Applied Mathematics 2570, Fall 2011
MWF 3-5:20pm, B&H 166

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Textbook

Numerical Solution of Partial Differential Equations by the Finite Element Method, by Claes Johnson.

Course Content:

- Finite Element method for second order problems, hyperbolic problems, and parabolic problems
- Type of methods: discontinuous Galerkin Methods, mixed methods, non-conforming methods

Grading Policy

- Homework 80%
- Final Project 20%

Homework Policy

Homework is due at the beginning of lecture on Monday. In order to get full credit, solutions to the homework must include all steps. Homework must be well organized, neat and stapled. You are allowed to work with classmates. However, everyone needs to turn in their own solution set. Some of the homework problems will involve coding. I require you write code using Matlab.

Final Project

The final project will consist of reading a paper related to finite element methods and presenting it to the class at the end of the semester.

- Bramble-Hilbert
- inf-sup
- Stokes / elasticity
- streamline diffusion
- céa's lemma

$$-\Delta u + \nabla p = f$$
$$\epsilon^2 \Delta p + \nabla \cdot u$$

9/12

1D Finite Elements

$$(P)\quad \begin{cases} -u''(x) = f(x) & 0<x<1 \\ u(0) = u(1) = 0 \end{cases}$$

Weak form을 찾아볼까!

문제에 따라 다른 weak form?

Multiply (P) by smooth ftn V and integrate (V satisfies $V(0) = V(1) = 0$).

$$\int_0^1 -u''(x)v(x)\,dx = \int_0^1 f(x)v(x)\,dx$$

Now, integration by parts

$$\int_0^1 u'(x)v'(x)\,dx - \left(u'(1)v(1) - u'(0)\cancel{v(0)}\right) = \int_0^1 f(x)v(x)\,dx$$

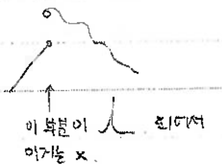
$$\int_0^1 u'(x)v'(x)\,dx = \int_0^1 f(x)v(x)\,dx \qquad \forall v \in H_0^1([0,1])$$

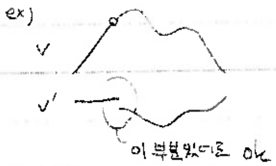
$$v \in H_0^1([0,1]) := \{m \in H^1([0,1]) : m(0) = 0 \quad m(1) = 0\}$$

$$H^1([0,1]) = \{m \in L^2([0,1]) : m' \in L^2([0,1])\}$$

$$L^2([0,1]) = \{m : \int_0^1 m^2(x)\,dx < \infty\}$$

⇒ m'가 모두 존재할 필요X





Weak form of problem P

Find $u \in H_0^1([0,1])$ st $\int_0^1 u'(x)v'(x)\,dx = \int_0^1 f(x)v(x)\,dx \quad \forall v \in H_0^1([0,1])$

Let $u \in H_0^1([0,1])$

$u = \sum_{i=1}^{\infty} c_i v_i \qquad v_i \in H_0^1([0,1])$ orthogonal decomposition.

$$\int_0^1 u'(x)v_i'(x)\,dx = \int_0^1 f(x)v_i(x)\,dx \quad \text{for every } i=1,2,\cdots$$

$$\int_0^1 \sum_{j=1}^{\infty} c_j v_j' v_i'\,dx = \int_0^1 f(x)v_i(x)\,dx$$

$$\begin{bmatrix} \int_0^1 v_1'v_1'\,dx & \int_0^1 v_1'v_2'\,dx & \cdots \\ \int_0^1 v_2'v_1'\,dx & \int_0^1 v_2'v_2'\,dx & \cdots \\ \vdots & & \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \int_0^1 f v_1\,dx \\ \int_0^2 f v_2\,dx \\ \vdots \end{bmatrix}$$

← Infinite를 풀 수 있으면 이렇게 할 수 있죠.

Finite Element Method

- We need a finite dimensional space $V_h \subseteq H_0^1([0,1])$
- Find $u_h \in V_h$ st $\int_0^1 u_h'v'\,dx = \int_0^1 f v\,dx \quad \forall v \in V_h$
- We would like that u_h approximates u well

☆ Uniqueness of finite element approximation (*)

pf) Set $f \equiv 0$

$$\int_0^1 u_h' v' \, dx = 0 \quad \forall v \in V_h$$

Let $v = u_h$

$$\int_0^1 (u_h')^2 dx = 0 \quad \Leftrightarrow \quad u_h' = 0 \quad (a.e)$$

$$u_h = \text{const} \in H_0^1([0,1]) \quad \Rightarrow \quad u_h = 0$$

Suppose $V_h = \text{span}\{v_1, v_2, \cdots v_N\}$

$u_h = \sum_{i=1}^{N} c_i v_i$

$$\begin{bmatrix} \int_0^1 v_1' v_1' dx & \int_0^1 v_1' v_2' dx & \cdots & \int_0^1 v_1' v_N' dx \\ \int_0^1 v_2' v_1' dx & \int_0^1 v_2' v_2' dx & & \\ \vdots & & & \\ \int_0^1 v_N' v_1' dx & & & \int_0^1 v_N' v_N' dx \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_N \end{bmatrix} = \begin{bmatrix} \int_0^1 f v_1 dx \\ \vdots \\ \int_0^1 f v_N dx \end{bmatrix}$$

↳ SPD ⇒ invertible

$$A\vec{c} = \vec{f}$$

Uniqueness 를 (*)에서 보였으므로 invertible 임을 예측 가능

We still have not argued that u_h will be a good approx of u

Space V_h should specify :

1) Good approximation :

Given any $u \in H_0^1([0,1])$

$\inf_{v \in V_h} \| u - v \|$ is small (norm $\|\cdot\|$ to be determined)

Q) 왜 inf이지요?
$\|u-v\|$가 작은 v이어도
있는거죠 in V_h
모든 $v \in V_h$가 small인 것이
아니잖아

2) $\begin{cases} \int_0^1 w' v' dx \\ \int_0^1 f v \, dx \end{cases}$ to be "inexpensive" to compute $\forall w, v \in V_h$

Example of V_h

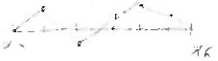
We first partition $[0,1]$

Let $0 = x_0 < x_1 < \cdots < x_{N+1} = 1$

$x_0 \quad x_1 \quad \cdots \quad x_N \quad x_{N+1}$

$I_j = (x_{j-1}, x_j) \qquad h_j = x_j - x_{j-1} \qquad h = \max h_j$

$V_h = \{ v \in C_0([0,1]) : v|_{I_j} \in P^1(I_j) \text{ for } j = 1, \cdots N \}$



N개 내부점만 알면 되죠.

Basis for $V_h = \text{span}\{\phi_1, \phi_2, \cdots, \phi_N\}$

For every $\phi_i \in V_h$, $\quad \phi_i(x_j) = \begin{cases} 0 & j \neq i \\ 1 & j = i \end{cases}$

ϕ_i

tent / hat fn

If $v \in V_h$, $\quad v(x) = v(x_1)\phi_1(x) + v(x_2)\phi_2(x) + \cdots + v(x_N)\phi_N(x)$

$\because \quad v(x_j) = v(x_j)\phi_j(x_j) = v(x_j)$

Good approximation

Given any $u \in H^2([0,1]) \cap H_0^1([0,1])$

$$\inf_{v \in V_h} \| u - v \|_{H^1([0,1])} \le Ch \| u'' \|_{L^2([0,1])}$$ ✓

$$\| v \|_{L^2[0,1]} = \sqrt{\int_0^1 u^2(x)\,dx}$$

$$\| v \|_{H^1[0,1]} = \| v \|_{L^2([0,1])} + \| v' \|_{L^2([0,1])}$$ ✓

$$\inf_{v \in V_h} \| u - v \|_{H^1([0,1])} \le \| u - \tilde{v} \|_{H^1([0,1])} \le Ch \| u'' \|_{L^2([0,1])}$$ ✓

bilinear form

$$a(u,v) := \int_0^1 u'(x)v'(x)\,dx$$

$$(u,v) := \int_0^1 u(x)v(x)\,dx$$

$$\Rightarrow \quad u_h \in V_h, \qquad a(u_h, v) = (f, v) \quad \forall v \in V_h$$

$$\underset{\overset{\shortparallel}{A}}{\begin{bmatrix} a(\phi_1,\phi_1) & \cdots & a(\phi_1,\phi_N) \\ \vdots & & \\ a(\phi_N,\phi_1) & & a(\phi_N,\phi_N) \end{bmatrix}} \begin{bmatrix} u_h(x_1) \\ \vdots \\ u_h(x_N) \end{bmatrix} = \begin{bmatrix} (f,\phi_1) \\ \cdot \\ (f,\phi_N) \end{bmatrix}$$

$$a(\phi_j, \phi_i) = \int_0^1 \phi_j'(x)\,\phi_i'(x)\,dx = 0 \qquad |i-j| \ge 2$$

$$a(\phi_i, \phi_i) = \int_{x_{i-1}}^{x_{i+1}} \phi_i'(x)\,\phi_i'(x)\,dx$$

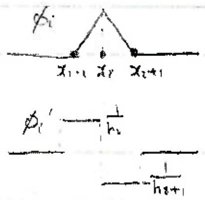
$$a(\phi_{i-1}, \phi_i) = \int_{x_{i-1}}^{x_i} \phi_i'\,\phi_{i-1}'\,dx$$

$$a(\phi_{i+1}, \phi_i) = \int_{x_i}^{x_{i+1}} \phi_i'\,\phi_{i+1}'\,dx$$

) 이부분만 계산하면 되는 것이 Fourier와 비교하여 장점

$w, v \in V_h$

$$a(w,v) = \vec{x}^T A \vec{y} \qquad \text{where} \quad \vec{x} = \begin{bmatrix} w(x_1) \\ \vdots \\ w(x_N) \end{bmatrix} \quad \vec{y} = \begin{bmatrix} v(x_1) \\ \vdots \\ v(x_N) \end{bmatrix}$$



$$a(\phi_i, \phi_i) = \int_{x_{i-1}}^{x_i} \left(\frac{1}{h_i}\right)^2 dx + \int_{x_i}^{x_{i+1}} \left(-\frac{1}{h_{i+1}}\right)^2 dx = \frac{1}{h_i} + \frac{1}{h_{i+1}}$$

$$a(\phi_{i-1}, \phi_i) = \int_{x_{i-1}}^{x_i} \left(-\frac{1}{h_i}\right)\left(\frac{1}{h_i}\right) dx = -\frac{1}{h_i}$$

$$a(\phi_{i+1}, \phi_i) = -\frac{1}{h_{i+1}}$$

$$A = \begin{bmatrix} \frac{1}{h_1}+\frac{1}{h_2} & -\frac{1}{h_2} & & & \\ -\frac{1}{h_2} & \frac{1}{h_2}+\frac{1}{h_3} & -\frac{1}{h_3} & & \\ & -\frac{1}{h_3} & \ddots & \ddots & \\ & & \ddots & \ddots & -\frac{1}{h_{N-1}} \\ & & & -\frac{1}{h_{N-1}} & \frac{1}{h_{N-1}}+\frac{1}{h_N} \end{bmatrix}$$

Uniform mesh $h_i = h \quad \forall i$, $\quad A = \frac{1}{h}\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}$

In 1D, $u_h(x_i) = u(x_i)$ FDM을 A는 같지만 이 성질 X

Properties of A.

- tridiagonal (sparse)
- symmetric
- positive definite $\vec{y}^T A \vec{y} > 0$ for any $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} \neq 0$

pf) Let $\vec{y}$ be given

Define $v(x) = y_1 \phi_1(x) + y_2 \phi_2(x) + \cdots + y_N \phi_N(x)$

$$\vec{y}^T A \vec{y} = a(v, v) = \int_0^1 v'(x) v'(x) \, dx > 0$$

Generalize.

$$V_h = \{ v \in C_0([0,1]) \quad v|_{I_j} \in P^k(I_j), \ j = 1, \ldots, N \}$$

$P^k(I_j)$ = space of all polynomials of degree k or less defined on I_j

$k\uparrow \Rightarrow$ better approx. but dimension increases with k

Good approximation

$(k+1)^{th}$ derivative

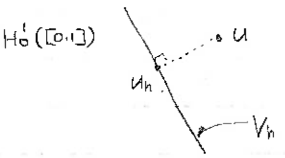
$$\inf \| u - v \|_{H^1([0,1])} \leq C h^k \| u^{(k+1)} \|_{L^2([0,1])}$$

But if u is not smooth, the bound ↑

Thm Let u solve (P) and

u_h be the finite element approximation using piecewise polynomial of order k

✓ Then,
$$\| (u - u_h)' \|_{L^2([0,1])} \leq \inf_{v \in V_h} \| (u - v)' \|_{L^2([0,1])} \leq \inf_{v \in V_h} \| u - v \|_{H^1([0,1])} \leq C h^k \| u^{(k+1)} \|_{L^2([0,1])}$$



mycourses에서 HW 확인!

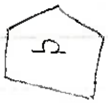
HW#4 Integrate right-hand side exactly (by hand or approximately but very exactly)

화요일 까지

9/19

FEM for Poisson's problem

We let Ω be a bounded polygonal domain



$$(P) \begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad \Delta u = \partial_{xx} u + \partial_{yy} u$$

Sobolev spaces.

$$H^1(\Omega) = \{ v \in L^2(\Omega) : \nabla v \in L^2(\Omega) \}$$
$$H_0^1(\Omega) = \{ v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega \}$$
$$L^2(\Omega) = \{ v : \int_\Omega v^2 \, dx < \infty \}$$

★ Derive weak formulation of problem P

First multiply (P1) by a ftn $v \in H_0^1(\Omega)$ and integrate

$$\int_\Omega -\Delta u(x) v(x) \, dx = \int_\Omega f(x) v(x) \, dx$$

Use IBP, $\int_\Omega \nabla u(x) \cdot \nabla v(x) \, dx = \int_\Omega f(x) v(x) \, dx$ ↙ Green's thm & Boundary = 0 on

↑ ? Integral by part ?

$u \in H_0^1(\Omega)$ solves $a(u, v) = (f, v) \quad \forall v \in H_0^1(\Omega)$

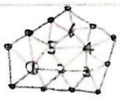
where $a(u, v) = \int_\Omega \nabla u \cdot \nabla v(x) \, dx$

$(f, v) := \int_\Omega f(x) v(x) \, dx.$

FEM : Find $u_h \in V_h$ st $a(u_h, v) = (f, v) \quad \forall v \in V_h$

where $V_h \subseteq H_0^1(\Omega)$ is finite dimensional space

Example of V_h



Let $\mathcal{T}_h$ be the collection of triangles st $\Omega = \bigcup_{T \in \mathcal{T}_h} T$

$h = \max_{T \in \mathcal{T}_h} \text{diameter}(T)$

$$V_h = \{ v \in C_0(\Omega) \quad v|_T \in P^1(T) \text{ for all } T \in \mathcal{T}_h \}$$

$P^s(T)$ = all polynomials defined on T of degree less than or equal to s

If $\exists v \neq w \in V_h$ satisfying $v = w$ for all nodes.

$v, w \in V_h \Rightarrow v - w \in V_h$

$\Rightarrow v - w = 0$ at all nodes

$\Rightarrow v - w = 0$ for all points.

M interior nodes $\vec{N}_1, \vec{N}_2, \cdots, \vec{N}_M$

Basis ftns are $\phi_i \in V_h$ $\quad \phi_i(\vec{N}_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad (i = 1, \cdots, M)$

If $v \in V_h$. $\quad v(\vec{x}) = \sum_{i=1}^{M} v(\vec{N}_i) \phi_i(\vec{x})$

↑

$$v(\vec{N}_j) = \sum_{i=1}^{M} v(\vec{N}_i) \phi_i(\vec{N}_j) = v(\vec{N}_j)$$

$$u_h \in V_h \quad \text{s.t.} \quad a(u_h, v) = (f, v) \quad \forall v \in V_h$$

$$\Leftrightarrow u_h \in V_h \quad \text{s.t.} \quad a(u_h, \phi_j) = (f, \phi_j) \quad j = 1, 2, \cdots, M$$

$$u_h = \sum_{i=1}^{M} u_h(\vec{N_i}) \phi_i$$

$$\Leftrightarrow a\left(\sum_{i=1}^{M} u_h(\vec{N_i})\phi_i, \phi_j\right) = (f, \phi_j) \quad j = 1, 2, \cdots, M$$

Recall $a(u, v) = \int_\Omega \nabla u \cdot \nabla v \Rightarrow$ bilinear

$$\sum_{i=1}^{M} \underbrace{u_h(\vec{N_i})}_{\text{unknowns}} a(\phi_i, \phi_j) = (f, \phi_j) \quad j = 1, 2, \cdots, M$$

$$\begin{bmatrix} a(\phi_1, \phi_1) & a(\phi_2, \phi_1) & \cdots & a(\phi_M, \phi_1) \\ a(\phi_1, \phi_2) & a(\phi_2, \phi_2) & & \vdots \\ \vdots & & \ddots & \\ a(\phi_1, \phi_M) & & & a(\phi_M, \phi_M) \end{bmatrix} \begin{bmatrix} u_h(\vec{N_1}) \\ u_h(\vec{N_2}) \\ \vdots \\ u_h(\vec{N_M}) \end{bmatrix} = \begin{bmatrix} (f, \phi_1) \\ (f, \phi_2) \\ \vdots \\ (f, \phi_M) \end{bmatrix}$$

$$A \vec{u} = \vec{f}$$

Need to compute A & $\vec{f}$

1) Triangulation

- vertex array

$$z = \begin{bmatrix} \\ z(i,1) & z(i,2) & z(i,3) \\ \\ \end{bmatrix}$$

$z(i,1)$ is the x value of node $\vec{N_i}$

$z(i,2)$ is the y value of node $\vec{N_i}$

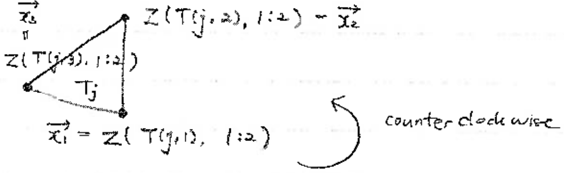
$$z(i,3) = \begin{cases} 0 & \text{if } \vec{N_i} \text{ is a boundary node} \\ \text{interior node numbers of node } \vec{N_i} & \text{if } \vec{N_i} \text{ is an interior node} \end{cases}$$

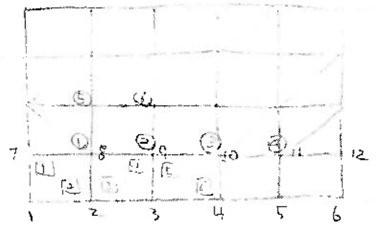
$= (M' \times 3)$ matrix where $M' = \#(\text{total nodes})$ interior + boundary

- Connectivity Matrix

$T = (@ \times 3)$ matrix where $@$ is the number of triangles

$$= \begin{bmatrix} \\ T(j,1) & T(j,2) & T(j,3) \\ \\ \end{bmatrix} \leftarrow j\text{th triangle}$$





2) $A_{ij} = a(\phi_i, \phi_j) = \int_\Omega \nabla\phi_i \cdot \nabla\phi_j \, dx = \sum_{n=1}^{@} \int_{T_n} \nabla\phi_i \cdot \nabla\phi_j \, dx \qquad \mathcal{T}_h = \{T_1, T_2, \cdots, T_@\}$

A : stiffness matrix

$A^{loc,n}$: the local stiffness matrix corresponding to triangle n (3×3) matrix

Define $\psi_\alpha(\beta) = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases}$ for $\alpha, \beta = 1, 2, 3$

$$A^{loc,n}_{\alpha\beta} = \int_{T_n} \nabla\psi_\alpha \cdot \nabla\psi_\beta \, dx$$

$d_x = \frac{dx}{du} du$

$dF(u) = J dv$

```
Initialize A is M×M
for n = 1, 2, ..., ⑥
    for α, β = 1, 2, 3
        if z(T(n,α), 3) ≠ 0 & z(T(n,β), 3) ≠ 0
            $A_{z(T(n,\alpha),3),\, z(T(n,\beta),3)} = A_{z(T(n,\alpha),3),\, z(T(n,\beta),3)} + A^{lc,n}_{\alpha,\beta}$
        end
    end
end
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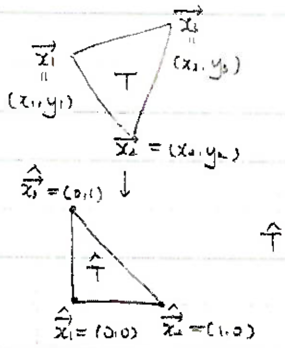
구하는 subroutine 필요



$\int_T \nabla\psi_1 \cdot \nabla\psi_2$

$\hat{T}$ is the reference triangle
$$\begin{cases} \hat{\psi}_1(\hat{x}, \hat{y}) = 1 - \hat{x} - \hat{y} \\ \hat{\psi}_2(\hat{x}, \hat{y}) = \hat{x} \\ \hat{\psi}_3(\hat{x}, \hat{y}) = \hat{y} \end{cases}$$

degree of ψ in T
= degree of $\hat{\psi}$ in $\hat{T}$

$F: \hat{T} \to T$ st. $F(\hat{x}) = B\hat{x} + \vec{x_1}$ where $B = \begin{bmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{bmatrix}$

Given $\psi(x, y)$ defined on T $\hat{\psi}(\hat{x}, \hat{y}) = \psi(x, y)$ $\Rightarrow$ $\nabla\hat{\psi} = B^T \nabla\psi$

where $\begin{pmatrix} x \\ y \end{pmatrix} = F\left(\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}\right)$ $\nabla\psi = (B^T)^{-1} \nabla\hat{\psi}$

$$\int_T \nabla\psi_1 \cdot \nabla\psi_2 \, dx = \int_T \left(B^{-T}\nabla\hat{\psi}_1\left(\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}\right)\right) \cdot B^{-T}\nabla\hat{\psi}_2\left(\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}\right) dxdy$$
$$= \int_T \left(B^{-T}\nabla\hat{\psi}_1\left(F^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right)\right) \cdot B^{-T}\nabla\hat{\psi}_2\left(F^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right) dxdy$$
$$= \int_{\hat{T}} \left(B^{-T}\nabla\hat{\psi}_1\left(\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}\right)\right) \cdot B^{-T}\nabla\hat{\psi}_2\left(\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}\right) |\det B| \, d\hat{x}d\hat{y}$$

If $\hat{\psi}_1, \hat{\psi}_2$ are linear,

$$= \underbrace{|\hat{T}|}_{\frac{1}{2}} |\det B| \, B^{-T} \cdot \nabla\hat{\psi}_1\begin{pmatrix} 0 \\ 0 \end{pmatrix} \cdot B^{-T}\nabla\hat{\psi}_2\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

9/

Example $\begin{cases} -\Delta u = f & \Omega \\ u = g & \partial\Omega \end{cases}$: Nonhomogeneous B.C

Let θ be any $H^1(\Omega)$ s.t. $\theta = g$ on $\partial\Omega$

$u = \theta + (u - \theta)$

Let $w = u - \theta \in H_0^1(\Omega)$

Let $v \in H_0^1(\Omega)$

$$a(w, v) = a(u - \theta, v) = a(u, v) - a(\theta, v) = (f, v) - a(\theta, v)$$

Find $w \in H_0^1(\Omega)$ satisfying

$a(w, v) = L(v) \quad \forall v \in H_0^1(\Omega)$

where $L(v) = (f, v) - a(\theta, v)$

$\downarrow$ piecewise linear, globally cont ftn
$V_h \subseteq V = H_0^1$

Find $\theta_h \in V_h$ s.t. $\theta_h = Ig$ for some interpolant of g, and $\theta_h = 0$ on interior nodes

Find $w_h \in V_h$ s.t.

$$a(w_h, v) = L_h(v) \quad \forall v \in V_h$$

where $L_h(v) = (f, v) - \underbrace{a(\theta_h, v)}_{\text{[illegible]}}$

Then, $u_h = \theta_h + w_h$

Neumann Problem

Note. Neumann의 주어지는 경계는 boundary 값을 모르는 것이므로 unknown 증가
└ Dirichlet 에서는 known

$$\begin{cases} -\Delta u + u = f & \Omega \\ \frac{\partial u}{\partial n} = 0 & \partial\Omega \end{cases}$$

$$\int_\Omega (-\Delta u + u) v \, dx = \int_\Omega f v \, dx$$

$$\int_\Omega (\nabla u \nabla v + uv)\, dx - \int_{\partial\Omega} \Big(\underbrace{\frac{\partial u}{\partial n}}_{0} v\Big)\, ds = \int_\Omega f v\, dx$$

$\Rightarrow$ v에 조건 필요x $\quad V = \{v \in H^1(\Omega)\}$

ex) $V_h = \{v \in C(\Omega) : v|_T \in P^1(T) \quad \forall T \in \mathcal{T}_h\}$

Find $u_h \in V_h$ satisfying

$$a(u_h, v) = (f, v) \quad \forall v \in V_h$$

where $a(u, v) = \int_\Omega (\nabla u \nabla v + uv)\, dx$

$$\begin{cases} -\Delta u + u = f & \Omega \\ \frac{\partial u}{\partial n} = g & \partial\Omega \end{cases}$$

$$\int_\Omega (\nabla u \nabla v + uv)\, dx - \int_{\partial\Omega} \Big(\frac{\partial u}{\partial n} \cdot v\Big)\, ds = \int_\Omega f v\, dx$$
$$= \int_{\partial\Omega} (g, v)\, dx$$

Weak formulation Find $u \in H^1(\Omega)$ s.t. $a(u, v) = \int_\Omega f v\, dx + \int_{\partial\Omega} (g, v)\, ds$

FEM Find $u_h \in V_h$ s.t. $a(u_h, v) = \int_\Omega f v\, dx + \int_{\partial\Omega} (g, v)\, ds \quad \forall v \in V_h$

Dirichlet & Neumann $\Rightarrow$ Mixed B.C.

$$\begin{cases} -\Delta u + u = f & \Omega & (*) \\ u = g_1 & \Gamma_1 \\ \frac{\partial u}{\partial n} = g_2 & \Gamma_2 & \Gamma_1, \Gamma_2 \subset \partial\Omega, \quad \Gamma_1 \cup \Gamma_2 = \partial\Omega \end{cases}$$

$V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_1\}$

Multiply $(*)$ with $v \in V$.

$$\int_\Omega (-\Delta u + u) v\, dx = \int f v\, dx$$

$$\int_\Omega (\nabla u \nabla v + uv)\, dx - \underline{\int_{\partial\Omega} \Big(\frac{\partial u}{\partial n} \cdot v\Big)\, ds} = \int_\Omega f v\, dx$$

$$= \int_{\Gamma_1} \frac{\partial u}{\partial n} v\, ds + \int_{\Gamma_2} \underbrace{\frac{\partial u}{\partial n}}_{g_2} v\, ds = \int_{\Gamma_2} g_2 v\, ds$$

$$\Rightarrow a(u, v) = \int_\Omega f v\, dx + \int_{\Gamma_2} g_2 v\, ds \quad \forall v \in V$$

Let $\theta \in H^1(\Omega)$ with $\theta = g_1$ on Γ_1, then $u - \theta \in V$

$$u = \theta + (u - \theta) =: \theta + w \qquad w = u - \theta$$

Given $v \in V$, $a(w, v) = a(u - \theta, v) = a(u, v) - a(\theta, v) = \int_\Omega f v\, dx + \int_{\Gamma_2} g_2 v\, ds - a(\theta, v) = L(v)$

Neumann Dirichlet 합친거

Find $w \in V$ satisfying $a(w, v) = L(v) \quad \forall v \in V$

Many PDE's are formulated in the following way

$$\begin{cases} \text{Find } u \in V \\ \quad a(u,v) = L(v) \quad \text{for all } v \in V \end{cases}$$

V is a inner-product space

a is a bilinear form $a : V \times V \longrightarrow \mathbb{R}$

$$a(\alpha v_1 + \beta v_2,\ u) = \alpha\, a(v_1, u) + \beta a(v_2, v)$$

$$a(u,\ \alpha v_1 + \beta v_2) = \alpha\, a(u, v_1) + \beta\, a(u, v_2)$$

$L : V \longrightarrow \mathbb{R}$ $\quad L(\alpha v_1 + \beta v_2) = \alpha L(v_1) + \beta L(v_2)$

Example $\quad V = H_0^1(\Omega)$

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \qquad L(v) = (f,v) = \int_\Omega f v$$

Example
$$\begin{cases} \Delta^2 u = f & \Omega \\ u = 0 & \partial\Omega \\ \dfrac{\partial u}{\partial n} = 0 & \partial\Omega \end{cases}$$

<2 조건 줘서 가능 ~

이건 Numerically matrix 어떻게??

$$V = H_0^2(\Omega) = \left\{ v \in H^2(\Omega) \mid v = 0,\ \frac{\partial v}{\partial n} = 0 \quad \partial\Omega \right\}$$

$$a(u,v) = \int_\Omega \Delta u \cdot \Delta v \, dx \qquad L(v) = \int_\Omega f v$$

General Assumptions

induced norm $\|v\|_V = \sqrt{(u,v)_V}$

V is an inner product space with inner product $(u,v)_V$ for any $u, v \in V$

There exist positive constants γ, α, Λ s.t. the following hold

(i) $a(\cdot,\cdot)$ continuous i.e. $|a(v,w)| \le \gamma \|v\|_V \|w\|_V$ for all $v, w \in V$

(ii) $a(\cdot,\cdot)$ is V-elliptic (coercive) i.e. $a(v,v) \ge \alpha \|v\|_V^2$

(iii) $|L(v)| \le \Lambda \|v\|_V$

Frobenius inner product

$$A : B = \sum_{i,j} A_{ij} B_{ij}$$

Example. $\quad a(u,v) = \int_\Omega \nabla u\ \nabla v \, dx \qquad L(v) = \int_\Omega f v \, dx$

For $V = H_0^2(\Omega)$,

$$(u,v)_V = \int_\Omega D^2 u : D^2 v + \nabla u \cdot \nabla v + uv$$

$$D^2 u = \begin{bmatrix} \partial_{xx} u & \partial_{xy} u \\ \partial_{xy} u & \partial_{yy} u \end{bmatrix}$$

Check (i) - (iii)

(i) $\quad a(v,w) = \int_\Omega \Delta v \Delta w \, dx \le \|\Delta v\|_{L^2(\Omega)} \|\Delta w\|_{L^2(\Omega)} \le \|v\|_V \|w\|_V$

$\Rightarrow$ (i) is satisfied with $\gamma = 1$

$$\|\Delta v\|_{L^2(\Omega)}^2 = \int_\Omega (v_{xx}^2 + 2 v_{xx} v_{yy} + v_{yy}^2) \le 2 \int (v_{xx}^2 + v_{yy}^2)$$

$$\|v\|_V^2 = \int_\Omega (v_{xx}^2 + 2 v_{xy}^2 + v_{yy}^2 + v_x^2 + v_y^2 + v^2) \, dx$$

$\int v_{xx} v_{yy} = - \int v_x v_{yyx} \, dx +$

이렇게 하면 $\gamma = 1$?

이거 [illegible]

(ii) $\alpha \int (D^2 v : D^2 v + \nabla v \cdot \nabla v + v v) \, dx \le \int (\nabla v)^2 dx$

If Ω is convex, $\quad \|v\|_{H^2(\Omega)} \le C \|\Delta v\|_{L^2} \qquad v \in H^2 \cap H_0^1(\Omega)$ $\quad$ elliptic regularity

$C = C(\Omega)$

$$\Rightarrow \alpha = \frac{1}{C}$$

(iii) $\quad |L(v)| \le \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \le \|f\|_{L^2(\Omega)} \|v\|_V$

$$\Rightarrow \Lambda = \|f\|_{L^2(\Omega)}$$

Finite Element

$V_h \in V$

Find $u_h \in V_h$ satisfying $a(u_h, v) = L(v) \quad \forall v \in V_h$

Galerkin Orthogonality

Lemma Suppose $u \in V$ is the solution to

$$a(u,v) = L(v) \quad \text{for all} \quad v \in V$$

Let $u_h \in V_h$ be its finite element approximation satisfying

$$a(u_h, v) = L(v) \quad \text{for all} \quad v \in V_h$$

Then, $a(u-u_h, v) = 0 \quad \forall v \in V_h$

pf) Let $v \in V_h \subset V$

$a(u,v) = L(v)$, $a(u_h, v) = L(v)$

$$\Rightarrow \quad a(u,v) - a(u_h, v) = \underset{\text{"0" orthogonal!}}{\underbrace{a(u-u_h, v) = 0}} \quad \forall v \in V_h$$

Thm Suppose (i) - (iii) hold , ⊛ (Cea's Lemma)

Then,

$$\| u - u_h \|_V \leq \frac{\gamma}{\alpha} \underset{v \in V_h}{\inf} \| u - v \|_V$$

approximation theory (V_h 가 얼마나 rich 하여야 u를 잘 approximate 하느냐 ...)
도메인에 따라 v 가 만족해야 하는 regularity를 알려줌

p.f) $u - u_h \in V$

$$\begin{aligned} \alpha \| u - u_h \|_V^2 &\leq a(u-u_h, u-u_h) && \text{by (ii)} \\ &= a(u-u_h, u) - \underbrace{a(u-u_h, u_h)}_{=0 \text{ 이므로}} \\ &= a(u-u_h, u) - a(u-u_h, v) && \underbrace{\forall v \in V_h}_{\text{로 바꿀 수 있다}} \quad \text{by Galerkin Orthogonality} \\ &= a(u-u_h, u-v) \\ &\leq \gamma \| u - u_h \|_V \| u - v \|_V && \text{by (i)} \end{aligned}$$

$$\| u - u_h \|_V \leq \frac{\gamma}{\alpha} \| u - v \|_V \quad \forall v \in V_h$$

(finite element 라서 inf 대신 min 으로 가능)

10/3

1) Formal Definition of Finite Elements. examples

2) Bramble - Hilbert Lemma (Approximation theory)

Def A Finite Element is a triple (K, P_K, Σ) ← Ciarlet이 만든 def.

where K is a geometric object. △ □ ◇

P_K is a finite dimensional space of function defined on K.

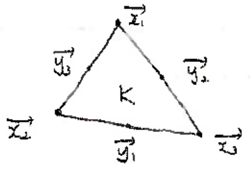
and Σ is a set of degrees of freedom that determine uniquely each function $v \in P_K$

review ✓

Example K = triangle

$P_K = P^2(K) = \text{span}\{1, x, y, xy, x^2, y^2\}$

Σ = value at vertices and values at midpoints of edges.



Claim: Given $v \in P_K$, we know exactly if we know $v(\vec{x_i}), v(\vec{y_i})$ $i = 1, 2, 3$

$\dim(P_K) = \dim(P^2(K)) = 6$ = # of given points $\vec{x_i}, \vec{y_i}$

We need to show that if $v \in P_K$ and $v(\vec{x_i}) = v(\vec{y_i}) = 0$ for $i = 1, 2, 3$ then $v \equiv 0$

pf) quadratic ftn | restrict to an edge | that vanishes at 3 points ⇒ zero function.

$v = \ell \lambda_1$ where λ_1 is unique linear function that is one on $\vec{x_1}$, and zero on $\vec{x_2}$ and $\vec{x_3}$.

and ℓ is a linear function

$0 = v(\vec{y_3}) = \ell(\vec{y_3})\lambda_1(\vec{y_3}) = \frac{1}{2}\ell(\vec{y_3})$

Similarly, $\ell(\vec{x_1}) = 0$ $\quad (v(\vec{x_1}) = \ell(\vec{x_1})\lambda_1(\vec{x_1}) = 0)$

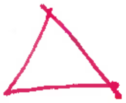
$v = c\lambda_1\lambda_3$ where $\lambda_3 \in P^1(K)$ $\quad \lambda_3(\vec{x_j}) = \begin{cases} 1, & j = 3 \\ 0 & j \neq 3 \end{cases}$

$0 = v(\vec{y_2}) = c\lambda_1(\vec{y_2})\lambda_3(\vec{y_2}) = \frac{1}{4}c$ $\quad \therefore c = 0$

$\therefore v = 0$

- barycentric coordinate of triangle $\lambda_i \in P^1(K)$, $\lambda_i(\vec{x_j}) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

$ax^2 + bxy + cy^2$
$+ dx + ey$
$+ f$



Example K = triangle

$P_K = P^2(K)$

Σ = values at nodes and average on each edge $\frac{1}{|e_i|}\int_{e_i} v$

$\pi \cdot V \longrightarrow V_h$

$\frac{1}{|K|}\int_K \partial_{x_i}(\pi v) = \frac{1}{|K|}\int_K \partial_{x_i} v$

Example $K = [0, 1]$

$P_K = P^5(K)$

Σ = values of function, first and second derivatives on 0 and 1 ⇒ C^2 continuity

$v(0) = v'(0) = v''(0) = v(1) = v'(1) = v''(0)$ ⇒ $v \equiv 0$

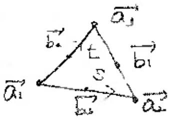
[illegible]

pf) similar to the above pf

[illegible] 1D example임

Example. $K =$ triangle

$P_K = P^5(K)$



$\Sigma = \; V(a_i),\; \nabla V(a_i),\; \nabla^2 V(a_i),\; \frac{\partial}{\partial n} V(b_i) \qquad i=1,2,3$

: Agyris-element $\quad (\Rightarrow C^1$ continuity$)$

각 edge로 restriction 하는 [illegible]
Σ가 $P^5(K) \ni v$ 를 uniquely 결정해보고
C^1 continuity가 나올까! 이것이 궁금?

- $\dim(P^p(K)) = \frac{1}{2}(p+1)(p+2)$ in 2-D.

$\Rightarrow \dim(P^5(K)) = 21$

$\qquad = 3 + 6 + 9 + 3$

- Suppose $v \in P^5(K)$, $\quad V(a_i) = \partial_{x_j} V(a_i) = \partial_{x_j}\partial_{x_k} V(a_i) = \frac{\partial V}{\partial n}(b_i) = 0$.

$\Rightarrow$ v vanishes at edges.

$\Rightarrow V = r\lambda_1\lambda_2\lambda_3$ where r is quadratic

First show r vanishes at vertices.

$$\partial_s\partial_t V = \partial_s\left(r\,\partial_t(\lambda_1\lambda_2\lambda_3) + (\partial_t r)\lambda_1\lambda_2\lambda_3 \right)$$

$$= r\,\partial_s\partial_t(\lambda_1\lambda_2\lambda_3) + (\partial_s r)\,\underbrace{\partial_t(\lambda_1\lambda_2\lambda_3)}_{0} + (\partial_s\partial_t r)\,\underbrace{\lambda_1\lambda_2\lambda_3}_{0.} + (\partial_t r)\,\underbrace{\partial_s(\lambda_1\lambda_2\lambda_3)}_{0}$$

$$\partial_s\partial_t V(\vec{a_1}) = r(\vec{a_1})\,\partial_s\partial_t(\lambda_1\lambda_2\lambda_3)$$

$$= r(\vec{a_1})\Big(\lambda_1\big(\underbrace{\partial_s\lambda_2\partial_t\lambda_3}_{\neq 0} + \underbrace{\partial_s\lambda_3}_{0.}\partial_t\lambda_2\big) + \underset{0}{\cancel{\lambda_2}}(\partial_s\lambda_1\partial_t\lambda_3 + \partial_s\lambda_3\partial_t\lambda_1) + \underset{0}{\cancel{\lambda_3}}(\partial_s\lambda_2\partial_t\lambda_1 + \partial_s\lambda_1\partial_t\lambda_2$$

$$\Rightarrow r(\vec{a_1}) = 0$$

Similarly, $r(\vec{a_2}) = r(\vec{a_3}) = 0$

$$\underbrace{\partial_n V(b_1)}_{0} = (\partial_n r(b_1))\,\underbrace{\lambda_1\lambda_2\lambda_3}_{0} + r(b_1)\,\underbrace{\partial_n(\lambda_1\lambda_2\lambda_3)}_{\neq 0} \quad \Rightarrow r(\vec{b_1}) = 0$$

$\Rightarrow r \equiv 0$.

$\Rightarrow V \equiv 0$.

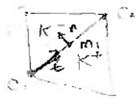
- Global finite element space

$V_h = \{ v : \; v|_K \in P^5(K) \;\; \forall K \in T_h$ such that $v, \nabla v, \nabla^2 v, \partial_n v$ agree from different triangles $\}$.

이 [illegible] C^1이 가능!

($v, \nabla v, \nabla^2 v$ agree at vertices
$\partial_n v$ agree at midpoints of triangulation)

$\Rightarrow V_h \subseteq H^2(\Omega) \quad V_h \subseteq C^1(\Omega)$



$e = \bar{K}_- \cap \bar{K}_+$

$V_+ = V|_{K_+}, \quad V_- = V|_{K_-}$

Let $W = V_+ - V_-$. We must show $W = 0$, $\nabla W = 0$ on e $\quad (\Rightarrow C^1)$

pf) $W_e \in P^5(e)$

$w(c_i) = 0, \; \nabla w(c_i) = 0, \; \nabla^2 w(c_i) = 0 \quad i = 1,2 \quad \Rightarrow W|_e \equiv 0$ on e $\rightarrow V_+ = V_e$ on e

$$\nabla W|_e = \underset{0}{(\nabla w \cdot \vec{t})}\,\vec{t}|_e + (\nabla w \cdot \vec{n})\,\vec{n}|_e$$

0 since $w \equiv 0$ along the line

Let $g = \nabla w \cdot \vec{n}|_e \in P^4(e)$

$$g(m_1) = g(c_1) = g'(c_1) = g(c_2) = g'(c_2) = 0. \Rightarrow g \equiv 0 \Rightarrow \nabla w|_e = 0$$

- $\dim(V_h) = \underbrace{6}_{v,\, \partial_x v,\, \partial_y x,\, \partial_{xx} v,\, \partial_{xy} v,\, \partial_{yy} v} \cdot$ number of vertices $+ \underbrace{1}_{\frac{\partial v}{\partial n}} \cdot$ number of edges.

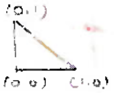
Example $V_h = \{ v \in H^1(\Omega) \quad v|_T \in P^k(T) \quad \forall T \in \mathcal{T}_h \}$

Given $u \in H^{k+1}(\Omega)$,

$$\inf_{v \in V_h} \| (u-v) \|_{H^1(\Omega)} \le c\, h^k |u|_{H^{k+1}(\Omega)} \quad : \text{Goal}$$

Bramble–Hilbert Lemma

Lemma Let $\hat{T}$ be the reference triangle



Let $u \in H^{k+1}(\hat{T})$

Then, we have

$$\inf_{v \in P^k(\hat{T})} |u-v|_{H^m(\hat{T})} \le c\, |u|_{H^{k+1}(\hat{T})} \qquad \forall\, m \le k+1$$

Note $\Big(u \in H^k \overset{\not}{\Rightarrow} |u|_{H^{k+1}(\hat{T})} = 0 \overset{\not}{\Rightarrow} ~~u \in P^k(\hat{T})~~ \Big)$ what?

$$|u|_{H^m(\hat{T})} = \sum_{\alpha_1+\alpha_2=m} \| \partial_x^{\alpha_1} \partial_y^{\alpha_2} u \|^2_{L^2(\hat{T})}$$

[illegible] Triangle, [illegible]

- Poincare Inequality. for ref triangle (holds for Lipschitz domains)

Let $u \in H^1(\hat{T})$

$$\Big\| w - \frac{1}{|\hat{T}|} \int_{\hat{T}} w \Big\|_{L^2(\hat{T})} \le c\, |w|_{H^1(\hat{T})}$$

★ ← preliminal proof.

pf of Lemma) ①

There $\boxed{\text{exists}}$ $v \in P^k(T)$ such that

i.e. find α_α st this holds

$$(*) \quad \frac{1}{|\hat{T}|} \int (\partial_x^{\alpha_1} \partial_y^{\alpha_2} V_k) = \frac{1}{|\hat{T}|} \int (\partial_x^{\alpha_1} \partial_y^{\alpha_2} u) \qquad \text{for any } \alpha_1, \alpha_2 \quad \alpha_1+\alpha_2 = k.$$

(k+1) conditions

pf $(\because)$ $V_k = a_k x^k y^0 + a_{k-1} x^{k-1} y + a_{k-2} x^{k-2} y^2 + \cdots + a_0 x^0 y^k$ ← (k+1) coefficients

$$\partial_x^{\alpha_1} \partial_y^{\alpha_2} V_k = a_{\alpha_1} (\alpha_1)!\,(\alpha_2)!$$

$$(*) \Leftrightarrow \frac{1}{|\hat{T}|} \int_{|\hat{T}|} a_{\alpha_1} (\alpha_1)!\,(\alpha_2)! = \frac{1}{|\hat{T}|} \int \partial_x^{\alpha_1} \partial_y^{\alpha_2} u$$

②

$$\boxed{a_{\alpha_1}} = \frac{1}{(\alpha_1)!\,(\alpha_2)!\,|\hat{T}|} \int_{|\hat{T}|} \partial_x^{\alpha_1} \partial_y^{\alpha_2} u$$

이와 같이 a_{α} 들을 계산하면 모든 u에 대하여 v 찾을 수 있다

$$|u - V_k|_{H^k(\hat{T})} = \Big(\sum_{\alpha_1+\alpha_2=k} \| \partial_x^{\alpha_1} \partial_y^{\alpha_2} (u - V_k) \|^2_{L^2(\hat{T})} \Big)^{1/2} \underset{\uparrow}{\le} C\, |u|_{H^{k+1}(\hat{T})}$$

Poincare

Find $V_{k-1} \in P^{k-1}(\hat{T})$ s.t.

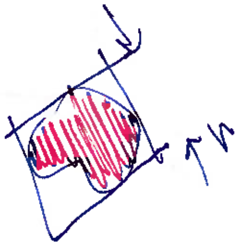
$$\operatorname{avg}(\partial_x^{\alpha_1} \partial_y^{\alpha_2} (u - V_k)) = \operatorname{avg}(\partial_x^{\alpha_1} \partial_y^{\alpha_2} V_{k-1}) \quad \text{for } \alpha_1+\alpha_2 = k-1$$

⋮

$V_0 \in P^0(\hat{T})$

$$\begin{aligned} |u - (V_k + V_{k-1} + \cdots + V_0)|_{H^m} &= |u - (V_k + \cdots + V_m)|_{H^m} \\ &= |(u - V_k - V_{k-1} - \cdots - V_{m+1}) - V_m|_{H^m} \\ &\le C\, |u - (V_k + V_{k-1} + \cdots + V_{m+1})|_{H^{m+1}} \\ &\le \cdots \le C\, |u|_{H^{k+1}(\hat{T})} \end{aligned}$$

$m \le k+1$



2D에서 증명하지만 3D에서도 hold!

10/17

① BH on $\hat{T}$

② BH on T

Approximation properties.

Last time we proved Bramble - Hilbert

$V_h^k = \{ v \in C_0(\Omega) \; v|_T \in P^k(T) \;,\; \forall T \in \mathcal{T}_h \}$ Generalize

The aim of this lecture is to prove

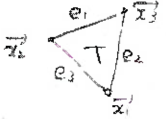
$$\boxed{\inf_{v \in V_h^k} |u - v|_{H^1(\Omega)} \le C h^k |u|_{H^{k+1}(\Omega)}}$$

$h = \max_{T \in \mathcal{T}_h} \operatorname{diam} T$

Define the global interpolant of u which we denote by $Iu \in V_h^k$

Locally $Iu|_T \in P^k(T)$ defined via the degree of freedom Σ of out finite element $(T, P^k(T), \Sigma)$

Essentially the degrees of Iu match those of u



$Iu|_T \in P^k(T)$

1) $Iu(\vec{x}_i) = u(\vec{x}_i)$ for $i = 1, 2, 3$. <D 이상 ignore

2) $\int_{e_i} (Iu) \xi \, ds = \int_{e_i} u \xi \, ds$ for $\xi \in P^{k-2}(e_i)$ $i = 1, 2, 3$

moment 같은 거라고 생각

3) $\int_T (Iu) w \, dx = \int uw \, dx$ $\forall w \in P^{k-3}(T)$

$\ell = [(Iu)|_{T^+} - (Iu)|_{T^-}]|_e$ $\quad \ell \in P^k(e)$

$\ell(\vec{x}_1) = 0$, $\ell(\vec{x}_2) = 0$

$\int_e \ell \xi \, ds = 0$ $\quad \forall \xi \in P^{k-2}(e)$ $\Rightarrow$ $\ell \equiv 0$ $\Rightarrow$ continuous through e

pf) Let $\ell \in P^k([0,1])$

$\ell(0) = \ell(1) = 0$

$\int_0^1 \ell \xi \, dx = 0$ $\quad \forall \xi \in P^{k-2}([0,1])$ $\quad (*)$

Since $\ell \in P^k([0,1])$, $\ell = x(1-x) m$ where $m \in P^{k-2}([0,1])$

$(*) \Rightarrow \int_0^1 \ell(x) m(x) \, dx = 0$

$\int_0^1 \underbrace{x(1-x)}_{\text{zero at } 0,1 \text{ only}} m^2(x) \, dx = 0$ $\quad \therefore m \equiv 0$ $\quad \therefore \ell \equiv 0$.

$$\boxed{\inf_{v \in V_h^k} |u - v|_{H^1(\Omega)} \le |u - Iu|_{H^1(\Omega)} = \Big(\sum_{T \in \mathcal{T}_h} |u - Iu|^2_{H^1(T)} \Big)^{1/2} \le C h^k \Big(\sum |u|^2_{H^{k+1}(T)} \Big)^{1/2} = C h^k |u|_{H^{k+1}(\Omega)}}$$

If the claim is true

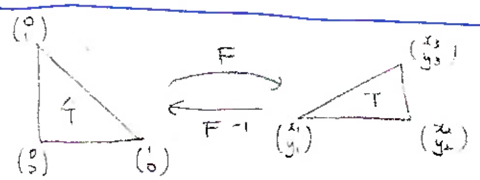
①

Claim : For any $T \in \mathcal{T}_h$,

$$|u - Iu|_{H^1(T)} \le C h^k |u|_{H^{k+1}(T)}$$

only depends on reference triangle & degree of polynomial

<— $h = 1$이면 Bramble - Hilbert로 바로 나옴 (reference triangle) 이렇게 하자.



$$\begin{pmatrix} x \\ y \end{pmatrix} = F\left(\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} \right) = B \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \text{where} \quad B = \begin{bmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{bmatrix}$$

Can't bound $|\hat{I}\hat{v}|_{H_1} \le |\hat{v}|_{H_1}$ $\wedge$ operator $\hat{w}\begin{pmatrix}\hat{x}\\\hat{y}\end{pmatrix} = w\left(F\begin{pmatrix}\hat{x}\\\hat{y}\end{pmatrix}\right)$

For any function $\hat{w}$ defined on $\hat{T}$, we have a corresponding function defined on T

$$w\left(F\begin{pmatrix}\hat{x}\\\hat{y}\end{pmatrix}\right) = \hat{w}\begin{pmatrix}\hat{x}\\\hat{y}\end{pmatrix} \qquad w\left(\begin{pmatrix}x\\y\end{pmatrix}\right) = \hat{w}\left(F^{-1}\begin{pmatrix}x\\y\end{pmatrix}\right)$$

Know

Lemma

$$\boxed{\begin{aligned} |\hat{w}|_{H^m(\hat{T})} &\le C\|B\|^m |\det B|^{-\frac{1}{2}} |w|_{H^m(T)} \\ |w|_{H^m(T)} &\le C\|B^{-1}\|^m |\det B^{-1}|^{-\frac{1}{2}} |\hat{w}|_{H^m(\hat{T})}\end{aligned}}$$

$$\|B\| = \sup_{\vec{z}\neq 0} \frac{\|B\vec{z}\|}{\|\vec{z}\|} \quad \text{where} \quad \|\vec{z}\| = \sqrt{z_1^2+z_2^2}$$

pf) $|\hat{w}|_{H^m(\hat{T})} = \left(\sum_{|\alpha+\beta|=m} \|\partial_{\hat{y}}^{\alpha}\partial_{\hat{x}}^{\beta}\hat{w}(\hat{x},\hat{y})\|_{L^2(\hat{T})}\right)^{\frac{1}{2}}$

$$\partial_{\hat{y}}^{\alpha}\partial_{\hat{x}}^{\beta}\hat{w}(\hat{x},\hat{y}) = \partial_{\hat{y}}^{\alpha}\partial_{\hat{x}}^{\beta} w\left(F\begin{pmatrix}\hat{x}\\\hat{y}\end{pmatrix}\right) = \partial_{\hat{y}}^{\alpha}\partial_{\hat{x}}^{\beta} w(b_{11}\hat{x} + b_{12}\hat{y} + x_1,\ b_{21}\hat{x} + b_{22}\hat{y} + y_1)$$

$$= \sum_{r=0}^{\alpha}\sum_{\xi=0}^{\beta} b_{11}^{\xi} b_{12}^{r} b_{21}^{\beta-\xi} b_{22}^{\alpha-r} \partial_y^{\alpha+\beta-(r+\xi)} \partial_x^{r+\xi} w\left(F\begin{pmatrix}\hat{x}\\\hat{y}\end{pmatrix}\right)$$

multiple chainrules

$$\|\partial_{\hat{y}}^{\alpha}\partial_{\hat{x}}^{\beta}\hat{w}\|_{L^2(\hat{T})}^2 = \int_{\hat{T}} (\partial_{\hat{y}}^{\alpha}\partial_{\hat{x}}^{\beta}\hat{w})^2\, d\hat{x}\,d\hat{y}$$

$$= \int_T \left(\sum_{r=0}^{\alpha}\sum_{\xi=0}^{\beta} b_{11}^{\xi} b_{12}^{r} b_{21}^{\beta-\xi} b_{22}^{\alpha-r} \partial_y^{\alpha+\beta-(r+\xi)} \partial_x^{r+\xi} w\begin{pmatrix}x\\y\end{pmatrix}\right)^2 |\det B|^{-1}\, dx\,dy$$

$$\le C\left(\max|b_{ij}|\right)^{2m} |\det B|^{-1} |w|^2_{H^m(T)}$$

$$\|\partial_{\hat{y}}^{\alpha}\partial_{\hat{x}}^{\beta}\hat{w}\|_{L^2(\hat{T})} \le \underset{\substack{\uparrow \\ \text{depends on } m}}{C} \underbrace{(\max|b_{ij}|)^m}\ |\det B|^{-\frac{1}{2}} |w|_{H^m(T)}$$

$$\left\|B\cdot\begin{pmatrix}1\\0\end{pmatrix}\right\| = \left\|\begin{pmatrix}b_{11}\\b_{21}\end{pmatrix}\right\| \le \sqrt{b_1^2+b_2^2}$$

"In finite dimensions all norms are equivalent" why

WTS $\boxed{|u - Iu|_{H_1(T)} \le h^K |u|_{H^{K+1}}}$

$$\le C\|B\|^m |\det B|^{-\frac{1}{2}} |w|_{H^m(T)} \quad \square$$

Back to the claim)

by lemma

$$|u - Iu|_{H^1(T)} \le C\|B^{-1}\| |\det B^{-1}|^{-\frac{1}{2}} |\widehat{u - Iu}|_{H^1(\hat{T})} = C\|B^{-1}\| |\det B^{-1}|^{-\frac{1}{2}} |\hat{u} - \hat{I}\hat{u}|_{H^1(\hat{T})}$$

$$\widehat{u - Iu} = \hat{u} - \widehat{Iu}$$

$$\widehat{Iu} = \hat{I}\hat{u} \quad \leftarrow \text{can prove}$$

$$\hat{I}:\ C(\hat{T}) \longrightarrow P^k(\hat{T})$$

ex?

$$\begin{cases} \hat{I}\hat{u}(\vec{x}_i) = \hat{u}(\vec{x}_i) \\ \int_{\hat{e}_i} \hat{I}\hat{u}\hat{\xi} = \int_{\hat{e}_i} \hat{u}\hat{\xi} & \forall \hat{\xi} \in P^{k-2}(\hat{e}_i) \\ \int_{\hat{T}} (\hat{I}\hat{u})w = \int_{\hat{T}} \hat{u}w & \forall w \in P^{k-3}(\hat{T}) \end{cases}$$

$$|\hat{I}| = \sup_{v} \frac{\|Iv\|}{\|v\|}$$

For any $\hat{v} \in P^k(\hat{T})$

$$|\hat{u} - \hat{I}\hat{u}|_{H^1(\hat{T})} \le |\hat{u} - \hat{v}|_{H^1(\hat{T})} + |\hat{v} - \hat{I}\hat{u}|_{H^1(\hat{T})} \qquad \hat{v} = \hat{I}\hat{v} \text{ by def of } \hat{I}$$

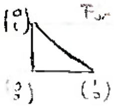
$$\le |\hat{u} - \hat{v}|_{H^1(\hat{T})} + |\hat{I}(\hat{v} - \hat{u})|_{H^1(\hat{T})} \qquad (\ast)$$

$$|\hat{I}(\hat{v} - \hat{u})|_{H^1(\hat{T})} \le \underset{\substack{\uparrow\\ \text{작아야 good}}}{\underbrace{|\hat{I}|}_{\text{operator norm}}} |\hat{v} - \hat{u}|_{H^1(\hat{T})}$$

max norm

note. $|\hat{I}\hat{w}|_{C(\hat{T})} \le C|\hat{w}|_{C(\hat{T})}$

$\hat{I}\hat{w}$는 $\hat{w}$의 value의 sort of average라서 $\hat{w}$로 bound. prove in HW



For $k=1$

$$|\hat{I}\hat{w}| \le \max_{i=1,2,3} |\hat{I}\hat{w}(\vec{x}_i)| = \max_{i=1,2,3} |\hat{w}(\vec{x}_i)| \le \max_{\vec{x}\in T} |\hat{w}(\vec{x})| = |\hat{w}|_{C(\hat{T})}$$

$$|\hat{I}(\hat{v} - \hat{u})|_{H^1(\hat{T})} \le C|\hat{I}(\hat{v} - \hat{u})|_{C(\hat{T})} \le C|\hat{v} - \hat{u}|_{C(\hat{T})}$$

Need to pick I carefully w/ a good norm $\|\cdot\|_m$ s.t $\|Iu\|_m = \|u\|_m \le \|u\|_{L^\infty}$

$$|\cdot|_{C(\hat{T})} = |\cdot|_{L^\infty}$$

★ Sobolev inequality

In two dimensions, ★ $\left(|\hat{w}|_{C(\hat T)} \le C \|\hat{w}\|_{H^2(\hat T)} \right.$

finite 이면 continuous하도 [illegible]

p15

$$(\star) \Rightarrow |\hat{u} - \hat{I}\hat{u}|_{H^1(\hat T)} \le |\hat{u} - \hat{v}|_{H^1(\hat T)} + C \|\hat{u} - \hat{v}\|_{H^2(\hat T)}$$

$$|u - Iu|_{H^1(T)} \le C \|B^{-1}\| \, |\det B^{-1}|^{-\frac{1}{2}} \inf_{\hat v \in P^k(\hat T)} \|\hat u - \hat v\|_{H^2(\hat T)}$$

H^1 semi H^2 semi apply 하면 ↙ B-H lemma.

$$\le C \|B^{-1}\| \, |\det B^{-1}|^{-\frac{1}{2}} \, |\hat u|_{H^{k+1}(\hat T)}$$

$$\le C \|B^{-1}\| \, |\det B^{-1}|^{-\frac{1}{2}} \|B\|^{k+1} |\det B|^{-\frac{1}{2}} |u|_{H^{k+1}(T)}$$

$$\boxed{|u - Iu|_{H^1(T)} \le C \|B\| \|B^{-1}\| \|B\|^k |u|_{H^{k+1}(T)}} \qquad (\star\star)$$

[illegible] 여기서 triangle 성질이 들어간다.

Let $\rho(T) = \sup\{\operatorname{diam}(S) : S \subseteq T, \ S \text{ is a circle}\}$ See figure at p14

$\hat h = \operatorname{diam}(\hat T) = \sqrt{2}$ $\qquad h = \operatorname{diam}(T) \quad (= h_T)$

$$\|B^{-1}\| \le \frac{\hat h}{\rho(T)} \qquad \|B\| \le \frac{h}{\rho(\hat T)}$$

$$(\because) \quad \|B^{-1}\| = \sup_{\vec z \ne 0} \frac{\|B^{-1}\vec z\|}{\|\vec z\|} = \frac{1}{\rho(T)} \sup_{\|\vec z\| = \rho(T)} \|B^{-1}\vec z\| \le \frac{\hat h}{\rho(T)}$$

Let $\vec z \in \mathbb{R}^2$, $\|\vec z\| = \rho(T)$

There exist points $\vec m, \vec g \in T$ s.t. $\vec m - \vec g = \vec z$

$$F^{-1}(\vec z) = F^{-1}(\vec m - \vec g) = F^{-1}(\vec m) - F^{-1}(\vec g) = (B^{-1}\vec m + c) - (B^{-1}\vec g + c) = B^{-1}(\vec m - \vec g)$$

$$\|F^{-1}(\vec z)\| \le \hat h \qquad (\because \ \hat T 에 있으므로) \qquad \|F^{-1}(\vec z)\| = \|B^{-1}(\vec z)\|$$

$$(\star\star) \Rightarrow |u - Iu|_{H^1(T)} \le C \frac{h}{\rho(\hat T)} \frac{\hat h}{\rho(T)} \left(\frac{h}{\rho(\hat T)}\right)^k |u|_{H^{k+1}(T)} = C \frac{h}{\rho(T)} h^k |u|_{H^{k+1}(T)}$$

(좋은 triangle [illegible] 이걸 [illegible])

shape regularity : $\frac{h}{\rho(T)} \le C \quad \forall T \in \mathcal{T}_h$

나쁜 triangle :

skinny triangle

10/24 Last time,

$$V_h = \{ v \in C_0(\Omega) : v|_T \in P^k(T), \ \forall T \in \mathcal{T}_h \}$$

Then let $u \in H^{k+1}(\Omega)$ convergence rate

$$\inf_{v \in V_h} |u - v|_{H^1(\Omega)} \le C h^{k} |u|_{H^{k+1}(\Omega)} \qquad C \text{ is independent of } u.$$

as long as we have $\frac{\operatorname{diam}(T)}{\rho_T} \le M$ for every $T \in \mathcal{T}_h$

independent of T.

Shape regularity of $\mathcal{T}_h$

Ask Seonmin

같은 방법으로 $L^2(\Omega)$ norm 에 대해서도 증명가능.

$$\Rightarrow \inf_{v\in V_h} \|u-v\|_{L^2(\Omega)} \leq Ch^{k+1} |u|_{H^{k+1}(\Omega)}$$

Generalize

$$\Rightarrow \inf_{v\in V_h} |u-v|_{H^m(\Omega)} \leq ch^{(k+1)-m} |u|_{H^{k+1}(\Omega)}$$

$m = 0, 1$ $\quad v \in H^2$

m이 더 크면 v 가 $H^{\textcircled{3}}$ 에 속하지 알 수 x $\quad$ (higher than 2) $\quad \underline{|u-v|_{\textcircled{H}^2}}$ 구할 수 x

But locally 각 triangle에서는 계산 가능

$$\|u-v\|_{H_1}^2 = \|u-v\|_{L_2}^2 + |u-v|_{H_1}^2$$
$$\leq c(h^{k+1} + h^k)\, |u-v|_{H_1}^2$$

Today : 1) a few applications of ↑

2) Elasticity

3) STOKES problem

$$\circledast \begin{cases} -\nabla\cdot(A\nabla u) + ru = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$$\int (\nabla^2 u)\, v\, dV = ?$$

Assumption 1) $v^T A(x) v > \alpha |v|^2 \quad \forall x$ pos def

2) $r(x) \geq 0 \quad \forall x$

3) $|r(x)| < c, \ |A(x)| < c$ (각 entry가 bounded)

$$\int \nabla\cdot(v\nabla u)\, dV = \int v\nabla u \cdot n\, dA$$

$$\int \nabla v \nabla u + v\nabla^2 u$$

Weak form of $\circledast$: $a(u,v) = L(v) \quad \forall v \in H_0^1(\Omega)$

where $a(u,v) = \int_\Omega (A\nabla u\cdot\nabla v + ruv)\, dx$

$L(v) = \int_\Omega f\cdot v\, dx$

$$\begin{cases} \text{i) } \|u\|_{H^1(\Omega)} \leq \beta\, a(u,u) & \forall u \in H^1(\Omega) \\ \text{ii) } a(u,v) \leq \gamma \|u\|_{H^1}\|v\|_{H^1} & \forall u,v \in H^1 \\ \text{iii) } L(v) \leq C\|v\|_{H^1} & \forall v \in H^1 \end{cases}$$

(Ω) 생략

이 세조건을 만족하면, ↓ 몇 수업에 prove

Let $V_h = \{v \in C_0(\Omega),\ v|_T \in P^k(T)\ \ \forall T \in \mathcal{T}_h\}$

Let $u_h \in V_h$ satisfy $a(u_h, v) = L(v)$ for all $v \in V_h$

$$\|u-u_h\|_{H^1(\Omega)} \leq C \inf_{v\in V_h} \|u-v\|_{H^1(\Omega)} \quad \text{Cea's}$$

생략

$$\leq Ch^k |u|_{H^{k+1}(\Omega)} \text{ as long as } \{\mathcal{T}_h\} \text{ are shape regular}$$

General BH result

$$|u-v|_{H^m} \leq C^{k-m+1} |u|_{H^k}$$

But, L^2 norm을 이용하면

$$\|u-u_h\|_{L^2(\Omega)} \underset{\substack{\uparrow \\ \text{not true}}}{\leq} C \inf_{v\in V_h} \|u-v\|_{L^2(\Omega)} \underset{\substack{\uparrow \\ \text{true}}}{\overset{\text{BH}}{\leq}} Ch^{k+1} |u|_{H^{k+1}(\Omega)}$$

but 비슷한 것 있음! 숙제 HW1 에서도 있음 1D case

예전에 나옴

$$\boxed{\begin{array}{l} \underline{\text{Avbin - Nitsche Lemma}} \\ \|u-u_h\|_{L^2(\Omega)} \leq Ch\, \|u-u_h\|_{H^1(\Omega)} \end{array}}$$

with assumption that Ω is convex

pf) FIRST solve a dual problem

Let $\psi \in H^1(\Omega)$ solve

$$\begin{cases} -\nabla\cdot(A\nabla\psi) + r\psi = u-u_h & \Omega \\ \psi = 0 & \partial\Omega \end{cases}$$

$$a(u,v) = (f,v) \xrightarrow{\text{dual}} a^*(w,v) = a(v,w)$$

위의 theorem이 [illegible]

모든 sufficiently smooth한 [illegible]

여기서는 interpolating [illegible]

Elliptic regularity : $\|\psi\|_{H^2(\Omega)} \le C\|u-u_h\|_{L^2(\Omega)}$

우리가 찾는 ψ가 $-\Delta\psi = f$ 이지요. 그래서 8) 와 비슷한 intuition!

$$\|u-u_h\|^2_{L^2(\Omega)} = \int_\Omega (u-u_h)\left[-\nabla\cdot(A\nabla\psi) + r\psi\right]dx = a(u-u_h,\psi)$$

$$= a(u-u_h, \psi - v) \quad \text{for any } v \in V_h$$

$$\le C\|u-u_h\|_{H^1(\Omega)}\|\psi - v\|_{H^1(\Omega)}$$

$$\Rightarrow \|u-u_h\|^2_{L^2(\Omega)} \le C\|u-u_h\|_{H^1(\Omega)} \inf_{v\in V_h}\|\psi - v\|_{H^1(\Omega)}$$

B-H Lemma

$$\le C\|u-u_h\|_{H^1(\Omega)}\, h\,\|\psi\|_{H^2(\Omega)}$$

Elliptic regularity

$$\le Ch\|u-u_h\|_{H^1(\Omega)}\|u-u_h\|_{L^2(\Omega)}$$

$$\Rightarrow \|u-u_h\|_{L^2(\Omega)} \le Ch\|u-u_h\|_{H^1(\Omega)}$$

⇒ L^2를 사용해도 비슷한 결과가 나온다! □

Plane Elasticity

$\vec{u}$: displacement

σ stresses

$$\varepsilon(\vec{u}) = \frac{\nabla\vec{u} + (\nabla\vec{u})^t}{2} \qquad \text{strain tensor}$$

where $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\nabla\vec{u} = \begin{bmatrix} \partial_x u_1 & \partial_y u_1 \\ \partial_x u_2 & \partial_y u_2 \end{bmatrix}$

Elasticity equations

$$\begin{cases} \sigma = \mu\,\varepsilon(\vec{u}) + \mu\frac{\nu}{1-2\nu}\operatorname{div} u\, I & \Omega \\ -\operatorname{div}\sigma = \vec{f} & \Omega \\ \vec{u} = 0 & \partial\Omega \end{cases}$$

μ, ν constant $\quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$\partial_i \sigma_{ij}$

$$\operatorname{div}\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} \partial_x\sigma_{11} + \partial_y\sigma_{12} \\ \partial_x\sigma_{21} + \partial_y\sigma_{22} \end{bmatrix}$$

Eliminate σ,

$$\begin{cases} -\operatorname{div}\left(\mu\,\varepsilon(\vec{u}) + \mu\frac{\nu}{1-2\nu}\operatorname{div} u\, I\right) = \vec{f} & \Omega \\ \vec{u} = 0 & \partial\Omega \end{cases}$$

$\vec{H}_0^1(\Omega) \equiv [H_0^1(\Omega)]^2$ (각 element가 $H_0^1(\Omega)$에 속한다)

$\vec{u} \in \vec{H}_0^1(\Omega)$ solves

$$a(\vec{u},\vec{v}) = \int_\Omega \vec{f}\cdot\vec{v} \qquad \forall\vec{v}\in\vec{H}_0^1(\Omega)$$

where $a(\vec{u},\vec{v}) = \int_\Omega \left(\mu\varepsilon(\vec{u}):\varepsilon(\vec{v}) + \mu\frac{\nu}{1-2\nu}\operatorname{div}\vec{u}\operatorname{div}\vec{v}\right)dx$

$$M : G = M_{11}G_{11} + M_{12}G_{12} + M_{21}G_{21} + M_{22}G_{22}$$

Natural finite element approximation

$$\vec{V}_h = \{\vec{v} \in [C_0(\Omega)]^2 : \vec{v}|_T \in [P^1(T)]^2 \quad \forall T \in \mathcal{T}_h\}$$

Find $\vec{u}_h \in \vec{V}_h$ s.t.

$$a(\vec{u}_h, \vec{v}) = \int_\Omega \vec{f}\cdot\vec{v} \qquad \forall\vec{v}\in\vec{V}_h$$

(i) - (iii) 만족하는지만 확인할 것

1) $\alpha\|u\|^2_{H^1(\Omega)} \le \mu\|\varepsilon(\vec{u})\|^2_{L^2(\Omega)} \le a(\vec{u},\vec{u})$ (coercivity)

$\boxed{\text{Korn's inequality}}$

by def

2) $a(\vec{u},\vec{v}) \le \mu \|\vec{u}\|_{H^1(\Omega)} \|\vec{v}\|_{H^1(\Omega)} + \mu \frac{\nu}{1-2\nu} \|\vec{u}\|_{H^1(\Omega)} \|\vec{v}\|_{H^1(\Omega)}$

$= \mu\left(1+\frac{\nu}{1-2\nu}\right) \|\vec{u}\|_{H^1(\Omega)} \|\vec{v}\|_{H^1(\Omega)}$

3) right-hand-size is bounded

$$\Rightarrow \|\vec{u}-\vec{u}_h\|_{H^1(\Omega)} \le \underset{\sim\sim}{C}\, h^k\, |\vec{v}|_{H^{k+1}(\Omega)}$$

depends on μ, ν

$\mu \to 0$, $1-2\nu \to 0$ 이면 문제

- $\nu \nearrow \frac{1}{2}$ $\Rightarrow$ c가 커지네

$\Rightarrow$ standard method does not do well when $\nu \nearrow \frac{1}{2}$

그래서 이런 문제가 있을 때 사용하는 새로운 방법들이 많이 생겼지요

1) Mixed methods
2) Non-conforming methods
3) Projection methods
4) Red interpolation methods
5) Discontinuous Galerkin

$$\mu \int_\Omega \varepsilon(\vec{u}):\varepsilon(\vec{v}) + \frac{\mu\nu}{1-2\nu}\int_\Omega \operatorname{div}\vec{u}\,\operatorname{div}\vec{v} = \int_\Omega \vec{f}\cdot\vec{v}$$

Define $p = \dfrac{\mu\nu}{1-2\nu}\operatorname{div}\vec{u}$

~~where's this come from?~~

$$\Rightarrow \begin{cases} \mu \int_\Omega \varepsilon(\vec{u}):\varepsilon(\vec{v}) + \int_\Omega p \operatorname{div}\vec{v} = \int_\Omega \vec{f}\cdot\vec{v} & \forall \vec{v} \in \vec{H}_0^1(\Omega) \\ \int_\Omega (\operatorname{div}\vec{u})\, q - \frac{1-2\nu}{\mu\nu}\int_\Omega p\, q = 0 & \forall q \in L^2(\Omega) \end{cases}$$

← 넌 어디서 왔니!
? define 해서 유용하고 ∫
($\nu = c$ / $\nu = 0$ 이면서 큰 문제점)

Limiting case $\nu \to \frac{1}{2}$,

$$\begin{cases} -\mu\operatorname{div}(\varepsilon(\vec{u})) - \nabla p = \vec{f} \\ \operatorname{div}(\vec{u}) = 0 \end{cases}$$

← integrate of unit [illegible]

: STOKES problem

Simplify $\rightarrow$ $\begin{cases} -\Delta\vec{u} - \nabla p = \vec{f} \\ \operatorname{div}\vec{u} = 0 \end{cases}$: saddle-point problem. (variational form으로 바로 봐야 saddle-point problem과 연관)

$$\int_\Omega \nabla\vec{u}\cdot\nabla\vec{v} + \int_\Omega p \operatorname{div}\vec{v} = \int_\Omega \vec{f}\cdot\vec{v} \quad \forall \vec{v} \in \vec{H}_0^1(\Omega)$$

$$\int_\Omega \operatorname{div}\vec{u}\, q = 0 \quad q \in L_0^2(\Omega) = \{ q \in L^2(\Omega) \quad \underset{\sim\sim\sim}{\int_\Omega q\, dx = 0} \}$$

uniqueness를 위해 필요

Stability result

Let $\vec{v} = \vec{u}$

$$\|\nabla\vec{u}\|_{L^2(\Omega)}^2 + \int_\Omega p \overset{0}{\operatorname{div}\vec{u}} = \int_\Omega \vec{f}\cdot\vec{u} \underset{\text{Cauchy-Schwartz}}{\le} \|\vec{f}\|_{L^2(\Omega)} \|\vec{v}\|_{L^2(\Omega)}$$

$$\le \|\vec{f}\|_{L^2(\Omega)}\, C\, \|\nabla\vec{u}\|_{L^2(\Omega)} \quad \text{↲ Poincare ineq.}$$

$$\Rightarrow \|\nabla\vec{u}\|_{L^2(\Omega)} \le \|\vec{f}\|_{L^2(\Omega)}.$$

f변화량이 u 변화 bound $\Rightarrow$ stable

Lemma Let $p \in L_0^2(\Omega)$

There exists $\vec{v} \in \vec{H}_0^1(\Omega)$ such that

$\operatorname{div} \vec{v} = p$ and $\|\vec{v}\|_{H^1(\Omega)} \le C\|p\|_{L^2(\Omega)}$ where C is independent of p

$$\int_\Omega p^2 dx \overset{\text{Lemma}}{=} \int_\Omega \operatorname{div}\vec{v}\, p = \int_\Omega \vec{f}\cdot\vec{v} - \int_\Omega \nabla\vec{u} : \nabla\vec{v}$$

$$\le \|\vec{f}\|_{L^2(\Omega)}\|\vec{v}\|_{L^2(\Omega)} + \|\nabla\vec{u}\|_{L^2(\Omega)}\|\nabla\vec{v}\|_{L^2(\Omega)}$$ ↲ Cauchy-Schwarz

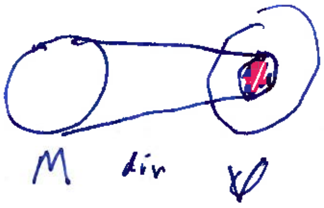
$$\le C\|\vec{f}\|_{L^2(\Omega)}\left(\|\vec{v}\|_{L^2(\Omega)} + \|\nabla\vec{v}\|_{L^2(\Omega)}\right)$$ ↲ 앞에서 prove

$$\le C\|\vec{f}\|_{L^2(\Omega)}\|\vec{v}\|_{H^1(\Omega)}$$ ↲ Lemma

$$\le C\|\vec{f}\|_{L^2(\Omega)}\|p\|_{L^2(\Omega)}$$

$$\Rightarrow \|p\|_{L^2(\Omega)} \le C\|\vec{f}\|_{L^2(\Omega)}$$

$$\Rightarrow \|\nabla u\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega)} \le C\|\vec{f}\|_{L^2(\Omega)}$$



Note ★ $\vec{H}_0^1(\Omega) \longrightarrow \vec{L}_0^2(\Omega)$

↑ div map이 onto로 mapping 합니다

roughly speaking, " $\vec{V}_h \xrightarrow[\text{div}]{} M_h$ " 이런 것을 하고 싶다, 마치 continuous case에서 했듯이

↑ velocity space ↑ pressure space

Let $\vec{V}_h \subseteq \vec{H}_0^1(\Omega)$, $M_h \subset \vec{L}_0^2(\Omega)$

FINITE ELEMENT approximation

FIND $\vec{u}_h \in \vec{V}_h$, $p_h \in M_h$ such that

$$\int_\Omega \nabla\vec{u}_h : \nabla\vec{v} + \int p_h \operatorname{div}\vec{v} = \int_\Omega \vec{f}\cdot\vec{v} \qquad \forall \vec{v} \in \vec{V}_h$$

$$\int (\operatorname{div}\vec{u}_h)\xi = 0 \qquad \xi \in M_h$$

symmetric but not positive definite

Uniqueness (uniqueness in a finite dimension $\overset{?}{\Rightarrow}$ existence)

p.f) Let $\vec{f} = 0$ $\vec{v} = \vec{u}_h$

$$\|\nabla\vec{u}_h\|_{L^2(\Omega)} + \int p_h \operatorname{div} u_h = 0.$$

$$\|\nabla\vec{u}_h\|_{L^2(\Omega)} = 0$$

$$\Rightarrow \nabla\vec{u}_h = 0$$

$\Rightarrow$ $\vec{u}_h$ = const. but vanishes at boundaries $\Rightarrow$ $\vec{u}_h \equiv 0$.

$\vec{u}_h = 0 \Rightarrow \int p_h \operatorname{div}\vec{v} = 0 \quad \forall \vec{v} \in \vec{V}_h$

If we can find $\vec{v} \in \vec{V}_h$ so that $\operatorname{div}\vec{v} = p_h$, then $p_h \equiv 0$

$\vec{V}_h \xrightarrow[\text{div}]{} M_h$ compatible condition

div map이 onto가 되어 같도록 만들고, 그러면 $p_h \equiv 0$ ⇒ uniqueness

eg) $\vec{V}_h$ = {piecewise linear} M_h = {piecewise constant}

이것도 onto map이 아니에요. difficult to show can find the counter ex

triangulation에 따라 onto가 되기도 / 안되기도.

Wrong, see next page

10/31

Stokes FINITE Elements

$$\begin{cases} -\Delta \vec{u} - \nabla p = \vec{f} & \Omega \\ \text{div } \vec{u} = 0 & \Omega \\ \vec{u} = 0 & \partial\Omega \end{cases}$$

$$\int \frac{\partial p}{\partial x} v = \int p \frac{\partial v}{\partial x}$$

$$\int \frac{\partial p}{\partial y} v = \int p \frac{\partial v}{\partial y}$$

$$\int \frac{\partial p}{\partial z} v = \int p \frac{\partial v}{\partial w}$$

$$p \in L_0^2(\Omega) = \{ \xi \in L^2(\Omega) \;\cdot\; \int_\Omega \xi \, dx = 0 \}$$

- Weak form:

$u \in H_0^1(\Omega)$, $p \in L_0^2(\Omega)$ satisfies

$$(\nabla \vec{u}, \nabla \vec{v}) + (p, \text{div } \vec{v}) = (\vec{f}, \vec{v})$$

$$(\text{div } \vec{u}, \xi) = 0 \quad \text{for all } \vec{v} \in H_0^1(\Omega) \text{ and } \xi \in L_0^2(\Omega)$$

- Finite element approximation

$\vec{u}_h \in \vec{V}_h$, $p_h \in M_h$ $\quad (\vec{V}_h \subseteq H_0^1(\Omega),\ M_h \subseteq L_0^2(\Omega))$ ← 이런 space 의 ㅌ...

(1) $(\nabla \vec{u}_h, \nabla \vec{v}) + (p_h, \text{div } \vec{v}) = (\vec{f}, \vec{v})$

(2) $(\text{div } u_h, \xi) = 0$ for all $\vec{v} \in \vec{V}_h$ and $\xi \in M_h$

- There has to be a compatible condition satisfied between two spaces $\vec{V}_h$ and M_h

FE approximation 이 (well-posed (exist & unique) / stable) 하도록 하는 condition 을 만족하는 spaces

Start with Uniqueness

Assume $\vec{f} = 0$

Let $\vec{v} = \vec{u}_h$ in (1)

$$\Rightarrow (\nabla \vec{u}_h, \nabla \vec{u}_h) + (p_h, \text{div } \vec{u}_h) = (f, \vec{u}_h) = 0$$

$$\|\nabla \vec{u}_h\|^2_{L^2(\Omega)} = -(p_h, \text{div } \vec{u}_h)$$

Let $\xi = p_h$ in (2)

$$\Rightarrow \|\nabla \vec{u}_h\|^2_{L^2(\Omega)} = 0. \Rightarrow \nabla u_h = 0 \Rightarrow \vec{u}_h = \text{const} \xrightarrow[\downarrow]{\vec{u}_h = 0 \text{ on border}} \vec{u}_h \equiv 0$$

$\vec{u}_h = 0$ implies $(\vec{p}_h, \text{div } \vec{v}) = 0$ for all $\vec{v} \in \vec{V}_h$

We are going to assume the following:

There exists a constant β such that given $\xi \in M_h$, there exists a $v \in \vec{V}_h$

$$\beta \|\xi\|_{L^2(\Omega)} \le \frac{(\xi, \text{div } \vec{v})}{\|\vec{v}\|_{H^1(\Omega)}}$$

이것이 M_h 와 $\vec{V}_h$ 를 연결하는 condition (compatibility) 이대 M_h, $\vec{V}_h$ 를 정해야 X

Let $\xi = p_h$ in assumption. Then there exists $\vec{v} \in \vec{V}_h$

$$\beta \|p_h\|_{L^2(\Omega)} \le \frac{(p_h, \text{div } \vec{v})}{\|\vec{v}\|_{H^1(\Omega)}} = 0$$

$$\Rightarrow \|p_h\|_{L^2(\Omega)} = 0$$

$$\Rightarrow p_h \equiv 0$$

Inf-sup stable spaces

saddle-point property

Def $\vec{V}_h$ and M_h are a pair of inf-sup stable spaces as long as there exists a constant $\beta > 0$ s.t.

$$\beta \|\xi\|_{L^2(\Omega)} \le \sup_{\substack{\vec{v} \in \vec{V}_h \\ \vec{v} \ne 0}} \frac{(\xi, \text{div } \vec{v})}{\|\vec{v}\|_{H^1(\Omega)}} \quad \forall \xi \in M_h$$

$$0 < \beta < \inf_{\substack{\xi \in M_h \\ \xi \ne 0}} \sup_{\substack{\vec{v} \in \vec{V}_h \\ \vec{v} \ne 0}} \frac{(\xi, \text{div } \vec{v})}{\|\xi\|_{L^2(\Omega)} \|\vec{v}\|_{H^1(\Omega)}}$$

↖ 이 statement에서 sup → max 로 바뀐 부분이 assumption과 동일
ξ∈Mh이고 finite space 이므로 max로 가도 되지만
다른 contexts 에서는 const 하기 위해 sup으로

Natural Choice of $\vec{V}_h$ & M_h

$$\begin{cases} \vec{V}_h = \{ \vec{v} \in H_0^1(\Omega) : \vec{v}|_T \in \vec{P}_{k+1}(T), \ \forall T \in \mathcal{T}_h \} \\ M_h = \{ q \in L^2(\Omega) : q|_T \in P_k(T), \ \forall T \in \mathcal{T}_h \} \\ \text{Given } q \in M_h, \ \exists \vec{v} \in \vec{V}_h \text{ s.t. } \operatorname{div} \vec{v} = q \end{cases}$$

div를 취하면 one dimension less 니까 ← 이렇게 잡으면 naturally 성립할 수 있는데..

: Not true always (not satisfy inf-sup stable spaces)

Example
$$\begin{cases} \vec{V}_h = \text{the same as above} \\ M_h = \{ m \in L^2(\Omega) \quad m|_T \in P_{k-1}(T), \ \forall T \in \mathcal{T}_h \} \end{cases}$$

$\operatorname{div} \vec{V}_h$ 되도록

← V_h를 increase 하는 것으로 볼 수 있음 이렇게 하면 ok.

지난시간 p20

Lemma Given $q \in L_0^2(\Omega)$, there exists $\vec{v} \in \vec{H}_0^1(\Omega)$ s.t. $\begin{cases} \operatorname{div} \vec{v} = q & \Omega \\ \|\vec{v}\|_{H^1(\Omega)} \le C \|q\|_{L^2(\Omega)} \end{cases}$ the constant is independent of q

Let $q \in M_h$ By Lemma, there exists $\vec{v} \in \vec{H}_0^1(\Omega)$

$$\begin{cases} \operatorname{div} \vec{v} = q & \Omega \\ \|\vec{v}\|_{H^1(\Omega)} \le C \|q\|_{L^2(\Omega)} \end{cases}$$

($\vec{v} \in \vec{V}_h$ 로 보이면 됨. but 그렇지 x, projection of $v = \pi v \in \vec{V}_h$ & satisfy 그리고...)

①. $\boxed{\text{Define}}$ $\pi \vec{v}|_T \in \vec{P}_{k+1}(T)$

s.t. $\pi\vec{v}(\vec{x}_i) = \vec{v}(x_i)$ for $i = 1, 2, 3$

(HW - Prob 2)

$$(☆)\begin{cases} \int_{e_i} (\pi\vec{v}) \cdot \vec{d} \, ds = \int_{e_i} \vec{v} \cdot \vec{d} & \text{for } i = 1, 2, 3 \quad \forall \vec{d} \in \vec{P}^{k-1}(e_i) \\ \int_T \pi\vec{v} \cdot \vec{w} \, dx = \int_T \vec{v} \cdot \vec{w} \, dx & \forall \vec{w} \in \vec{P}^{k-2}(T) \end{cases}$$

(여기에 있듯이 continuity로 증명가능)

$\Rightarrow \pi\vec{v} \in \vec{V}_h$

Part1

Condition to get π

$(\operatorname{div} \pi\vec{v} - \operatorname{div} \vec{v}, r) = 0$ for any $r \in M_h$

$$= \int_\Omega \operatorname{div}(\pi\vec{v} - \vec{v}) \, r \, dx = \sum_{T \in \mathcal{T}_h} \int_T \operatorname{div}(\pi\vec{v} - \vec{v}) \, r \, dx \underset{\substack{\uparrow \\ \text{integration by parts}}}{=} - \sum_{T \in \mathcal{T}_h} \int_T (\pi\vec{v} - \vec{v}) \overset{0}{\nabla r} \, dx + \sum_{T \in \mathcal{T}_h} \underset{\text{by } (☆)}{\int_{\partial T} (\pi\vec{v} - \vec{v}) \cdot \vec{n} \, r \, d}$$

$\Rightarrow (\operatorname{div} \pi\vec{v}, r) = (\operatorname{div} \vec{v}, r) = (q, r) \quad \forall r \in M_h$

If $r = q$,

$$\|q\|^2_{L^2(\Omega)} = (q, q) = (\operatorname{div} \pi\vec{v}, q)$$

$$\|q\|_{L^2(\Omega)} = \frac{(\operatorname{div} \pi\vec{v}, q)}{\|q\|_{L^2(\Omega)}} = \frac{(q, \operatorname{div} \pi\vec{v})}{\|\pi\vec{v}\|_{H^1(\Omega)}} \boxed{\frac{\|\pi\vec{v}\|_{H(\Omega)}}{\|q\|_{L^2(\Omega)}}}$$

Part 2

Find norm so we can bound

Assume $\dfrac{\|\pi\vec{v}\|_{H(\Omega)}}{\|q\|_{L^2(\Omega)}} \le \dfrac{1}{\beta}$ $(\beta > 0)$ ← need to prove.

Then, $\beta \|q\|_{L^2(\Omega)} \le \dfrac{(q, \operatorname{div} \pi\vec{v})}{\|\pi\vec{v}\|_{H(\Omega)}}$ (satisfy inf-sup condition)

Need to prove : $\|\pi\vec{v}\|_{H^1(\Omega)} \le \frac{1}{\beta} \|q\|_{L^2(\Omega)}$

$\|\pi\vec{v}\|_{H^1(\Omega)} \le C_1 \|\vec{v}\|_{H^1(\Omega)} \underset{\substack{\uparrow \\ \text{given}}}{\le} C_1 C \|q\|_{L^2(\Omega)}$. take $\beta = \frac{1}{C_1 C}$

이부분 증명 필요.

What's this say?

Π define에 문제!

$\vec{V}\in H_0^1(\Omega)$ 는 continuity 보장 × $\Rightarrow$ point value가 안 정해지므로

$\Rightarrow$ $\Pi\vec{V}(\vec{x_i})=\vec{V}(x_i)$ for $i=1,2,3$ 는 문제 있음

continuous 이면 이 point value 를 빗금친 부분의 average로 approximate 할 수 있겠지요 (다른 방법도 있겠지요)
$=P_i$
edge에서의 값을 average 할수도

$$\longrightarrow\ R\vec{V}(x_i)=\frac{1}{|P_i|}\int_{P_i}\vec{V}(\vec{x_i})\,dx$$

★ $\Pi\vec{V}(x_i)=R\vec{V}(x_i)$ 로 바꾸자

Clement, Scott-Zhang 등이 고안.
매우 중요! In FEM

If $\vec{x_i}\in\partial\Omega$, $R\vec{V}(\vec{x_i})=0$ 로 정의

$\Rightarrow$ 이렇게 하면 example 로 주어진 것이 inf-sup 만족

There are many other inf-sup stable spaces.

For example, $M_h\subseteq C_0(\Omega)$ [Taylor-Hood]

General Theory for stable points problems

$$\begin{cases} a(u,v)+b(p,v)=L_1(v)\\ \qquad\quad\ \ b(q,v)=L_2(q)\end{cases}$$

equation이 변하면 inf-sup condition도 변한다 우리가 한 것은 for STOKES prob.

u_h p_h가 best approximation이라는 의미!

$$\circledast\le C\left(h^{k+1}\|u\|_{H^{k+2}(\Omega)}+h^{k}\|p\|_{H^{k+1}(\Omega)}\right)$$

not balanced
바꾸면 too much work
balance 하고 [illegible]

<u>Thm</u> Let $\vec{u}\in H_0^1(\Omega)$, p solve

$$-\Delta\vec{u}-\nabla p=f\quad\Omega$$
$$\operatorname{div}\vec{u}=0\quad\partial\Omega$$

and let $\vec{u_h}\in\vec{V_h}$, $p_h\in M_h$ be ~~finite element~~ approximation where $\vec{V_h}\in M_h$ are <u>inf-sup stable</u> pair of spaces

Then. $\|\nabla(\vec{u}-\vec{u_h})\|_{L^2(\Omega)}+\|p-p_h\|_{L^2(\Omega)}\le C\inf_{\substack{\vec{v}\in\vec{V_h}\\ q\in M_h}}\left(\|\nabla(\vec{u}-\vec{v})\|_{L^2(\Omega)}+\|p-q\|_{L^2(\Omega)}\right)\le\circledast$

pf) $\|\nabla(\vec{u}-\vec{u_h})\|^2_{L^2(\Omega)}=\left(\nabla(\vec{u}-\vec{u_h}),\ \nabla(\vec{u}-\vec{u_h})\right)$

$\qquad=\left(\nabla(\vec{u}-\vec{u_h}),\ \nabla(\vec{u}-\vec{v})\right)+\left(\nabla(\vec{u}-\vec{u_h}),\ \nabla(\vec{v}-\vec{u_h})\right)^{★}$ where $\vec{v}\in\vec{V_h}$

weak form : $(\nabla\vec{u},\nabla\vec{v})+(p,\operatorname{div}\vec{v})=(\vec{f},\vec{v})$, finite-element app. : $(\nabla\vec{u_h},\nabla\vec{v})+(p_h,\operatorname{div}\vec{v})=(\vec{f},\vec{v})$

$(\operatorname{div}\vec{u},q)=0\quad\forall\vec{v}\in\vec{H}_0^1(\Omega),\ q\in L_0^2(\Omega)\qquad(\operatorname{div}\vec{u_h},q)=0\quad\forall\vec{v}\in\vec{V_h},\ q\in M_h$

(*)
$\Rightarrow\ (\nabla(\vec{u}-\vec{u_h}),\nabla\vec{v})+(p-p_h,\operatorname{div}\vec{v})=0$

$\qquad(\operatorname{div}(\vec{u}-\vec{u_h}),q)=0$ for all $\vec{v}\in\vec{V_h}$, $q\in M_h$

$\Rightarrow^{★}\ (\nabla(\vec{u}-\vec{u_h}),\nabla(\vec{v}-\vec{u_h}))=-(p-p_h,\operatorname{div}(\vec{v}-\vec{u_h}))$ since $\vec{v}-\vec{u_h}\in\vec{V_h}$

$\qquad=-(p-p_h,\operatorname{div}(\vec{v}-\vec{u}))-(p-p_h,\operatorname{div}(\vec{u}-\vec{u_h}))$

$\qquad=-(p-p_h,\operatorname{div}(\vec{v}-\vec{u}))-(p-q,\operatorname{div}(\vec{u}-\vec{u_h}))\quad\forall q\in M_h$

$\Rightarrow\ \|\nabla(\vec{u}-\vec{u_h})\|^2_{L^2(\Omega)}\le\|\nabla(\vec{u}-\vec{u_h})\|_{L^2(\Omega)}\|\nabla(\vec{u}-\vec{v})\|_{L^2(\Omega)}+\|p-p_h\|_{L^2(\Omega)}\|\operatorname{div}(\vec{v}-\vec{u})\|_{L^2(\Omega)}$
$\qquad+\|p-q\|_{L^2(\Omega)}\|\operatorname{div}(\vec{u}-\vec{u_h})\|_{L^2(\Omega)}$

Assume for a second, $\|p-p_h\|_{L^2(\Omega)}\le C\|\nabla(\vec{u}-\vec{u_h})\|_{L^2(\Omega)}+C\|p-q\|_{L^2(\Omega)}$ (**)

Then, $\|\nabla(\vec{u}-\vec{u_h})\|^2_{L^2(\Omega)}\le C\left(\|\nabla(\vec{u}-\vec{v})\|_{L^2(\Omega)}+\|p-q\|_{L^2(\Omega)}\right)\quad\forall\vec{v}\in V_h,\ q\in M_h$

(∵) $\sqrt{a^2+b^2+c^2+d^2}\le C(a+b+c+d)$ 이라는 점이 나타남. 해보면

$$-(\nabla(\vec{u}-\vec{u_h}), \nabla\vec{v}) \quad \text{from } (*) \text{ p23}$$

$$\|p-p_h\|_{L^2(\Omega)} \le \|p-q\|_{L^2(\Omega)} + \|q-p_h\|_{L^2(\Omega)}$$

$\exists\, \vec{v} \in \vec{V_h}$ so that

$$\beta\|q-p_h\|_{L^2(\Omega)} \underset{\overset{\uparrow}{\text{inf-sup}}}{\le} \frac{(q-p_h, \operatorname{div}\vec{v})}{\|\vec{v}\|_{H^1(\Omega)}} = \frac{(q-p, \operatorname{div}\vec{v})}{\|\vec{v}\|_{H^1(\Omega)}} + \frac{(p-p_h, \operatorname{div}\vec{v})}{\|\vec{v}\|_{H^1(\Omega)}}$$

$$\underset{\text{Cauchy-Schwartz}}{\le} \|q-p\|_{L^2(\Omega)} \frac{\|\operatorname{div}\vec{v}\|_{L^2}}{\|\vec{v}\|_{H^1(\Omega)}} + \|\nabla(\vec{u}-\vec{u_h})\|_{L^2(\Omega)} \frac{\|\nabla(\vec{v})\|_{L^2(\Omega)}}{\|\vec{v}\|_{H^1(\Omega)}}$$

$$\le \|q-p\|_{L^2(\Omega)} + \|\nabla(\vec{u}-\vec{u_h})\|_{L^2(\Omega)}$$

$$\Rightarrow \|p-p_h\|_{L^2(\Omega)} \le \left(1+\frac{1}{\beta}\right)\|p-q\|_{L^2(\Omega)} + \frac{1}{\beta}\|\nabla(\vec{u}-\vec{u_h})\|_{L^2(\Omega)}$$

$\rightarrow$ (**) in p23

(항상 이와 같이 ① u의 bound를 찾고 ② inf-sup으로 p를 bound) ← 이 순서대로!

<u>Back to Elasticity</u>

$$(\varepsilon(\vec{u}), \varepsilon(\vec{v})) + \frac{1}{\lambda}(\operatorname{div}\vec{u}, \operatorname{div}\vec{v}) = (\vec{f}, \vec{v})$$

Define $p = \frac{1}{\lambda}\operatorname{div}(\vec{u})$

$$(\varepsilon(\vec{u}), \varepsilon(\vec{v})) + (p, \operatorname{div}\vec{u}) = (\vec{f}, \vec{v})$$

$$(\operatorname{div}\vec{u}, q) - \underbrace{\lambda(p, q)}_{} = 0.$$

$\lambda \to 0$ then STOKES problem

Let $\vec{V_h}$ and M_h be a stable inf-sup pair

Find $\vec{u_h} \in \vec{V_h}$, $p_h \in M_h$ satisfying

$$(\varepsilon(\vec{u_h}), \varepsilon(\vec{v})) + (p_h, \operatorname{div}\vec{u_h}) = (\vec{f}, \vec{v}) \quad \text{— (***)}$$

$$(\operatorname{div}\vec{u_h}, q) - \underbrace{\lambda(p_h, q)}_{} = 0 \qquad \forall v \in \vec{V_h},\ q \in M_h$$

이 term이 있지만 같은 결과를 얻을 수 있음

$$\Rightarrow \|\nabla(\vec{u}-\vec{u_h})\|_{L^2(\Omega)} + \|p-p_h\|_{L^2(\Omega)} \le \underbrace{C}_{\text{independent of } \lambda} \inf\left(\|\nabla(u-v)\|_{L^2} + \|p-q\|_{L^2}\right)$$

[illegible] 나올거야

여러 method가 있는데 다 똑같아요...

$$(\operatorname{div}\vec{u_h} - \lambda p_h, q) = 0 \quad \text{for all } q \in M_h$$

Define $\mathbb{P} : L_0^2(\Omega) \longrightarrow M_h$ s.t. $(\mathbb{P}r, q) = (r, q)$ $\forall q \in M_h$ (L^2 projection)

$$\Rightarrow \mathbb{P}(\operatorname{div}\vec{u_h} - \lambda p_h) = 0$$

$$\Rightarrow \mathbb{P}(\operatorname{div}\vec{u_h}) = \mathbb{P}(\lambda p_h) = \lambda p_h$$

$$(***) \Rightarrow (\varepsilon(\vec{u_h}), \varepsilon(\vec{v})) + \frac{1}{\lambda}(\mathbb{P}\operatorname{div}\vec{u_h}, \mathbb{P}\operatorname{div}\vec{v}) = (\vec{f}, \vec{v}) \quad \forall v \in \vec{V_h}$$

and $\mathbb{P}$ is the L^2 projection onto M_h and $\vec{V_h}$ and M_h is a stable inf-sup pair.

Projection method

$$(\varepsilon(\vec{u_h}), \varepsilon(\vec{v})) + \frac{1}{\lambda}(\operatorname{div} u_h, \operatorname{div} v) = 0.$$

$$\vec{V_h} = \{\vec{v} \in \vec{V_h},\ \vec{v} \in P(T)\}$$

$$(\varepsilon(\vec{u_h}), \varepsilon(\vec{v})) + \frac{1}{\lambda}\sum_{T\in T_h}\int_T (\operatorname{div} u_h)(\operatorname{div}\vec{v})\,dx = (\vec{f}, \vec{v})$$

$$(\varepsilon(\vec{u_h}), \varepsilon(\vec{v})) + \frac{1}{\lambda}\sum_{T\in T_h}|T|(\operatorname{div} u_h)(m_T)\operatorname{div}\vec{v}(m_T) = (\vec{f}, \vec{v})$$

midpoint

midpoint를 쓰면 underintegration

not correct 하게 work

L² projection 이용 했다 비슷.

Underintegration method

$$\begin{cases} -\Delta u = f & \Omega \\ u = 0 & \partial\Omega \end{cases}$$

Ω is a polygon.

Let $\{\mathcal{T}_h\}_h$ is a family of triangulations of Ω.

So far, we have only considered methods such that $V_h \subseteq H_0^1(\Omega)$

Can we have ~~that~~ a method s.t. $V_h \nsubseteq H_0^1(\Omega)$? Yes! (지금부터...)

Why's inf-sup important

Stokes

stability & error analysis?

controling p w/ v

etc

쿠제 라비아

Example of a non-conforming method (Croueix - Raviar 1973)

Let $\mathcal{E}_h$ = the collection of edges of $\mathcal{T}_h$

$\mathcal{E}_h = \mathcal{E}_h^I \cup \mathcal{E}_h^B$

where $\mathcal{E}_h^I$ = all interior edges of $\mathcal{T}_h$

$\mathcal{E}_h^B$ = all boundary edges of $\mathcal{T}_h$

C-R space $V_h = \{ v \in L^2(\Omega) : v|_T \in P_1(T) \ \forall T \in \mathcal{T}_h$

v is continuous at midpoint of all edges $e \in \mathcal{E}_h^I$

and $v = 0$ at midpoint of all edges $e \in \mathcal{E}_h^B \}$

(#(edges) > #(vertex) : Euler 공식에서...)

⇒ V_h의 dof이 edge # 라서 vertex 값을 dof 결정하는 것으로 가지는 것보다 dof이 크다.

$v^+ = v|_{T^+}$, $v^- = v|_{T^-}$

$$v^+(x_m) = v^-(x_m) \ \underset{?}{\Longleftrightarrow} \ \int_e v^+ ds = \int_e v^- ds$$

Yes

$$\overset{\parallel}{m(e)\, v^+(x_m)} \qquad \overset{\parallel}{m(e)\, v^-(x_m)}$$ since $v|_T \in P_1(T)$ linear 라서 midpoint가 exact value

Non-conforming approximation $u_h \in V_h$ satisfies

$$a(u_h, v) = (f, v) \qquad \forall v \in V_h$$

(derivative  across triangles

gradient

이런 것들 생각할 수 X ⇒ $a(u_h, v)$의 새로운 정의 필

where $a(u_h, v) = \int_\Omega \nabla_h u_h \cdot \nabla_h v \, dx$, $(f, v) = \int_\Omega f v \, dx$

$$\nabla_h v|_T = [\nabla(\underset{\sim}{v|_T})]|_T$$

내부에서 ∇을 구한 것

* The method is well-posed (existence, uniqueness, the sol depends continuously on the data)

The system defining u_h is a square finite dimensional system. ⇐ ?

Therefore, it is enough to show uniqueness

Let $f \equiv 0$, we need to show ~~that~~ $u_h \equiv 0$

Let $v = u_h$, $a(u_h, u_h) = 0$

$$\int_\Omega \nabla_h u_h \cdot \nabla_h u_h = 0 \Rightarrow \nabla u_h = 0$$

$u_h|_T$ = constant $\forall T \in \mathcal{T}_h$

⇒ $u_h \equiv C$ on Ω by the continuity at midpoint

⇒ $u_h \equiv 0$ since u_h vanishes at midpoint of the boundary

Note. uniqueness 에서는 midpoint에서 continuity를 가질 필요 X. 이는 임의의 점에서 continuity를 주어도 ok

→ accuracy에 영향

Inconsistency

Let $u \in H_0^1(\Omega)$ solve $-\Delta u = f$

Does it follow that

$$a(u,v) = (f,v) \qquad \forall v \in V_h$$

→ Not true (was true when $V_h \subseteq H_0^1(\Omega)$)

let $v \in V_h$

$$\int_\Omega -\Delta u\, v\, dx = \int_\Omega f v\, dx$$

$$\sum_{T \in \mathcal{T}_h} \left(-\int_T \Delta u\, v\, dx\right) = \int_\Omega f v\, dx$$

$$-\int_T \Delta u\, v\, dx = \int_T \nabla u \cdot \nabla v - \int_{\partial T} \frac{\partial u}{\partial n} v\, ds$$

$$\to \sum_{T \in \mathcal{T}_h} \left\{ \int_T \nabla_h u \cdot \nabla_h v\, dx - \int_{\partial T} \frac{\partial u}{\partial n} v\, ds \right\} = \int_\Omega f v\, dx$$

$$a(u,v) - \underbrace{\sum_{T \in \mathcal{T}_h} \int_{\partial T} \frac{\partial u}{\partial n} v\, ds}_{\text{additional term}} = (f,v)$$

inconsistency but not important…

variational CRIME.

Error Analysis

$$\|\nabla_h(u-u_h)\|^2_{L^2(\Omega)} = a(u-u_h,\, u-u_h)$$

Galerkin orthogonality는 zero 였는데 여기서는 적용 X

$$= a(u-u_h,\, u-v) + a(u-u_h,\, v-u_h) \quad \text{for any } v \in V_h$$

$$= a(u-u_h,\, u-v) + a(u,\, v-u_h) - a(u_h,\, v-u_h)$$

$$= a(u-u_h,\, u-v) + a(u,\, v-u_h) - (f,\, v-u_h) \quad \text{since } v-u_h \in V_h$$

$$\le |a(u-u_h,\, u-v)| + \frac{|a(u,v-u_h) - (f,v-u_h)|}{\|\nabla_h(v-u_h)\|_{L^2(\Omega)}} \|\nabla_h(v-u_h)\|_{L^2(\Omega)}$$

$$\le |a(u-u_h,\, u-v)| + \underbrace{\sup_{w \in V_h} \frac{|a(u,w) - (f,w)|}{\|\nabla_h w\|_{L^2(\Omega)}}}_{I} \|\nabla_h(v-u_h)\|_{L^2(\Omega)}$$

Cauchy-Schwartz

$$\le \|\nabla_h(u-u_h)\|_{L^2(\Omega)} \|\nabla_h(u-v)\|_{L^2(\Omega)} + I\left(\|\nabla_h(v-u)\|_{L^2(\Omega)} + \|\nabla_h(u-u_h)\|_{L^2(\Omega)}\right)$$

$$\le \|\nabla_h(u-u_h)\|_{L^2(\Omega)}\left(\|\nabla_h(u-v)\|_{L^2(\Omega)} + I\right) + I\|\nabla_h(v-u)\|_{L^2(\Omega)}$$

$2ab \le a^2+b^2$

$$\le \frac{1}{2}\|\nabla_h(u-u_h)\|^2_{L^2(\Omega)} + \frac{1}{2}\left(\|\nabla_h(u-v)\|_{L^2(\Omega)} + I\right)^2 + I\|\nabla_h(v-u)\|_{L^2(\Omega)}$$

$$\frac{1}{2}\|\nabla_h(u-u_h)\|^2_{L^2(\Omega)} \le \frac{1}{2}\left(\|\nabla_h(u-v)\|_{L^2(\Omega)} + I\right)^2 + I\|\nabla_h(v-u)\|_{L^2(\Omega)}$$

$$\le \|\nabla_h(u-v)\|^2_{L^2(\Omega)} + I^2 + I\|\nabla_h(v-u)\|_{L^2(\Omega)} \le \left(\|\nabla_h(u-v)\|_{L^2(\Omega)} + I\right)^2$$

$\frac{1}{2}(a+b)^2 \le a^2+b^2$

$$\|\nabla_h(u-u_h)\|_{L^2(\Omega)} \le \sqrt{2}\,\|\nabla_h(u-v)\|_{L^2(\Omega)} + \sqrt{2}\, I \quad \text{for any } v \in V_h$$

$$\le \sqrt{2} \inf_{v \in V_h} \|\nabla_h(u-v)\|_{L^2(\Omega)} + \sqrt{2}\, I$$

was zero for conforming methods

이 term이 conforming에서 없어지면서 how non-conforming

HW) One can show that $\inf_{v \in V_h} \|\nabla_h(u-v)\|_{L^2(\Omega)} \le Ch|u|_{H^2(\Omega)}$ 예전 approximation properties 를 이용 but with new V_h

Also we will show

$$I \le Ch|u|_{H^2(\Omega)}$$

Estimate for I

Let $w \in V_h$

$$|a(u,w) - (f,w)| = \Big| \sum_{T\in\mathcal{T}_h} \int_{\partial T} \frac{\partial u}{\partial n} w \, ds \Big|$$

If w is a scalar function

w define the jump on an edge $e = T^+ \cap T^-$

$$[\![w]\!] = w|_{T^+} n^+ + w|_{T^-} n^-$$
$$= n^+ (w|_{T^+} - w|_{T^-})$$

$$\sum_{T\in\mathcal{T}_h} \int_{\partial T} \frac{\partial u}{\partial n} w \, ds = \sum_{e\in\mathcal{E}_h^I} \int \underset{\sim\sim}{\{\nabla u\}} \cdot [\![w]\!] \, ds + \sum_{e\in\mathcal{E}_h^B} \nabla u \cdot (w\vec{n}) \, ds$$

average of ∇u $\quad (\{\vec{\xi}\}|_e = \frac{1}{2} (\vec{\xi}|_{T^+} + \vec{\xi}|_{T^-}))$

$$\left(\int_{\partial T^+} \frac{\partial u}{\partial n} w + \int_{\partial T^-} \frac{\partial u}{\partial n} w = \int_e \nabla u \cdot n^+ w|_{T^+} + \int_e \nabla u \cdot n^- w|_{T^-} \right.$$
$$\left. = \int_e \nabla u \cdot (n^+ w|_{T^+} + n^- w|_{T^-}) + \text{other edges} \right)$$

Since $w \in V_h$, $\int_e \vec{c} \cdot [\![w]\!] \, ds = 0$ for all $e \in \mathcal{E}_h^I$ where $\vec{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

(midpoint가 같은 것 = jump를 integral 하면 0)

$$\sum_{T\in\mathcal{T}_h} \int_{\partial T} \frac{\partial u}{\partial n} w \, ds = \sum_{e\in\mathcal{E}_h^I} \int_e \{\nabla u - \nabla_h v\} \; [\![w]\!] \, ds + \sum_{e\in\mathcal{E}_h^B} \int_e \nabla_h (u - v) \cdot w\vec{n} \, ds \qquad v \in V_h$$

$$\sum_{e\in\mathcal{E}_h^I} \int_e \{\nabla u - \nabla_h v\} [\![w]\!] \, ds \le \sum_{e\in\mathcal{E}_h^I} \| \nabla u - \nabla_h v \|_{L^2(e)} \| [\![w]\!] \|_{L^2(e)}$$

Recall Trace Inequality : $\|\phi\|_{L^2(\partial\hat{T})} \le C |\phi|_{H^1(\hat{T})}$ for $\phi \in H^1(\hat{T})$ $\quad \hat{T}$: the reference triangle

★
$$\Rightarrow \|\phi\|_{L^2(\partial T)} \le C \left(\frac{1}{h_T^{\frac{1}{2}}} \|\phi\|_{L^2(T)} + h_T^{\frac{1}{2}} | \nabla \phi |_{\underset{\uparrow}{H^1(T)}} \right)$$

very important! in D-G method

(◎) check $|\cdot|$ semi norm)

$$\| \{ \nabla_h (u - v) \} \|_{L^2(e)} = \| \tfrac{1}{2} \nabla_h (u - v)|_{T^+} + \tfrac{1}{2} \nabla_h (u - v)|_{T^-} \|_{L^2(e)} \qquad \text{where } e = T^+ \cap T^-$$
$$\le \tfrac{1}{2} \| \nabla_h (u - v)|_{T^+} \|_{L^2(e)} + \tfrac{1}{2} \| \nabla_h (u - v)|_{T^-} \|_{L^2(e)}$$
$$\le \tfrac{1}{2} \| \nabla_h (u - v) \|_{L^2(\partial T^+)} + \tfrac{1}{2} \| \nabla_h (u - v) \|_{L^2(\partial T^-)}$$
$$\overset{\text{Trace Ineq.}}{\le} C \left(\frac{1}{h_T^{\frac{1}{2}}} \| \nabla_h (u - v) \|_{L^2(T^+ \cup T^-)} + h_T^{\frac{1}{2}} | \nabla_h (u) |_{H^1(T^+ \cup T^-)} \right)$$

$\| [\![w]\!] \|_{L^2(e)}$ $\qquad$ Poincare & Inverse inequality

$\le \underset{\uparrow}{c h_e} \| D_e [\![w]\!] \|_{L^2(e)}$ where D_e is the gradient on edge e

reference domain → original triangle 할 때 나오는 term

$h_e = |e|$ $\qquad$ $\left(\begin{array}{ll} h_e \le h_{T^+} , \; h_e \le h_{T^-} & \text{by shape regularity} \\ h_{T^+} \le C h_e \quad h_{T^-} \le C h_e & \end{array} \right)$

$$\le C h_e \| D_e (w|_{T^+}) n^+ + D_e (w|_{T^-}) n^- \|_{L^2(e)}$$
$$\le C h_e \left(\| D_e (w|_{T^+}) n^+ \|_{L^2(\partial T^+)} + \| D_e (w|_{T^-}) n^- \|_{L^2(\partial T^-)} \right)$$
$$\le C h_e \| \nabla_h w \|_{L^2(\partial T^+)} + C h_e \| \nabla_h w \|_{L^2(\partial T^-)}$$
$$\underset{\text{trace}}{\le} C h_e \left(\frac{1}{h_e^{\frac{1}{2}}} \| \nabla_h w \|_{L^2(T^+ \cup T^-)} + h_e^{\frac{1}{2}} | \nabla_h u |_{H^{\circ}(T^+ \cup T^-)} \right) = C h_e^{\frac{1}{2}} \| \nabla w \|_{L^2(T^+ \cup T^-)}$$

$$\sum_{e\in\mathcal{E}_h^I}\int_e \{\nabla_h(u-v)\}[\![w]\!]\,ds \le \sum_{T\in\mathcal{T}_h}\left(\|\nabla_h(u-v)\|_{L^2(T)}\|\nabla_h w\|_{L^2(T)} + h|u|_{H^2(T)}\|\nabla_h w\|_{L^2(T)}\right)$$

$$\overset{\text{true}}{\le} \left(\|\nabla_h(u-v)\|_{L^2(\Omega)} + |u|_{H^2(\Omega)}\right)\|\nabla_h w\|_{L^2(\Omega)}$$

$$\|\nabla_h(w)\|_{L^2(T^+\cup T^-)}\|\nabla_h(u-v)\|_{L^2(T^-\cup T^+)} \le \left(\|\nabla_h(w)\|_{L^2(T^+)} + \|\nabla_h w\|_{L^2(T^-)}\right)\left(\|\ \ \|_{L^2(T^+)} + \|\ \ \|_{L^2(T^-)}\right)$$

이건 진실이 아니야. what the

- $$\sum_{e\in\mathcal{E}_h}\left(\|\nabla_h(u-h)\|_{L^2(T^+\cup T^-)} + |u|_{H^2(T^+\cup T^-)}\right)\|\nabla_h w\|_{L^2(T^+\cup T^-)}$$

$$\overset{?}{\le} \left(\sum_{e\in\mathcal{E}_h}\left(\|\nabla_h(u-v)\|_{L^2(T^+\cup T^-)} + |u|_{H^2(T^+\cup T^-)}\right)^2\right)^{\frac{1}{2}}\left(\sum_{e\in\mathcal{E}_h}\|\nabla_h(w)\|^2_{L^2(T^+\cup T^-)}\right)^{\frac{1}{2}}$$

- $$\sum_{e\in\mathcal{E}_h}\|\nabla_h(w)\|^2_{L^2(T^-\cup T^+)} = \sum_{e\in\mathcal{E}_h}\int_{T^+\cup T^-}(\nabla_h w)^2dx \le 3\sum_{T\in\mathcal{T}_h}\int_T(\nabla_h w)^2dx$$
$$= \sum_{e\in\mathcal{E}_h}\left(\int_{T^+}(\nabla_h w)^2 + \int_{T^-}(\nabla_h w)^2\right)$$

$$\left|\sum_{T\in\mathcal{T}_h}\int_T\frac{\partial u}{\partial n}w\,ds\right| \le \left(\|\nabla_h(u-v)\|_{L^2(\Omega)} + h|u|_{H^2(\Omega)}\right)\|\nabla_h w\|_{L^2(\Omega)}$$

$$\sup_{w\in V_h}\frac{\left|\sum_{T\in\mathcal{T}_h}\int_T\frac{\partial u}{\partial n}w\,ds\right|}{\|\nabla_h w\|_{L^2(\Omega)}} \le c\left(\|\nabla_h(u-v)\|_{L^2(\Omega)} + h|u|_{H^2(\Omega)}\right) \quad \text{for any } v\in V_h$$

$$\le c\inf_{v\in V_h}\|\nabla_h(u-v)\|_{L^2(\Omega)} + ch|u|_{H^2(\Omega)}$$

$$\Rightarrow \|\nabla_h(u-u_h)\|_{L^2(\Omega)} \le \sqrt{2}\inf\|\nabla_h(u-v)\|_{L^2(\Omega)} + \sqrt{2}\,\mathrm{I}$$
$$\le C\inf_{v\in V_h}\|\nabla_h(u-v)\|_{L^2(\Omega)} + ch|u|_{H^2(\Omega)} \le Ch|u|_{H^2(\Omega)}$$

우리가 보이려한 final goal

복습하세요....

여기서는 $V_h \not\subset H_0^1(\Omega)$ 에서 inconsistency 발생.

실제 numerical implementation 에서는 numerical integration 에서도 inconsistency 발생.

accuracy에 영향을 주지 않으려면 integration을 얼마나 정확히 해야하나 고려해서 integration 해야한다.

History

<Elliptic problem>

G. Baker (Biharmonic 70's)

|

Doolas Arnold (interior penalty method 79')

<Transport problem>

(Los Alamos and 70's)

cockburn - Lo Foch 80's

Cockburn - Shu 80's

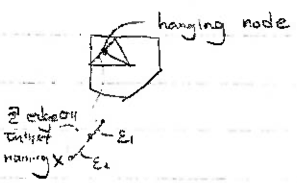
elliptic transport

← $-\varepsilon\Delta u + \beta\cdot\nabla u$ 둘 다 포함

Local DG [Cockburn - Shu early 90's]

In class

Many DG methods : "Interior penalty methods," Local discontinuous Galerkin method, ...



$\mathcal{T}_h$: a triangulation

$\mathcal{E}_h$: the collection of edges.

$$\mathcal{E}_h = \mathcal{E}_h^B \cup \mathcal{E}_h^I$$

where $\mathcal{E}_h^I$ interior edges

$\mathcal{E}_h^B$ boundary edges

Let $\vec{V}$ be a vector

$$[\![\vec V]\!]|_e = \vec{V_+}\cdot\vec{n_+} + \vec{V_-}\cdot\vec{n} \qquad e = T_+\cap T_-$$

$$\{\vec V\}|_e = \tfrac12\vec{V_+} + \tfrac12\vec{V_-} \qquad e\in\mathcal{E}_h^I$$
$$= \vec V \qquad e\in\mathcal{E}_h^B$$

Let v be a scalar

$$[\![v]\!]|_e = \begin{cases} v_+\vec{n_+} + v_-\vec{n} & e\in\mathcal{E}_h^I \\ v\vec{n} & e\in\mathcal{E}_h^B \end{cases}$$

$$\{v\}|_e = \begin{cases} \tfrac12 v_+ + \tfrac12 v_- & e\in\mathcal{E}_h^I \\ v & e\ \mathcal{E}_h^B \end{cases}$$

$$H^2(\mathcal{T}_h) = \{v\in L^2(\Omega) : v|_T\in H^2(T),\ \forall T\in\mathcal{T}_h\}$$

DG space :

$$V_h^k = \{v\in L^2(\Omega) : v|_T\in P^k(T),\ \forall T\in\mathcal{T}_h\} \subseteq H^2(\mathcal{T}_h)$$

Problem : $-\Delta u = f \quad \Omega$

$u = 0 \quad \partial\Omega$

What if $v\in H^2(\mathcal{T}_h)$ is my test ftn

$$\int_\Omega -\Delta u\cdot v = \int_\Omega f\cdot v$$

바로 integration by parts $\int_\Omega$ 에서 불가능! because of discontinuity of v.

$$-\int_\Omega \Delta u\cdot v = \sum_{T\in\mathcal{T}_h} -\int_T \Delta u\cdot v\,dx = \sum_{T\in\mathcal{T}_h}\left\{\int_T \nabla u\cdot\nabla v\,dx - \int_{\partial T}\nabla u\cdot\vec n\, v\,ds\right\}$$
$$= (\nabla_h u, \nabla_h v) - \sum_{T\in\mathcal{T}_h}\int_{\partial T}\nabla u\cdot\vec n\, v\,ds$$

defined in the last class

$$\sum_{T\in\mathcal{T}_h}\int_{\partial T}(\nabla u\,\vec n)v\,ds = \sum_{e\in\mathcal{E}_h}\int_e \{\nabla u\}\cdot[\![v]\!]\,ds + \sum_{e\in\mathcal{E}_h^I}\int_e [\![\nabla u]\!]\{v\}\,ds$$
$$= \sum_{e\in\mathcal{E}_h}\int_e \tfrac12(\nabla u_+ + \nabla u_-)\cdot(v_+\vec{n_+} + v_-\vec{n_-}) + \sum_{e\in\mathcal{E}_h}\int_e(\nabla u_+ n_+ + \nabla u_- n_-)\tfrac12(v_+ + v_-)\,ds + \cdots \sum_{e\in\mathcal{E}_h^B}$$

$$\int_e(\nabla u_+\cdot\vec{n_+}\,v_+ + \nabla u_-\cdot\vec{n_-}\,v_-)\,ds$$

e가 나뉘는 부분

equal

$$\tfrac12(\nabla u_+\cdot\vec{n_+}v_+ + \nabla u_-\cdot\vec{n_-}v_- + \nabla u_+\cdot\vec{n_-}v_- + \nabla u_-\cdot\vec{n_+}v_+)$$
$$+\tfrac12(\nabla u_+\vec{n_+}v_+ + \nabla u_+\vec{n_+}v_- + \nabla u_-\cdot\vec{n_-}v_+ + \nabla u_-\cdot\vec{n_-}v_-)$$

ground system ??

$$\underset{\overset{\parallel}{(f,v)}}{-\int \Delta u \cdot v} = (\nabla_h u, \nabla_h v) - \langle \{\nabla_h u\}, [\![v]\!] \rangle_{\mathcal{E}_h} - \langle [\![\nabla u]\!], \{v\} \rangle_{\mathcal{E}_h^I} \quad \forall v \in H^2(\mathcal{T}_h)$$

$$\langle \vec{M}, \vec{g} \rangle_{\mathcal{E}_h} = \sum_{e \in \mathcal{E}_h} \int_e \vec{M}\, \vec{g}\, ds.$$

$$\langle r, s \rangle_{\mathcal{E}_h} = \sum_{e \in \mathcal{E}_h} \int_e rs\, ds$$

$$\tilde{a}_h(u,v) = (\nabla_h u, \nabla_h v) - \langle \{\nabla_h u\}, [\![v]\!] \rangle_{\mathcal{E}_h} - \langle [\![\nabla u]\!], \{v\} \rangle_{\mathcal{E}_h^I}$$

$\Rightarrow \tilde{a}_h(u,v) = (f,v) \quad \forall v \in H^2(\mathcal{T}_h)$ since $u \in H^2$ by assumption $[\![\nabla u]\!] = 0$

positive definite 를 보이기 어렵다

$$\tilde{a}_h(u,v) = (\nabla_h u, \nabla_h v) - \langle \{\nabla_h u\}, [\![v]\!] \rangle_{\mathcal{E}_h} + \langle \{\nabla v\}, [\![u]\!] \rangle_{\mathcal{E}_h} + \langle [\![u]\!], [\![v]\!] \rangle$$

$\tilde{a}_h(v,v) = (\nabla_h v, \nabla_h v) + \langle [\![v]\!], [\![v]\!] \rangle \Rightarrow$ positive definite

not symmetric $\Rightarrow$ CG convergence 가 느리다.

$\frac{\alpha}{h}$를 추가하면 Nonsymmetric IP

Interior Penalty method :

$$a_h(u,v) = (\nabla_h u, \nabla_h v) - \langle \{\nabla_h u\} [\![v]\!] \rangle_{\mathcal{E}_h} - \langle \{\nabla_h v\}, [\![u]\!] \rangle + \frac{\alpha}{h} \langle [\![u]\!], [\![v]\!] \rangle_{\mathcal{E}_h}$$

We assume for simplicity that we have a quasi-linear mesh.

i.e. $\exists$ a constant C_a s.t. $\max_{T \in \mathcal{T}_h} h_T = h \le C_a h_T \quad \forall T \in \mathcal{T}_h$

α is a parameter to be chosen by the user. (should be large enough to be PD)

But $\alpha \uparrow \Rightarrow$ ill-posed, closed to continuous Galerkin.

Find $u_h \in V_h^k$ s.t. $a_h(u_h, v) = (f, v) \quad \forall v \in V_h^k$

Preliminaries

- Trace inequality :

  If $v \in H^1(T)$, $\quad \|v\|_{L^2(\partial T)} \le C_{tr} \left(\frac{1}{h^{\frac{1}{2}}} \|v\|_{L^2(T)} + h^{\frac{1}{2}} \|\nabla v\|_{L^2(T)} \right)$

- Inverse estimate :

  1) If $v \in P^k(T)$, $\quad \|\nabla v\|_{L^2(T)} \le \frac{C_1}{h} \|v\|_{L^2(T)}$

  finite dimension

  2) $\quad \|v\|_{L^2(\partial T)} \le C_{inv} \frac{1}{h^{\frac{1}{2}}} \|v\|_{L^2(T)}$

Coercivity

Lemma $\exists \beta > 0$ such that $\beta \|v\|^2_{H_h^1(\Omega)} \le a_h(v,v) \quad \forall v \in V_h^k$ provided α is large enough.

Here $\|v\|^2_{H_h^1(\Omega)} = (\nabla_h v, \nabla_h v) + h \langle \{\nabla v\}, \{\nabla v\} \rangle_{\mathcal{E}_h} + \frac{1}{h} \langle [\![v]\!], [\![v]\!] \rangle_{\mathcal{E}_h}$

pf) Let $v \in V_h^k$

$$a_h(v,v) = (\nabla_h v, \nabla_h v) - 2 \langle \{\nabla_h v\}, [\![v]\!] \rangle_{\mathcal{E}_h} + \frac{\alpha}{h} \langle [\![v]\!], [\![v]\!] \rangle_{\mathcal{E}_h}$$

By Cauchy-Schwartz.

$$2 \langle \{\nabla_h v\}, [\![v]\!] \rangle_{\mathcal{E}_h} \le \sqrt{h \langle \{\nabla_h v\} \{\nabla_h v\} \rangle_{\mathcal{E}_h}} \sqrt{\langle [\![v]\!], [\![v]\!] \rangle_{\mathcal{E}_h} \frac{1}{h}}$$

$$h \langle \{\nabla_h v\} \{\nabla_h v\} \rangle_{\mathcal{E}_h} = h \sum_{e \in \mathcal{E}_h} \int_e \{\nabla v\}^2 ds \le \sum_{e \in \mathcal{E}_h} \int_e \frac{1}{2} C_{inv}^2 \|\nabla_h v\|^2_{L^2(M(e))} \le \frac{1}{2} N C_{inv}^2 \sum_{T \in \mathcal{T}_h} \|\nabla_h v\|^2_{L^2(T)} = \frac{1}{2} N C_{inv}^2 (\nabla_h v, \nabla_h v$$

For each edge e, let $M(e)$ be the triangles that have e as an edge

N_T = the number of edges T has from $\mathcal{E}_h$

$N = \max_{T \in \mathcal{T}_h} N_T$

$$a_h(v,v) \geq (\nabla_h v, \nabla_h v) - \underbrace{\sqrt{\frac{1}{h}\langle [\![\nabla_h v]\!], [\![\nabla_h v]\!]\rangle_{\mathcal{E}_h}}\sqrt{\frac{1}{2}NC_{inv}^2}}_{b}\underbrace{\sqrt{(\nabla_h v, \nabla_h v)}}_{a} + \frac{\alpha}{h}\langle [\![v]\!], [\![v]\!]\rangle_{\mathcal{E}_h}$$

$$ab = \sqrt{2(1-\varepsilon)}\,a\,\frac{b}{\sqrt{2(1-\varepsilon)}}$$

$$\geq (\nabla_h v, \nabla_h v) - (1-\varepsilon)(\nabla_h v, \nabla_h v) - \frac{NC_{inv}^2}{4(1-\varepsilon)h}\langle [\![v]\!], [\![v]\!]\rangle_{\mathcal{E}_h} + \frac{\alpha}{h}\langle [\![v]\!], [\![v]\!]\rangle_{\mathcal{E}_h}$$

$$= \varepsilon(\nabla_h v, \nabla_h v) + \frac{1}{h}\left(\alpha - \frac{NC_{inv}^2}{4(1-\varepsilon)}\right)\langle [\![v]\!], [\![v]\!]\rangle_{\mathcal{E}_h}$$

Let $\varepsilon = \frac{1}{2}$

$$= \frac{1}{2}(\nabla_h v, \nabla_h v) + \frac{1}{h}\left(\alpha - \frac{NC_{inv}^2}{2}\right)\langle [\![v]\!], [\![v]\!]\rangle_{\mathcal{E}_h}$$

If $\alpha \geq \frac{1}{2}NC_{inv}^2 + \frac{1}{2}$, then

$$a_h(v,v) \geq \frac{1}{2}(\nabla_h v, \nabla_h v) + \frac{1}{2}\frac{1}{h}\langle [\![v]\!], [\![v]\!]\rangle_{\mathcal{E}_h}$$

$$\|v\|^2_{H_h^1(\Omega)} \leq (\nabla_h v, \nabla_h v) + \frac{1}{2}NC_{inv}^2(\nabla_h v, \nabla_h v) + \frac{1}{h}\langle [\![v]\!], [\![v]\!]\rangle_{\mathcal{E}_h}$$

$$\leq \left(1 + \frac{1}{2}NC_{inv}^2\right)\left[(\nabla_h v, \nabla_h v) + \frac{1}{h}\langle [\![v]\!], [\![v]\!]\rangle_{\mathcal{E}_h}\right] \leq \left(1 + \frac{1}{2}NC_{inv}^2\right)a_h(v,v)$$

$\therefore\ \beta\|v\|^2_{H_h^1(\Omega)} \leq a_h(v,v)$ where $\beta = \dfrac{1}{1+\frac{1}{2}NC_{inv}^2}$ ■

Lemma (Continuity) $\quad a_h(u,v) \leq C_c\|u\|_{H_h^1(\Omega)}\|v\|_{H_h^1(\Omega)} \quad \forall u,v \in H^2(\mathcal{T}_h)\ (\supset H_h^1(\Omega))$

pf) Trace, Cauchy-Schwartz

Lemma (Galerkin Orthogonality)

If u satisfies
$$-\Delta u = f \quad \Omega$$
$$u = 0 \quad \partial\Omega$$
and u_h is the IPDG approximation, then (Interior Penalty Discontinuous Galerkin)
$$a_h(u-u_h, v) = 0 \quad \forall v \in V_h^k$$
← non-conforming 에서는 없는 성질

Thm Assume previous results hold.
$$\|u-u_h\|_{H_h^1(\Omega)} \leq C\inf_{v\in V_h^k}\|u-v\|_{H_h^1(\Omega)}$$

pf) $\|u-u_h\|_{H_h^1(\Omega)} \leq \|u-v\|_{H_h^1(\Omega)} + \|v-u_h\|_{H_h^1(\Omega)} \quad \forall v \in V_h^k$

By coercivity, $\quad$, 0 by the Galerkin Orthogonality

$$\beta\|v-u_h\|^2_{H_h^1(\Omega)} \leq a_h(v-u_h, v-u_h) = a(u-u_h, v-u_h) + a(v-u, v-u_h)$$

$$\leq C_c\|v-u\|_{H_h^1(\Omega)}\|v-u_h\|_{H_h^1(\Omega)} \quad \text{by continuity}$$

$$\|v-u_h\|_{H_h^1(\Omega)} \leq \frac{C_c}{\beta}\|v-u\|_{H_h^1(\Omega)}$$

$$\Rightarrow \|u-u_h\|_{H_h^1(\Omega)} \leq \left(1+\frac{C_c}{\beta}\right)\|v-u\|_{H_h^1(\Omega)} \quad \forall v \in V_h^k$$

$$\leq \left(1+\frac{C_c}{\beta}\right)\inf_{v\in V_h^k}\|u-v\|_{H_h^1(\Omega)} \quad \underbrace{\leq Ch^k|u|_{H^{k+1}(\Omega)}}_{\text{True!}}$$

$$\|u-v\|^2_{H_h^1(\Omega)} = (\nabla_h(u-v), \nabla_h(u-v)) + h < \{\nabla_h(u-v)\}, \{\nabla_h(u-v)\} >_{\mathcal{E}_h} + \frac{1}{h} < [\![u-v]\!], [\![u-v]\!] >_{\mathcal{E}_h}$$

$$\underset{\substack{\uparrow \\ \text{Trace}}}{\leq} C \sum_{T\in\mathcal{T}_h} \left[\frac{1}{h^2}\|u-v\|^2_{L^2(T)} + \|\nabla(u-v)\|^2_{L^2(T)} + h^2 \|D^2(u-v)\|_{L^2(T)} \right]$$

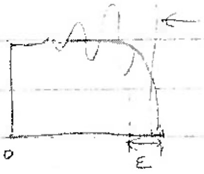
$$\underset{\substack{\uparrow \\ \text{Bram-Hilbert} \\ \text{Take infimum}}}{\leq} C\, h^k |u|_{H^{k+1}(\Omega)}$$

continuous Galerkin 의 경우.

Potential Topics for next class ..

$-\varepsilon u'' + u' = 0 \quad [0,1] \qquad \varepsilon$: very small.

$u(0) = 1, \quad u(1) = 0.$



← ε이 작을수록 oscillate ↑

textbook에 나와있음.

→ artificial / streamline diffusion (Thomas Hughes) 휴즈

upwinding FD.

Discontinuous Galerkin

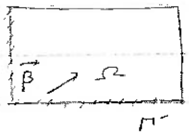
$$-(\varepsilon + h)\tilde{u}'' + \tilde{u}' = 0$$

mesh size ↓ (h); ↑ smooth solution ($\tilde{u}$) $\quad (\varepsilon \sim 10^{-8}, \ h \sim 10^{-2})$

(숙) $u_\beta + u = f$. 다 하면 모든 부분이 쉬워지도 in textbook

11/21

$$\begin{cases} u_\beta = f & \Omega \\ u = 0 & \Gamma \end{cases}$$



$\Gamma^+ = \{x \in \partial\Omega. \ \vec{\beta}\cdot\vec{n} < 0\}$

$\Gamma^- = \{x \in \partial\Omega: \ \vec{\beta}\cdot\vec{n} > 0\}$

$u_\beta = \nabla u \cdot \vec{\beta} \qquad \vec{\beta}$ is a fixed vector

$V_h = \{v \in H^1(\Omega). \ \ v|_T \in P^1(T) \quad v = 0 \ \ \Gamma^-\}$

$\int u_\beta \cdot v = \int f \cdot v \qquad \forall v \in L^2(\Omega)$

Standard Galerkin method :

Find $u_h \in V_h$ s.t. $\int_\Omega (u_h)_\beta \cdot v = \int_\Omega f \cdot v \qquad v \in V_h$

squared system ⇒ uniqueness implies well-posed ness ??

$a_h(u_h, v) = (f, v) \qquad \forall v \in V_h$

$a_h(u_h, v) = \int_\Omega (u_h)_\beta v \, dx$

Uniqueness : If $f \equiv 0$, want to show $u_h \equiv 0$

i.e. $a_h(u_h, v) = 0 \quad \forall v \in V_h$

$$0 = a_h(u_h, u_h) = \int_\Omega (u_h)_\beta u_h = \frac{1}{2}\int_\Omega \partial_\beta (u_h)^2 = \frac{1}{2}\int_{\partial\Omega} u_h^2 \, \vec{\beta}\cdot\vec{n} \overset{\substack{u=0 \ \Gamma^- \\ ? \\ \downarrow}}{=} \frac{1}{2}\int_{\Gamma^+} u_h^2 \, \overset{\text{why } |\cdot| ?}{|\vec{\beta}\cdot\vec{n}|}$$

$(u_h)_\beta \notin V_h$ 라서 v에 $(u_h)_\beta$를 넣을 수 x

uniqueness를 보일 수 x — 그래서 streamline diffusion method

Streamline diffusion method

$$u_\beta = f$$

$$\int_\Omega u_\beta (v + h v_\beta) = \int_\Omega f (v + h v_\beta) \qquad \forall v \in H^1$$

Find $u_h \in V_h$

$$a(u_h, v) = (f, v + h v_\beta) \qquad \forall v \in V_h$$

where $a(u_h, v) = \int_\Omega [h (u_h)_\beta v_\beta + (u_h)_\beta v]$

- Uniqueness

Let $f = 0$

$$0 = a_h(u_h, u_h) = h \int_\Omega (u_h)_\beta^2 + \int_\Omega (u_h)_\beta u_h = h \int_\Omega (u_h)_\beta^2 + \int_{\Gamma^+} (u_h)^2 |\beta \cdot n|$$

$$\Rightarrow \quad u_h^2 |\beta \cdot n| = 0 \quad \Gamma^+$$

$$(u_\beta)_\beta = 0 \quad \Omega$$

$$u(x) = \underbrace{u(x_0)}_{0} + \underbrace{\int u_\beta}_{0} \qquad \Rightarrow u(x) = 0 \text{ in 빗금}$$

빗금친 부분 0 ⇒ 모두 0 $\quad v|_T \in P^1(T)$

Lemma $\|u_h\|_\beta \le c \|f\|_{L^2(\Omega)}$ "discrete poincare type inequality"

p.f) $h \int_\Omega (u_h)_\beta^2 + \int_{\Gamma^+} u_h^2 |\beta \cdot n| = a_h(u_h, u_h) = (f, u_h + h(u_h)_\beta) \le \|f\| \left(\|u_h\|_{L^2} + h \|(u_h)_\beta\|_{L^2} \right)$

$$\le \frac{1}{\varepsilon} \|f\| + \varepsilon c (h \int u_\beta^2)$$

$$\|v\|_{L^2(\Omega)}^2 \le c \left(h \int_\Omega (v)_\beta^2 + \int_{\Gamma^+} v^2 |\beta \cdot n| \right)$$

Choose $\varepsilon c < \frac{1}{2}$,

$$\Rightarrow \frac{1}{2} h \int_\Omega (u_h)_\beta^2 + \int_{\Gamma^+} u_h^2 |\beta \cdot n| \le C \|f\|_{L^2(\Omega)}^2$$

$$\left(\|u\|_\beta^2 := h \int_\Omega u_\beta^2 + \int_{\Gamma^+} u^2 |\beta \cdot n| \right)$$

$\|v\|_{L^2(T)} \le h \|v_\beta\|_{L^2(T)}$ for Γ^-

first layer

second layer

$$\|v\|_{L^2(T)} \le h \|v_\beta\|_{L^2(T)} + \underbrace{h^{\frac{1}{2}} \|v\|_{L^2(T)}}$$

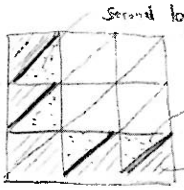
0이 아니라서 추가.. 뭐 이런식으로....

recap....

If $v \in P^1|_T$ then $\|v\|_{L^2(T)} \le c(h_T \|v_\beta\|_{L^2(T)} + h_T^{\frac{1}{2}} \|v\|_{L^2(T_-)})$

→ reference triangle에서 둘이 norm 이라는 것을 보여야한다

→ 다시 triangle로 바꾸면서 h_T 들이 나온다. (scaling argument)



$$\|v\|_{L^2(T_1)} \le c\, h_T \|v_\beta\|_{L^2(T)}$$

trace or inverse ↓

$$\|v\|_{L^2(T_2)} \le c \left(h_T \|v_\beta\|_{L^2(T_2)} + h_T^{\frac{1}{2}} \|v\|_{L^2(T_2^-)} \right) \le C h_T \|v_\beta\|_{L^2(T_2)} + C h_T \|v_\beta\|_{L^2(T_1)}$$

$$\|v\|_{L^2(T_m)}$$

squared everything & summation $\qquad M(T) = T_1 \cup T_2 \cup \cdots \cup T_M$

$$\sum_{T \in \mathcal{T}_h} \|v\|_{L^2(T)}^2 \le \sum_{T \in \mathcal{T}_h} \sum h_T^2 \|v_\beta\|_{L^2(M(T))}^2$$

$$\le N \sum_{T \in \mathcal{T}_h} h_T^2 \|v_\beta\|_{L^2(T)}^2$$

first layer 는 $N \sim \frac{1}{h}$ 개 [illegible]

$$= c \sum h_T \|v_\beta\|^2 = c h \int (v_\beta)^2$$

$a_h(u, v) = h\int (u_\beta)\cdot v_\beta + \int u_\beta \cdot v$ weak form

$u_\beta = f \longrightarrow u_\beta - \underbrace{h u_{\beta\beta}}_{} = f$ 이 form의 weak form으로 볼 수 있다.

diffusion term을 넣어서 푼 것! to stabilize problem.

이런 것 많이 해요

$-\underset{\sim\sim}{\varepsilon \Delta u^\varepsilon} + u_t^\varepsilon + (f(u^\varepsilon))_x = 0$

adding term stabilize 하기 위해 artificial diffusion을 넣어놓고 $\rightarrow u^\varepsilon$

and show $\|u^\varepsilon - u\|_{L^1} \leq \sqrt{\varepsilon}$ u^ε가 u와 가까운 것을 보여요.

Consider $-\varepsilon \Delta u + u_\beta = f$ Ω

$u = 0$ $\partial\Omega$

$V_h = \{v \in H_0^1(\Omega) : v|_T \in P^1(T)\}$

$\varepsilon \int_\Omega \nabla u \cdot \nabla v + \int_\Omega u_\beta \cdot v = \int_\Omega f v$

Find $u_h \in V_h$

$a(u_h, v) = (f, v) \quad \forall v \in V_h$

where $a(u, v) = \varepsilon \int_\Omega \nabla u \cdot \nabla v + \int u_\beta v$

$\|u_h\|_{L^2}^2 \leq c\, h \int (u_\beta)^2 dx$

stability : $a_h(u, u_h) = (f, u_h)$

$\varepsilon \|\nabla u_h\|_{L^2(\Omega)}^2 + \frac{1}{2}\int_{\Gamma^+} u_h^2 |\beta\cdot n| = (f, u_h) \leq \|f\|_{L^2} \|u_h\|_{L^2} \leq \|f\|_{L^2(\Omega)} (c h^{\frac{1}{2}} \|\nabla u_h\|_{L^2(\Omega)})$

$\leq \frac{1}{2}\varepsilon \|\nabla u_h\|^2 + \frac{c h}{2\varepsilon}\|f\|^2$

when $\varepsilon \to 0$, not stable

$\varepsilon \ll h$

⇒ Instead of ε, USE h.

$\int (-\varepsilon \Delta u + u_\beta)(v + h v_\beta) = \int f (v + h v_\beta)$ when $u = \underline{u_h}$ continuous piecewise linear $\Delta u|_T = 0$

$\varepsilon \int_\Omega \nabla u_h \nabla v + h\int_\Omega (u_h)_\beta v_\beta + \int_\Omega (u_h)_\beta v - \int_\Omega \varepsilon \Delta u_h h v_\beta = \int f (v + h v_\beta) \quad \forall v \in V_h$

$\varepsilon \|\nabla u_h\|_{L^2(\Omega)}^2 + h \|(u_h)_\beta\|_{L^2(\Omega)}^2 \leq c \|f\|_{L^2(\Omega)}^2$

$\|(u_h)_\beta\|_{L^2(\Omega)} \leq \frac{c}{\sqrt{h}} \|f\|_{L^2(\Omega)}$

h 있는 norm bound를 얻을 수 있었다!

$\sqrt{\int_L \left(\frac{1}{h}\right)^2} = \underset{\sqrt{h}}{\underset{\sim}{|L|}}\frac{1}{h} = \frac{1}{\sqrt{h}}$

$\sqrt{\int_L \left(\frac{1}{h}\right)} = |L|\frac{1}{\sqrt{h}} = 1$ → L^1 norm을 이용하여 h가 사라져요

Solve by DG

$$u_\beta = f \qquad \Omega$$
$$u = 0 \qquad \partial\Omega$$

$V_h = \{ v \in L^2(\Omega), \quad v|_T \in P^k(T) \quad \forall T \in \mathcal{T}_h \}$

$$\sum_{T\in\mathcal{T}_h} \int u_\beta \cdot v = \int f v$$
$$\sum_{T\in\mathcal{T}_h} \int_T u v_\beta + \int_{\partial T} u v \vec{\beta}\cdot\vec{n} = \int_\Omega f v$$

DG method

Find $u_h \in V_h$ s.t. $a_h(u_h, v) = (f, v) \quad \forall v \in V_h$

where $a_h(u,v) = \sum_{T\in\mathcal{T}_h} -\int_T u v_\beta + \int_{\partial T} \hat{u} v \vec{\beta}\cdot\vec{n}$

$\hat{u}$ is the numerical flux on each edge it is single valued

for example, $\hat{u} = \dfrac{u^+ + u^-}{2}$

or

$\hat{u} = u^+$, etc... (stability에서 $\hat{u}$를 어떻게 choose 하는가가 결정)

What is a good choice of $\hat{u}$?

$a_h(v,v) > 0$ for any $v \in V_h$

$$a_h(v,v) = \sum_{T\in\mathcal{T}_h} -\int_T v v_\beta + \int_{\partial T} \hat{v} v \vec{\beta}\cdot\vec{n} = \sum_{T\in\mathcal{T}_h} \int_{\partial T} (\hat{v} - \tfrac{1}{2}v) v \vec{\beta}\cdot\vec{n}$$
$$-\int_T v v_\beta = -\frac{1}{2}\int_T (v^2)_\beta = -\frac{1}{2}\int v^2 \vec{\beta}\cdot\vec{n}$$

$$v^{\pm}(\vec{x}) = \lim_{s\to 0^{\mp}} (x + s\vec{\beta}) \Rightarrow a_h(v,v) = \sum_{e\in\mathcal{E}_h^I} \int_e \left[(\hat{v} - \tfrac{v^-}{2}) v^- |\vec{\beta}\cdot\vec{n}| - (\hat{v} - \tfrac{v^+}{2}) v^+ |\vec{\beta}\cdot\vec{n}| \right]$$
$$+ \sum_{e\in\mathcal{E}_h^B} \int_e (\hat{v} - \tfrac{1}{2}v) v |\vec{\beta}\cdot\vec{n}|$$

$$\Rightarrow a_h(v,v) = \sum_{e\in\mathcal{E}_h^I} \int_e |\beta\cdot n| \hat{v}(v^- - v^+) + \tfrac{1}{2}((v^+)^2 - (v^-)^2)|\beta\cdot n| + \sum_{e\in\mathcal{E}_h^B} \int_e (\hat{v} - \tfrac{1}{2}v) v |\vec{\beta}\cdot\vec{n}|$$
$$= \sum_{e\in\mathcal{E}_h^I} \int_e |\beta\cdot n| (v^- - v^+) [\hat{v} - \tfrac{1}{2}(v^+ + v^-)] + \sum_{e\in\mathcal{E}_h^B} \int_e (\hat{v} - \tfrac{1}{2}v) v |\vec{\beta}\cdot\vec{n}|$$

Choose $\hat{v} = v^-$ or $\frac{1}{2}(v^+ + v^-) + \alpha(v^- - v^+)$

$\rightarrow \frac{1}{2}(v^- - v^+)^2$

$\hat{v} = 0$ on Γ^- $\quad \frac{1}{2}\int_{\Gamma^-} v^2 |\beta\cdot n| + \frac{1}{2}\int_{\Gamma^+} v^2 |\beta\cdot n|$

$\hat{v} = v$ on Γ^+

$$\Rightarrow \hat{v} = \begin{cases} v^- & \\ 0 & \Gamma^- \\ v & \Gamma^+ \end{cases}$$

$$a_h(v,v) = \sum_{e\in\mathcal{E}_h^I} \frac{1}{2}\int_e (v^- - v^+)^2 |\beta\cdot n| + \sum_{e\in\mathcal{E}_h^B} \frac{1}{2}\int_e v^2 |\beta\cdot n|$$

$(v_\beta) \in V_h$

also in V_h since V_h continuous 조건 x

* Mistake ↑ this lecture.

$\|u\|_{L^2} \le \sqrt{h}\,\|u_\beta\|_{L^2} + \int_\Gamma$ to prove stability for streamline diffusion method

$$\begin{cases} -\varepsilon\Delta^2 u + u_\beta + u = f \\ \qquad u = 0 \end{cases}$$

추가해야 stability 성립 수 있음

u 없는 경우 실제로 u가 있는 곳은 [illegible]

12/5

FEM for parabolic problems.

$$\begin{cases} \dot{u} - \Delta u = f & \Omega \times (0,T] \\ u = 0 & \partial\Omega \times [0,T] \\ u(x\ 0) = u^0(x) & \Omega \end{cases}$$

Let $V = H_0^1(\Omega)$

Fix time $0 < t \le T$

$$\int_\Omega \dot{u}(t) v\,dx - \int_\Omega \Delta u(t) v\,dx = \int_\Omega f(t) v\,dx$$

$$\int_\Omega \dot{u}(t) v\,dx + \int_\Omega \nabla u(t) \nabla v\,dx = \int_\Omega f(t) v\,dx$$

$$(\dot{u}(t), v) + a(u(t), v) = (f(t), v) \qquad \forall v \in V, \quad 0 < t \le T$$

where $a(u, v) = (\nabla u, \nabla v)$

First we discritize in space only.

$V_h \subseteq V$

eg. $V_h = \{ v \in C_0(\Omega) : v|_T \in P^k(T) \quad \forall T \in \mathcal{T}_h \}$

Find $u_n(t) \in V_h$ s.t.

$$(*) \begin{cases} (\dot{u}_n(t), v) + a(u_n(t), v) = (f(t), v) & \forall v \in V_h, \quad 0 < t \le T \\ (u_n(0), v) = (u^0, v) & \forall v \in V_h \qquad \leftarrow \text{IC.} \end{cases}$$

This turns out to be a system of ODE's

$(\because)$ Any function $v \in V_h$. $V = \sum_{i=1}^{M} z_i \phi_i$ where $\{\phi_i\}_{i=1}^{M}$ is a basis of V

$\Rightarrow u_n(t) = \sum_{i=1}^{M} z_i(t) \phi_i$

: we need to find the functions $z_i : [0,T] \longrightarrow \mathbb{R}$.

$\dot{u}_n(t) = \sum_{i=1}^{M} \dot{z}_i(t) \phi_i$

$$(*) \Rightarrow \left(\sum_{i=1}^{M} \dot{z}_i(t) \phi_i, \phi_j \right) + a\left(\sum_{i=1}^{M} z_i(t) \phi_i, \phi_j \right) = (f(t), \phi_j) \qquad \forall j$$

$$\sum_{i=1}^{M} \left[(\phi_i, \phi_j) \dot{z}_i(t) + a(\phi_i, \phi_j) z_i(t) \right] = (f(t), \phi_j)$$

Let $B_{ij} = (\phi_i, \phi_j)$, $A_{ij} = a(\phi_i, \phi_j)$

$$B = \begin{bmatrix} (\phi_1, \phi_1) & & & \\ (\phi_2, \phi_1) & & & \\ \vdots & & & \\ (\phi_n, \phi_1) & \cdots & & (\phi_m, \phi_m) \end{bmatrix}$$

$$\Rightarrow \begin{cases} B\dot{\vec{Z}}(t) + A\vec{Z}(t) = \vec{F}(t) \\ \text{where} \quad \vec{Z}(t) = \begin{bmatrix} z_1(t) \\ \\ z_M(t) \end{bmatrix}, \quad \vec{F}(t) = \begin{bmatrix} (f(t), \phi_1) \\ \vdots \\ (f(t), \phi_M) \end{bmatrix} \\ B\vec{Z_i}(0) = \vec{U^0} \end{cases}$$

I.C : $\sum_{i=1}^{M} z_i(0) (\phi_i, \phi_j) = (u^0, \phi_j)$

$B \vec{Z_i}(0) = \vec{U^0}$

eigenvalues 들이 많이 흩어져있으면 steep. ⇒ RungeKutta, Backward Euler.. 등 잘 골라야해요.

Semi-discrete approximation

Stability (Assume $f \equiv 0$)

$$(\dot{u}_n(t), v) + a(u_n(t), v) = 0$$

Let $v = u_n(t)$

$$(\dot{u}_n(t), u_n(t)) + a(u_n(t), u_n(t)) = \frac{1}{2} \int_\Omega \frac{d}{dt}(u_n^2)\,dx + a(u_n(t), u_n(t)) = 0$$

$$\frac{1}{2} \frac{d}{dt} \|u_n\|^2 + a(u_n(t), u_n(t)) = 0$$

$$\frac{1}{2} \int_0^T \frac{d}{dt} \|u_n(t)\|^2 + \int_0^T a(u_n(t), u_n(t))\,dt = 0$$

$$\frac{1}{2} \left(\|u_n(T)\|^2 - \|u_n(0)\|^2 \right) \le 0$$

$$\|u_n(T)\| \le \|u_n(0)\| \le \|u^0\|$$

Error estimate for semi-discrete scheme

We need to define the elliptic projection

Given w, define $P_{ht}w(t) \in V_h$

$$(\nabla(P_{ht}w(t)), \nabla v) = (\nabla w(t), \nabla v) \quad \forall v \in V_h$$

$- \Delta w = \underbrace{- \Delta (P_{ht} w(t))}$

할 수 없어서 ↑ 정한 것

$$\cdot \forall v \in V_h, \ (\nabla(\partial_t P_{ht}w(t)), \nabla V) = (\partial_t \nabla (P_{ht}w(t)), \nabla V) = \partial_t(\nabla(P_{ht}w(t)), \nabla V) \overset{\text{by def.}}{=} \partial_t(\nabla w(t), \nabla V)$$
$$= (\nabla(\partial_t w(t)), \nabla V).$$
$$= (\nabla(P_{ht}(\partial_t w(t))), \nabla V)$$

$$\Rightarrow \partial_t P_{ht} w(t) = P_{ht}(\partial_t w(t)) \quad -(*)$$

- $\|\nabla(P_{ht}w(t) - w(t))\|_{L^2(\Omega)} \le Ch^k \|w(t)\|_{H^{k+1}(\Omega)}$

$(\because)$ $\| \ \|^2_{L^2(\Omega)} = (\nabla(P_{ht}w(t) - w(t)), \nabla(P_{ht}w(t) - w(t)))$

$$= (\nabla(P_{ht}w(t) - w(t)), \nabla(Iw(t) - w(t)))$$

$$\|\nabla(P_h w(t) - w(t)\|_{L^2(\Omega)} \le \|\nabla(Iw(t) - w(t))\| \le Ch^k \|w(t)\|_{H^{k+1}(\Omega)}.$$

$\rightarrow$ $\|\nabla(P_{ht}\dot{w}(t) - \dot{w}(t)\|_{L^2(\Omega)} \underset{\substack{\uparrow \\ \text{by } (*)}}{=} \|\nabla(P_h\dot{w}(t) - \dot{w}(t))\|_{L^2(\Omega)} \le ch^k \|\dot{w}(t)\|_{H^{k+1}(\Omega)}$

$$(\dot{u}_h(t), v) + a(u_h(t), v) = (f(t), v) \quad \forall v \in V_h$$
$$(\dot{u}(t), v) + a(u(t), v) = (f(t), v) \quad \forall v \in V$$
$$\Rightarrow (\dot{u}_h - \dot{u}, v) + a(u_h - u, v) = 0 \quad \forall v \in V_h$$

$$(\dot{u}_h - [\dot{P_{ht}u}], v) + a(u_h - P_{ht}u, v) = (u - P_{ht}\dot{u}, v) + \overset{,0}{a(u - P_{ht}u, v)} \quad \forall v \in V_h$$

Define $e_h(t) = u_h(t) - P_{ht}u(t) \in V_h$

$$(\dot{e}_h, v) + a(e_h, v) = ((u - P_{ht}\dot{u})(t), v) \quad \forall v \in V_h$$

Let $v = e_h(t)$

$$(\dot{e}_h, e_h) + a(e_h, e_h) = ((u - P_{ht}\dot{u})(t), e_h)$$

$$\frac{1}{2}\frac{\partial}{\partial t}\|e_h(t)\|^2_{L^2(\Omega)} + a(e_h, e_h) \le \|(u - P_{ht}\dot{u})(t)\|_{L^2(\Omega)} \|e_h(t)\|_{L^2(\Omega)} \le \frac{1}{2}C_p\|e_h(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{1}{C_p}\|(\dot{u} - P\dot{u})(t)\|^2_{L^2}$$

$$\frac{1}{2}\frac{\partial}{\partial t}\|e_h(t)\|^2_{L^2(\Omega)} + C_p\|e_h\|^2_{L^2(\Omega)} \underset{\substack{\uparrow \\ \text{Poincare}}}{\le} \frac{1}{2}\frac{\partial}{\partial t}\|e_h\|^2_{L^2(\Omega)} + \underset{= \int_\Omega \nabla e_h \nabla e_h}{\underbrace{a(e_h, e_h)}}$$

$$\frac{1}{2}\partial_t\|e_h(t)\|^2_{L^2(\Omega)} + \frac{1}{2}C_p\|e_h(t)\|^2_{L^2(\Omega)} \le \frac{1}{2C_p}\|(\dot{u} - P\dot{u})(t)\|^2_{L^2(\Omega)}$$
$$\partial_t\|e_h(t)\|^2_{L^2(\Omega)} \le -C_p\|e_h(t)\|^2_{L^2(\Omega)} - \frac{1}{C_p}\|(\dot{u} - P\dot{u})(t)\|^2_{L^2(\Omega)}$$

<u>Recall</u> (Gronwall's inequality)

$$\phi'(t) \le \beta\phi(t) + f(t)$$
$$\phi(t) \le e^{\beta t}(\phi(0) + \int_0^t f(s)\,ds)$$

$(\because)$ $e^{-\beta t}\phi'(t) \le e^{-\beta t}\beta\phi(t) + e^{-\beta t}f(t)$

Since $(e^{-\beta t}\phi)' = -\beta e^{-\beta t}\phi + e^{-\beta t}\phi'$,

$$(e^{-\beta t}\phi)' \le e^{-\beta t}f(t)$$
$$\int_0^t (e^{-\beta s}\phi)(s)\,ds \le \int_0^t e^{-\beta s}f(s)\,ds$$
$$e^{-\beta t}\phi(t) - \phi(0) \le \int_0^t e^{-\beta s}f(s)\,ds \quad \le 1 \text{ if } \beta > 0$$
$$\phi(t) \le e^{\beta t}(\phi(0) + \int_0^t e^{-\beta s} f(s)\,ds)$$

$\sqrt{a^2+b^2} \le a+b$

Apply Gronwall's ineq.

$$\|e_h(t)\|^2_{L^2(\Omega)} \le e^{-c_p t}\|e_h(0)\|^2_{L^2(\Omega)} + \underbrace{\frac{e^{-c_p t}}{c_p}\int_0^t e^{c_p s}\|(\dot{u}-P_t \dot{u})(t)\|^2_{L^2(\Omega)}}_{} \le e^{-c_p t}\|e_h(0)\|^2_{L^2(\Omega)} + \frac{I}{c_p^2}(1-e^{-c_p t}).$$

$$\| e_h(t) \| = U_h(t) - P_{ht} u(t)$$

$$\le \frac{e^{-c_p t}}{c_p} \underbrace{\max_{0\le s\le t}\|(\dot{u}-P_t \dot{u})(s)\|^2_{L^2(\Omega)}}_{I} \int_0^t e^{c_p s}\,ds = \frac{e^{-c_p t}}{c_p} I\left(\frac{1}{c_p}e^{c_p t} - \frac{1}{c_p}\right)$$

$$= \frac{I}{c_p^2}(1-e^{-c_p t})$$

$$\|u(t)-U_h(t)\|_{L^2(\Omega)} \le \|u(t)-P_{ht}u(t)\|_{L^2(\Omega)} + \|e_h\|_{L^2(\Omega)}$$

$$\le C h^{k+1}\|u(t)\|_{H^{k+1}(\Omega)} + e^{-\frac{c_p}{2}t}\underbrace{\|e_h(0)\|_{L^2(\Omega)}}_{\le Ch^{k+1}\|u(0)\|_{H^{k+1}}} + \frac{Ch^{k+1}}{c_p^2}\max_{0\le s\le t}\|\dot{u}(s)\|_{H^{k+1}(\Omega)}$$

(앞에서 봤던 것과 비교 → 모 있어서 h^k가 아닌 h^{k+1})
duality 이용해서 증명

$$\|e_h(0)\|_{L^2} = \|u(0) + P_{ht}(0)\|_{L^2} + \|u(0) - U_h(0)\|_{L^2(\Omega)}$$

$$\le C h^{k+1}\|u(0)\|_{H^{k+1}(\Omega)}.$$

Fully discrete approximation

Start with backward Euler

The semi-discrete : $(\dot{U}_h, v) + a(U_h, v) = (f(t), v)$

$[0, T]$

$0 = t_0 < t_1 < \cdots < t_N = T$, $t_i - t_{i-1} = K$ $t_i = iK$

BE
Backward Euler $\left(\frac{U_h^n - U_h^{n-1}}{K}, v\right) + a(U_h^n, v) = (f(t_n), v)$ $\forall v \in V_h$

where $U_h^n \in V_h$, $U_h^n = u(t_n, \cdot)$

FE
Forward $\left(\frac{U_h^n - U_h^{n-1}}{K}, v\right) + a(U_h^{n-1}, v) = (f(t_{n-1}), v)$ $\forall v \in V_h$

$U_h^n = \sum_{i=1}^{M} Z_i^n \phi_i$

BE : $B\vec{Z}^n + KA\vec{Z}^n = B\vec{Z}^{n-1} + K\vec{F}^n$ $(B+KA)\vec{Z}^n = B\vec{Z}^{n-1} + K\vec{F}^n$

where $\vec{Z}^n = \begin{bmatrix} Z_1^n \\ \vdots \\ Z_M^n \end{bmatrix}$ $\vec{F}^n = \begin{bmatrix} f(t_n, \phi_1) \\ \vdots \\ f(t_n, \phi_M) \end{bmatrix}$

FE : $B\vec{Z}^n + KA\vec{Z}^{n-1} = B\vec{Z}^{n-1} + K\vec{F}^{n-1}$

$\underset{\sim}{B}\vec{Z}^n = (B - KA)\vec{Z}^{n-1} + K\vec{F}^{n-1}$

eigenvalue가 1에 모여있어서 condition number가 좋아서 (≈1), 빨리 계산 가능

or mass lumping method (diagonal term만 남기는 방법). error는 좀 증가하겠죠

discontinuous Galerkin 은 저거 $\begin{bmatrix} \square & & \\ & \square & \\ & & \square \end{bmatrix}$ block diagonal matrix가 나옴.
element 끼리 communicate 하지 X → $\square$ block을 invert 하면 되니까 계산
but more degree of freedom.

Stability of BE. $(f_n \equiv 0)$

Let $V = U_h^n$

가지고 증명하면 better result

$$(U_h^n, U_h^n) + \underbrace{Ka(U_h^n, U_h^n)} = (U_h^{n-1}, U_h^n)$$

$$\|U_h^n\|_{L^2(\Omega)}^2 \leq \|U_h^{n-1}\|_{L^2(\Omega)} \|U_h^n\|_{L^2(\Omega)}$$

$$\|U_h^n\|_{L^2(\Omega)} \leq \|U_h^{n-1}\|_{L^2(\Omega)}$$ unconditionally stable

prelim ★

FE $\|U_h^n\|_{L^2(\Omega)}^2 + \|U_h^n - U_h^{n-1}\|_{L^2(\Omega)}^2$

$= \|U_h^n\|_{L^2(\Omega)}^2 + (U_h^n, U_h^n) + (U_h^{n-1}, U_h^{n-1}) - 2(U_h^n, U_h^{n-1})$

$= 2(U_h^n, U_h^n) - 2(U_h^n, U_h^{n-1}) + (U_h^{n-1}, U_h^{n-1})$

$= 2(U_h^n, U_h^n - U_h^{n-1}) + (U_h^{n-1}, U_h^{n-1})$ $\quad (\because)\ (U_h^n, V) = -Ka(U_h^{n-1}, V) + (U_h^{n-1}, V)$

$= -2Ka(U_h^{n-1}, U_h^n - U_h^{n-1}) + 2(U_h^{n-1}, U_h^n - U_h^{n-1}) + (U_h^{n-1}, U_h^{n-1})$

$= -2Ka(U_h^{n-1}, U_h^n - U_h^{n-1}) - (U_h^{n-1}, U_h^{n-1}) + 2(U_h^{n-1}, U_h^n)$

$= -2Ka(U_h^{n-1}, U_h^n) + 2Ka(U_h^{n-1}, U_h^{n-1}) - (U_h^{n-1}, U_h^{n-1}) + 2(U_h^{n-1}, U_h^n)$

$(U_h^n, V) = -Ka(U_h^{n-1}, V) + (U_h^{n-1}, V)$ FE

$= (U_h^{n-1}, U_h^{n-1}) - 2Ka(U_h^{n-1}, U_h^n)$

$\|U_h^n\|_{L^2(\Omega)}^2 + \|U_h^n - U_h^{n-1}\|_{L^2(\Omega)}^2 + \underbrace{2Ka(U_h^{n-1}, U_h^n)} = (U_h^{n-1}, U_h^{n-1}) = \|U_h^{n-1}\|_{L^2(\Omega)}^2$

$2Ka(U_h^n, U_h^n) + 2Ka(U_h^{n-1} - U_h^n, U_h^n)$

$\|U_h^n\|_{L^2(\Omega)}^2 + \|U_h^n - U_h^{n-1}\|_{L^2(\Omega)}^2 + 2Ka(U_h^n, U_h^n) = \|U_h^{n-1}\|_{L^2(\Omega)}^2 - 2Ka(U_h^{n-1} - U_h^n, U_h^n)$

$\leq \|U_h^{n-1}\|_{L^2(\Omega)}^2 + C_{inv}^2 \frac{k}{h^2} \|U_h^{n-1} - U_h^n\|_{L^2(\Omega)}^2 + K\|\nabla U_h^n\|_{L^2(\Omega)}^2$

$-2Ka(U_h^{n-1} - U_h^n, U_h^n) = -2K(\nabla(U_h^{n-1} - U_h^n), \nabla U_h^n)$

$\leq \sqrt{2K}\, \|\nabla(U_h^{n-1} - U_h^n)\|_{L^2(\Omega)} \sqrt{2K}\, \|\nabla U_h^n\|_{L^2(\Omega)}$

$\leq \frac{C_{inv}}{h} \sqrt{2K}\, \|U_h^{n-1} - U_h^n\|_{L^2(\Omega)} \sqrt{2K}\, \|\nabla U_h^n\|_{L^2(\Omega)}$

$\leq \frac{C_{inv}^2}{2h^2} 2K \|U_h^{n-1} - U_h^n\|_{L^2(\Omega)}^2 + \frac{2K}{2} \|\nabla U_h^n\|_{L^2(\Omega)}^2$

$\rightarrow \|U_h^n\|_{L^2(\Omega)}^2 + \left(1 - C_{inv}^2 \frac{k}{h^2}\right) \|U_h^{n-1} - U_h^n\|_{L^2(\Omega)}^2 + K\|\nabla U_h^n\|_{L^2(\Omega)}^2 \leq \|U_h^{n-1}\|_{L^2(\Omega)}^2.$

$C_{inv}^2 \frac{k}{h^2} \leq 1$ i.e $k \leq \frac{h^2}{C_{inv}^2}$

big limitation.