1) $y(t) = t^3 + \int_0^t \operatorname{sen}(t-r)\, y(r)\, dr$

Observe que a integral é uma convolução de sen(t) e Y. Aplicando a transformada de Laplace temos:

$$Y(s) = \frac{6}{s^4} + \left(\frac{1}{s^2+1}\right) Y(s)$$

$$Y(s) - \left(\frac{1}{s^2+1}\right) Y(s) = \frac{6}{s^4}$$

$$Y(s)\left(1 - \frac{1}{s^2+1}\right) = \frac{6}{s^4}$$

$$Y(s)\left(\frac{s^2+1-1}{s^2+1}\right) = \frac{6}{s^4}$$

$$Y(s)\left(\frac{s^2}{s^2+1}\right) = \frac{6}{s^4}$$

$$Y(s) = \frac{6}{s^4} \Big/ \frac{s^2}{s^2+1}$$

$$Y(s) = \frac{6s^2+6}{s^6} \Rightarrow \frac{6s^2}{s^6} + \frac{6}{s^6}$$

$$Y(s) = \frac{6}{s^4} + \left(\frac{6}{120} \cdot \frac{120}{s^6}\right)$$

$$Y(s) = \frac{3!}{s^4} + \frac{1}{20}\left(\frac{5!}{s^6}\right)$$

Aplicando L^{-1} temos:

$$L^{-1}[Y(s)] = L^{-1}\left[\frac{3!}{s^4}\right] + L^{-1}\left[\frac{1}{20}\left(\frac{5!}{s^6}\right)\right]$$

$$\boxed{Y(t) = t^3 + \frac{t^5}{20}}$$

2) $L\{f\}(s) = F(s) = \frac{1}{s(s+3)}$

$H(s) = \frac{1}{s}$ e $G(s) = \frac{1}{s+3}$

$f(t) = (H * G)(t) = \int_0^t e^{-3r} \cdot 1 \, dr$

$f(t) = (H * G)(t) = -\frac{1}{3} e^{-3r} \Big|_0^t$

$f(t) = (H * G)(t) = -\frac{1}{3} e^{-3t} - \left(-\frac{1}{3} e^{-3 \cdot 0}\right)$

$$\boxed{f(t) = (H * G)(t) = \frac{1}{3} - \frac{1}{3} e^{-3t}}$$

3) $\begin{cases} x'(t) = x + y \\ y'(t) = y \end{cases}$

$\begin{cases} x' = x + y & (1) \\ y' = y & (2) \end{cases} \Rightarrow \begin{array}{l} y = x' - x \ (3) \\ y'' = x'' - x' \ (4) \end{array}$

$x'' - x' = x' - x$

$x'' - 2x' + x = 0$

Aplicando a equação auxiliar:

$a = 1$

$b = -2$

$c = 1$

$\Delta = (-2)^2 - 4 \cdot 1 \cdot 1$

$\Delta = 4 - 4$

$\Delta = 0$

$r = \frac{-(-2) \pm 0}{2} \qquad r_1 = r_2 = 1$

$x_1(t) = e^t \ ; \ x_2(t) = t e^t$

$$\boxed{x(t) = c_1 e^t + c_2 t e^t}$$

$$\boxed{x'(t) = c_1 e^t + c_2 e^t + c_2 t e^t}$$

$$Y(t) = c_1 e^t + c_2 e^t + c_2 t e^t - (c_1 e^t + c_2 t e^t)$$

$$\boxed{Y(t) = c_2 e^t}$$

Solução:

$$\begin{vmatrix} X(t) \\ Y(t) \end{vmatrix} = \begin{vmatrix} c_1 e^t + c_2 t e^t \\ c_2 e^t \end{vmatrix}$$