Name:- Rahul.D
Roll: 180102002

Assignment 4

Q1) $f(\theta) = 6\theta(1-\theta)$ $\quad 0 < \theta < 1$

$p(1) = \theta \quad p(0) = 1-\theta$

$\theta, X \sim \text{Binomial}(N, \theta)$

$f(x|\theta) = {}^{n}C_x\, \theta^{x} (1-\theta)^{n-x}$

here x is no of ones

a) $L(x;\theta) = \log(f(x;\theta))$

$= x\log\theta + (N-x)\log(1-\theta) + \log({}^{n}C_x)$

$$\left.\frac{\partial L}{\partial \theta}\right|_{\hat{\theta}_{MLE}} = \frac{x}{\theta} - \frac{N-x}{1-\theta} = 0$$

$$\frac{x}{\hat{\theta}_{MLE}} = \frac{N-x}{1-\hat{\theta}_{MLE}}$$

$$\frac{1-\hat{\theta}_{MLE}}{\hat{\theta}_{MLE}} = \frac{N-x}{x}$$

$$\frac{1}{\hat{\theta}_{MLE}} = \frac{N}{x}$$

$$\boxed{\hat{\theta}_{MLE} = \frac{x}{N}}$$

b) $f(x|\theta) = {}^{n}C_x\, \theta^{x}(1-\theta)^{n-x}$

& $f(\theta) = 6\theta(1-\theta)$

$$f(\theta|x) = \frac{f(x|\theta) f(\theta)}{f(x)}$$

$L(\theta|x) = \log f(x|\theta) + \log f(\theta) - \log f(x)$

$= x\log\theta + (N-x)\log(1-\theta) + \log {}^{n}C_x$

$+ \log 6 + \log\theta + \log(1-\theta) - \log f(x)$

classmate
Date
Page

$$\left.\frac{\partial L}{\partial \theta}\right|_{\hat{\theta}_{MAA}} = \left[\frac{x}{\theta} - \frac{N-x}{1-\theta} + \frac{1}{\theta} - \frac{1}{1-\theta}\right] = 0$$

$$\cancel{\frac{x+1}{N-x+1} = \frac{\hat{\theta}_{MAP}}{1-\hat{\theta}_{MAP}}}$$

$$\cancel{\frac{\hat{\theta}_{MAP}}{1-\hat{\theta}_{MAP}} = \frac{x+1}{N-x+1}}$$

$$\frac{x+1}{\theta} = \frac{N-x+1}{1-\theta}$$

$$\frac{1-\theta}{\theta} = \frac{N-x+1}{x+1}$$

$$\frac{1}{\theta} = \frac{N+2}{x+1}$$

$$\boxed{\hat{\theta}_{MAP} = \frac{x+1}{N+2}}$$

2) $X \sim UNIF(0,\theta)$ with $\theta \sim UNIF(0,1)$

$$f(x|\theta) = \frac{U(x) - U(x-\theta)}{\theta}$$

$$f(\theta) = U(\theta) - U(\theta-1)$$

a) For $\hat{\theta}_{MLE} \rightarrow f(x|\theta) \rightarrow f(x;\theta)$

$$\therefore f(x;\theta) = \frac{U(x)-U(x-\theta)}{\theta}$$

Clearly this likelihood is maximum when we take $\theta = x$

But here is an attempt to prove it

classmate

Date ______

Page ______

$$\left. \frac{\partial f(x;\theta)}{\partial \theta} \right|_{\hat{\theta}_{MLE}} = 0$$

$$\frac{[\delta(x) + \delta(x-\theta)]\theta - [u(x) - u(x-\theta)]}{\theta^2} = 0$$

$$\hat{\theta}_{MLE} = u(x) - u(x - \theta_{MLE})$$

defined only at ~~x~~ $\theta = x$ & $\cancel{x} = 0$.

$$\boxed{\hat{\theta}_{MLE} = x}$$

b) $f(x|\theta) = \frac{1}{\theta}[u(x) - u(x-\theta)]$

$$f(\theta) = u(\theta) - u(\theta - 1)$$

$$f(x;\theta) = f(x|\theta) f(\theta) = \frac{1}{\theta}[u(x) - u(x-\theta)] \times [u(\theta) - u(\theta - 1)]$$

Now, $f(x) = \int_{-\infty}^{\infty} f(x|\theta) f(\theta) d\theta$

$$= \int_{-\infty}^{\infty} \frac{1}{\theta}[u(x) - u(x-\theta)] \cdot [u(\theta) - u(\theta - 1)] d\theta$$

$f(x|\theta) = 0$ where $x > \theta$ & ~~θ~~ $\theta \geq 1$

$$f(x) = \int_{x}^{1} \frac{u(x) - u(x-\theta)}{\theta} d\theta$$

$$= \int_{x}^{1} u(x) d\theta$$

$\Rightarrow f(x) = \ln 1 - \ln x = -\ln x$

Scanned with CamScanner

$$f(\theta/x) = \frac{f(x;\theta)}{f(x)} = \frac{[U(x) - U(x-\theta)]\,(U(\theta) - U(\theta-1))}{\theta \ln x}$$

$$\hat{\theta}_{MMSE} = \int_{-\infty}^{\infty} \theta \, f(\theta/x)\, d\theta$$

$$= \int_{-\infty}^{\infty} \frac{[U(x) - U(x-\theta)]}{\ln x} [U(\theta) - U(\theta-1)]\, d\theta$$

$$\Rightarrow \hat{\theta}_{MMSE} = \int_1^{x} \frac{d\theta}{\ln x} = \boxed{\frac{x-1}{\ln x}}$$

3) $x_1, x_2 \ldots x_N$ = iid, x_i exp(λ)

$f_{x_i} = \lambda e^{-\lambda x}$ $\quad$ λ to be estimated

$$f_x(x) = \prod_{i=1}^{N} f_{x_i}(x) = \lambda^N e^{-\lambda \sum_0^N x_i}$$

By method of moments

$$E(x_1) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_0^{\infty} x \lambda e^{-\lambda x} dx$$

$$= \frac{1}{\lambda} \int_0^{\infty} k e^{-k} dk = \frac{1}{\lambda}$$

$$\Rightarrow \hat{\lambda}_{mm} \rightarrow \hat{\lambda}_{m} = \frac{1}{E(X_1)}$$

$$\therefore \hat{\lambda}_{mm} = \frac{1}{\bar{X}} \text{ where } \bar{X} = \frac{1}{N} \sum_{i=1}^{N} X_i$$

classmate
Date
Page

$f(x;\lambda) = \lambda^N e^{-\lambda \sum_1^N x_i}$

$L(x;\lambda) = \log(f(x;\lambda)) = N \log\lambda - \lambda \sum_{i=1}^N x_i$

$\frac{\partial L}{\partial \lambda} = 0$

$\frac{N}{\lambda} - \sum_1^N x_i = 0$

$\lambda_{MLE} = \frac{1}{\frac{1}{N}\sum_1^N x_i} = \boxed{\frac{1}{\bar{x}}}$

b) $\hat{\lambda}_{MLE} = \frac{1}{\bar{x}}$

$E(\hat{\lambda}_{MLE}) = E\left(\frac{1}{\bar{x}}\right) = N\, E\left(\frac{1}{\sum_1^N x_i}\right)$

$\frac{1}{\bar{x}}$ is convex so $E\left(\frac{1}{\bar{x}}\right) \geq \frac{1}{E(\bar{x})} = \frac{N}{E(\sum_1^N x_i)}$

$= \frac{N}{N\, 1/\lambda}$

$\boxed{= \lambda}$

$E(\hat{\lambda}_{MLE}) \geq \lambda$

$N \to \infty \quad E(\hat{\lambda}_{MLE}) \to \frac{1}{E(\bar{x})} \to \lambda$

$\Rightarrow \hat{\lambda}_{MLE}$ is asymptotically unbiased estimator

classmate
Date
Page

4) $X_1, X_2 \dots X_n \sim$ iid

$X \sim N(0; \sigma^2)$

$$f_{X_i}(x; \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-x_i^2}{2\sigma^2}\right)$$

$$f_X(x; \sigma^2) = \frac{1}{(\sqrt{2\pi\sigma^2})^N} \exp\left(\frac{-\sum_{i=1}^{N} x_i^2}{2\sigma^2}\right)$$

a) $L(x; \sigma^2) = \log(f_X(x; \sigma^2))$

$$= -\frac{N}{2}(\log \cancel{2\pi} + \log \sigma^2) - \frac{\sum_{i=1}^{N} x_i^2}{2\sigma^2}$$

$$\frac{\partial L}{\partial \sigma^2} = -\frac{N}{2\sigma^2} + \frac{\sum_{i=1}^{N} x_i^2}{2\sigma^2} = 0$$

$$\hat{\sigma}^2_{ML} = \boxed{\frac{\sum_{i=1}^{N} x_i^2}{N}}$$

b) $\hat{\sigma}^2_{MLE} = \frac{\sum_{i=1}^{N} x_i^2}{N}$

$$E(\hat{\sigma}^2_{MLE}) = \frac{E(\sum x_i^2)}{N}$$

$$= \frac{\cancel{N} N \operatorname{Var}(x_i^2) + \cancel{E(x)^2}}{N} \quad (\because E(X_i) = 0)$$

$$= \sigma^2$$

$\Rightarrow \hat{\sigma}^2_{MLE}$ is an unbiased estimator

$$\frac{\partial L(x; \sigma^2)}{\partial \sigma^2} = 0 = 0 \times (\hat{\sigma}^2 - \sigma^2)$$

$\Rightarrow$ $\boxed{\text{efficient estimator}}$

c) $\beta = \frac{1}{2}(\sigma^2 - 1)$ θ is σ^2 here.

$f(\hat{\sigma}^2_{MLE})$ is MLE of β.

$\Rightarrow \beta_{MLE} = \frac{1}{2}(\hat{\sigma}^2_{MLE} - 1)$

$= \frac{1}{2}\left(\frac{1}{N}\sum_1^N x_i^2 - 1\right)$

$$\boxed{\beta_{MLE} = \frac{1}{2N}\sum_1^N (x_i^2 - 1)}$$