End Semester Solutions MTH 112 2022-2023(I)

1 Find the unit tangent vector, unit normal, curvature of the curve C with the parametrisation $R(t) = (6\cos 2t, 6\sin 2t, 5t), t \in \mathbb{R}$. 6 marks

Solution:

- We have $R'(t) = (-12\sin 2t, 12\cos 2t, 5)$, and $||R'(t)|| = 13$

  So we have the unit tangent vector :

$$T(t) = \frac{R'(t)}{||R'(t)||} = (-\frac{12}{13}\sin 2t, \frac{12}{13}\cos 2t, \frac{5}{13}).$$

2 marks

- The unit normal

$$N(t) = \frac{T'(t)}{||T'(t)||},$$

  so

$$T'(t) = (-\frac{24}{13}\cos 2t, -\frac{24}{13}\sin 2t, 0)$$

  and

$$||T'(t)|| = \frac{24}{13}.$$

  So

$$N(t) = (-\cos 2t, -\sin 2t, 0)$$

2 marks

- Curvature

$$\kappa = \frac{||T'(t)||}{||R'(t)||} = \frac{24}{169}.$$

2 marks

2 Find an absolute maximum and absolute minimum of the function $f(x, y) = (x^2 - 4x)\sin y$ over the region $\{(x, y) \in \mathbb{R}^2 : 1 \leq x \leq 3, -\frac{3\pi}{4} \leq y \leq \frac{3}{4}\pi\}$.

Solution:

- As

$$f_x(x, y) = (2x - 4)\sin y$$

  and

$$f_y(x, y) = (x^2 - 4x)\cos y,$$

  the critical points are $(2, \frac{\pi}{2})$ and $(2, -\frac{\pi}{2})$. As $f_{x,x}(2, \frac{\pi}{2}) > 0$ and determinant of the Hessian matrix is greater than zero, $(2, \frac{\pi}{2})$ is a local minimum of f.

  Similarly $(2, -\frac{\pi}{2})$ is a local maximum and we have $f(2, -\frac{\pi}{2}) = 4$ and $f(2, \frac{\pi}{2}) = -4$ 2 marks

- We now inspect the boundaries of the region. $f(1, y) = -3siny$ for $y \in [-\frac{3}{4}\pi, \frac{3}{4}\pi]$. Here we see that the maximum value of f is 3 and minimum value is -3. Similarly for $f(3, y) = -3siny$. 1 mark

- $f(x, -\frac{3}{4}\pi) = (x^2 - 4x)(-\frac{\sqrt{2}}{2})$ for $x \in [1, 3]$. Here $2\sqrt{2}$ is the maximum and $3\frac{\sqrt{2}}{2}$ is the minimum value. 1 mark
- For the function $f(x, \frac{3}{4}\pi) = (x^2 - 4x)(\frac{\sqrt{2}}{2})$ for $y \in [1, 3]$, $-2\sqrt{2}$ is the minimum and $-3\frac{\sqrt{2}}{2}$ maximum value. 1 mark
- Hence we have the points $(2, \frac{\pi}{2})$ and $(2, -\frac{\pi}{2})$ as absolute minimum and absolute maximum on the region. 1 mark

3 (a) Evaluate $\iint_T \exp(\frac{x-y}{x+y})\ dxdy$ where T is the region inside the triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$. 4 marks

(b) Find the values of $a, b, c, d, e, g, h, i, j, k, l$ in the triple integrals 6 marks

$$\int_0^6 \int_0^{x^2} \int_0^{36-y} dz\,dy\,dx = \int_0^{36} \int_a^b \int_c^d dz\,dx\,dy = \int_0^{36} \int_e^f \int_g^h dx\,dy\,dz = \int_0^{36} \int_i^j \int_k^l dx\,dz\,dy$$

(for part(b):write the values of $a, b, c, d, e, f, g, h, i, j, k, l$ without justification)

Solution:

- (a)Define

$$u := x - y \text{ and } v := x + y$$

So we have

$$x = \frac{u+v}{2} = f(u, v)$$

and

$$y = \frac{v-u}{2} = g(u, v).$$

We have

$$\iint_T \exp(\frac{x-y}{x+y})\ dxdy = \iint_R \exp(\frac{u}{v})|J(u, v)|\ dudv,$$

where R is the image of T under the map $(x, y) \mapsto (x - y, x + y)$ and which is a region enclosed by the triangle with vertices $(0, 0), (1, 1)$ and $(-1, 1)$ 1 mark

- Jacobian of transformation

$$J(u, v) = \frac{\partial(f, g)}{\partial(u, v)} = \det \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix} = \frac{1}{2}$$

- $$\iint_T exp(\frac{x-y}{x+y})\ dxdy = \frac{1}{2}\int_0^1 \int_{-v}^{v} \exp(\frac{u}{v})\ dudv \quad \text{2 marks}$$

$$= \frac{e - e^{-1}}{4}$$

1 mark

(b) The region of the integration is $\{(x, y, z) : 0 \leq x \leq 6, 0 \leq y \leq x^2, 0 \leq z \leq 36 - y\}$ can be written as

$$\{(x, y, z) : 0 \leq y \leq 36, \sqrt{y} \leq x \leq 6, 0 \leq z \leq 36 - y\}$$

or

$$\{(x, y, z) : 0 \leq z \leq 36, 0 \leq y \leq 36 - z, \sqrt{y} \leq z \leq 6\}$$

or

$$\{(x, y, z) : 0 \leq y \leq 36, 0 \leq z \leq 36 - y, \sqrt{y} \leq z \leq 6\}.$$

So the tripple integrals can be written as:

$$\int_0^6 \int_0^{x^2} \int_0^{36-y} dz\, dy\, dx = \int_0^{36} \int_{\sqrt{y}}^{6} \int_0^{36-y} dz\, dx\, dy = \int_0^{36} \int_0^{36-z} \int_{\sqrt{y}}^{6} dx\, dy\, dz$$

$$= \int_0^{36} \int_0^{36-y} \int_{\sqrt{y}}^{6} dx\, dz\, dy.$$

- The value of $a = \sqrt{y}, b = 6, c = 0, d = 36 - y$. 2 marks only if all 4 values are correct otherwise zero.
- The value of $e = 0, f = 36 - z, g = \sqrt{y}, h = 6$. 2 marks only if all 4 values are correct otherwise zero.
- The value of $i = 0, j = 36 - y, k = \sqrt{y}, l = 6$. 2 marks only if all 4 values are correct otherwise zero.

4 Consider the triangular region T bounded by the lines joining the points $(0, 0), (5, 4)$ and $(5, 8)$. Calculate the volume of the solid generated by revolving the region T

(a) (i) about x axis by washer method, (ii) about y axis by shell method.

(b) Using Pappus theorem find the centroid of T. 4+2 marks

Solution:

- Let V_1 be the volume of solid obtained from revolving the triangle about y axis. By shell method we have

$$V_1 = \int_0^5 2\pi x(\frac{8}{5}x - \frac{4}{5}x)dx = \frac{200}{3}\pi$$

2 marks

- Let V_2 be the volume of solid obtained from revolving the triangle about x axis. Then by washer method

$$V_2 = \int_0^5 \left[\pi(\frac{8}{5}x)^2 - \pi(\frac{4}{5}x)^2\right] dx = 80\pi.$$

2 marks

- Let $(\bar{x}, \bar{y})$ denote the centroid of the triangle. Let

$$A = (0, 0), B = (5, 4), C = (5, 8).$$

Then by Pappus theorem

$$V_1 = 2\pi \bar{x} A = \frac{200}{3}\pi$$

and

$$V_2 = 2\pi\bar{y}A = 80\pi$$

where A is the area of $\triangle ABC$.

Let $D = (5,0)$, then area of $\triangle ABC$ = area of $\triangle ADC$ - area of $\triangle ADB$.

Area of the triangle $A = 10$. and we have $(\bar{x}, \bar{y}) = (\frac{10}{3}, 4)$. 2 mark

5 (a) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function and $X_0 \in \mathbb{R}^3$. Give the definition of differentiability of f at X_0.

(b) Let $G(x,y) = \begin{cases} \frac{xy^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{otherwise.} \end{cases}$

Prove that G is continuous at $(0,0)$. Is G is differentiable at $(0,0)$? Justify with reasons.

(c) Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a function which satisfies the following two conditions:

- F is continuous at $\mathbf{0}$.
- F is differentiable on $\mathbb{R}^3 \setminus \{\mathbf{0}\}$.

If $\lim\limits_{\mathbf{P}\to\mathbf{0}} \frac{\partial F}{\partial x}(\mathbf{P}) = \lim\limits_{\mathbf{P}\to\mathbf{0}} \frac{\partial F}{\partial y}(\mathbf{P}) = \lim\limits_{\mathbf{P}\to\mathbf{0}} \frac{\partial F}{\partial z}(\mathbf{P}) = \mathbf{0}$, prove or disprove that F is differentiable at $\mathbf{0}$. 2 +6+8 marks

solution

- Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function and $X_0 \in \mathbb{R}^3$. The function f is said to be differentiable at X_0 if there exists $\alpha \in \mathbb{R}^3$ such that the error function

$$\epsilon(H) = \frac{f(X_0 + H) - f(X_0) - \alpha \cdot H}{||H||}$$

  tends to 0 as H tends to 0. 2 marks

- (a)Consider the function $G(x,y) = \frac{xy^2}{x^2+y^2}$ when $(x,y) \neq (0,0)$ and $G(0,0) = 0$. First note that G is continuous at $(0,0)$ as for $(x,y) \neq (0,0)$ with both $x \neq 0$ and $y \neq 0$, $|G(x,y)| = \frac{|xy^2|}{x^2+y^2} \leq \frac{|xy^2|}{2|xy|} \leq \frac{1}{2}|y|$. Also we have $G(x,0) = G(0,y) = G(0,0) = 0 \leq \frac{1}{2}|y|$. Hence wh have $|G(x,y)| \leq \frac{1}{2}|y|$ for all $(x,y) \in \mathbb{R}^2$ and so when (x,y) tends to zero $G(x,y)$ tends to zero. 2 marks
- If G is differentiable at $(0,0)$ then $G'(0,0) = (G_x(0,0), G_y(0.0))$ and $\lim_{(h,k)\to(0,0)} \frac{G(h,k)-G(0,0)-(h,k)(G_x(0,0),G_y(0.0))}{\sqrt{h^2+k^2}} = 0$. 1 mark

  But $G_x(0,0) = \lim_{h\to 0} \frac{G(h,0)-G(0,0)}{h} = 0 = G_y(0,0)$. 1 mark

  Then $\lim_{(h,k)\to(0,0)} \frac{hk^2}{(h^2+k^2)^{\frac{3}{2}}}$ which does not converge to zero: as for $h = k$ the $\lim_{h\to 0} \frac{h^3}{2^{\frac{3}{2}}|h|^3}$ does not converge. 2 marks
- (b) With the given hypothesis F will be differentiable at $(0,0,0)$. It is sufficient to prove that there exists $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$ such that for $H = (h_1, h_2, h_3) \in \mathbb{R}^3$ the limit $\lim_{H\to(0,0,0)} \frac{F(H)-F((0,0,0)-H.\alpha}{||H||}$ is zero. 2 marks

- Now as F is differentiable on $\mathbb{R}^3 \setminus \{0\}$, applying mean value theorem on the line segment L joining the points H and $(0,0,0)$ (note that to apply mean value theorem function on the line segment function need not be differentiable on the end points), we have $F(H) = F(0,0,0) + F'(C)H$ where C lies on L and $C \neq H$ and $C \neq 0$. 3 marks
- Claim for $\alpha = (0,0,0)$ the error function in the definition will tend to zero. Taking $\alpha = (0,0,0)$ we have:
$$\lim_{H \to (0,0,0)} \frac{F(H) - F((0,0,0) - H.\alpha}{||H||} = \lim_{H \to (0,0,0)} \frac{F'(C).H}{||H||}$$
which tends to zero as as $H \to (0,0,0)$ implies $C \to (0,0,0)$ and by given hypothesis $F'(C)$ tends to zero and $\frac{H}{||H||}$ is bounded. 3 marks

6 Let $F = -yz\vec{i} + (6y+1)\vec{j} + xy\vec{k}$ be a vector field.

(a) Let C be the circle of radius 3 centered at $(0,4,0)$ with the parametrisation $(3\cos t, 4, 3\sin t), 0 \leq t \leq 2\pi$. Evaluate $\int_C F \cdot dR$. 4 marks

(b) Verify Stokes theorem by calculating the surface integral $\iint_S \text{Curl}F \cdot n\, dS$ where S is the surface $y = 13 - x^2 - z^2, 4 \leq y \leq 13$ and n is the unit outward normal to the surface S. 6 marks

(c) Use divergence theorem to calculate $\iint_S F \cdot n\; dS$, where S is the surface $y = 13 - x^2 - z^2, 4 \leq y \leq 13$, and n is the unit outward normal to the surface S. 6 marks

Solution:

- (a)We have
$$\int_0^{2\pi} F(R(t)R'(t)dt = \int_0^{2\pi} (-12\text{sin}t(-3\text{sin}t) + 0 + 12\text{cos}t(3\text{cos}t))dt$$
2 marks.
- $$= \int_0^{2\pi} 36dt = 72\pi$$
. 2 marks
- (b)Observe that the circle in part (a) is the boundary of the surface S. Let $g(x,y,z) = x^2 + z^2 + y - 13 = 0$ be the level surface S. We have curl $F = (x, -2y, z)$ and $n = \frac{(2x,1,2z)}{||\nabla g||}$ which is an outward normal to the surface S. We note 2 marks
- curl F.$n = \frac{(2x^2 - 2y + 2z^2)}{||\nabla g||} = \frac{(-26 + 4x^2 + 4z^2)}{||\nabla g||}$ 1 mark
- $\iint_S$ curl F.$n\, dS = \iint_S \frac{(-26+4x^2+4z^2)}{||\nabla g||}\, dS$
Using cylindrical coordinate system : $x = r\text{cos}\theta, z = r\text{sin}\theta$ $y = y$ we have

$$\int_0^{2\pi} \int_0^3 \frac{(-26+4r^2)}{||\nabla g||}(r)||\nabla g||dr d\theta \quad \text{2marks}$$

- $= -72\pi$. The negative sign is justified since the boundary which is a circle is parametrised with the orientation with respect to the inward normal to the surface. 1 mark

  (c)
- By divergence theorem we have
  $$\int\int\int_D \text{div}(F)dV = \iint_{S_1} F.n_1\, dS + \iint_{S_2} F.n_2\, dS,$$
  where D is solid bounded by the surface $S = S_1 := y = 13 - x^2 - z^2, , 4 \leq y \leq 13$ and $S_2 : y = 4$, and n_1 and n_2 are outward unit normals of S_1 and S_2 respectively. 1 mark
- Hence $\iint_S \; F.n\, dS = \int\int\int_D \text{div}(F)dV - \iint_{S_2} F.n_2\, dS$
- $$\int\int\int_D \text{div}(F)dV = \int_0^3 \int_0^{2\pi} \int_4^{13-r^2} 6dV = 243\pi$$
  2 marks
- $$\iint_{S_2} F.n_2\, dS = \int_0^3 \int_0^{2\pi} -25 r d\theta dr = -225\pi$$
  2 marks
- Hence
  $$\iint_S \; F.n\, dS = 243\pi + 225\pi = 468\pi.$$
  1 mark