Practice

From Math 20630

HW 6: Sets

1. a) $\mathbf{A} \cup (\mathbf{B} \cap \mathbf{C}) \subseteq (\mathbf{A} \cup \mathbf{B}) \cap (\mathbf{A} \cup \mathbf{C})$

Say $x \in A \cup (B \cap C)$. Then either $x \in A$ or $x \in B \cap C$. If $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$. By definition of intersections, $x \in (A \cup B) \cap (A \cup C)$. If $x \in B \cap C$, then $x \in B$ and $x \in C$. It follows that $x \in A \cup B$ and $x \in A \cup C$, respectively, so $x \in (A \cup B) \cap (A \cup C)$.

$(\mathbf{A} \cup \mathbf{B}) \cap (\mathbf{A} \cup \mathbf{C}) \subseteq \mathbf{A} \cup (\mathbf{B} \cap \mathbf{C})$

Say $x \in (A \cup B) \cap (A \cup C)$. So $x \in A \cup B$ and $x \in A \cup C$. If $x \in A \cup B$, either $x \in A$ or $x \in B$. If $x \in A$, $x \in A \cup (B \cap C)$. Take the case $x \in B$. Since $x \in A \cup C$, then, if $x \in C$ (the case where $x \in A$ is already covered), then $x \in B$ and $x \in C$, so $x \in B \cap C$. It follows $x \in A \cup (B \cap C)$.

b) $\mathbf{A} \cap (\mathbf{B} \cup \mathbf{C}) \subseteq (\mathbf{A} \cap \mathbf{B}) \cup (\mathbf{A} \cap \mathbf{C})$

Let $x \in A \cap (B \cup C)$. So $x \in A$ and $x \in B \cup C$. Since $x \in B \cup C$, either $x \in B$ or $x \in C$. If $x \in B$, then $x \in A$ and $x \in B$ and so $x \in A \cup B$, and accordingly $x \in (A \cap B) \cup (A \cap C)$. Similarly, if $x \in C$, then $x \in A$ and $x \in C$ and so $x \in A \cup C$, and accordingly $x \in (A \cap B) \cup (A \cap C)$.

$(\mathbf{A} \cap \mathbf{B}) \cup (\mathbf{A} \cap \mathbf{C}) \subseteq \mathbf{A} \cap (\mathbf{B} \cup \mathbf{C})$

Let $x \in (A \cap B) \cup (A \cap C)$. Either $x \in A \cap B$ or $x \in A \cap C$). If $x \in A \cap B$, then $x \in A$ and $x \in B$. Since $x \in B$, then $x \in B \cup C$. It follows $x \in A$ and $x \in B \cup C$, so $x \in A \cap (B \cup C)$. If $x \in A \cap C$, then $x \in A$ and $x \in C$. Since $x \in C$, then $x \in B \cup C$. Similar to above, then $x \in A \cap (B \cup C)$ as desired.

c) $(\mathbf{A} \cup \mathbf{B})^{\mathbf{c}} \subseteq \mathbf{A}^{\mathbf{c}} \cap \mathbf{B}^{\mathbf{c}}$

Let $x \in (A \cup B)^c$. Then $x \notin A$ and $x \notin B$. This is exactly what we want; it follows directly that $x \in A^c \cup B^c$.

$\mathbf{A}^{\mathbf{c}} \cap \mathbf{B}^{\mathbf{c}} \subseteq (\mathbf{A} \cup \mathbf{B})^{\mathbf{c}}$

Let $x \in A^c \cup B^c$. Either $x \in A^c$ or $x \in B^c$. If $x \in A^c$, $x \notin A$. If $x \in B^c$, $x \notin B$. So, $x \notin A$ and $x \notin B$. From above, it follows that $x \in (A \cup B)^c$.

2. $(\mathbf{A} \cup \mathbf{B}) \setminus \mathbf{C} \subseteq [\mathbf{A} \setminus (\mathbf{B} \cup \mathbf{C})] \cup [\mathbf{B} \setminus (\mathbf{A} \cap \mathbf{C})]$

Let $x \in (A \cup B) \setminus C$. This means $x \in A$ or $x \in B$ and $x \notin C \equiv x \in C^c$. Consider the case when $x \in A$ and $x \in C^c$. Then $x \in A \cap C^c$, so $x \notin A \cap C$. Further suppose $x \in B$. Since $x \in B$ and $x \notin A \cap C$, then finally $x \in B \setminus (A \cap C)$.

The next case is when $x \notin B$. Since $x \in A \cap B^c$, $x \notin A \cap B$. Since $x \in C^c$, then $x \notin C$ either. Now we have $x \in A$ but $x \notin B$ and $x \notin C$, so $A \cup B \setminus C \subseteq [A \setminus (B \cup C)]$ and the rest of the formula follows.

Now consider $x \in B$ and $x \in C^c$. Since $x \in C^c, x \notin A \cap C$. So, $x \in B \setminus (A \cap C)$ and the rest of the formula follows. This is true no matter whether $x \in A$ or $x \notin A$.

$[\mathbf{A} \setminus (\mathbf{B} \cup \mathbf{C})] \cup [\mathbf{B} \setminus (\mathbf{A} \cap \mathbf{C})] \nsubseteq (\mathbf{A} \cup \mathbf{B}) \setminus \mathbf{C}$

Let $x \in A \setminus (B \cup C)$. Then $x \in A$ and $x \notin B \cup C$.

Next, let $x \in B \setminus (A \cap C)$. Then $x \in B$ and $x \notin A \cap C$.

Now consider both the above cases. $x \in A$ and $x \in B$, but $x \notin B \cup C$ and $x \notin A \cap C$. It follows $x \in A$ and $x \notin A$, but this is a contradiction, so equality does not hold.

3. a) There are 10 elements in [10], so there are 2^{10} many subsets of [10]. Let $A =$ the set of all elements in [10] that are not odd. $|A| = 5$, so the number of subsets that do not contain an odd integer is 2^5. So the number of subsets containing an odd integer is $2^{10} - 2^5 = 992$.

b) MISSISSIPPI has 11 letters. There are $11 - 4 + 1 = 8$ ways of arranging the S's consecutively. For each way, there are 7 letters to be rearranged to make a unique string (4 I's, 2P's, and 1 M). $8 \cdot \frac{7!}{4!2!} = 840$ ways of repositioning the letters.

4. Assume the statement is false. That is, there is an n for which for all k

$$\binom{n}{k} < \frac{2^n}{n+1}$$

Notice that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$. Adjusting the above statement,

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} &< \sum_{k=0}^{n} \frac{2^n}{n+1} \\
2^n &< \sum_{k=0}^{n} \frac{2^n}{n+1} \\
2^n &< \frac{2^n(n+1)}{n+1} \\
2^n &< 2^n
\end{aligned}$$

This is false, and so the original statement is true.

5. a) Say you have n people and want to form a committee of k members with a sub-committee of j members.

   From the left side of the equation, there are $\binom{n}{k}$ ways to choose k members out of n people. For each of these ways of picking k members, there is $\binom{k}{j}$ ways of picking a sub-committee of j people from those k members that already form a committee to accomplish the above task.

   From the right side of the equation, there are $\binom{n}{j}$ ways to choose a sub-committee of j people out of n people. For each of these ways of picking a sub-committee, there are $\binom{n-j}{k-j}$ ways of picking a committee from the remaining people. j is subtracted since they have already been chosen for the sub-committee.

   b) Say you want to find the total number of ways to choose a committee with a president from a group of people whose size ranges from 1 to n.

   From the left side of the equation, a president can be chosen out of k people. For each of these presidents, $\binom{n}{k}$ committees can be formed, making $k\binom{n}{k}$ many possible committees. We sum them all up to find the total number of ways to form these committees of group size k, $1 \leq k \leq n$.

   From the right side of the equation, one president must be chosen. Now the remaining number of people is $n-1$, and consequently there are 2^{n-1} ways of configuring a committee out of them. For each of these teams, a unique president can be chosen from the original n people, so there are n many possible presidents. The total number of possible committees is therefore $n \cdot 2^{n-1}$.