Assignment 2

(COMP 3601 - Design and Analysis of Algorithms, 2022-2023)

Date Available: 8AM, Wednesday, March 01, 2023
Due Date: 11.50 PM, Wednesday, March 15, 2023
Total Mark: 100 marks (weighted 16% out of 100%)

Answer ALL Questions

INSTRUCTIONS
1. Type or write your answers neatly.
2. Show all working of your answers.
3. Your solutions must be your own. You must not share your working or solutions with your peers.
4. You are not permitted to copy, summarize, or paraphrase the work of others in your solutions.
5. Submit your answers in a single zipped file named A2_ID.zip to the email comp3601daa@gmail.com, where ID is replaced with your student ID. The file A2_ID.zip contains
- a single PDF file containing all of your typed, handwritten, and screenshots answers.
- a signed and dated UWI Plagiarism Declaration indicating that the work submitted is your own.

Question 1 [20 marks]
a. [10 marks] The exponentiation a^n can be computed by using the squaring algorithm as follows.

$$a^n = \begin{cases} 1 & \text{if } n = 0 \\ a & \text{if } n = 1 \\ (a^2)^{n/2} & \text{if } n \text{ is even and } n > 0 \\ (a^2)^{\lfloor n/2 \rfloor} \cdot a & \text{if } n \text{ is odd and } n > 1 \end{cases}$$

where a is a positive integer and n is a nonnegative integer.
Set up and solve a recurrence relation by the backward substitution method for the number of multiplications made by the squaring algorithm for $n = 2^k$.

b. [10 marks] You are given a sorted array $A[\ell]$, $A[\ell+1]$, ..., $A[r]$ whose values increase linearly, the interpolation search that is used to find whether a search key v in the array A compares the key v with the element $A[x]$ whose index x is computed as

$$x = \ell + \left\lfloor \frac{(v - A[\ell])(r - \ell)}{A[r] - A[\ell]} \right\rfloor,$$

where $\lfloor \cdot \rfloor$ is the floor function. Derive the above formula.

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Question 2 [20 marks]

a. [10 marks] Solve the recurrence relation $T(n) = T(\sqrt{n}) + 1$, where $T(n)$ is a constant c for $n \le 2$, by using the backward substitution method.

b. [2 marks] Prove the equality $a^{\log_b c} = c^{\log_b a}$.

c. [8 marks] Show that a^{62} can be computed with only eight multiplications.

Question 3 [20 marks]

a. [10 marks] The best-case recurrence relation for the number of key comparisons made by the merge sort algorithm is $C_b(n) = 2C_b(n / 2) + n/2$ for $n > 1$, $C_b(1) = 0$.
Solve the above recurrence by the backward substitution method for $n = 2^k$.

b. [10 marks] Use mathematical induction to prove that $n! > 2^n$ for all integers $n \ge 4$.

Question 4 [20 marks]

a. [10 marks] Use the mathematical induction method to prove the following equality

$$\begin{bmatrix} F(n-1) \\ F(n) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ for } n \ge 1,$$

where $F(n)$ is the nth Fibonacci number, $F(n) = F(n-1) + F(n-2)$ for $n \ge 2$, $F(0) = 0$, $F(1) = 1$.

b. [10 marks] The average-case number of key comparisons made by the insertion sort algorithm is

$$C_a(n) = \sum_{i=1}^{n-1}\left(\frac{i}{2} + \frac{i}{i+1}\right).$$

Show that $C_a(n) \approx \frac{n(n+3)}{4} - \ln n$, where $\ln n = \log_e n$, $e \approx 2.718$.

Question 5 [20 marks]

The average-case number of key comparisons made by the bubble sort algorithm is

$$C_a(n) = \frac{1}{n-1}\sum_{i=1}^{n-1} C(i),$$

where $C(i) = \sum_{j=n-1}^{n-i} j$. Show that $C_a(n) = \frac{n^2}{3} - \frac{n}{6}$.

End of Assignment 2