Problem Set 1

CSCE 629 (Dr. Klappenecker)

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Resources

- https://app.perusall.com/courses/23-spring-csce-411-503-design-analy-algorithms/dm_ch11-270748931
- https://www.youtube.com/watch?v=-VBQZYGu-Tc
- https://www.learnlatex.org/
- https://www.geeksforgeeks.org/arrow-symbols-in-latex/
- https://jblevins.org/log/greek
- https://tex.stackexchange.com/questions/337351/multiple-lines-one-side-of-equation-with-a-curly-bracket

On my honor, as an Aggie, I have neither given nor received any unauthorized aid on any portion of the academic work included in this assignment. Furthermore, I have disclosed all resources (people, books, web sites, etc.) that have been used to prepare this homework. The work shown here is entirely my own, and is written in my own words.

Signature: Ajain

Problem 1.

Solution.

Let f and g be functions from the set of positive integers to the set of real numbers. We say that f is **asymptotically equal** to g if and only if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$$

holds. In other words, two functions f and g are asymptotically equal if and only if the relative error $\frac{f(n)-g(n)}{g(n)}$ between these functions vanishes for large n, meaning that the functions f and g have the same growth for large n. Since Ernie and Bert are interested in asymptotic equality, i.e. the growth of functions, it is important to realize that the relative error $\frac{f(n)-g(n)}{g(n)}$ should be considered, not the difference of the functions $f(n) - g(n)$.

$$\frac{f(n)-g(n)}{g(n)} = \frac{n^2+2n-n^2}{n^2} = \frac{2n}{n^2} = \frac{2}{n}$$

Further,

$$\lim_{n\to\infty} \frac{2}{n} = 0$$

Clearly, the relative error between these functions vanishes for large n. Hence, Ernie is right in claiming that the functions are asymptotically equal.

Problem 2.

Solution.

As per the definition (given in Perusall), a real number a is an accumulation value of a function $f : \mathbf{N}_1 \to \mathbf{R}$ if and only if for each real number $\epsilon > 0$ there exist an infinite number of function values $f(n)$ such that

$$|f(n) - a| < \epsilon.$$

If the limit $l = \lim_{n\to\infty} f(n)$ exists, then l is an accumulation value of f. The real number u is an **upper accumulation point** of f if and only if the following two conditions are met:

1. For each real number $\epsilon > 0$ there exist infinitely many positive integers n such that $f(n) > u - \epsilon$,

2. For each real number $\epsilon > 0$ there exist at most finitely many positive integers such that $f(n) > u + \epsilon$.

The real number l is an **lower accumulation point** of f if and only if the following two conditions are met:

1. For each real number $\epsilon > 0$ there exist infinitely many positive integers n such that $f(n) < l + \epsilon$,

2. For each real number $\epsilon > 0$ there exist at most finitely many positive integers such that $f(n) < l - \epsilon$.

(a) $f(n) = (-1)^n$
If n is odd, $f(n) = -1$ and if n is even, $f(n) = 1$, which can be stated as

$$f(n) = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$$

Since the function $f(n)$ takes only two values -1 and 1, there exists no real number a for which infinite number of function values $f(n)$ exists such that $|f(n) - a| < \epsilon$. Hence, $f(n) = (-1)^n$ does not have any accumulation points. The lower and upper accumulation points are also not defined for this function since no accumulation points exist.

(b) $f(n) = 4 + \frac{(-1)^n n}{n+10}$
Upon simplifying, the function can be written as

$$f(n) = \begin{cases} 4 - \frac{n}{n+10} & \text{if } n \text{ is odd} \\ 4 + \frac{n}{n+10} & \text{if } n \text{ is even} \end{cases}$$

Applying limit to the function f,

$$\lim_{n\to\infty} f(n) = \lim_{n\to\infty} 4 + \frac{(-1)^n n}{n+10} = \begin{cases} \lim_{n\to\infty} 4 - \frac{n}{n+10} & \text{if } n \text{ is odd} \\ \lim_{n\to\infty} 4 + \frac{n}{n+10} & \text{if } n \text{ is even} \end{cases} = \begin{cases} 3 & \text{if } n \text{ is odd} \\ 5 & \text{if } n \text{ is even} \end{cases}$$

Hence, 3 and 5 are the accumulation values of f.
$\limsup_{n\to\infty} f(n) = 5$, therefore 5 is upper accumulation point of $f(n)$. $\liminf_{n\to\infty} f(n) = 3$, therefore 3 is lower accumulation point of $f(n)$.

(c) $f(n) = ((-1)^n + (-1)^{\lfloor n/2 \rfloor})(1 + 1/n)$.
This is a complex function but upon substituting values 2,3,4,5 we can observe a pattern. Whenever n is one of $[3, 7, 11, \cdots]$, i.e. of the form $4x - 1$, then $f(n) = -2(1 + 1/n)$. Whenever n is one of $[5, 9, 13, \cdots]$, i.e. of the form $4x + 1$, then $f(n) = 0$. Whenever n is one of $[2, 6, 10, \cdots]$, i.e. of the form $4x - 2$, then $f(n) = 0$. Whenever n is one of $[4, 8, 12, \cdots]$, i.e. of the form $4x$, then $f(n) = 2(1 + 1/n)$. So the function can be re-written as

$$f(n) = \begin{cases} -2(1 + 1/n) & \text{if n = 4x - 1} \\ 0 & \text{if n = 4x + 1} \\ 0 & \text{if n = 4x - 2} \\ 2(1 + 1/n) & \text{if n = 4x} \end{cases}$$

where x is an integer. Applying limit to the function f,

$$\lim_{n\to\infty} f(n) = \lim_{n\to\infty}((-1)^n+(-1)^{\lfloor n/2\rfloor})(1+1/n) = \begin{cases} \lim_{n\to\infty} -2(1+1/n) & \text{if n = 4x - 1} \\ \lim_{n\to\infty} 0 & \text{if n = 4x + 1} \\ \lim_{n\to\infty} 0 & \text{if n = 4x - 2} \\ \lim_{n\to\infty} 2(1+1/n) & \text{if n = 4x} \end{cases}$$

$$= \begin{cases} -2 & \text{if n = 4x - 1} \\ 0 & \text{if n = 4x + 1} \\ 0 & \text{if n = 4x - 2} \\ 2 & \text{if n = 4x} \end{cases}$$

So, -2 and 2 are the accumulation points of $f(n)$.
$\limsup_{n\to\infty} f(n) = 2$, therefore 2 is upper accumulation point of $f(n)$.
$\liminf_{n\to\infty} f(n) = -2$, therefore -2 is lower accumulation point of $f(n)$.

Problem 3.

Solution.

(a) Let's consider an arbitrary function f such that $\log_b n$ is an asymptotically tight bound for f, i.e. $f(n) \in \Theta(\log_b n)$. By limit rule, we can say that

$$\lim_{n\to\infty} \frac{f(n)}{\log_b n} = C$$

for some positive real constant C. We also know that,

$$\lim_{n\to\infty} \frac{\log_b n}{\log_d n} = \lim_{n\to\infty} \frac{\log_b n}{\log_b n/\log_b d} = \lim_{n\to\infty} \log_b d = \log_b d$$

Since b and d are positive real numbers not equal to 1, so $0 < \log_b d < \infty$. Therefore, (multiplying and dividing by $\log_b n$)

$$\lim_{n\to\infty} \frac{f(n)}{\log_d n} = \lim_{n\to\infty} \frac{f(n)}{\log_b n} \cdot \frac{\log_b n}{\log_d n} = \left(\lim_{n\to\infty} \frac{f(n)}{\log_b n}\right)\left(\lim_{n\to\infty} \frac{\log_b n}{\log_d n}\right) = C \cdot \log_b d$$

and $0 < C \cdot \log_b d < \infty$.
Using limit rule again, $f(n) \in \Theta(\log_d n)$.
Similarly, it can be shown that for any arbitrary function f such that $f(n) \in \Theta(\log_d n)$, it must be the case that $f(n) \in \Theta(\log_b n)$.
Hence, $\Theta(\log_b n) = \Theta(\log_d n)$, so we can write $\Theta(\log\ n)$ using a baseless logarithm without confusion.

(b) Let's consider an arbitrary function f such that $n^{\log_b n}$ is an asymptotically tight bound for f, i.e. $f \in \Theta(n^{\log_b n})$. So, we can say that

$$cn^{\log_b n} \le f(n) \le Cn^{\log_b n}$$

for some positive real constants c and C. Let g be another arbitrary function such that $n^{\log_d n}$ is an asymptotically tight bound for g, i.e. $g \in \Theta(n^{\log_d n})$. So, we can say that

$$c_1 n^{\log_d n} \leq g(n) \leq C_1 n^{\log_d n}$$

or some positive real constants c_1 and C_1. Clearly, when $b \neq d$, then $\log_b n \neq \log_d n$, meaning the tight bound will vary for f and g since they will have different powers of n. Hence, $\Theta(n^{\log_b n}) = \Theta(n^{\log_d n})$ is false and does not hold in general.

Problem 4.

Solution.

We can get the lower bound Ω and upper bound O individually and combine them to get the asymptotically tight bound. Let's get the lower bound first.
$1^k + 2^k + \cdots + n^k \geq \lceil n/2 \rceil^k + \cdots + (n-1)^k + n^k$ (removing the first half terms)
$\geq \lceil n/2 \rceil^k + \cdots + \lceil n/2 \rceil^k + \lceil n/2 \rceil^k$ (replacing the remaining terms by $\lceil n/2 \rceil^k$)
$= \lceil (n+1)/2 \rceil \cdot \lceil n/2 \rceil^k$
$\geq (n/2)(n/2)^k$
$= (n/2)^{k+1}$
$= n^{k+1}/2^{k+1}$
So, $\mathbf{1^k + 2^k + \cdots + n^k \geq n^{k+1}/2^{k+1} = \Omega(n^{k+1})}$
Now, let's get the upper bound.
$1^k + 2^k + \cdots + n^k \leq n^k + n^k + \cdots + n^k$ (replacing each integer by n)
$= n \cdot n^k$
$= n^{k+1}$
$= O(n^{k+1})$
So, $\mathbf{1^k + 2^k + \cdots + n^k = O(n^{k+1})}$
Hence, we can conclude that $1^k + 2^k + \cdots + n^k = \Theta(n^{k+1})$.

Problem 5.

Solution.

We know that for two functions f and g from the set of positive integers to real numbers, if the limit

$$\lim_{n\to\infty} \frac{|f(n)|}{|g(n)|}$$

exists and is finite, then $f \in O(g)$.
Let's take $f(n) = n^{\ln n}$ and
$g(n) = e^{(\ln n)^2} = e^{\ln n \cdot \ln n} = (e^{\ln n})^{\ln n} = n^{\ln n}$. So,

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{n^{\ln n}}{e^{(\ln n)^2}} = \lim_{n\to\infty} \frac{n^{\ln n}}{n^{\ln n}} = \lim_{n\to\infty} 1 = 1$$

Since the limit exists and is finite, therefore we can say that $f \in O(g)$, i.e. $n^{\ln n} \in O(e^{(\ln n)^2})$ is true.