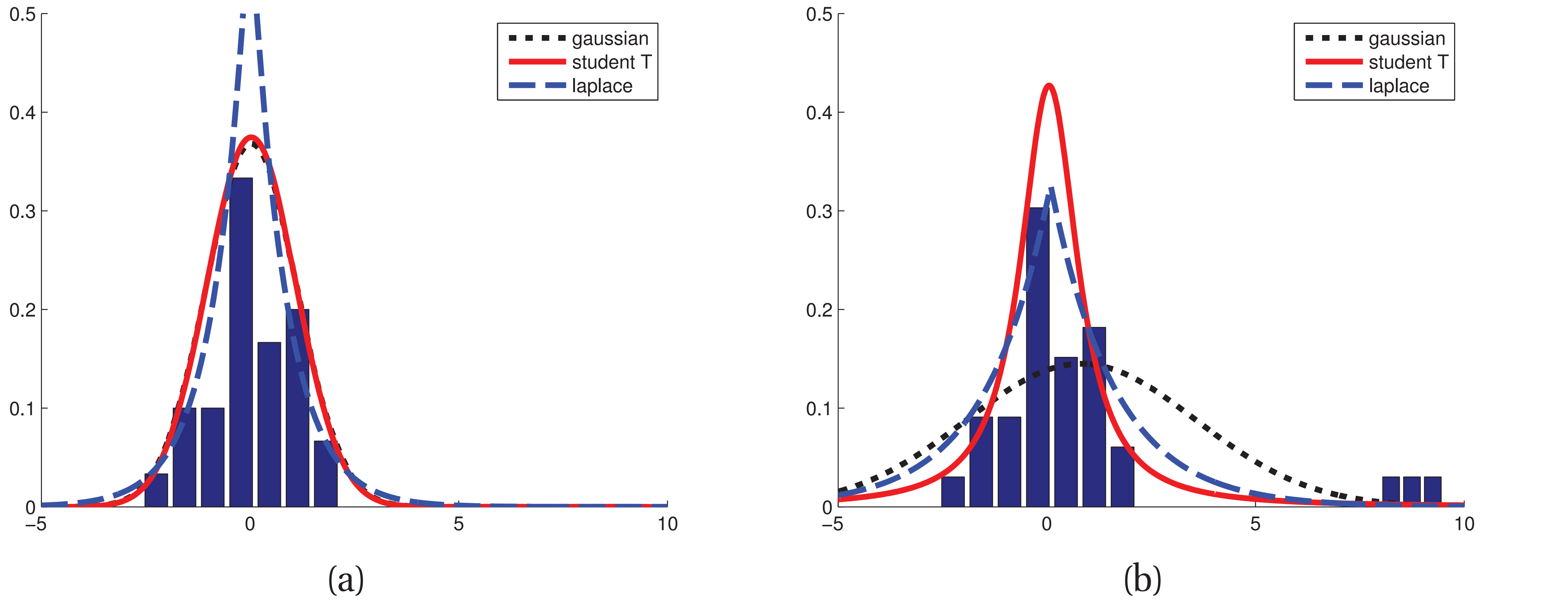


Figure 2.8 Illustration of the effect of outliers on fitting Gaussian, Student and Laplace distributions. (a) No outliers (the Gaussian and Student curves are on top of each other). (b) With outliers. We see that the Gaussian is more affected by outliers than the Student and Laplace distributions. Based on Figure 2.16 of (Bishop 2006a). Figure generated by `robustDemo`.