Expectation / Variance

Angermueller Christof

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Contents

1 Notations

The following document summarizes rules for computing the expected value, variance, and covariance of random variables and random vectors. x_i denotes a random variable with expectation value $m_i = \mathrm{E}[x_i]$, variance $s_i = \mathrm{Var}[x_i]$ and covariance $s_{ij} = \mathrm{Cov}[x_i, x_j]$. a_i and b_i are constant scalars. $\boldsymbol{x}_i$ denotes a random vector with expectation vector $\boldsymbol{m}_i = \mathrm{E}[\boldsymbol{x}_i]$, variance matrix $S_i = \mathrm{Var}[\boldsymbol{x}_i]$ and covariance matrix $S_{ij} = \mathrm{Cov}[\boldsymbol{x}_i, \boldsymbol{x}_j]$. A_i and $\boldsymbol{b}_i$ is a constant matrix and vector.

2 Expectation

2.1 Random variable

$$
\begin{aligned}
\mathrm{E}[x] &= \int p(x) x \, dx \\
\mathrm{E}[ax+b] &= am+b \\
\mathrm{E}[a_1 x_1 + a_2 x_2 + b] &= a_1 m_1 + a_2 m_2 + b \\
\mathrm{E}[\prod_{i=1}^{N} x_i] &= \prod_{1 \le i \le N} m_i + \sum_{1 \le i < j \le N} s_{ij} \\
\mathrm{E}[x_1 x_2] &= m_1 m_2 + s_{12} \\
\mathrm{E}[x^2] &= m^2 + s
\end{aligned}
$$

2.2 Random vector

$$
\begin{aligned}
\mathrm{E}[\boldsymbol{x}] &= \int p(\boldsymbol{x}) \boldsymbol{x} \, d\boldsymbol{x} \\
\mathrm{E}[A\boldsymbol{x}+\boldsymbol{b}] &= A\boldsymbol{m}+\boldsymbol{b} \\
\mathrm{E}[A_1 \boldsymbol{x}_1 + A_2 \boldsymbol{x}_2 + \boldsymbol{b}] &= A_1 \boldsymbol{m}_1 + A_2 \boldsymbol{m}_2 + \boldsymbol{b} \\
\mathrm{E}[(\boldsymbol{x}_1 - \boldsymbol{b}_1)^T A (\boldsymbol{x}_2 - \boldsymbol{b}_2)] &= (\boldsymbol{m}_1 - \boldsymbol{b}_1)^T A (\boldsymbol{m}_2 - \boldsymbol{b}_2) + \mathrm{Tr}[A^T S_{12}] \\
\mathrm{E}[\boldsymbol{x}_1^T A \boldsymbol{x}_2] &= \boldsymbol{m}_1^T A \boldsymbol{m}_2 + \mathrm{Tr}[A^T S_{12}] \\
\mathrm{E}[\boldsymbol{x}_1^T \boldsymbol{x}_2] &= \boldsymbol{m}_1^T \boldsymbol{m}_2 + \mathrm{Tr}[S_{12}] \\
\mathrm{E}[X_1^T A X_2]_{ij} &= \mathrm{E}[X_{1:i}]^T A \, \mathrm{E}[X_{2:j}] + \mathrm{Tr}[A^T \, \mathrm{Cov}[X_{1:i}, X_{2:j}]] \\
\mathrm{E}[(\boldsymbol{x}_1 - \boldsymbol{b}_1)(\boldsymbol{x}_2 - \boldsymbol{b}_2)^T] &= (\boldsymbol{m}_1 - \boldsymbol{b}_1)(\boldsymbol{m}_2 - \boldsymbol{b}_2)^T + S_{12} \\
\mathrm{E}[\boldsymbol{x}_1 \boldsymbol{x}_2^T] &= \boldsymbol{m}_1 \boldsymbol{m}_2^T + S_{12}
\end{aligned}
$$

3 Variance

3.1 Random variable

$$\begin{aligned}
\mathrm{Var}[x] &= \int p(x)(x - \mathrm{E}[x])^2 \, dx = \mathrm{E}[(x - \mathrm{E}[x])^2] \\
\mathrm{Var}[ax + b] &= a^2 s \\
\mathrm{Var}[a_1 x_1 + a_2 x_2 + b] &= a_1^2 s_1 + a_2^2 s_2 + 2 s_{12} \\
\mathrm{Var}[a_1 x_1 - a_2 x_2 + b] &= a_1^2 s_1 + a_2^2 s_2 - 2 s_{12}
\end{aligned}$$

If x_1 and x_2 and independent:

$$\mathrm{Var}[x_1 x_2] = \mathrm{E}[x_1^2] \, \mathrm{E}[x_2^2] - \mathrm{E}[x_1]^2 \, \mathrm{E}[x_2]^2$$

3.2 Random vector

$$\begin{aligned}
\mathrm{Var}[\boldsymbol{x}] &= \int p(\boldsymbol{x})(\boldsymbol{x} - \mathrm{E}[\boldsymbol{x}])^2 \, dx \\
&= \mathrm{E}[(\boldsymbol{x} - \mathrm{E}[\boldsymbol{x}])^2] \\
\mathrm{Var}[A\boldsymbol{x} + \boldsymbol{b}] &= A \, \mathrm{Var}[x] A^T \\
\mathrm{Var}[A_1 \boldsymbol{x}_1 + A_2 \boldsymbol{x}_2 + \boldsymbol{b}] &= A_1 \, \mathrm{Var}[x_1] A_1^T + A_2 \, \mathrm{Var}[x_2] A_2^T + 2 A_1 S_{12} A_2^T
\end{aligned}$$

4 Covariance

4.1 Random variable

$$\begin{aligned}
\mathrm{Cov}[x_1, x_2] &= \int p(x_1, x_2)(x_1 - \mathrm{E}[x_1])(x_2 - \mathrm{E}[x_2]) \, dx \\
&= \mathrm{E}[(x_1 - \mathrm{E}[x_1])(x_2 - \mathrm{E}[x_2])] \\
\mathrm{Cov}[a_1 x_1 + b_1, a_2 x_2 + b_2] &= a_1 a_2 s_{12}
\end{aligned}$$

4.2 Random vector

$$\begin{aligned}
\mathrm{Cov}[\boldsymbol{x}_1, \boldsymbol{x}_2] &= \int p(\boldsymbol{x}_1, \boldsymbol{x}_2)(\boldsymbol{x}_1 - \mathrm{E}[\boldsymbol{x}_1])(\boldsymbol{x}_2 - \mathrm{E}[\boldsymbol{x}_2])^T \, dx \\
&= \mathrm{E}[(\boldsymbol{x}_1 - \mathrm{E}[\boldsymbol{x}_1])(\boldsymbol{x}_2 - \mathrm{E}[\boldsymbol{x}_2])^T] \\
\mathrm{Cov}[A_1 \boldsymbol{x}_1 + \boldsymbol{b}_1, A_2 \boldsymbol{x}_2 + \boldsymbol{b}_2] &= A_1 s_{12} A_2^T
\end{aligned}$$