DET Assignment 4

Name: Ujjwal Rustagi
Roll No: 180108057

Q-1. Probability of generating 1 = θ
Probability of generating 0 = $1-\theta$

$f(\theta) = 6\theta(1-\theta) \qquad 0<\theta<1$

$X \sim$ number of 1's observed in a block of N data
Clearly $X \sim Bi(N, \theta)$

a) $\theta_{MLE} = ?$

We know $f(x/\theta) = {}^{n}C_{x}\, \theta^{x} (1-\theta)^{n-x}$

$x = 0, 1, 2, \ldots, n$

$$\therefore \left.\frac{\partial \ln(f(x/\theta))}{\partial \theta}\right|_{\hat{\theta}_{MLE}} = 0$$

$$\ln(f(x/\theta)) = \ln({}^{n}C_{x}) + x \ln\theta + (n-x)\ln(1-\theta)$$

$$\frac{\partial \ln(f(x/\theta))}{\partial \theta} = \frac{x}{\theta} - \frac{n-x}{1-\theta} = 0.$$

$$\Rightarrow \frac{x}{\theta} = \frac{n-x}{1-\theta}$$

$$\frac{1-\theta}{\theta} = \frac{n-x}{x}$$

Applying C & D, we get

$$\frac{1-\theta+\theta}{\theta} = \frac{n-x+x}{x}$$

$$\Rightarrow \boxed{\theta = \frac{x}{n}}$$

$$\boxed{\text{Thus } \hat{\theta}_{MLE} = \frac{x}{n}}$$ Ans 1(a)

where x = no. of 1's in block of N data

(b) $\hat{\theta}_{MAP} = ?$

$$\left.\frac{\partial \ln(f(\theta))}{\partial\theta}\right|_{\hat{\theta}_{MAP}} + \left.\frac{\partial \ln(f(x/\theta))}{\partial\theta}\right|_{\hat{\theta}_{MAP}} = 0. \quad - (I)$$

$$\ln(f(\theta)) = \ln(6) + \ln(\theta) + \ln(1-\theta)$$

$$\frac{\partial \ln(f(\theta))}{\partial\theta} = \frac{1}{\theta} - \frac{1}{1-\theta} = \frac{1-\theta-\theta}{\theta-\theta^2}$$

$$= \frac{1-2\theta}{\theta(1-\theta)} \quad - (II)$$

From prev. ques 1(a)

$$\frac{\partial \ln(f(x|\theta))}{\partial\theta} = \frac{x}{\theta} - \frac{n-x}{1-\theta} \quad - (III)$$

$\therefore$ (I) evaluates to [using (II) and (III)]

$$\left(\frac{1}{\theta} - \frac{1}{1-\theta}\right) + \left(\frac{x}{\theta} - \frac{n-x}{1-\theta}\right) = 0$$

$$\left(\frac{1+x}{\theta}\right) - \left(\frac{1+n-x}{1-\theta}\right) = 0.$$

$$\frac{1-\theta}{\theta} = \frac{1+n-x}{1+x}$$

Applying C&D, we get

$$\frac{1-\theta+\theta}{\theta} = \frac{1+n-x+1+x}{1+x}$$

$$\frac{1}{\theta} = \frac{n+2}{1+x}$$

$$\Rightarrow \boxed{\hat{\theta}_{MAP} = \frac{1+x}{2+n}} \quad \text{Ans 1(b)}$$

Q-2. X is uniformly distributed between 0 and θ. Where $\theta \sim U(0,1)$.

~~(a) $\hat{\theta}_{MLE} = ?$~~

$\Rightarrow f(\theta) = \frac{1}{1-0} = 1$ and $f(x|\theta) = \frac{1}{\theta}$

~~$\because X$ is u~~

where $x <= \theta < 1$

(a) $\hat{\theta}_{MLE} = ?$

This likelihood will be maximum if we take $\theta = x$ ~~and~~ as $\frac{1}{\theta}$ is decreasing for

$\Rightarrow \boxed{\hat{\theta}_{MLE} = x} \longrightarrow$ Ans. 2(a)

Similarly for N observations $x_1, x_2, \dots, x_N$ as the product of pdf is θ^{-n} which is a decreasing function for $\theta \geq \max(x_1, x_2, \dots, x_N)$
It is decreasing function because $\frac{\partial \ln L}{\partial \theta} = \frac{-n}{\theta} < 0$

$\Rightarrow \boxed{L(x|\theta) \text{ is maximized at } \theta = \max(x_1, x_2, \dots, x_N).}$

(b) $\hat{\theta}_{MMSE} = ?$

$$f(\theta/x) = \frac{f(\theta) \cdot f(x/\theta)}{f(x)}$$

$$= \frac{1 \cdot \frac{1}{\theta}}{f(x)} = \frac{1}{\theta \cdot f(x)} \quad \text{— (I)}$$

Now $f(x) = \int_{-\infty}^{\infty} f(\theta) \cdot f(x/\theta) d\theta$

$$= \int_{-\infty}^{\infty} 1 \times \frac{1}{\theta} d\theta$$

$$= \int_{x}^{1} \frac{1}{\theta} d\theta = [\ln \theta]_x^1 = -\ln x \quad \text{— (II)}$$

Using (II) in (I), we have $\boxed{f(\theta/x) = \frac{-1}{\theta \cdot \ln x}}$

Now $\hat{\theta}_{MMSE} = \int_{-\infty}^{\infty} \theta f(\theta/x)\, d\theta$

$= \int_{-\infty}^{\infty} \theta \cdot \frac{-1}{\theta \cdot \ln x} d\theta = \int_{x}^{1} \theta \cdot \frac{-1}{\theta \ln x} d\theta$

$= \frac{-1}{\ln x} \int_{x}^{1} d\theta$

$= \frac{x-1}{\ln x}$

$\Rightarrow \boxed{\hat{\theta}_{MMSE} = \frac{x-1}{\ln x}}$ Ans. 2(b)

Q-3. X_i's are iid and $X_i \sim exp(\lambda)$. $i = 1, 2, 3, 4, \dots, N$

(a) To show: $\hat{\lambda}_{MM} = \hat{\lambda}_{MLE} = \frac{1}{\overline{X}}$

where $\overline{X} = \frac{1}{N} \sum_{i=1}^{N} X_i$

pdf is given by, $f(x) = \lambda e^{-\lambda x}$ $x > 0, \lambda > 0$.

Using Method of Moment Estimator of λ, (λ_{MM})

$\cancel{\frac{1}{N}\sum x_i}$ $\frac{1}{N}\sum_{i=1}^{N} x_i = E(X)$ [1st sample moment equals to corresponding population moment]. (A)

(i) $E(X) = \int_{-\infty}^{\infty} x \lambda e^{-\lambda x} dx$

$= \lambda \int_{-\infty}^{\infty} e^{-\lambda x} x\, dx$

$= \lambda \int x^{2-1} e^{-\lambda x} dx$ [$\because$ Gamma distribution]

$$\cancel{= \frac{\lambda\sqrt{2}}{\lambda^2} = \frac{\sqrt{2}}{\lambda}}$$

$$\therefore \cancel{E(X) = \frac{\sqrt{2}}{\lambda}}$$

$$= \frac{\lambda\sqrt{2}}{\lambda^2}$$

$$= \frac{\sqrt{2}}{\lambda}$$

~~symbol use~~

$$\because \sqrt{P} = P-1\sqrt{P-1}$$
$$\sqrt{2} = 1\sqrt{1} = 1$$

$$\Rightarrow \boxed{E(X) = \frac{1}{\lambda}}$$

From (A) $\frac{1}{N}\sum_{i=1}^{N} x_i = E(X) = \frac{1}{\lambda}$

$$\Rightarrow \boxed{\hat{\lambda}_{MM} = \frac{1}{\bar{X}} \quad \text{where } \bar{X} = \frac{1}{N}\sum_{i=1}^{N} x_i} \quad (B)$$

Now calculating maximum likelihood estimator function of λ, $\hat{\lambda}_{MLE}$

So, likelihood function $L(\theta) = \lambda^n e^{\lambda \sum_i x_i}$

$$\therefore \log L(\theta) = n \log \lambda - \lambda \sum_i x_i$$

$$\Rightarrow \frac{d \log(L(\theta))}{d\theta} = 0 \Rightarrow \frac{n}{\lambda} - \sum x_i = 0$$

$$\Rightarrow \frac{n}{\lambda} = \sum_{i=1}^{N} x_i$$

$$\Rightarrow \lambda = \frac{1}{\bar{X}} \quad \text{where } \bar{X} = \sum_{i=1}^{n} x_i$$

$$\therefore \boxed{\hat{\lambda}_{MLE} = \frac{1}{\bar{X}} \quad \text{where } \bar{X} = \sum_{i=1}^{N} x_i} \quad (C)$$

From (B) and (C), we conclude

$$\boxed{\hat{\lambda}_{MM} = \hat{\lambda}_{MLE} = \frac{1}{\bar{X}}}$$

Hence verified.

(b) To check: $\hat{\lambda}_{MLE}$ is unbiased.

$$E[\hat{\lambda}_{MLE}] = E\left[\frac{1}{\bar{X}}\right]$$

Using Jensen's inequality

$$E\left[\frac{1}{\bar{X}}\right] \leq \frac{1}{E[\bar{X}]} = \frac{1}{E(X)}$$

$$\Rightarrow E[\hat{\lambda}_{MLE}] = \frac{1}{1/\lambda} = \lambda$$

$$\Rightarrow \boxed{E[\hat{\lambda}_{MLE}] = \lambda} \text{ Ans.}$$

Thus we can say that $\hat{\lambda}_{MLE}$ is unbiased

Q.4. Given: X_i's are iid $X_i \sim N(0, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-x^2}{2\sigma^2}\right) \quad x \in (-\infty, \infty)$$
$$\sigma > 0.$$

(a) $\hat{\sigma}^2_{MLE} = ?$

Likelihood function:

$$L(\sigma^2) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \exp\left(-\frac{\Sigma x_i^2}{2\sigma^2}\right)$$

$$\log(L(\sigma^2)) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{\Sigma x_i^2}{2\sigma^2}$$

$$\frac{\partial \log(L(\sigma^2))}{\partial\sigma^2} = 0.$$

$$\Rightarrow \quad -\frac{n}{2}\frac{2\pi}{2\pi\sigma^2} + \frac{\sum_i x_i^2}{2\sigma^4} = 0.$$

$$\frac{n}{2\sigma^2} = \frac{\sum_i x_i^2}{2\sigma^4}$$

$$\Rightarrow \boxed{\sigma^2 = \frac{1}{n}\sum_i x_i^2}$$

$$\therefore \boxed{\hat{\sigma}^2_{MLE} = \frac{1}{n}\sum_{i=1}^{n} x_i^2} \quad \text{Ans 4(a)}$$

(b) To show: $\hat{\sigma}^2_{MLE}$ is efficient.

By using Cramer's Rao Bound inequality, the lower bound of ~~estim~~ MVUE is

$$\frac{\partial \log L(\theta)}{\partial\theta} = -\frac{n}{2} \times \frac{2\pi}{2\pi\sigma^2} + \frac{\sum x_i^2}{2\sigma^4}$$

$$= \frac{\sum x_i^2}{2\sigma^4} - \frac{n}{2\sigma^2}$$

$$= \frac{2\sigma^4}{n}\left[\frac{\sum x_i^2}{n} - \sigma^2\right]$$

$\Rightarrow$ The variance $\hat{\sigma}^2_{MLE} = \dfrac{2\sigma^4}{n}$

$\boxed{\text{By CR inequality, we can say that } \hat{\sigma}_{MLE} \text{ is efficient.}}$

Hence verified.

(c) $\beta = \frac{1}{2}(\theta - 1)$

To find: $\hat{\beta}_{MLE}$.

$$\therefore \hat{\beta}_{MLE} = \beta(\hat{\theta}_{MLE})$$

$$\hat{\beta}_{MLE} = \frac{1}{2}(\hat{\sigma}^2_{MLE} - 1)$$

$$= \frac{1}{2}\left[\frac{1}{N}\Sigma x_i^2 - 1\right]$$

$$= \frac{1}{2N}\left[\Sigma(x_i^2 - 1)\right]$$

Hence,

$$\boxed{\hat{\beta}_{MLE} = \frac{1}{2N}\Sigma(x_i^2 - 1)}$$ Ans (c).