Homework 3: Getting comfortable with the calculus of variations

ENM5320 Spring 2025, UPenn

Due date: Feb. 26^{th} by midnight (11:59pm)

This is an **individual assignment**; you are encouraged to use any/all resources but you must attribute any resources used explicitly (collaborators/textbooks/journal papers/ChatGPT). Submit a handwritten or latexed report in Canvas, along with Jupyter notebooks, figures, or any other supporting files needed. Code should be documented sufficiently that we can run the code without modification and reproduce your results. Late submissions will follow the guidance outlined in the course syllabus.

This assignment is meant to check that you can apply some of the simple rules in the calculus of variations that we have been using in class. If you have gaps in your mathematical background, make sure you start the assignment early enough to get feedback at OH. For these problems you will find ample resources online that you could copy the solution from; I'd advise you to attempt the problem before you seek out resources, and to make sure you cite any resources used following the UPenn code of conduct.

Instructions:

1. **Q1. Continuous Euler-Lagrange equations.** For a linearly elastic material, the displacement field is denoted u_i, the strain tensor $\epsilon_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$, and the stress tensor σ_{ij} is related to ϵ_{ij} via the closure $\sigma_{ij} = C_{ijkl}\epsilon_{kl}$, where C_{ijkl} is the elasticity tensor. In three dimensions, there are particular constraints on the elasticity tensor which drastically reduce its dimension from $\mathbb{R}^{d\times d\times d\times d}$ down to just the two Lame coefficients. For the purposes of this class, we will only consider the case of transverse waves passing through a 1D bar (i.e. taking $d = 1$), so that all vector and tensor quantities reduce to scalar counterparts and one-dimensional derivatives.

   Given the Lagrangian density

   $$\mathcal{L}(u, \dot{u}) = \frac{1}{2}\rho\dot{u}^2 - \frac{C}{2}(\partial_x u)^2, \tag{1}$$

   apply the least action principle to derive the elastic wave equation

   $$\rho\partial_{tt}u = C\partial_{xx}u \tag{2}$$

2. **Q2. Identifying discrete adjoint operators.** Given a stencil function

$$D_+ u_i = \frac{u_{i+1} - u_i}{h} \tag{3}$$

defined on a periodic lattice with spacing h, derive the adjoint operator D^*. As we did on class Wednesday, apply the discrete variational principle to the functional

$$F[u] = \frac{1}{2} \sum_i (D_+ u_i)^2 \tag{4}$$

Show that the discrete variation with respect to F yields

$$\delta_u F = D^* D_+ u_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} \tag{5}$$