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- I replace the lowest midterm score with the final exam score
- I will grade it this weekend.
- I can give extension to the HW due yesterday.
- HW 4 Due Next Wednesday / Thursday.
- Ch. 2 Rate of Change  
Differentiation, Derivatives.
- Gilbert Strang's Calculus  
Tarasov's Calculus  
Skim through Ch 1 of Strang  
Ch. 8, 9 of Tarasov

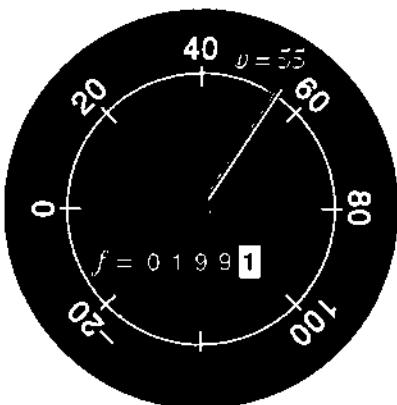


# Introduction to Calculus

## 1.1 Velocity and Distance

The right way to begin a calculus book is with calculus. This chapter will jump directly into the two problems that the subject was invented to solve. You will see what the questions are, and you will see an important part of the answer. There are plenty of good things left for the other chapters, so why not get started?

The book begins with an example that is familiar to everybody who drives a car. It is calculus in action—the driver sees it happening. The example is the relation between the *speedometer* and the *odometer*. One measures the speed (or *velocity*); the other measures the *distance traveled*. We will write  $v$  for the velocity, and  $f$  for how far the car has gone. The two instruments sit together on the dashboard:



**Fig. 1.1** Velocity  $v$  and total distance  $f$  (at one instant of time).

Notice that the units of measurement are different for  $v$  and  $f$ . The distance  $f$  is measured in kilometers or miles (it is easier to say miles). The velocity  $v$  is measured in km/hr or *miles per hour*. A unit of *time* enters the velocity but not the distance. Every formula to compute  $v$  from  $f$  will have  $f$  divided by time.

*The central question of calculus is the relation between  $v$  and  $f$ .*

Can you find  $v$  if you know  $f$ , and vice versa, and how? If we know the velocity over the whole history of the car, we should be able to compute the total distance traveled. In other words, if the speedometer record is complete but the odometer is missing, its information could be recovered. One way to do it (without calculus) is to put in a new odometer and drive the car all over again at the right speeds. That seems like a hard way; calculus may be easier. But the point is that the information is there. If we know everything about  $v$ , there must be a method to find  $f$ .

What happens in the opposite direction, when  $f$  is known? If you have a complete record of distance, could you recover the complete velocity? In principle you could drive the car, repeat the history, and read off the speed. Again there must be a better way.

The whole subject of calculus is built on the relation between  $v$  and  $f$ . The question we are raising here is not some kind of joke, after which the book will get serious and the mathematics will get started. On the contrary, I am serious now—and the mathematics has already started. We need to know how to find the velocity from a record of the distance. (That is called *differentiation*, and it is the central idea of *differential calculus*.) We also want to compute the distance from a history of the velocity. (That is *integration*, and it is the goal of *integral calculus*.)

Differentiation goes from  $f$  to  $v$ ; integration goes from  $v$  to  $f$ . We look first at examples in which these pairs can be computed and understood.

### CONSTANT VELOCITY

Suppose the velocity is fixed at  $v = 60$  (miles per hour). Then  $f$  increases at this constant rate. After two hours the distance is  $f = 120$  (miles). After four hours  $f = 240$  and after  $t$  hours  $f = 60t$ . We say that  $f$  increases *linearly* with time—its graph is a straight line.

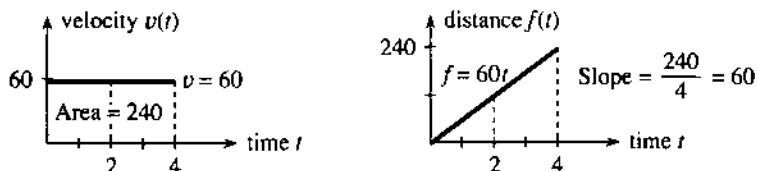


Fig. 1.2 Constant velocity  $v = 60$  and linearly increasing distance  $f = 60t$ .

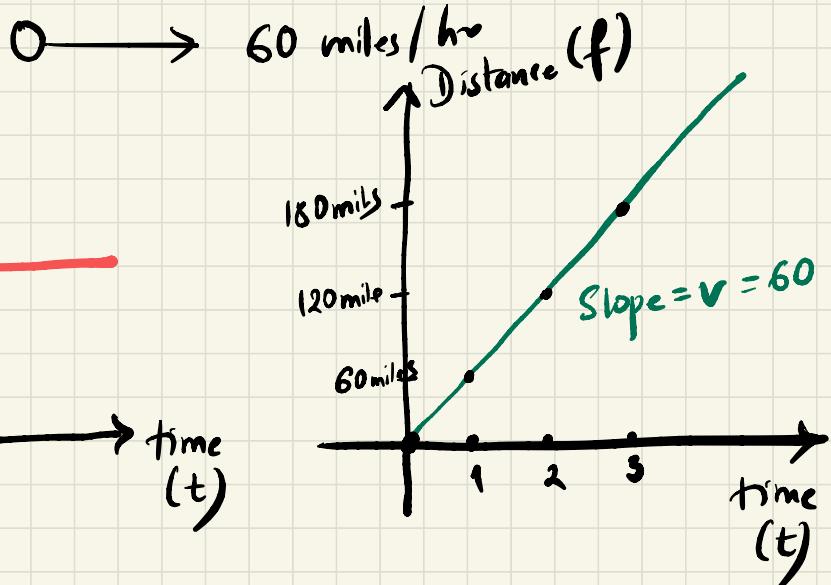
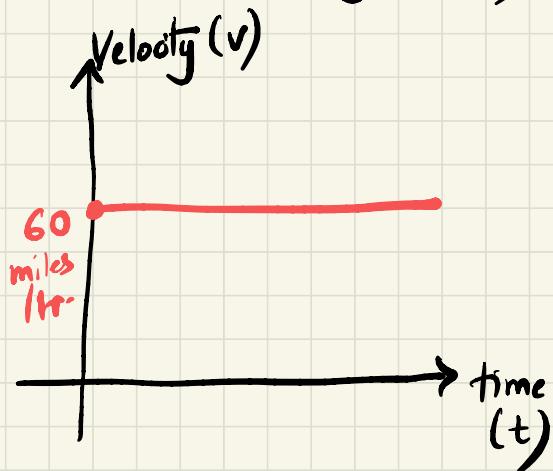
Notice that this example starts the car at full velocity. No time is spent picking up speed. (The velocity is a “step function.”) Notice also that the distance starts at zero; the car is new. Those decisions make the graphs of  $v$  and  $f$  as neat as possible. One is the horizontal line  $v = 60$ . The other is the sloping line  $f = 60t$ . This  $v, f, t$  relation needs algebra but not calculus:

if  $v$  is constant and  $f$  starts at zero then  $f = vt$ .

The opposite is also true. When  $f$  increases linearly,  $v$  is constant. The division by time gives the slope. The distance is  $f_1 = 120$  miles when the time is  $t_1 = 2$  hours. Later  $f_2 = 240$  at  $t_2 = 4$ . At both points, the ratio  $f/t$  is 60 miles/hour. Geometrically, the velocity is the slope of the distance graph:

$$\text{slope} = \frac{\text{change in distance}}{\text{change in time}} = \frac{vt}{t} = v.$$

velocity  $v = 60$  miles/hr



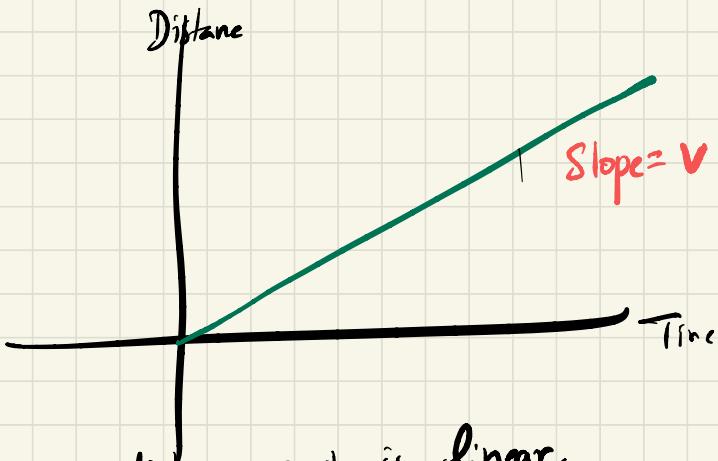
For an object moving at constant velocity,

$$v = \frac{f}{t} \quad \frac{\text{(distance travelled)}}{\text{(time taken)}}$$

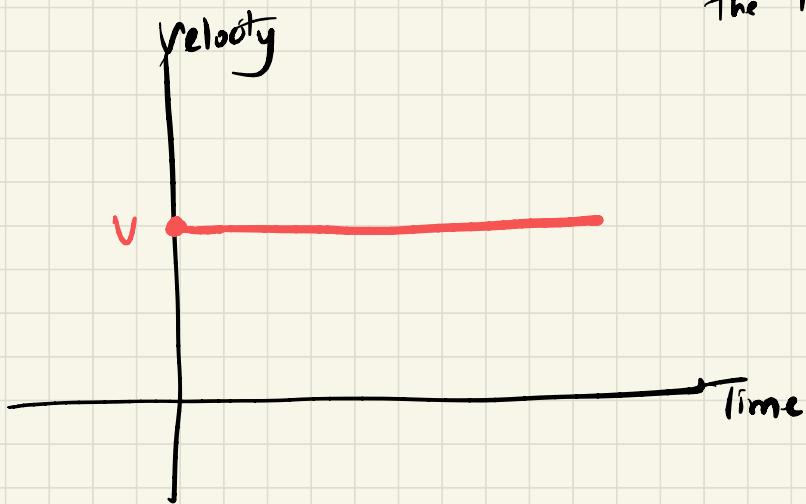
$$\Rightarrow f = vt$$

$$y = vx$$

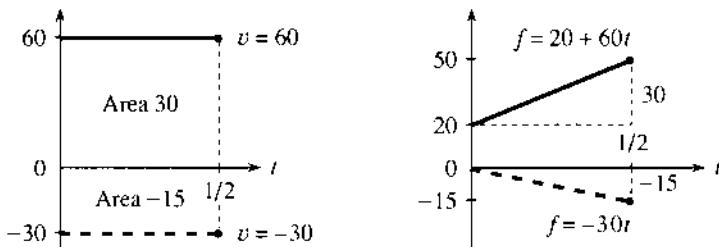
Straight line passing through origin and with slope  $v$



If the distance graph is linear,  
Velocity is constant = Slope of  
the line



## 1.1 Velocity and Distance



**Fig. 1.3** Straight lines  $f = 20 + 60t$  (slope 60) and  $f = -30t$  (slope  $-30$ ).

The slope of the  $f$ -graph gives the  $v$ -graph. Figure 1.3 shows two more possibilities:

1. The distance starts at 20 instead of 0. The distance formula changes from  $60t$  to  $20 + 60t$ . The number 20 cancels when we compute change in distance—so the slope is still 60.
2. When  $v$  is negative, the graph of  $f$  goes downward. The car goes backward and the slope of  $f = -30t$  is  $v = -30$ .

I don't think speedometers go below zero. But driving backwards, it's not that safe to watch. If you go fast enough, Toyota says they measure "absolute values"—the speedometer reads +30 when the velocity is  $-30$ . For the odometer, as far as I know it just stops. It should go backward.<sup>†</sup>

### VELOCITY vs. DISTANCE: SLOPE vs. AREA

How do you compute  $f$  from  $v$ ? The point of the question is to see  $f = vt$  on the graphs. We want to start with the graph of  $v$  and discover the graph of  $f$ . Amazingly, the opposite of slope is area.

**The distance  $f$  is the area under the  $v$ -graph.** When  $v$  is constant, the region under the graph is a rectangle. Its height is  $v$ , its width is  $t$ , and its area is  $v$  times  $t$ . This is integration, to go from  $v$  to  $f$  by computing the area. We are glimpsing two of the central facts of calculus.

**1A** The slope of the  $f$ -graph gives the velocity  $v$ . The area under the  $v$ -graph gives the distance  $f$ .

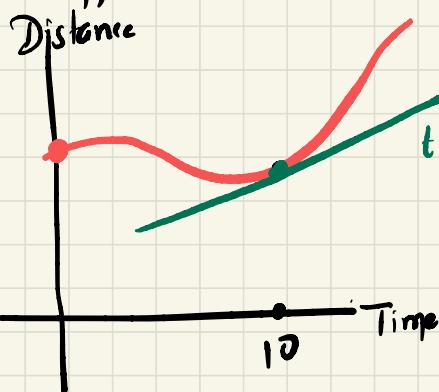
That is certainly not obvious, and I hesitated a long time before I wrote it down in this first section. The best way to understand it is to look first at more examples. The whole point of calculus is to deal with velocities that are *not* constant, and from now on  $v$  has several values.

**EXAMPLE (Forward and back)** There is a motion that you will understand right away. The car goes forward with velocity  $V$ , and comes back at the same speed. To say it more correctly, the *velocity in the second part is  $-V$* . If the forward part lasts until  $t = 3$ , and the backward part continues to  $t = 6$ , *the car will come back where it started*. The total distance after both parts will be  $f = 0$ .

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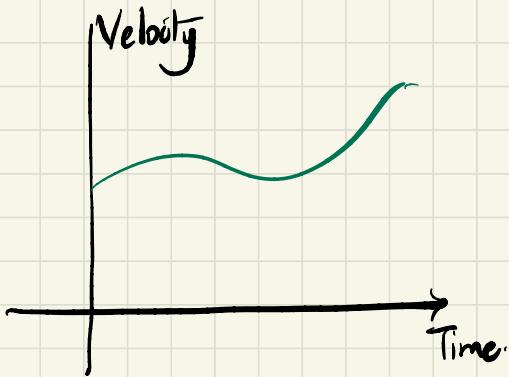
<sup>†</sup>This actually happened in *Ferris Bueller's Day Off*, when the hero borrowed his father's sports car and ran up the mileage. At home he raised the car and drove in reverse. I forgot if it worked.

## Differentiation



You want to find  
the velocity vs. time  
graph.

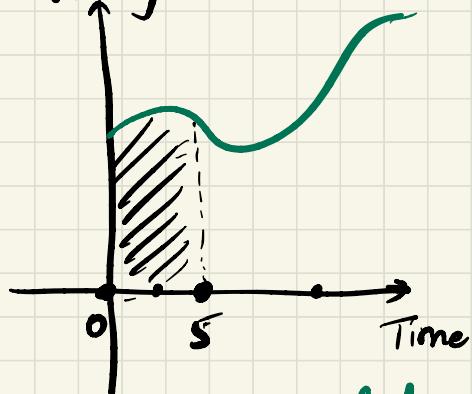
Take derivatives



$$f \rightarrow v$$

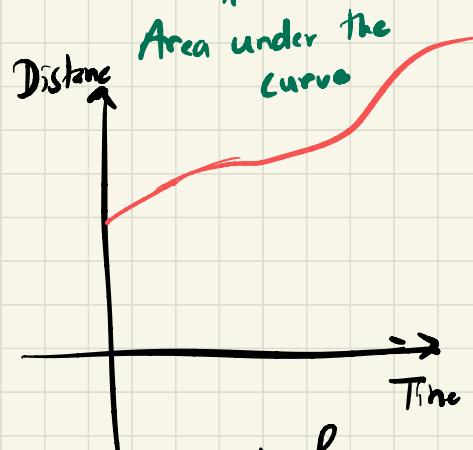
Differentiation

## Integration



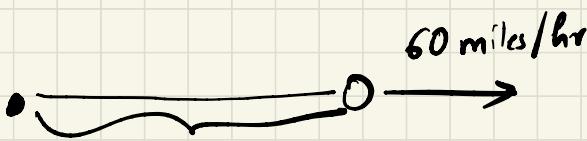
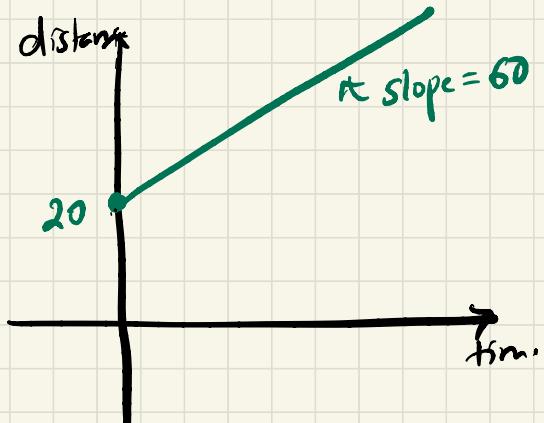
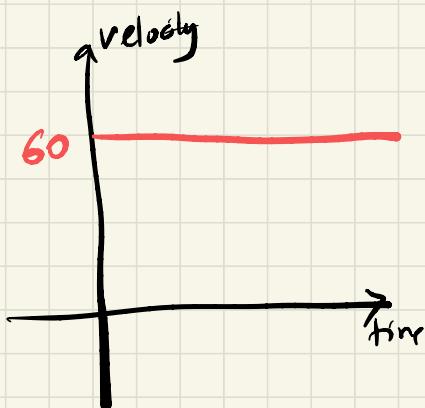
You want to find  
the distance vs time  
time.

Take integral



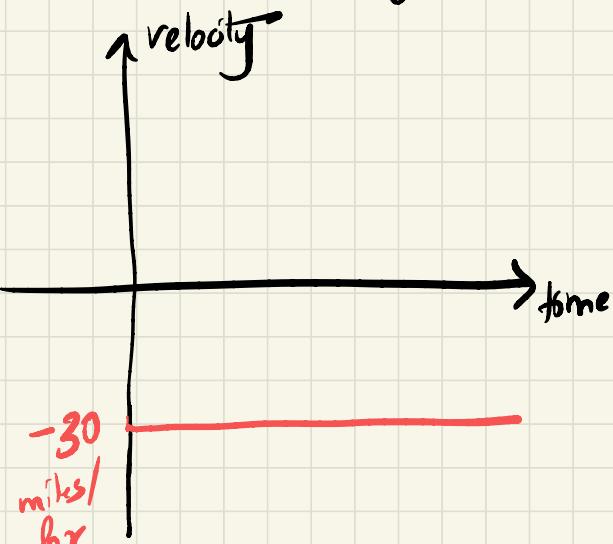
$$v \rightarrow f$$

Integration



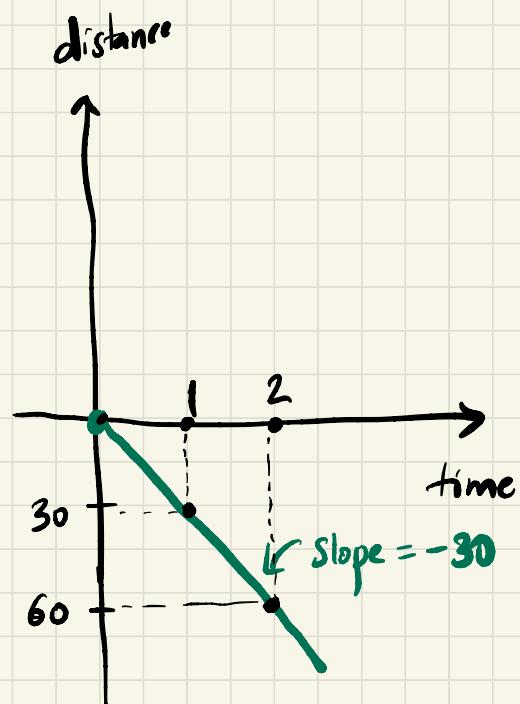
20 miles  
away from  
the origin

velocity



-30  
miles/  
hr

$\leftarrow$  30 miles/hr  
in opposite direction



$0 \rightarrow 0$

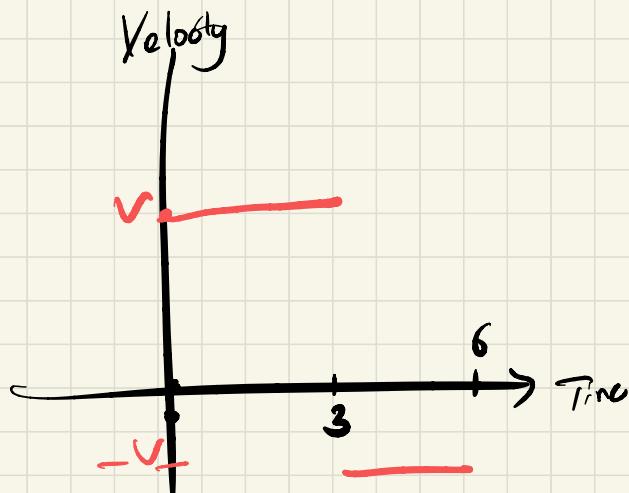
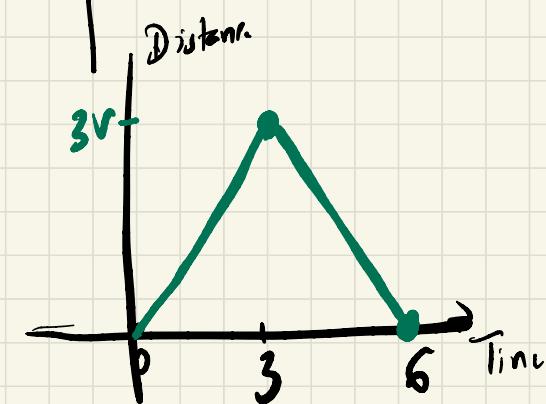
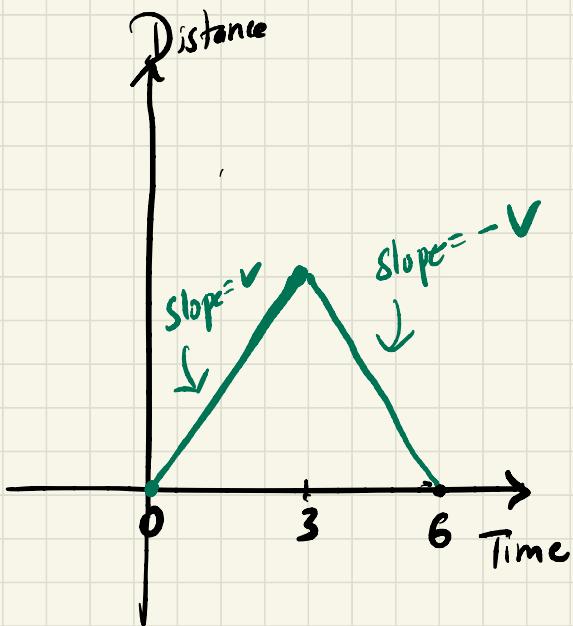
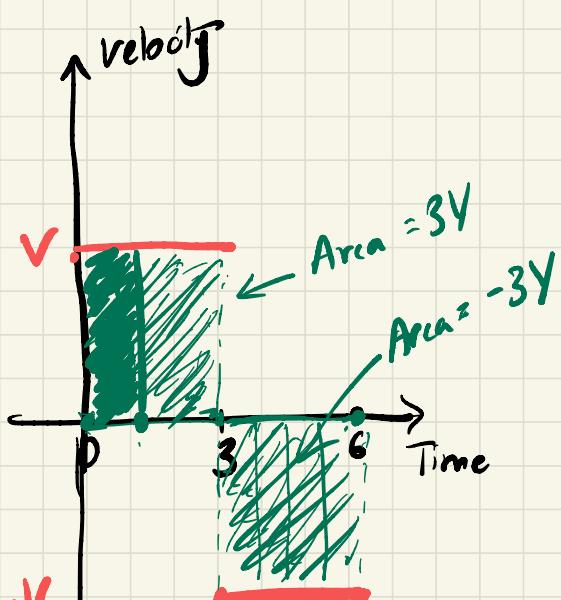
$V$  miles/hr.

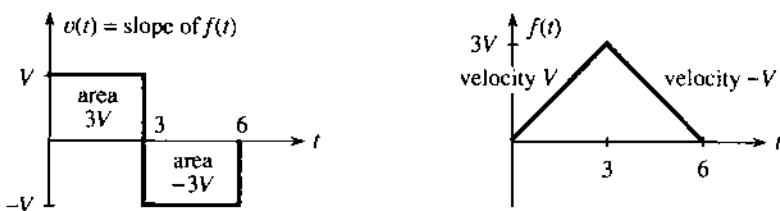
3 hrs.

$0 \leftarrow 0$

$-V$  miles/hr

3 hrs.





**Fig. 1.4** Velocities  $+V$  and  $-V$  give motion forward and back, ending at  $f(6)=0$ .

The  $v$ -graph shows velocities  $+V$  and  $-V$ . The distance starts up with slope  $+V$  and reaches  $f=3V$ . Then the car starts backward. The distance goes down with slope  $-V$  and returns to  $f=0$  at  $t=6$ .

Notice what that means. The total area “under” the  $v$ -graph is zero! A negative velocity makes the distance graph go *downward* (*negative slope*). The car is moving backward. *Area below the axis in the  $v$ -graph is counted as negative.*

## FUNCTIONS

This forward-back example gives practice with a crucially important idea—the concept of a “*function*.” We seize this golden opportunity to explain functions:

*The number  $v(t)$  is the value of the function  $v$  at the time  $t$ .*

The time  $t$  is the *input* to the function. The velocity  $v(t)$  at that time is the *output*. Most people say “ $v$  of  $t$ ” when they read  $v(t)$ . The number “ $v$  of 2” is the velocity when  $t = 2$ . The forward-back example has  $v(2) = +V$  and  $v(4) = -V$ . The function contains the whole history, like a memory bank that has a record of  $v$  at each  $t$ .

It is simple to convert forward-back motion into a formula. Here is  $v(t)$ :

$$v(t) = \begin{cases} +V & \text{if } 0 < t < 3 \\ ? & \text{if } t = 3 \\ -V & \text{if } 3 < t < 6 \end{cases}$$

The right side contains the instructions for finding  $v(t)$ . The input  $t$  is converted into the output  $+V$  or  $-V$ . The velocity  $v(t)$  depends on  $t$ . In this case the function is “discontinuous,” because the needle jumps at  $t = 3$ . *The velocity is not defined at that instant.* There is no  $v(3)$ . (You might argue that  $v$  is zero at the jump, but that leads to trouble.) The graph of  $f$  has a corner, and we can’t give its slope.

The problem also involves a second function, namely the distance. The principle behind  $f(t)$  is the same:  $f(t)$  is the *distance at time  $t$* . It is the net distance forward, and again the instructions change at  $t = 3$ . In the forward motion,  $f(t)$  equals  $Vt$  as before. In the backward half, a calculation is built into the formula for  $f(t)$ :

$$f(t) = \begin{cases} Vt & \text{if } 0 \leq t < 3 \\ V(6-t) & \text{if } 3 \leq t \leq 6 \end{cases}$$

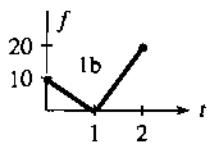
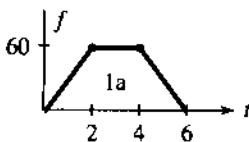
At the switching time the right side gives two instructions (one on each line). This would be bad except that they agree:  $f(3) = 3V$ .† The distance function is “con-

†A function is only allowed one value  $f(t)$  or  $v(t)$  at each time  $t$ .

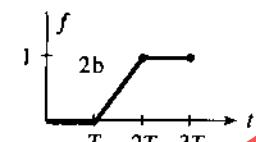
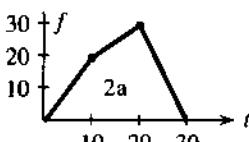
The set of inputs to a function is its F. The set of outputs is its G. The functions  $f(t) = 7 + 3(t - 2)$  and  $f(t) = vt + C$  are H. Their graphs are I with slopes equal to J and K. They are the same function, if  $v = \underline{L}$  and  $C = \underline{M}$ .

**Draw the velocity graph that goes with each distance graph.**

1



2



3 Write down three-part formulas for the velocities  $v(t)$  in Problem 2, starting from  $v(t) = 2$  for  $0 < t < 10$ .

4 The distance in 1b starts with  $f(t) = 10 - 10t$  for  $0 \leq t \leq 1$ . Give a formula for the second part.

5 In the middle of graph 2a find  $f(15)$  and  $f(12)$  and  $f(t)$ .

6 In graph 2b find  $f(1.4T)$ . If  $T = 3$  what is  $f(4)$ ?

7 Find the *average speed* between  $t = 0$  and  $t = 5$  in graph 1a. What is the speed at  $t = 5$ ?

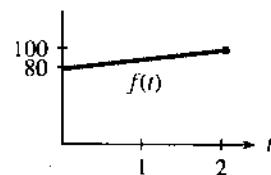
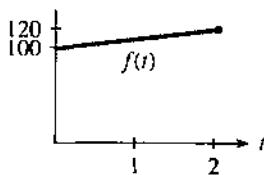
8 What is the average speed between  $t = 0$  and  $t = 2$  in graph 1b? The average speed is zero between  $t = \frac{1}{2}$  and  $t = \underline{\hspace{2cm}}$ .

9 (recommended) A car goes at speed  $v = 20$  into a brick wall at distance  $f = 4$ . Give two-part formulas for  $v(t)$  and  $f(t)$  (before and after), and draw the graphs.

10 Draw any reasonable graphs of  $v(t)$  and  $f(t)$  when

- the driver backs up, stops to shift gear, then goes fast;
- the driver slows to 55 for a police car;
- in a rough gear change, the car accelerates in jumps;
- the driver waits for a light that turns green.

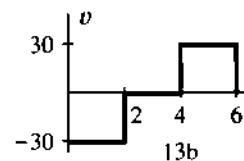
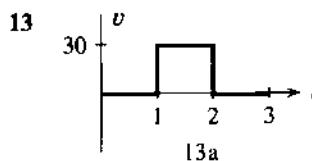
11 Your bank account earns simple interest on the opening balance  $f(0)$ . What are the interest rates per year?



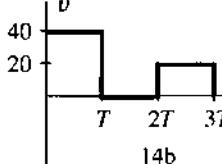
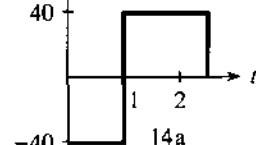
12 The earth's population is growing at  $v = 100$  million a year, starting from  $f = 5.2$  billion in 1990. Graph  $f(t)$  and find  $f(2000)$ .

**Draw the distance graph that goes with each velocity graph. Start from  $f = 0$  at  $t = 0$  and mark the distance.**

13



4



15 Write down formulas for  $v(t)$  in Problem 14, starting with  $v = -40$  for  $0 < t < 1$ . Find the average velocities to  $t = 2.5$  and  $t = 3T$ .

16 Give 3-part formulas for the areas  $f(t)$  under  $v(t)$  in 13.

17 The distance in 14a starts with  $f(t) = -40t$  for  $0 \leq t \leq 1$ . Find  $f(t)$  in the other part, which passes through  $f = 0$  at  $t = 2$ .

18 Draw the velocity and distance graphs if  $v(t) = 8$  for  $0 < t < 2$ ,  $f(t) = 20 + t$  for  $2 \leq t \leq 3$ .

19 Draw rough graphs of  $y = \sqrt{x}$  and  $y = \sqrt{x-4}$  and  $y = \sqrt{x-4}$ . They are "half-parabolas" with infinite slope at the start.

20 What is the break-even point if  $x$  yearbooks cost \$1200 +  $30x$  to produce and the income is  $40x$ ? The slope of the cost line is            (cost per additional book). If it goes above            you can't break even.

21 What are the domains and ranges of the distance functions in 14a and 14b—all values of  $t$  and  $f(t)$  if  $f(0) = 0$ ?

22 What is the range of  $v(t)$  in 14b? Why is  $t = 1$  not in the domain of  $v(t)$  in 14a?

**Problems 23–28 involve linear functions  $f(t) = vt + C$ . Find the constants  $v$  and  $C$ .**

23 What linear function has  $f(0) = 3$  and  $f(2) = -11$ ?

24 Find two linear functions whose domain is  $0 \leq t \leq 2$  and whose range is  $1 \leq f(t) \leq 9$ .

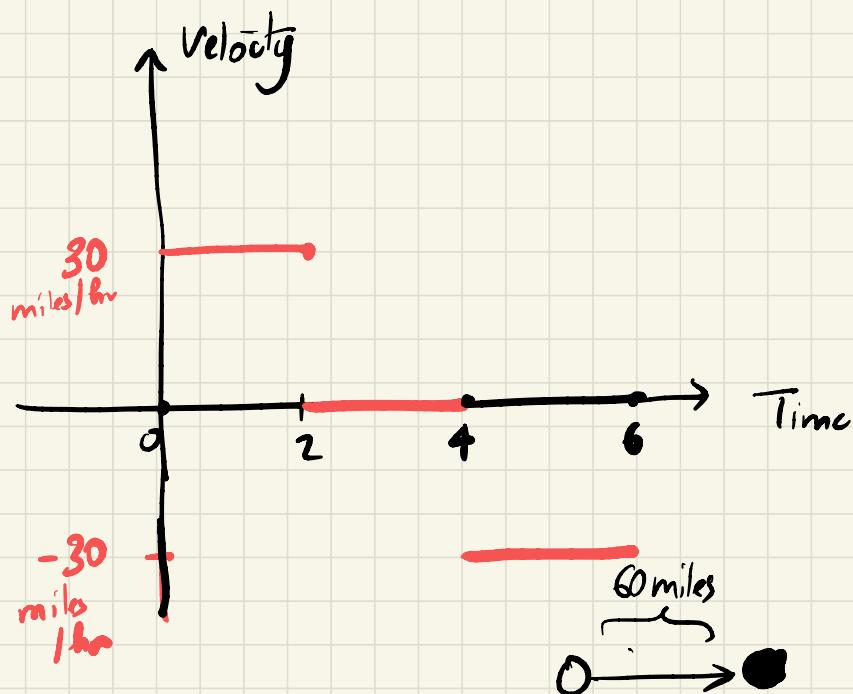
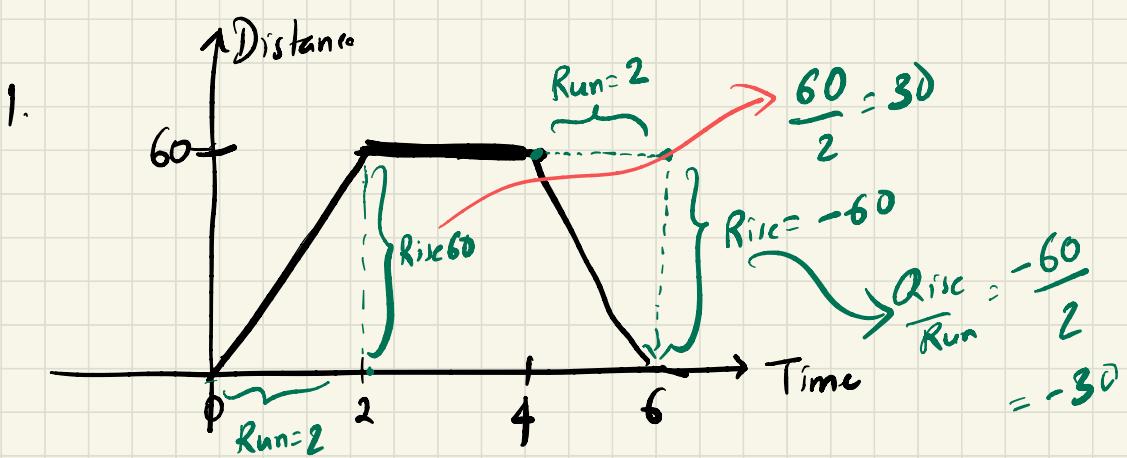
25 Find the linear function with  $f(1) = 4$  and slope 6.

26 What functions have  $f(t+1) = f(t) + 2$ ?

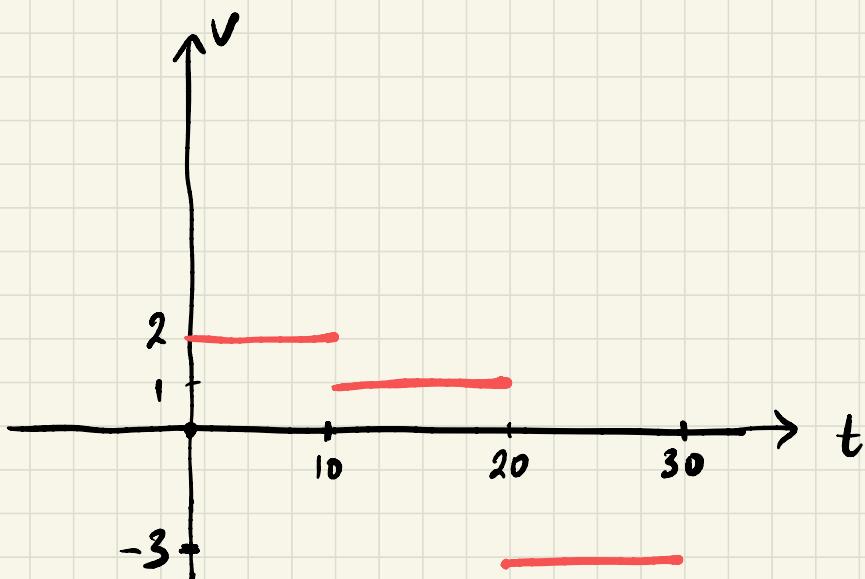
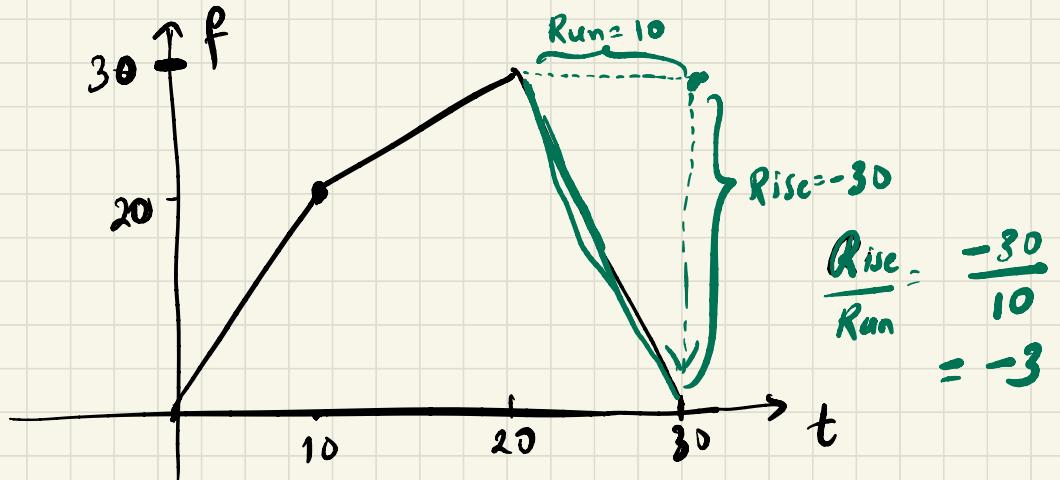
27 Find the linear function with  $f(t+2) = f(t) + 6$  and  $f(1) = 10$ .

28 Find the only  $f = vt$  that has  $f(2t) = 4f(t)$ . Show that every  $f = \frac{1}{2}at^2$  has this property. To go            times as far in twice the time, you must accelerate.

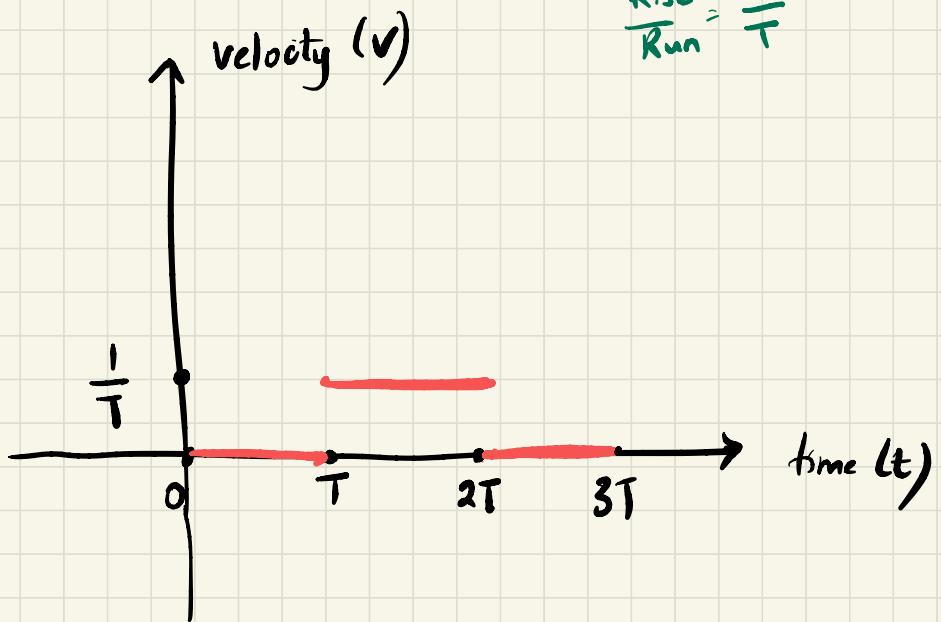
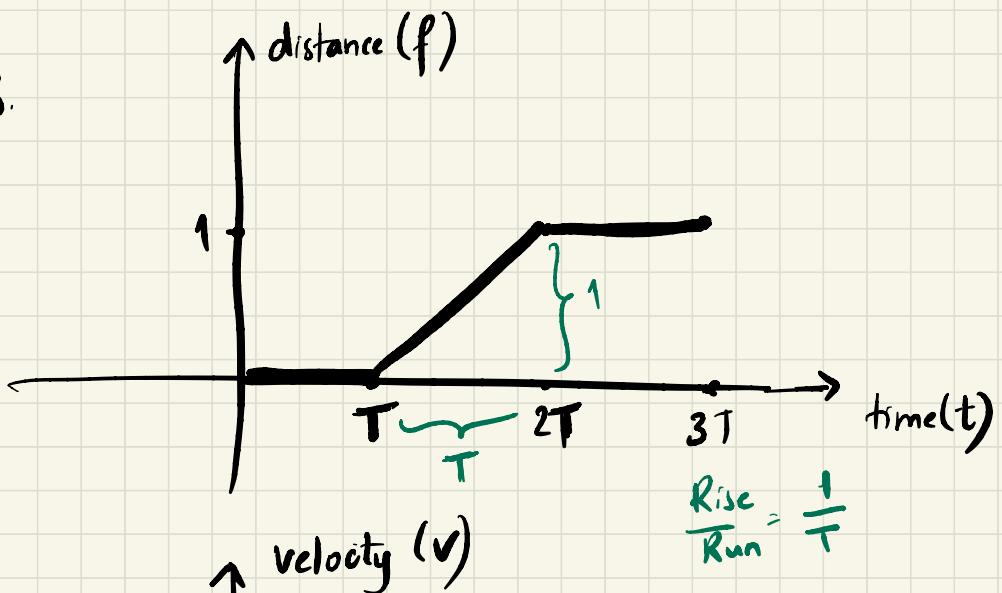
You  
can  
try  
some  
of  
these  
problems

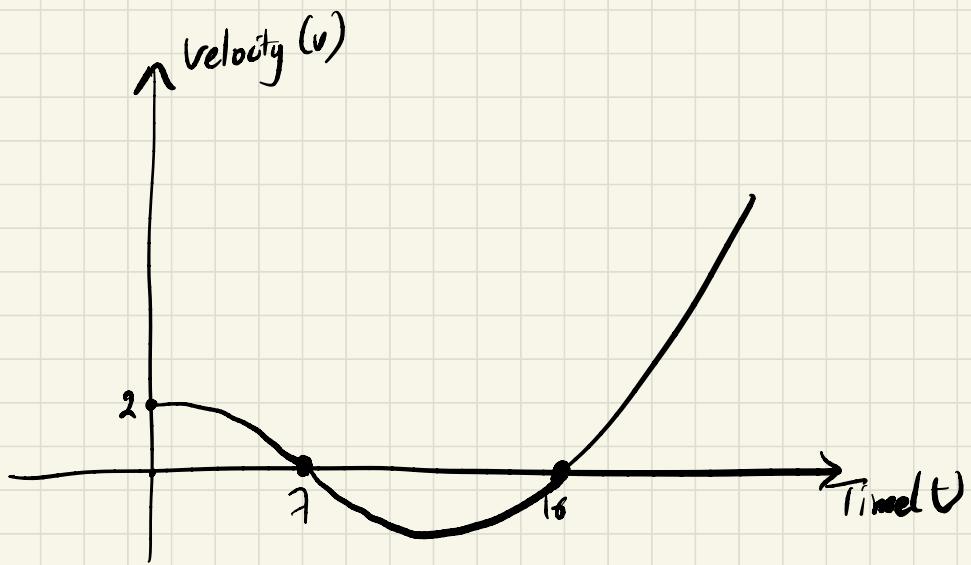
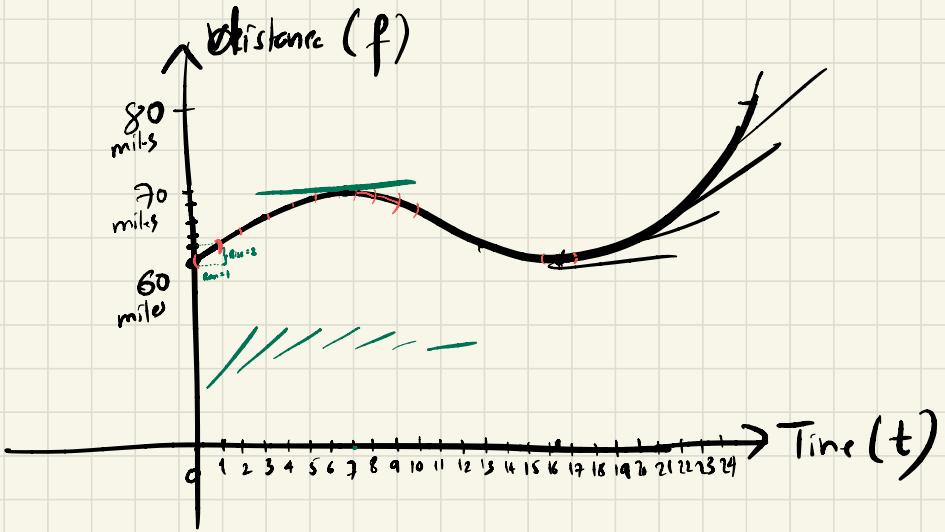


2.



3.



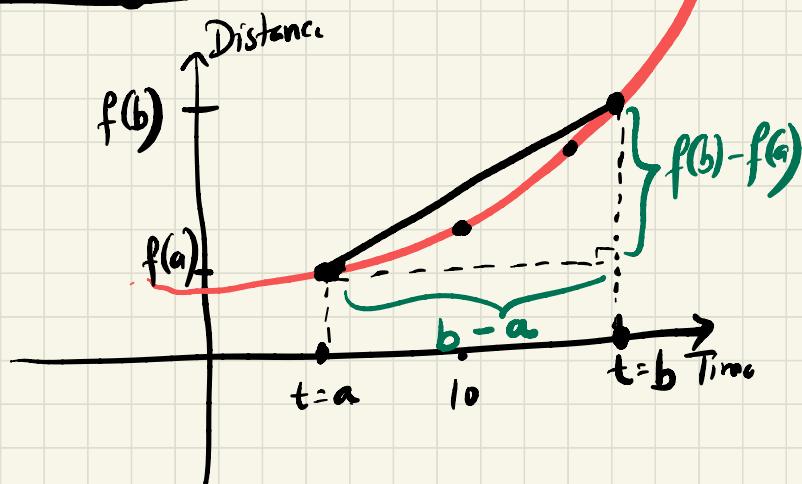


## Instantaneous velocity

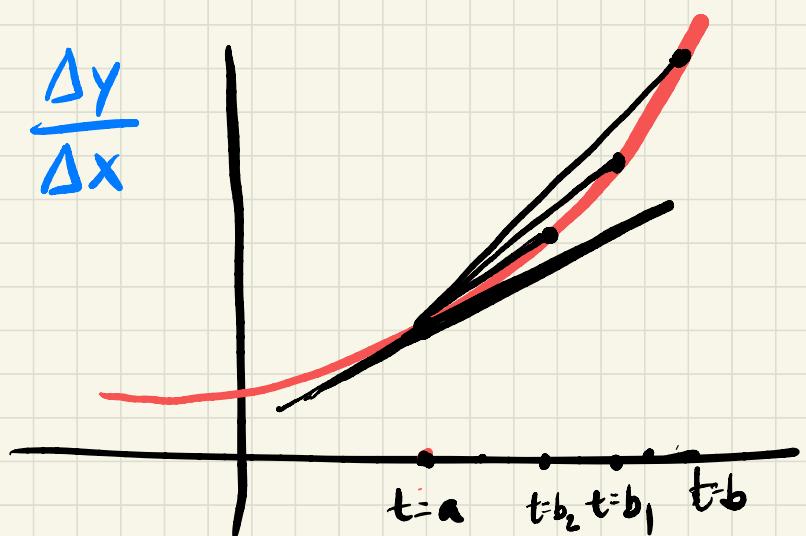
Recall

Average  
velocity

$$\frac{f(b) - f(a)}{b - a}$$



$$\frac{\text{Rise}}{\text{Run}} = \frac{\Delta Y}{\Delta X}$$



$$\frac{\Delta Y}{\Delta X}$$

As  $\Delta X \rightarrow 0$  becomes smaller and smaller, you get that the average velocity approaches the instantaneous velocity.

The limit of these values is the instantaneous velocity.

Instantaneous velocity = slope of the tangent at the graph at time  $t$   
= limit of the average velocities as time interval decreases.

Hw4 Due Next Wednesday / Thursday -

