# Expectation / Variance

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June 9, 2014

## Contents



## 1 Notations

The following document summarizes rules for computing the expected value, variance, and covariance of random variables and random vectors. $x_i$ denotes a random variable with expectation value $m_i = \mathrm{E}[x_i]$, variance $s_i = \mathrm{Var}[x_i]$ and covariance $s_{ij} = \mathrm{Cov}[x_i, x_j]$. $a_i$ and $b_i$ are constant scalars. $\boldsymbol{x}_i$ denotes a random vector with expectation vector $\boldsymbol{m}_i = \mathrm{E}[\boldsymbol{x}_i]$, variance matrix $S_i = \mathrm{Var}[\boldsymbol{x}_i]$ and covariance matrix $S_{ij} = \mathrm{Cov}[\boldsymbol{x}_i, \boldsymbol{x}_j]$. $A_i$ and $\boldsymbol{b}_i$ is a constant matrix and vector.

# 2 Expectation

## 2.1 Random variable

$$\begin{aligned}
\mathrm{E}[x] &= \int p(x)x\,dx \\
\mathrm{E}[ax+b] &= am+b \\
\mathrm{E}[a_1x_1+a_2x_2+b] &= a_1m_1+a_2m_2+b \\
\mathrm{E}[\prod_{i=1}^{N} x_i] &= \prod_{1\le i\le N} m_i + \sum_{1\le i<j\le N} s_{ij} \\
\mathrm{E}[x_1x_2] &= m_1m_2+s_{12} \\
\mathrm{E}[x^2] &= m^2+s
\end{aligned}$$

## 2.2 Random vector

$$\begin{aligned}
\mathrm{E}[\boldsymbol{x}] &= \int p(\boldsymbol{x})\boldsymbol{x}\,d\boldsymbol{x} \\
\mathrm{E}[A\boldsymbol{x}+\boldsymbol{b}] &= A\boldsymbol{m}+\boldsymbol{b} \\
\mathrm{E}[A_1\boldsymbol{x}_1+A_2\boldsymbol{x}_2+\boldsymbol{b}] &= A_1\boldsymbol{m}_1+A_2\boldsymbol{m}_2+\boldsymbol{b} \\
\mathrm{E}[(\boldsymbol{x}_1-\boldsymbol{b}_1)^TA(\boldsymbol{x}_2-\boldsymbol{b}_2)] &= (\boldsymbol{m}_1-\boldsymbol{b}_1)^TA(\boldsymbol{m}_2-\boldsymbol{b}_2)+\mathrm{Tr}[A^TS_{12}] \\
\mathrm{E}[\boldsymbol{x}_1^TA\boldsymbol{x}_2] &= \boldsymbol{m}_1^TA\boldsymbol{m}_2+\mathrm{Tr}[A^TS_{12}] \\
\mathrm{E}[\boldsymbol{x}_1^T\boldsymbol{x}_2] &= \boldsymbol{m}_1^T\boldsymbol{m}_2+\mathrm{Tr}[S_{12}] \\
\mathrm{E}[X_1^TAX_2]_{ij} &= \mathrm{E}[X_{1:i}]^TA\,\mathrm{E}[X_{2:j}]+\mathrm{Tr}[A^T\,\mathrm{Cov}[X_{1:i},X_{2:j}]] \\
\mathrm{E}[(\boldsymbol{x}_1-\boldsymbol{b}_1)(\boldsymbol{x}_2-\boldsymbol{b}_2)^T] &= (\boldsymbol{m}_1-\boldsymbol{b}_1)(\boldsymbol{m}_2-\boldsymbol{b}_2)^T+S_{12} \\
\mathrm{E}[\boldsymbol{x}_1\boldsymbol{x}_2^T] &= \boldsymbol{m}_1\boldsymbol{m}_2^T+S_{12}
\end{aligned}$$

# 3 Variance

## 3.1 Random variable

$$
\begin{aligned}
\operatorname{Var}[x] &= \int p(x)(x-\operatorname{E}[x])^2 \, dx = \operatorname{E}[(x-\operatorname{E}[x])^2] \\
\operatorname{Var}[ax+b] &= a^2 s \\
\operatorname{Var}[a_1x_1 + a_2x_2 + b] &= a_1^2 s_1 + a_2^2 s_2 + 2s_{12} \\
\operatorname{Var}[a_1x_1 - a_2x_2 + b] &= a_1^2 s_1 + a_2^2 s_2 - 2s_{12}
\end{aligned}
$$

If $x_1$ and $x_2$ and independent:

$$
\operatorname{Var}[x_1x_2] = \operatorname{E}[x_1^2]\operatorname{E}[x_2^2] - \operatorname{E}[x_1]^2\operatorname{E}[x_2]^2
$$

## 3.2 Random vector

$$
\begin{aligned}
\operatorname{Var}[\boldsymbol{x}] &= \int p(\boldsymbol{x})(\boldsymbol{x}-\operatorname{E}[\boldsymbol{x}])^2 \, dx \\
&= \operatorname{E}[(\boldsymbol{x}-\operatorname{E}[\boldsymbol{x}])^2] \\
\operatorname{Var}[A\boldsymbol{x}+\boldsymbol{b}] &= A\operatorname{Var}[x]A^T \\
\operatorname{Var}[A_1\boldsymbol{x}_1 + A_2\boldsymbol{x}_2 + \boldsymbol{b}] &= A_1\operatorname{Var}[x_1]A_1^T + A_2\operatorname{Var}[x_2]A_2^T + 2A_1S_{12}A_2^T
\end{aligned}
$$

Variance of quadratic form:

$$
\begin{aligned}
\operatorname{Var}[\boldsymbol{x}^TA\boldsymbol{x}] &= 4\boldsymbol{m}^TA'SA'\boldsymbol{m} + 2\operatorname{Tr}[A'SA'S] \\
A' &= \frac{1}{2}(A+A^T) \\
\operatorname{Var}[\boldsymbol{x}^T\boldsymbol{x}] &= 4\boldsymbol{m}^TS\boldsymbol{m} + 2\operatorname{Tr}[SS]
\end{aligned}
$$

If $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are independent:

$$
\begin{aligned}
\operatorname{Var}[\boldsymbol{x}_1^T\boldsymbol{x}_2] &= \sum_i \operatorname{E}[{x_1}_i{}^2]\operatorname{E}[{x_2}_i{}^2] + \operatorname{E}[x_{1i}]^2\operatorname{E}[x_{2i}]^2 \\
&= \sum_i \operatorname{E}[x_{1i}]^2\operatorname{Var}[x_{2i}] + \operatorname{E}[x_{2i}]^2\operatorname{Var}[x_{1i}] + \operatorname{Var}[x_{1i}]\operatorname{Var}[x_{2i}]
\end{aligned}
$$

# 4 Covariance

## 4.1 Random variable

$$
\begin{aligned}
\mathrm{Cov}[x_1, x_2] &= \int p(x_1, x_2)(x_1 - \mathrm{E}[x_1])(x_2 - \mathrm{E}[x_2])\, dx \\
&= \mathrm{E}[(x_1 - \mathrm{E}[x_1])(x_2 - \mathrm{E}[x_2])] \\
\mathrm{Cov}[a_1 x_1 + b_1, a_2 x_2 + b_2] &= a_1 a_2 s_{12}
\end{aligned}
$$

## 4.2 Random vector

$$
\begin{aligned}
\mathrm{Cov}[\boldsymbol{x}_1, \boldsymbol{x}_2] &= \int p(\boldsymbol{x}_1, \boldsymbol{x}_2)(\boldsymbol{x}_1 - \mathrm{E}[\boldsymbol{x}_1])(\boldsymbol{x}_2 - \mathrm{E}[\boldsymbol{x}_2])^T\, dx \\
&= \mathrm{E}[(\boldsymbol{x}_1 - \mathrm{E}[\boldsymbol{x}_1])(\boldsymbol{x}_2 - \mathrm{E}[\boldsymbol{x}_2])^T] \\
\mathrm{Cov}[A_1 \boldsymbol{x}_1 + \boldsymbol{b}_1, A_2 \boldsymbol{x}_2 + \boldsymbol{b}_2] &= A_1 s_{12} A_2^T
\end{aligned}
$$