Assignment - 04

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Q-1] $f(\theta) = 6\theta(1-\theta) \quad 0 < \theta < 1$

$p(1) = \theta, \; p(0) = 1-\theta$

$\theta X \sim \text{Binomial}(N, \theta)$

$f(x|\theta) = {}^nC_x \, \theta^x (1-\theta)^{n-x}$ where x denotes number of ones

a) $L(x;\theta) = \log(f(x;\theta))$

$= x\log\theta + (N-x)\log(1-\theta) + \log({}^nC_x)$

$$\left.\frac{\partial L}{\partial \theta}\right|_{\hat{\theta}_{MLE}} = \frac{x}{\theta} - \frac{N-x}{1-\theta} = 0$$

$$\frac{x}{\hat{\theta}_{MLE}} = \frac{N-x}{1-\hat{\theta}_{MLE}}$$

$$\frac{1}{\hat{\theta}_{MLE}} = \frac{N}{x}$$

$$\boxed{\hat{\theta}_{MLE} = \frac{x}{N}}$$

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b) $f(x|\theta) = {}^{n}C_x\, \theta^x (1-\theta)^{n-x}$, $f(\theta) = 6\theta(1-\theta)$

$$f(\theta|x) = \frac{f(x|\theta) \cdot f(\theta)}{f(x)}$$

$$L(\theta|x) = \log f(x|\theta) + \log f(\theta) - \log(f(x))$$

$$L(\theta|x) = x\log\theta + (N-x)\log(1-\theta) + \log {}^{n}C_x + \log 6 + \log\theta + \log(1-\theta) - \log f(x)$$

$$\left.\frac{\partial L}{\partial \theta}\right|_{\hat{\theta}_{MAP}} = \left[\frac{x}{\theta} - \frac{N-x}{1-\theta} + \frac{1}{\theta} - \frac{1}{1-\theta}\right] = 0$$

$$\frac{x+1}{\hat{\theta}_{MAP}} = \frac{N-x+1}{1-\hat{\theta}_{MAP}}$$

$$\boxed{\hat{\theta}_{MAP} = \frac{x+1}{N+2}}$$

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Q-3] $X \sim UNIF(0, \theta]$ with $\theta \sim UNIF(0,1)$

$$f(x|\theta) = \frac{U(x) - U(x-\theta)}{\theta}$$

$$f(\theta) = U(\theta) - U(\theta-1)$$

(a) for $\hat{\theta}_{MLE} \rightarrow f(x|\theta) \rightarrow f(x;\theta)$

$$\therefore \quad f(x;\theta) = \frac{U(x) - U(x-\theta)}{\theta}$$

Clearly shows likelihood is maximum when we take $\theta = x$. To prove it

$$\left.\frac{\partial f(x;\theta)}{\partial\theta}\right|_{\hat{\theta}_{MLE}} = 0$$

$$\frac{\left[\delta(x) + \delta(x-\theta)\right]\theta - \left[U(x) - U(x-\theta)\right]}{\theta^2} = 0$$

$$\hat{\theta}_{MLE} = U(x) - U(x - \hat{\theta}_{MLE})$$

defends only when $\theta = x$ and $x \neq 0$.

$$\boxed{\hat{\theta}_{MLE} = x}$$

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(b) $\hat{\theta}_{MMSE} \Rightarrow ?$

$$f(\theta|x) = \frac{f(\theta) \cdot f(x|\theta)}{f(x)}$$

$$= \frac{1 \cdot 1/\theta}{f(x)} = \frac{1}{\theta \cdot f(x)} \Rightarrow (i)$$

Now

$$f(x) = \int_{-\infty}^{\infty} f(\theta) \cdot f(x|\theta) \cdot d\theta$$

$$f(x) = \int_{-\infty}^{\infty} 1 \times \frac{1}{\theta} \, d\theta = -\ln x \Rightarrow (ii)$$

Using (ii) in (i), $f(\theta|x) = \frac{-1}{\theta \cdot \ln x}$

$$\hat{\theta}_{MMSE} = \int_{-\infty}^{\infty} \theta \cdot f(\theta|x) \cdot d\theta$$

$$= \int_{-\infty}^{\infty} \theta \cdot \frac{-1}{\theta \cdot \ln x} = \frac{-1}{\ln x} \int_{x}^{1} d\theta$$

$$\boxed{\hat{\theta}_{MMSE} = \frac{x-1}{\ln x}}$$

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Q-3] X_i's are iid and $X_i \sim \exp(\lambda)$ $i = 1, 2, 3, \ldots N$

(a) pdf is given by :

$$f(x) = \lambda e^{-\lambda x} \quad x > 0, \lambda > 0$$

Using method of moment estimator of $\lambda (\lambda_{MM})$

$$\frac{1}{N}\sum_{i=1}^{N} X_i = E(X)$$ [1st sample moment corresponds population moment]

(i) $$E(X) = \int_{-\infty}^{\infty} x \cdot \lambda e^{-\lambda x} dx$$

$$= \lambda \int_{-\infty}^{\infty} e^{-\lambda x} x\, dx$$

$$= \lambda \int x^{2-1} e^{-\lambda x} dx$$ (gamma distribution)

$$= \frac{\lambda \Gamma 2}{\lambda^2}$$

$$= \frac{\Gamma 2}{\lambda}$$

$$\Rightarrow \boxed{E(X) = \frac{1}{\lambda}}$$

Thus, $\hat{\lambda}_{MM} = \frac{1}{\bar{X}}$ where $\bar{X} = \frac{1}{N}\sum_{i=1}^{N} X_i$

as $E(X) = 1/\lambda$.

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Now, Calculating maximum likelihood estimator function of λ, $\hat{\lambda}_{MLE}$.

So, $L(\theta) = \lambda^n e^{-\lambda \sum_i x_i}$

$\therefore \log L(\theta) = n \log \lambda - \lambda \sum_i x_i$

$\Rightarrow \frac{d \log(L(\theta))}{d\theta} = 0 \Rightarrow \frac{n}{\lambda} - \sum x_i = 0$

$\Rightarrow \frac{n}{\lambda} = \sum_{i=1}^{n} x_i, \quad \lambda = \frac{1}{\bar{x}}$

$\therefore \hat{\lambda}_{MLE} = \frac{1}{\bar{x}}$ where $\bar{x} = \sum_{i=1}^{N} x_i$

Thus, $\boxed{\hat{\lambda}_{MM} = \hat{\lambda}_{MLE} = \frac{1}{\bar{x}}}$ Verified.

(b) To check : $\hat{\lambda}_{MLE}$ is unbiased.

$E[\hat{\lambda}_{MLE}] = E\left[\frac{1}{\bar{x}}\right]$

Using Jensen's inequality

$E\left[\frac{1}{\bar{x}}\right] \leq \frac{1}{E[\bar{x}]} = \frac{1}{E[x]}$

$\Rightarrow E[\hat{\lambda}_{MLE}] = \frac{1}{1/\lambda} = \lambda.$

$\therefore E[\hat{\lambda}_{MLE}] = \lambda.$

Thus, $\boxed{\hat{\lambda}_{MLE} \text{ is unbiased}}$

Q-4) Given X_i's are iid.

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(\frac{-x^2}{2\sigma^2}\right) \quad x \in (-\infty, \infty), \ \sigma > 0$$

(a) $\hat{\sigma}^2_{MLE} = ?$

Likelihood function:

$$L(\sigma^2) = \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^2 \exp\left(-\frac{\sum x_i^2}{2\sigma^2}\right)$$

$$\log(L(\sigma^2)) = \frac{-n}{2}\log(2\pi\sigma^2) - \frac{\sum x_i^2}{2\sigma^2}$$

$$\frac{\partial \log(L(\sigma^2))}{\partial \sigma^2} = 0.$$

$$\Rightarrow -\frac{n}{2}\frac{2\pi}{2\pi\sigma^2} + \frac{\sum x_i^2}{2\sigma^4} = 0$$

$$\frac{n}{2\sigma^2} = \frac{\sum x_i^2}{2\sigma^4}$$

$$\sigma^2 = \frac{1}{n}\sum x_i^2.$$

$$\therefore \boxed{\hat{\sigma}^2_{MLE} = \frac{1}{n}\sum_{i=1}^{n} x_i^2}$$

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(b) $\hat{\sigma}^2_{MLE} = \frac{\sum x_i^2}{N}$

$E(\hat{\sigma}^2_{MLE}) = E\left(\frac{\sum x_i^2}{N}\right)$

$= \frac{N \times Var(x_i^2)}{N}$ $[\because E x_i = 0]$

$= \sigma^2.$

$\Rightarrow \hat{\sigma}^2_{MLE}$ is unbiased estimator.

$\frac{\partial L}{\partial \sigma^2}(x_i; \sigma^2) = 0 = 0 \times (\hat{\sigma}^2 - \hat{\sigma}^2).$

$\Rightarrow$ $\boxed{\text{Efficient estimator}}$

(c) $\beta = \frac{1}{2}(\theta - 1)$

$\hat{\beta}_{MLE} = \beta(\hat{\theta}_{MLE}).$

$= \frac{1}{2}(\hat{\sigma}^2_{MLE} - 1)$

$= \frac{1}{2}\left[\frac{1}{N} \times \sum x_i^2 - 1\right]$

$= \frac{1}{2N}\left[\sum (x_i^2 - 1)\right]$

Hence, $\boxed{\hat{\beta}_{MLE} = \frac{1}{2N}\sum (x_i^2 - 1)}$