Gustafsson §A.5

Stability Analysis for Nonlinear Problems

- Consider a nonsingular $n \times n$ matrix A
- Then $Ax = b$ has a unique soln.
- Now let $f(x)$ be a smooth vector valued function, $f : \mathbb{R}^n \to \mathbb{R}^n$

- Want to solve

$$(*) \quad Ax + \varepsilon f(x) = b$$

- A simple iterative scheme is Picard / fixed point iteration

$$A x^{(n+1)} + \varepsilon f(x^{(n)}) = b \qquad n = 0, 1, \ldots$$

$$x^0 = A^{-1} b$$

<u>Thm</u> Assume $f(x)$ is uniformly Lipschitz i.e $\exists\ L$ such that for all x, y

$$|f(x) - f(y)| \le L |x - y|$$

If $\gamma = \varepsilon |A^{-1}| L < 1$ then $(*)$ has a unique soln, $\tilde{x}$ and the iterative scheme converges $\lim_{n \to \infty} x_n = \tilde{x}$

Pf To show uniqueness, we'll assume two solutions x, y and show they must be equal

$$Ax + \varepsilon f(x) = b$$
$$Ay + \varepsilon f(y) = b$$

subtract $$A(x-y) + \varepsilon [f(x) - f(y)] = 0$$

abs both sides
$$x - y = -\varepsilon A^{-1}(f(x) - f(y))$$
$$|x-y| = \varepsilon |A^{-1}[f(x) - f(y)]|$$
$$\leq \varepsilon |A^{-1}| \, |f(x) - f(y)|$$

Lipschitz Def
$$\leq \varepsilon |A^{-1}| \, L \, |x-y|$$

Pct
$$|x-y| \leq \gamma |x-y|$$
$$(1-\gamma)|x-y| \leq 0 \qquad \boxed{\text{if } \gamma < 1}$$
$$|x-y| \leq 0$$
$$\Rightarrow x = y$$

We can extend the same argument to analyze convergence of the scheme

$$A x^n + \varepsilon f(x^{n-1}) = b$$

by taking $(x, y) = (x^{n+1}, x^{n})$

$$x^{n+1} - x^n = -\varepsilon A^{-1}\left(f(x^{n}) - f(x^{n-1})\right)$$

$$\begin{aligned}
|x^{n+1} - x^n| &\le \gamma |x^n - x^{n-1}| \\
&\le \gamma^2 |x^{n-1} - x^{n-2}| \\
&\;\vdots \\
&\le \gamma^n |x^1 - x^0| \qquad ★★
\end{aligned}$$

This is an argument refered to as a Cauchy sequence

For $m > n$

$$|x^m - x^n| = \left|\sum_{i=n}^{m-1} (x^{i+1} - x^i)\right|$$

$$\cancel{\le \left(\sum_{i=n}^{m-1} \gamma^i\right) |x^1 - x^0|}$$

$$\le \left(\sum_{i=n}^{m-1} \gamma^i\right) |x^1 - x^0|$$

ie for each i between $n, m-1$ we get a γ^i from ★★

$$\le \frac{\gamma^n}{1-\gamma} |x^1 - x^0|$$

Geometric series formula

$$\sum_{k=0}^{n} a r^k = a\left(\frac{1-r^{n+1}}{1-r}\right)$$