Wheldon, Kirk, and Finlay Work-Precision Diagrams

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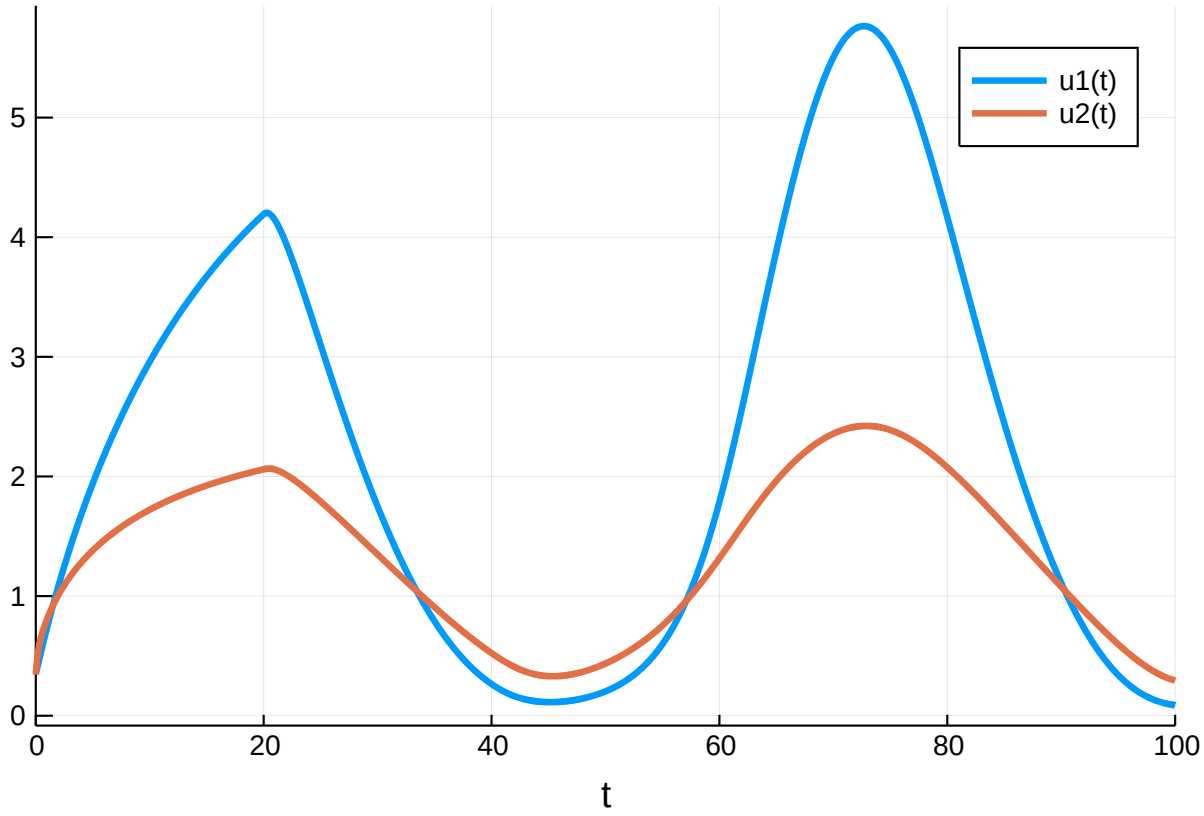
1 Wheldon, Kirk, and Finlay

We study algorithms for solving constant delay differential equations with a test problem from W.H. Enright and H. Hayashi, "The evaluation of numerical software for delay differential equations", 1997. It is a model of chronic granulocytic leukemia that was published by T. Wheldon, J. Kirk and H. Finlay in "Cyclical granulopoiesis in chronic granulocytic leukemia: A simulation study", 1974, and is given by

$$y_1'(t) = \frac{1.1}{1 + \sqrt{10} y_1(t-20)^{5/4}} - \frac{10 y_1(t)}{1 + 40 y_2(t)} \tag{1}$$

$$y_2'(t) = \frac{100 y_1(t)}{1 + 40 y_2(t)} - 2.43 y_2(t) \tag{2}$$

```
using DelayDiffEq, DiffEqDevTools, DiffEqProblemLibrary, Plots
using DiffEqProblemLibrary.DDEProblemLibrary: importddeproblems; importddeproblems()
import DiffEqProblemLibrary.DDEProblemLibrary: prob_dde_wheldon
gr()

sol = solve(prob_dde_wheldon, MethodOfSteps(Vern9(), max_fixedpoint_iters=1000);
    reltol=1e-14, abstol=1e-14)
test_sol = TestSolution(sol)
plot(sol)
```

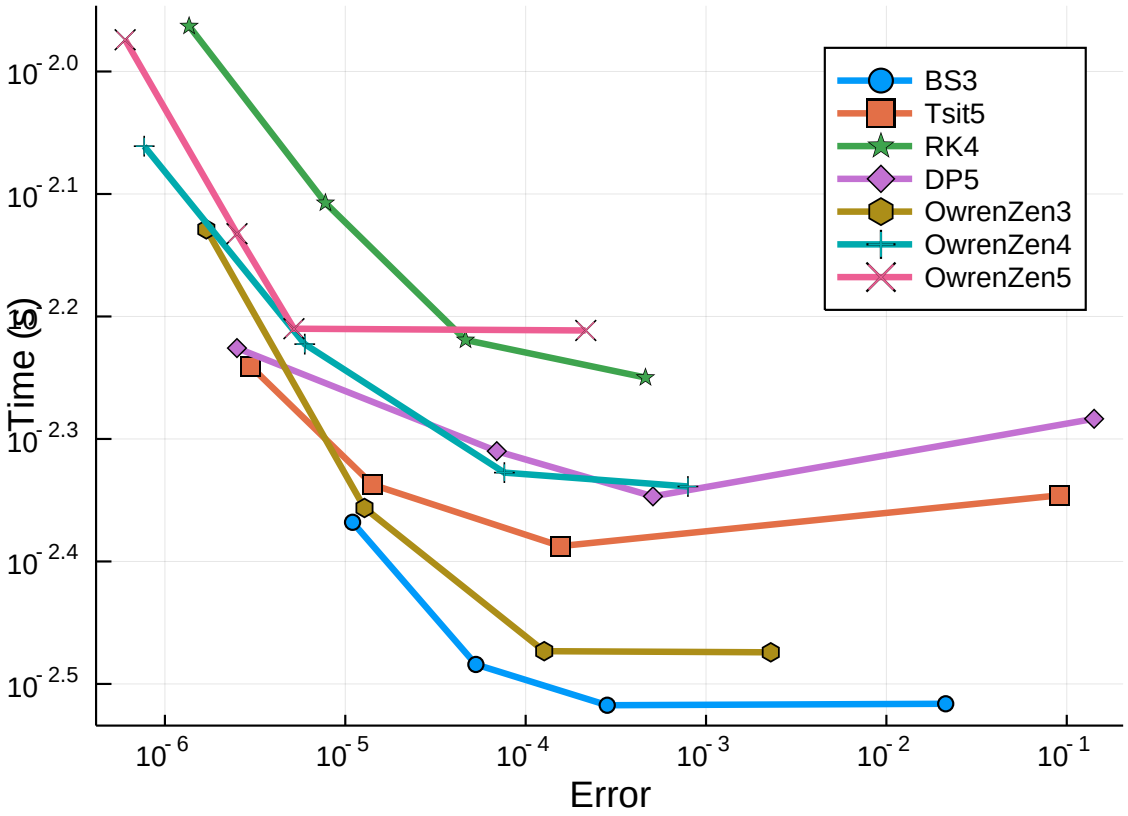


1.1 Low order RK methods

1.1.1 High tolerances

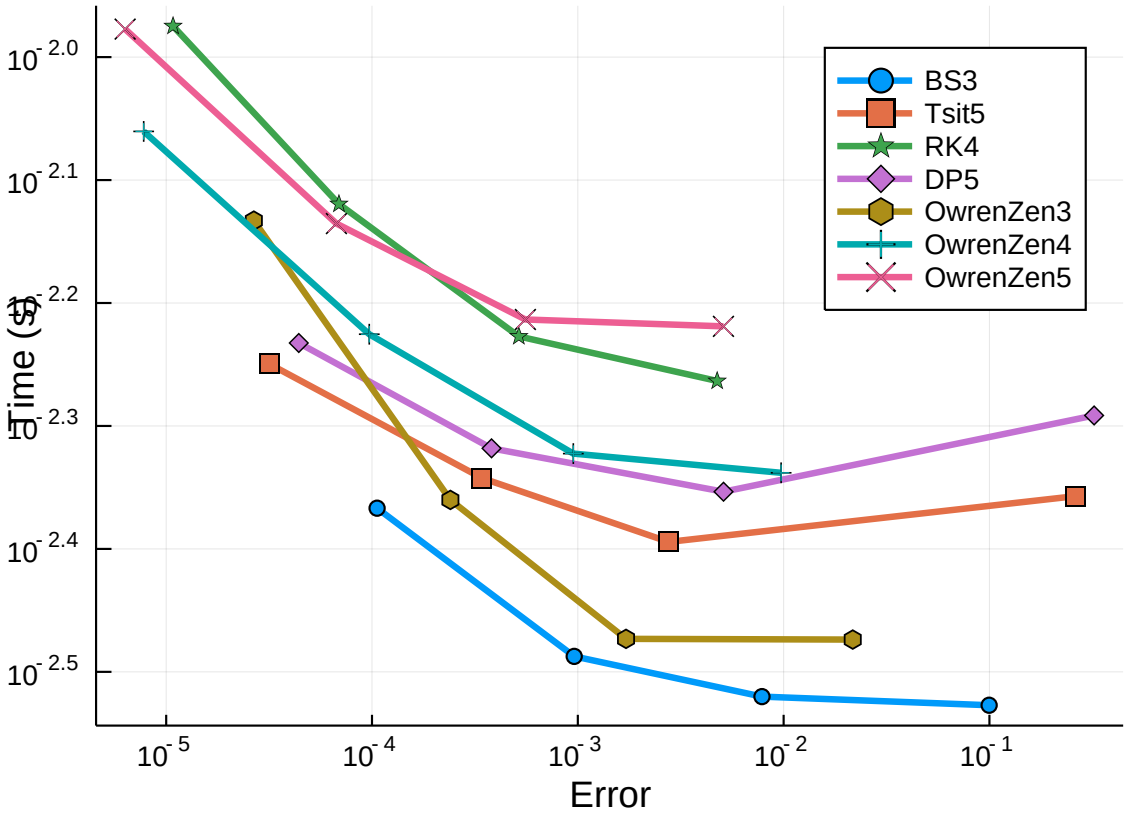
First we compare final errors of solutions with low order RK methods at high tolerances.

```
abstols = 1.0 ./ 10.0 .^ (4:7)
reltols = 1.0 ./ 10.0 .^ (1:4)

setups = [Dict(:alg=>MethodOfSteps(BS3())),
          Dict(:alg=>MethodOfSteps(Tsit5())),
          Dict(:alg=>MethodOfSteps(RK4())),
          Dict(:alg=>MethodOfSteps(DP5())),
          Dict(:alg=>MethodOfSteps(OwrenZen3())),
          Dict(:alg=>MethodOfSteps(OwrenZen4())),
          Dict(:alg=>MethodOfSteps(OwrenZen5()))]
names = ["BS3", "Tsit5", "RK4", "DP5", "OwrenZen3", "OwrenZen4", "OwrenZen5"]
wp = WorkPrecisionSet(prob_dde_wheldon,abstols,reltols,setups;names=names,
                      appxsol=test_sol,maxiters=Int(1e5),error_estimate=:final)
plot(wp)
```



Next we test interpolation errors:

```
abstols = 1.0 ./ 10.0 .^ (4:7)
reltols = 1.0 ./ 10.0 .^ (1:4)

setups = [Dict(:alg=>MethodOfSteps(BS3())),
          Dict(:alg=>MethodOfSteps(Tsit5())),
          Dict(:alg=>MethodOfSteps(RK4())),
          Dict(:alg=>MethodOfSteps(DP5())),
          Dict(:alg=>MethodOfSteps(OwrenZen3())),
          Dict(:alg=>MethodOfSteps(OwrenZen4())),
          Dict(:alg=>MethodOfSteps(OwrenZen5()))]
names = ["BS3", "Tsit5", "RK4", "DP5", "OwrenZen3", "OwrenZen4", "OwrenZen5"]
wp = WorkPrecisionSet(prob_dde_wheldon,abstols,reltols,setups;names=names,
                      appxsol=test_sol,maxiters=Int(1e5),error_estimate=:L2)
plot(wp)
```

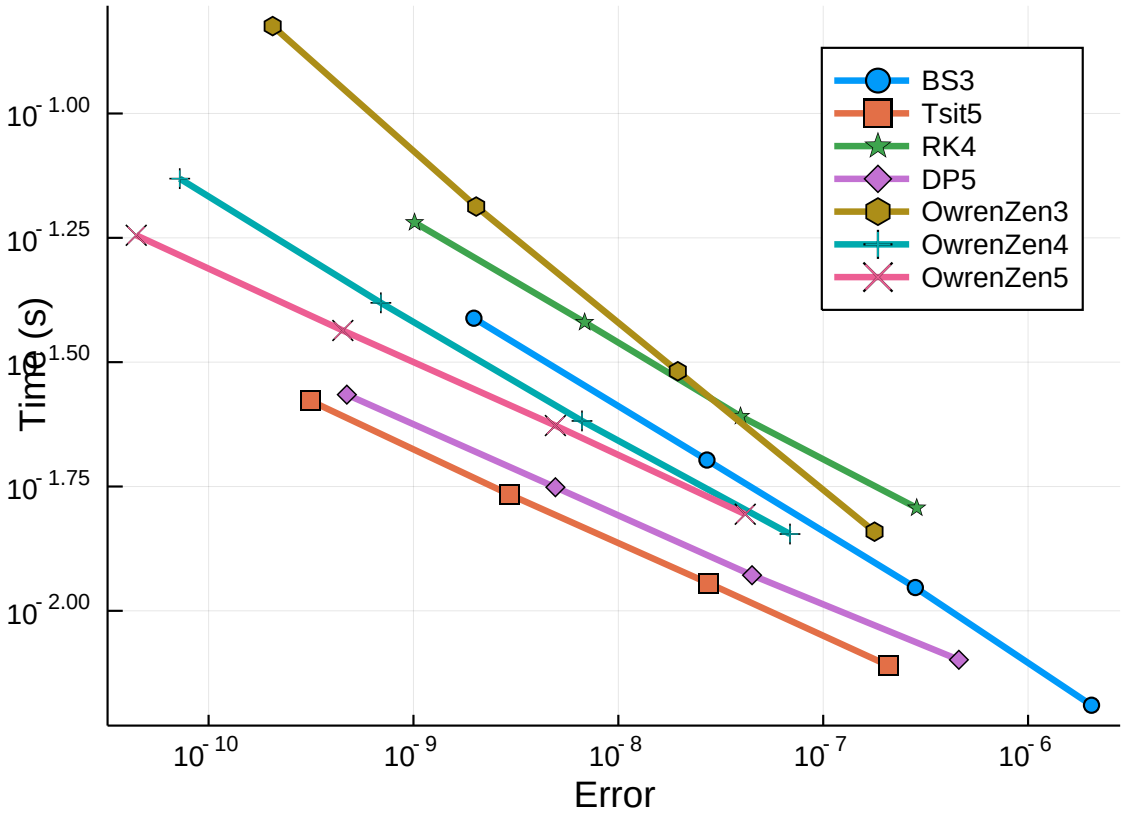


Both interpolation tests and tests of final error show similar results. `BS3` does quite well but only `OwrenZen4`, `OwrenZen5`, and `RK4` achieve interpolation errors of about 1e-5.

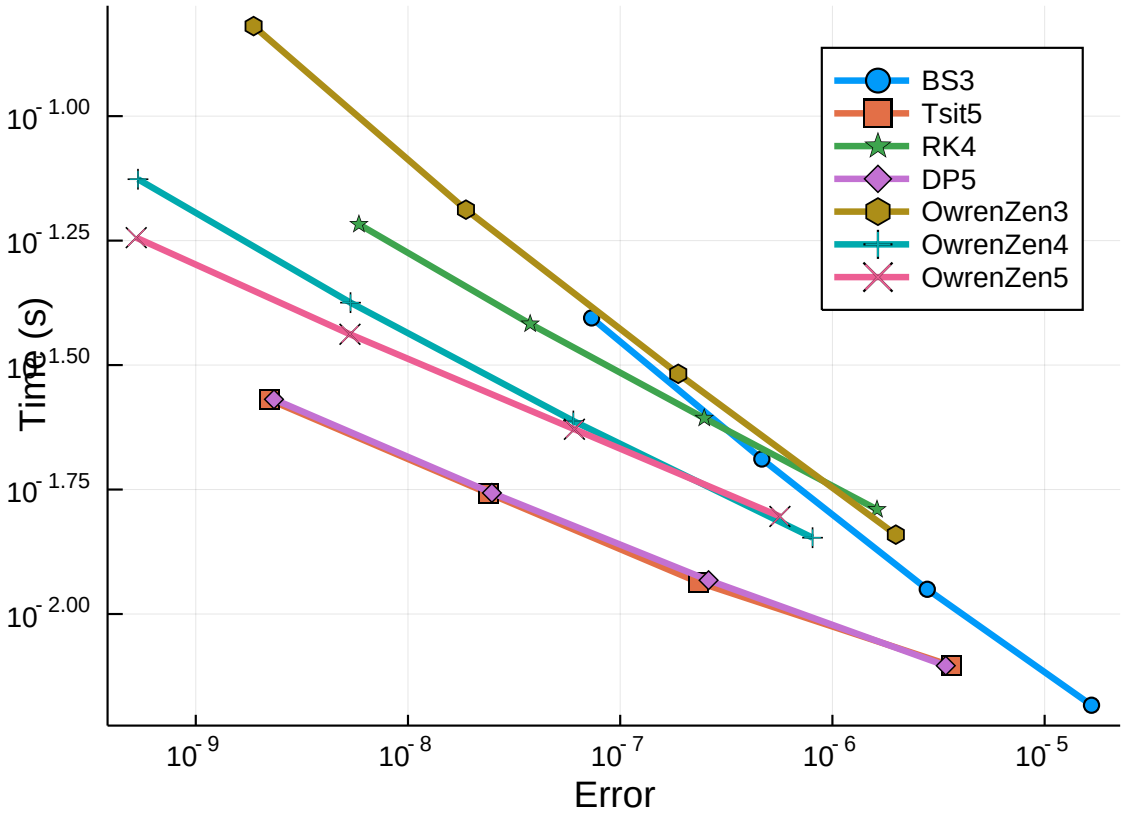
1.1.2 Low tolerances

We repeat our tests at low tolerances.

```
abstols = 1.0 ./ 10.0 .^ (8:11)
reltols = 1.0 ./ 10.0 .^ (5:8)

setups = [Dict(:alg=>MethodOfSteps(BS3())),
          Dict(:alg=>MethodOfSteps(Tsit5())),
          Dict(:alg=>MethodOfSteps(RK4())),
          Dict(:alg=>MethodOfSteps(DP5())),
          Dict(:alg=>MethodOfSteps(OwrenZen3())),
          Dict(:alg=>MethodOfSteps(OwrenZen4())),
          Dict(:alg=>MethodOfSteps(OwrenZen5()))]
names = ["BS3", "Tsit5", "RK4", "DP5", "OwrenZen3", "OwrenZen4", "OwrenZen5"]
wp = WorkPrecisionSet(prob_dde_wheldon,abstols,reltols,setups;names=names,
                      appxsol=test_sol,maxiters=Int(1e5),error_estimate=:final)
plot(wp)
```




```
abstols = 1.0 ./ 10.0 .^ (8:11)
reltols = 1.0 ./ 10.0 .^ (5:8)

setups = [Dict(:alg=>MethodOfSteps(BS3())),
          Dict(:alg=>MethodOfSteps(Tsit5())),
          Dict(:alg=>MethodOfSteps(RK4())),
          Dict(:alg=>MethodOfSteps(DP5())),
          Dict(:alg=>MethodOfSteps(OwrenZen3())),
          Dict(:alg=>MethodOfSteps(OwrenZen4())),
          Dict(:alg=>MethodOfSteps(OwrenZen5()))]
names = ["BS3", "Tsit5", "RK4", "DP5", "OwrenZen3", "OwrenZen4", "OwrenZen5"]
wp = WorkPrecisionSet(prob_dde_wheldon,abstols,reltols,setups;names=names,
                      appxsol=test_sol,maxiters=Int(1e5),error_estimate=:L2)
plot(wp)
```



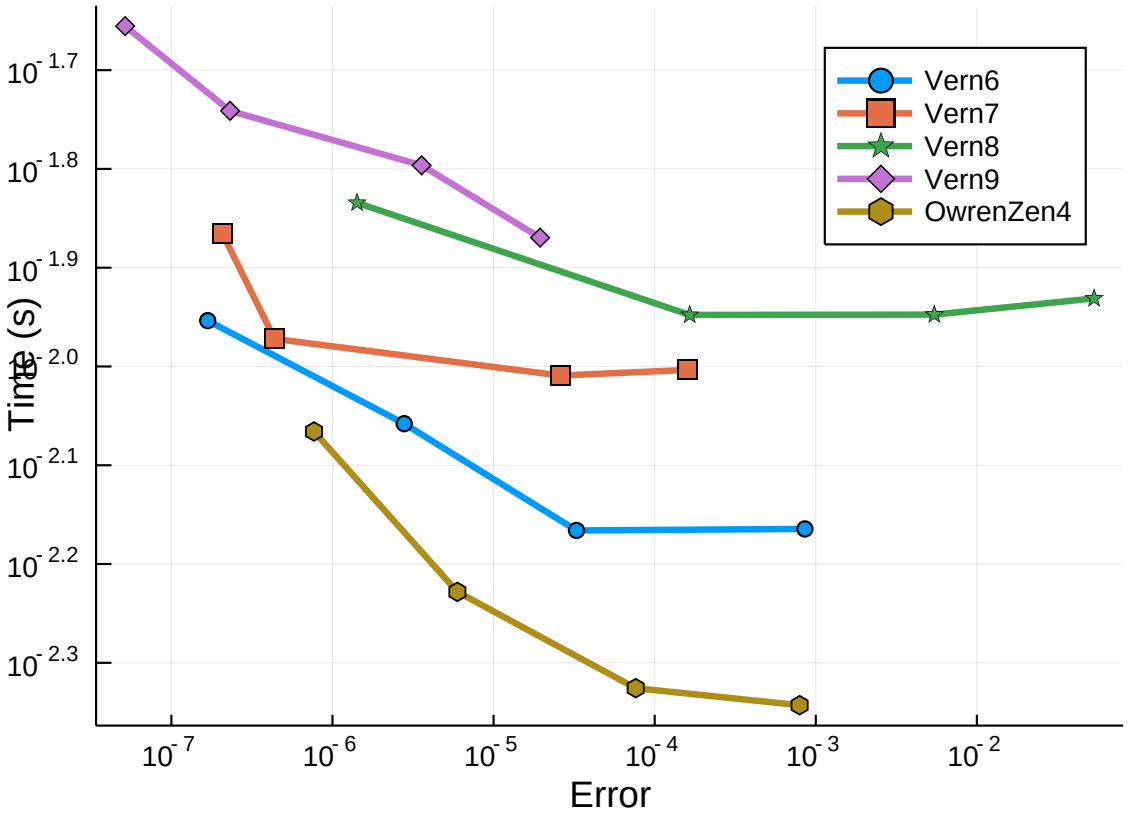
Out of the compared methods, `Tsit5`, `DP5`, and `OwrenZen5` seem to be the best methods for this problem at low tolerances, but also `OwrenZen4` performs similarly well. `OwrenZen5` and `OwrenZen4` can even achieve interpolation errors below 1e-9.

1.2 Lazy interpolants

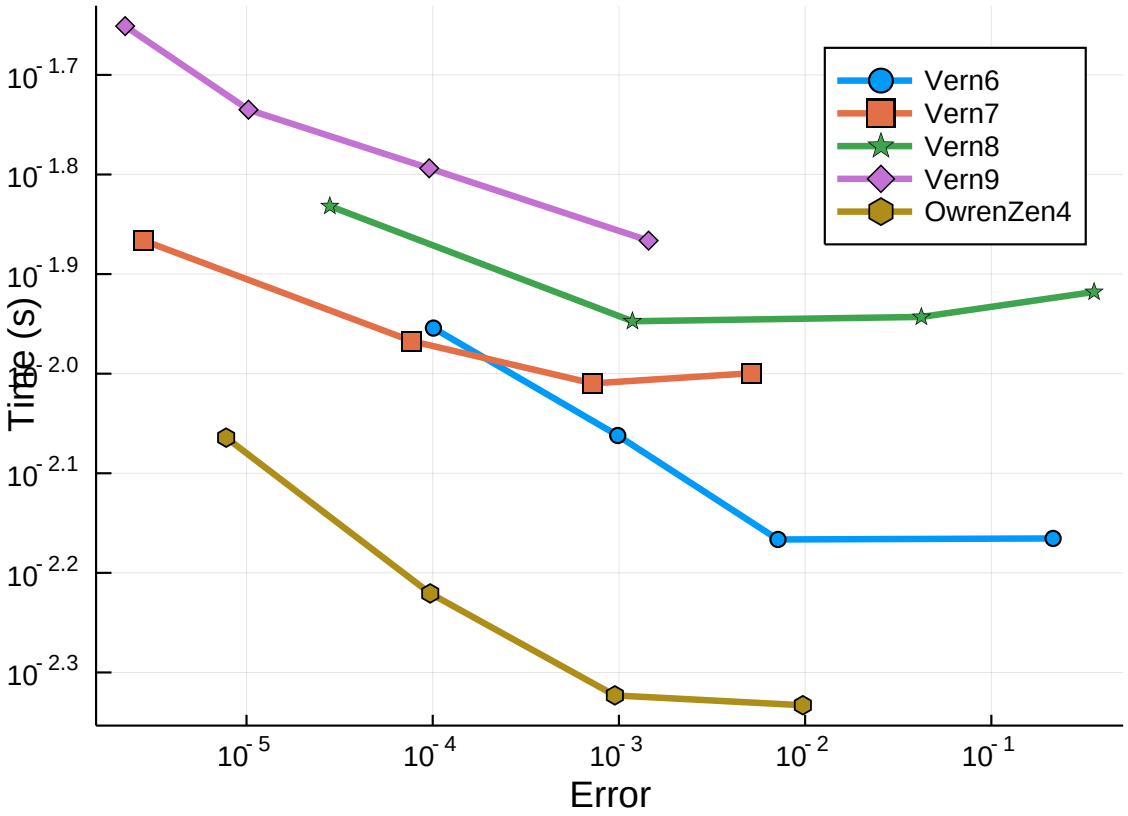
1.2.1 High tolerances

We compare the Verner methods, which use lazy interpolants, at high tolerances. As reference we include `OwrenZen4`.

```
abstols = 1.0 ./ 10.0 .^ (4:7)
reltols = 1.0 ./ 10.0 .^ (1:4)

setups = [Dict(:alg=>MethodOfSteps(Vern6())),
          Dict(:alg=>MethodOfSteps(Vern7())),
          Dict(:alg=>MethodOfSteps(Vern8())),
          Dict(:alg=>MethodOfSteps(Vern9())),
          Dict(:alg=>MethodOfSteps(OwrenZen4()))]
names = ["Vern6", "Vern7", "Vern8", "Vern9", "OwrenZen4"]
wp = WorkPrecisionSet(prob_dde_wheldon,abstols,reltols,setups;names=names,
                      appxsol=test_sol,maxiters=Int(1e5),error_estimate=:final)
plot(wp)
```



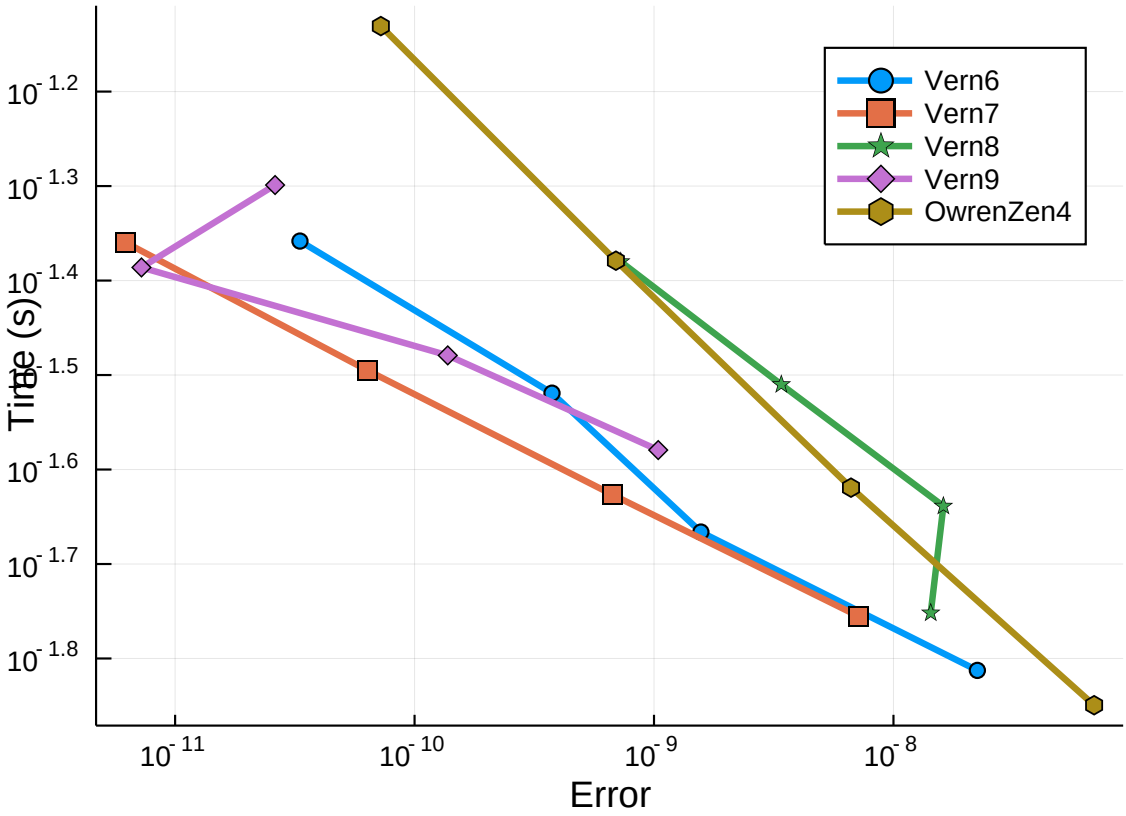

```
abstols = 1.0 ./ 10.0 .^ (4:7)
reltols = 1.0 ./ 10.0 .^ (1:4)

setups = [Dict(:alg=>MethodOfSteps(Vern6())),
          Dict(:alg=>MethodOfSteps(Vern7())),
          Dict(:alg=>MethodOfSteps(Vern8())),
          Dict(:alg=>MethodOfSteps(Vern9())),
          Dict(:alg=>MethodOfSteps(OwrenZen4()))]
names = ["Vern6", "Vern7", "Vern8", "Vern9", "OwrenZen4"]
wp = WorkPrecisionSet(prob_dde_wheldon,abstols,reltols,setups;names=names,
                      appxsol=test_sol,maxiters=Int(1e5),error_estimate=:L2)
plot(wp)
```



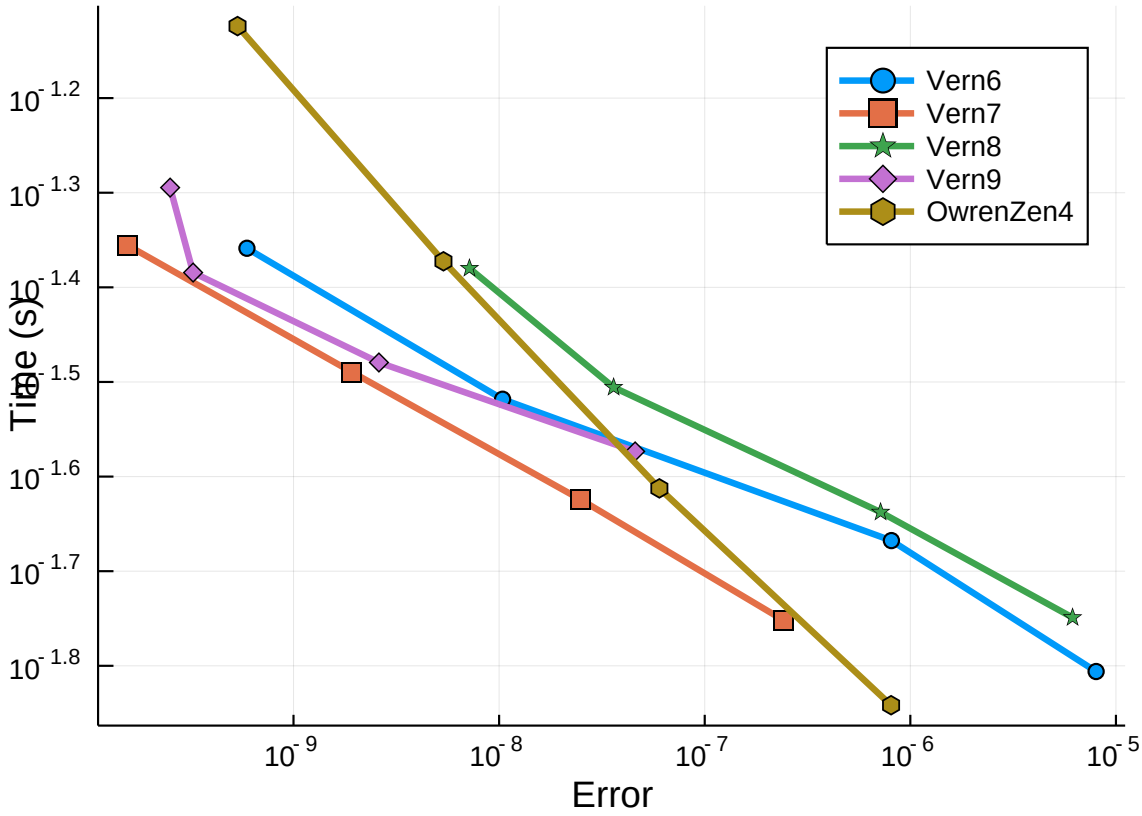
1.2.2 Low tolerances

We repeat these tests and compare the Verner methods also at low tolerances.

```
abstols = 1.0 ./ 10.0 .^ (8:11)
reltols = 1.0 ./ 10.0 .^ (5:8)

setups = [Dict(:alg=>MethodOfSteps(Vern6())),
          Dict(:alg=>MethodOfSteps(Vern7())),
          Dict(:alg=>MethodOfSteps(Vern8())),
          Dict(:alg=>MethodOfSteps(Vern9())),
          Dict(:alg=>MethodOfSteps(OwrenZen4()))]
names = ["Vern6", "Vern7", "Vern8", "Vern9", "OwrenZen4"]
wp = WorkPrecisionSet(prob_dde_wheldon,abstols,reltols,setups;names=names,
                      appxsol=test_sol,maxiters=Int(1e5),error_estimate=:final)
plot(wp)
```




```
abstols = 1.0 ./ 10.0 .^ (8:11)
reltols = 1.0 ./ 10.0 .^ (5:8)

setups = [Dict(:alg=>MethodOfSteps(Vern6())),
          Dict(:alg=>MethodOfSteps(Vern7())),
          Dict(:alg=>MethodOfSteps(Vern8())),
          Dict(:alg=>MethodOfSteps(Vern9())),
          Dict(:alg=>MethodOfSteps(OwrenZen4()))]
names = ["Vern6", "Vern7", "Vern8", "Vern9", "OwrenZen4"]
wp = WorkPrecisionSet(prob_dde_wheldon,abstols,reltols,setups;names=names,
                      appxsol=test_sol,maxiters=Int(1e5),error_estimate=:L2)
plot(wp)
```



It seems `Vern6` and `Vern7` are both well suited for the problem at low tolerances and outperform `OwrenZen4`, whereas at high tolerances `OwrenZen4` is more efficient.

```
using DiffEqBenchmarks
DiffEqBenchmarks.bench_footer(WEAVE_ARGS[:folder],WEAVE_ARGS[:file])
```

1.3 Appendix

These benchmarks are a part of the DiffEqBenchmarks.jl repository, found at: https://github.com/JuliaDi

To locally run this tutorial, do the following commands:

```
using DiffEqBenchmarks
DiffEqBenchmarks.weave_file("NonStiffDDE","Wheldon_Kirk_Finlay_wpd.jmd")
```

Computer Information:

```
Julia Version 1.1.0
Commit 80516ca202 (2019-01-21 21:24 UTC)
Platform Info:
  OS: Linux (x86_64-pc-linux-gnu)
  CPU: Intel(R) Xeon(R) CPU E5-2680 v4 @ 2.40GHz
  WORD_SIZE: 64
  LIBM: libopenlibm
  LLVM: libLLVM-6.0.1 (ORCJIT, haswell)
```

Package Information:

```
Status: `/home/crackauckas/.julia/environments/v1.1/Project.toml`
[c52e3926-4ff0-5f6e-af25-54175e0327b1] Atom 0.8.5
[bcd4f6db-9728-5f36-b5f7-82caef46ccdb] DelayDiffEq 5.2.0
[bb2cbb15-79fc-5d1e-9bf1-8ae49c7c1650] DiffEqBenchmarks 0.1.0
[459566f4-90b8-5000-8ac3-15dfb0a30def] DiffEqCallbacks 2.5.2
[f3b72e0c-5b89-59e1-b016-84e28bfd966d] DiffEqDevTools 2.7.2+
[055956cb-9e8b-5191-98cc-73ae4a59e68a] DiffEqPhysics 3.1.0
[a077e3f3-b75c-5d7f-a0c6-6bc4c8ec64a9] DiffEqProblemLibrary 4.1.0
[0c46a032-eb83-5123-abaf-570d42b7fbaa] DifferentialEquations 6.3.0
[b305315f-e792-5b7a-8f41-49f472929428] Elliptic 0.5.0
[e5e0dc1b-0480-54bc-9374-aad01c23163d] Juno 0.7.0
[7f56f5a3-f504-529b-bc02-0b1fe5e64312] LSODA 0.4.0
[c030b06c-0b6d-57c2-b091-7029874bd033] ODE 2.4.0
[54ca160b-1b9f-5127-a996-1867f4bc2a2c] ODEInterface 0.4.5
[09606e27-ecf5-54fc-bb29-004bd9f985bf] ODEInterfaceDiffEq 3.1.0
[1dea7af3-3e70-54e6-95c3-0bf5283fa5ed] OrdinaryDiffEq 5.5.0
[65888b18-ceab-5e60-b2b9-181511a3b968] ParameterizedFunctions 4.1.1
[91a5bcdd-55d7-5caf-9e0b-520d859cae80] Plots 0.24.0
[d330b81b-6aea-500a-939a-2ce795aea3ee] PyPlot 2.8.1
[90137ffa-7385-5640-81b9-e52037218182] StaticArrays 0.10.3
[c3572dad-4567-51f8-b174-8c6c989267f4] Sundials 3.3.0+
[92b13dbe-c966-51a2-8445-caca9f8a7d42] TaylorIntegration 0.4.1
[44d3d7a6-8a23-5bf8-98c5-b353f8df5ec9] Weave 0.9.0
```