Practice

From Math 20630

HW 4: More Functions Woot

1. a) Given f is a bijection, we need to show $f^{-1} : B \to A$ is both an injection and surjection. To show it is an injection, we want to show for $x, y \in A$, $f^{-1}(x) = f^{-1}(y) \implies x = y$. Since f is a bijection, then there is only one $a \in A$ and one corresponding $b \in B$ such that $f(a) = b$. It follows that for $x = y \in A$, $f(x) = f(y)$. It follows that $f^{-1}(f(x)) = f^{-1}(f(y))$ and so $x = y$.

   b) We want to show that $f^{-1} \circ g^{-1}$ is the inverse of $g \circ f$. We can demonstrate this:

$$\begin{aligned}
&(f^{-1} \circ g^{-1} \circ (g \circ f))(x) \\
&= (f^{-1} \circ (g^{-1} \circ g) \circ f)(x) \\
&= (f^{-1} \circ I_B \circ f)(x) \\
&= ((f^{-1} \circ I_B) \circ f))(x) \\
&= (f^{-1} \circ f)(x) \\
&= (I_A)(x) \\
&= x
\end{aligned}$$

   It's clear how the same strategy would work for composing the two functions in different order. Hence we have shown that $f^{-1} \circ g^{-1}$ is an inverse of $g \circ f$, and since a function with an inverse has a unique inverse, then it is the only inverse.

2. a) If A and B are the same size, f remains an injection. If B has more elements than A, then f remains an injection. If B has fewer elements than A, then there is some $a \in A$ such that $f(a)$ is undefined. WLOG, let A be $[n+1]$ and B be $[n]$. There is no element in B that $n+1$ in A maps to, so f is no longer injective. From these cases, $|A| \leq |B|$.

   b) If A and B are the same size, f remains a surjection. If A has more elements than B, then f remains a surjection since not all elements in A need to be mapped to an element in B. However, if A has fewer elements than B, then f cannot be a surjection. WLOG, use the sets $A = [n]$, $B = [n+1]$. What $a \in A$ maps to $n+1 \in B$? If there is none, f is not surjective. If there is, then, WLOG, we have $n \in A$ mapping to both n and $n+1$, and so f is not surjective. So, f can't be surjective if $|A| < |B|$, and therefore $|A| \geq |B|$.

   c) From part a), since f is an injection, $|A| \leq |B|$. Similarly, since g is an injection, then also $|B| \leq |A|$. So, $|A| = |B|$. We know f is an injection, so to show it is a bijection, we need to show it is also a surjection. Assume it is not a surjection. Then, from b), $|A| < |B|$. But we know $|A| = |B|$, which is a contradiction. So f is a surjection. The same argument can be applied to g.

3. a) We can define f for sets $A = \{a_1, a_2, \cdots, a_m\}$ and $B = \{b_1, b_2, \cdots, b_n\}$