Practice

From Math 20630

HW 3: Functions

1. a) **For every $x \in X$, there exists y such that $f(x) = y$**

   $A = [n]$ and B_n is the set of binary strings of length n. Take a subset S of A of size n. By definition of B_n, there is a binary string $b_0, b_1, \cdots, b_n$ in B_n.

   For every $x \in X$, $f(x) \in Y$

   See above.

   For every $x \in X$, there is only one $y \in Y$ such that $f(x) = y$

   Take $S \subset A$ of size n. Then there is a binary string $T \subset B_n$ whose digits correspond to $b_1 b_2 \dots b_n$ where $b_i = 1$ if $i \in S$, and $b_i = 0$ if $i \notin S$. WLOG, if there were a binary string $U = T$ and therefore U would not be unique, then its binary string representation would be $c_1 c_2 \dots c_k$ where $c_k = 1$ if $k \in S$, and $c_k = 0$ if $k \notin S$. U cannot be different from T since this would mean $c_k = 0$ even though $i \in S$ or $c_k = 1$ even though $i \notin S$ for some k. Thus $U \neq T$.

   b) See above (third point).

   c) B_n contains all binary strings of length n. To make a corresponding $S \subset A$, take all i for which $b_i = 1$ and put it in S. Since i is maximally n and minimally 0, any set made this way must have between 0 and n elements, and so it must be in A.

2. a) We want to show that, for injections F and G, $F \circ G$ is also an injection. That is, $F(G(a)) = F(G(b)) \implies a = b$. Since F is an injection, then $G(a) = G(b)$. Similarly, since G is an injection, then $a = b$. It follows $F \circ G$ is also an injection.

   b) We want to show the above but for surjections $F, G, F \circ G$. That is, for every y in $F \circ G$, there is a corresponding x such that $F(G(x)) = y$. Since F is a surjection, there is an g such that $F(g) = y$. Further, Since G is a surjection, there is an x such that $G(x) = g$. With this x, it follows that $F \circ G$ is a surjection. To verify, $G(x) = g, F(g) = y$, so $F(G(x)) = F(g) = y$.

3. a) Since h is injective, then for $x, y \in A$ such that $h(x) = h(y)$, $x = y$. Since $x = y$, then $g(f(x)) = g(f(y))$ since both f and g are well-defined. Hence $f(x) = f(y)$ and is an injection.

   b) False. If there are two elements in B that g maps to A, there could be only one element of A that maps to g. Hence g need not be injective, but h would still be injective.

   c) This is also false. There could be elements in g that F does not map to.

   d) Since h is surjective, then for any $x \in A$, there is a corresponding $y \in A$ such that $h(x) = y$. In other words, $g(f(x)) = y$. Suppose g is not surjective for y. So, $f(x) \in B$ does not exist, and so $g(f(x))$ does not exist, and so h does not map $g(f(x))$ to y and is not surjective. By contraposition, g must be surjective.

   More simply, we want to find $y \in B$ such that $g(y) = x, x \in A$. Since h is surjective, then for some $z \in A$, $g(f(z)) = x$. By definition, g maps $f(z)$ to x. We have found our $y \in B$, $f(z)$, so g must be surjective.