SINGULAR AND CRITICAL LOCUS

Orlov:

(Derived cat. ... 2009)

$X \subset \mathbb{P}^n$ nonsingular hypersurface of degree d

and the defining equation ω.

1. If $d < n+1$, there is a fully faithful functor

   $MF(\omega) \longrightarrow D^b coh(X)$ $\approx$ vector bundles

2. If $d = n+1$ then there is an equivalence

   $D^b coh(X) \cong MF(\omega)$

3. If $\omega > n+1$, then there is a fully faithful exact functor $D^b coh(X) \longrightarrow MF(\omega)$

Dyckerhoff:

Generalization of Orlov to hypersurfaces with isolated singularities ??

One application:

$$HH_*(MF(R,w)) \cong R/(\partial_1 w, \dots, \partial_n w)$$

Singularities

Consider a hypersurface $S = \{(x_1, \dots, x_n) \mid f(x_1 \dots x_n) = 0\}$

$f \in C^\infty(\mathbb{R}^n)$.

Pick a point $a = (a_1, \dots, a_n)$ in S and apply Taylor's formula for $x = (x_1, \dots, x_n) = a + h$ (h small)

$$f(x_1, \dots, x_n) = \sum \frac{\partial f}{\partial x_i}\bigg|_{(a_1, \dots, a_n)} h_i + \frac{1}{2} \sum \frac{\partial^2 f}{\partial x_i \partial x_j}\bigg|_{(a_1 \dots a_n)} h_i h_j + \dots$$

Tangent plane at $a \in S$ is the hyperplane

$$\sum \frac{\partial f}{\partial x_i}\bigg|_{(a_1, \dots, a_n)} (x_i - a_i) = 0$$

Definition Let $Y \subseteq \mathbb{A}^n$ be an affine variety and let $f_1, \ldots, f_t \in A = k[x_1, \ldots, x_n]$ be a set of generators for the ideal of Y. Y is nonsingular at a point $P \in Y$ if the rank of the matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \end{pmatrix}$$

is $n - r$, where r is the dimension of Y.

Idea

$$F : Y \longrightarrow \mathbb{A}^n$$

$F = (f_1, \ldots, f_t)$ is a submersion if the above condition is satisfied.

We know from topology, that preimage of a regular value is a smooth manifold.

Thm. Let $Y \subseteq \mathbb{A}^n$ be an affine variety. Let $P \in Y$ be a point. Y is nonsingular at P if and only if the local ring $\mathcal{O}_{P,Y}$ is a regular local ring.

Proof. $P = (a_1, \dots, a_n)$

$$\mathfrak{a}_P = (x_1 - a_1, \dots, x_n - a_n) \text{ (maximal ideal)}$$

Define

$$\theta : A \longrightarrow k^n$$

$$\theta(f) = \left(\frac{\partial f}{\partial x_1}(P), \dots, \frac{\partial f}{\partial x_n}(P) \right)$$

Note $\theta(x_i - a_i)$ form a basis of k^n.

$$\theta(\mathfrak{a}_P^2) = 0$$

$$\operatorname{Ker} = \mathfrak{a}_P^2$$

$$\therefore \theta : \mathfrak{a}_P / \mathfrak{a}_P^2 \xrightarrow{\simeq} k^n$$

Consider $I(Y)$ the vanishing ideal of Y.

Say $I(Y) = \langle f_1, \dots, f_t \rangle$.

The rank of Jacobian is dimension of $\theta(I(Y))$ as a subspace of k^n.

Using the isomorphism θ', this is the dimension of

$$(I(Y) + \mathfrak{a}_p^2) / \mathfrak{a}_p^2 \subseteq \mathfrak{a}_p / \mathfrak{a}_p^2$$

But the local ring $\mathcal{O}_p = A/I(Y)_{\mathfrak{a}_p}$

Thus if $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_p$,

$$\mathfrak{m}/\mathfrak{m}^2 \cong \mathfrak{a}_p / (I(Y) + \mathfrak{a}_p^2)$$

we have $\dim \mathfrak{m}/\mathfrak{m}^2 + \operatorname{rank} J = n$

Let $\dim Y = r$. Then $\dim \mathcal{O}_p = r$.

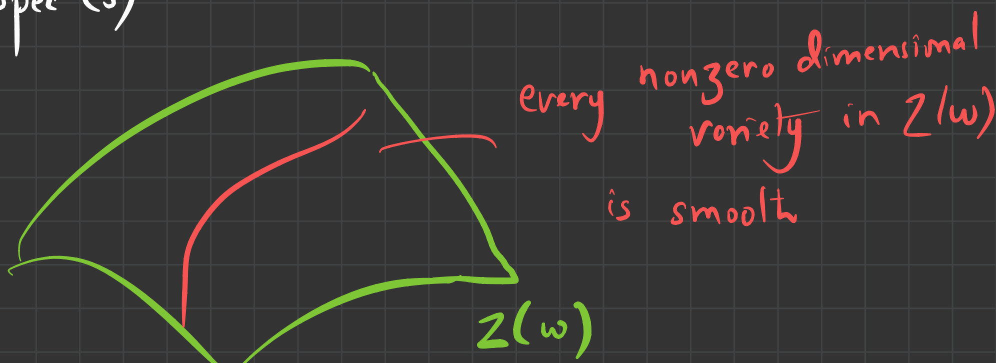
so $\mathcal{O}_p$ is regular iff $\operatorname{rank} J = n - r$

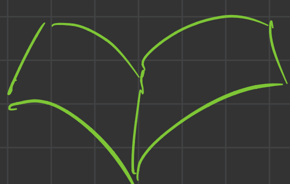
$\Updownarrow$

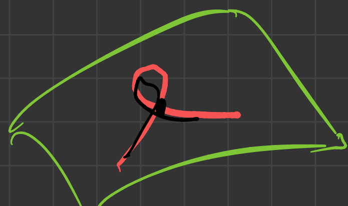
P nonsingular

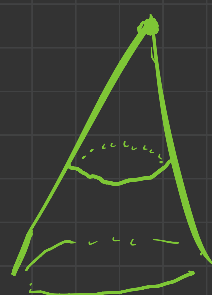
Prop 3.1 Let k be a field of characteristic 0. Consider a polynomial $\omega \in R = k[x_1, \dots, x_n]$. Let Z be the scheme-theoretic zero locus of the 1-form $d\omega$ on $\mathbb{A}^n$ (i.e. the critical locus of ω) and denote the hypersurface algebra $R/(\omega)$ by S. TFAE:

1) The hypersurface $\operatorname{Spec}(S)$ has isolated singularities, i.e. $S_{\mathfrak{p}}$ is regular for every non-maximal prime $\mathfrak{p} \in \operatorname{Spec}(S)$

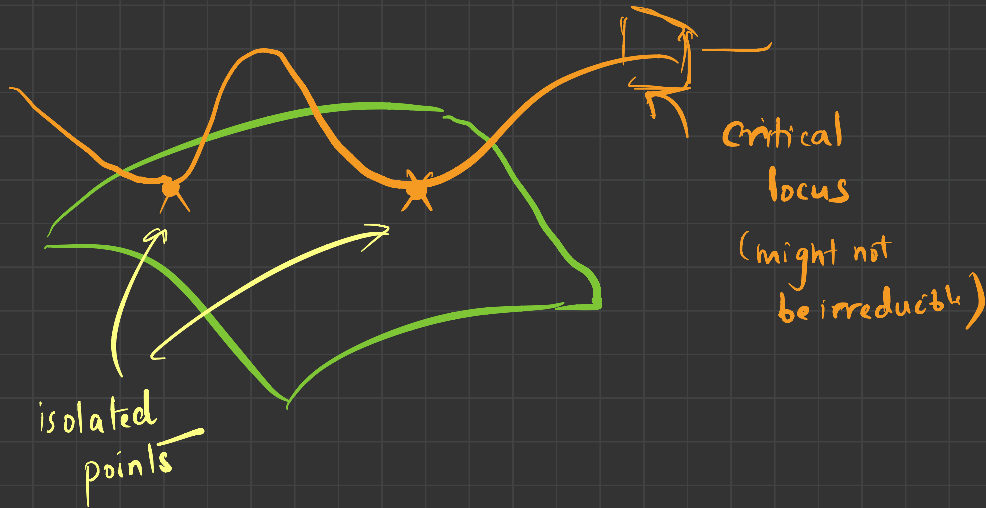








2. The restriction of the critical locus Z to $\mathrm{Spec}(S)$ is a 0-dimensional scheme



3. For each singular point $m \in \mathrm{Spec}(S)$, the restriction of the critical locus Z to $\mathrm{Spec}(R_m)$ is a 0-dim. scheme.

Thm (Jacobian Criterion)

Let $A = k[x_1, \dots, x_n]$. Let $I = \langle f_1, \dots, f_s \rangle$ be an ideal. and $R = A/I$. Let $\mathfrak{p}$ be a prime ideal of A containing I. Let c be the codimension of $I_{\mathfrak{p}}$ in $S_{\mathfrak{p}}$.

Then $J = (\partial f_i / \partial x_j)$ modulo $\mathfrak{p}$ has rank $\leq c$

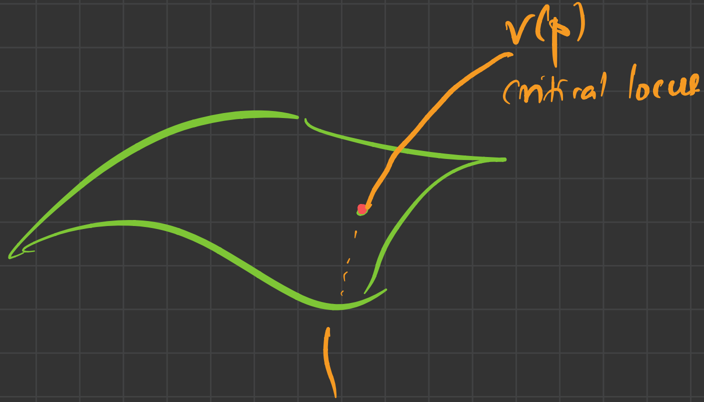
ideal has pure codimension c if all its minimal prims have codimension c.

Corollary Let $R = k[x_1, \dots, x_n]/I$. Suppose I has pure codimension c. Suppose $I = \langle f_1, \dots, f_s \rangle$. If J is the ideal generated by the $c \times c$ minors of the Jacobian, then J defines the singular locus of R in the sense that a prime $\mathfrak{p}$ contains J iff $R_{\mathfrak{p}}$ is not a regular local ring.

Proof $1 \Leftrightarrow 2$ follows from Jacobian criterion.

$(2) \Rightarrow (3)$

Assume (2) and $\exists$ a $\mathfrak{m} \in \operatorname{Spec}(S)$ st. the critical locus does not restrict to a 0-dimensional scheme of $\operatorname{Spec}(R_{\mathfrak{m}})$.



$\exists$ a non-maximal prime $\mathfrak{p} \subset \mathfrak{m}$ along which dw vanishes.

Claim: w must be constant along $V(\mathfrak{p})$ and hence zero.

If w were nonconstant on $V(\mathfrak{p})$ it would map $V(\mathfrak{p})$ into a dense subset U of $\mathbb{A}^1$.

Since $d\omega$ vanishes along $V(p)$, the Jacob.

$\Rightarrow$ every fiber of ω over U is singular.

This contradicts generic smoothness

Algebraic Sard's thm

$f: X \to Y$ is a dominant morphism between

varieties Then there is an open dense

set subvariety U of X s.t. $U \to Y$ is

smooth

<u>Prop.</u> $f : X \to Y$ affine varieties.

$\varphi : \mathcal{O}(Y) \to \mathcal{O}(X)$

$W \subseteq X$ be closed. Then

$$I_Y(\overline{f(W)}) = \varphi^{-1}(I_X(W))$$

$$I_Y(\overline{f(W)}) = I_Y(f(W))$$

$$= \{ g \in \mathcal{O}(Y) \mid g(f(u)) = 0 \ \forall u \in W \}$$

$$= \{ g \in \mathcal{O}(Y) \mid (\varphi(g))(u) = 0 \ \forall u \in W \}$$

$$= \varphi^{-1}(I_X(W))$$

Generators in matrix factorization categories

Fix (R, w) containg $(R, \mathfrak{m}, \Bbbk)$ a regular local ring of Krull dimension n containing $\Bbbk$ and $w \in \mathfrak{m}$. We assume (R, w) has isolated singular locus.

We enlarge the category to $MF^{\infty}(R, w)$ where we allow the R-modules to have infinite rank.

Def. Triangulated category

Let $\mathcal{T}$ be a triangulated category admitting infinite coproducts. Let $X \in ob(\mathcal{T})$.

Def X is called compact if the functor $Hom(X, -)$ commutes with infinite coproducts

Def X is called the generator of $\mathcal{T}$ if the smallest triangulated subcategory of $\mathcal{T}$ containing X and closed under coproducts and isomorphisms is $\mathcal{T}$ itself.

Thm. Assume that (R, ω) has isolated singular locus and consider the residue field k as an S module. Then k^{stab} is a compact generator of the triangulated category

$$[MF^{\infty}(R, \omega)]$$

Stabilization

S coordinate ring of the hypersurface

$S = R/(w)$

L S-module.

$L = R/I$

$I = \langle f_1, \dots, f_m \rangle$ regular sequence

Geometric meaning:

At every step the hypersurface $Z(f_i)$ cuts every component of $Z(f_1, \dots, f_{i-1})$ beyond hence reduces the dimension by 1.

$K = (\Lambda^* V, s_0)$

$V = R^m$

s_0 contraction with $(f_1, \dots, f_m) \in \mathrm{Hom}_R(V, R)$

Koszul Complex

N be R-module

ΛN free algebra $R \oplus N \oplus N \otimes N \oplus \cdots$

$x \otimes y = -y \otimes x$

$x \otimes x = 0$

"graded algebra"

$a \wedge b = (-1)^{(\deg a)(\deg b)}\, b \wedge a$

$x \in N$

$K(x): \quad 0 \to R \to N \to \Lambda^2 N \to \cdots \Lambda^i N \xrightarrow{d_x} \Lambda^{i+1} N \to \cdots$

$d_x(a) = x \wedge a$

$\not{p}$

$K(f_1, \dots, f_m)$

$$0 \to R \to R^m \to \Lambda^2 R^m \to \cdots \cdots$$

Stab.

$$K(s_0): \quad 0 \to \Lambda^n V \longrightarrow \Lambda^{n-1} V \to \cdots \to \Lambda^2 V \to V \to R \to 0$$