Practice

From Math 20630

HW 1: Induction

1. We start with the base case $n = 1$.

$$P_1 : \sum_{i=1}^{1}(2-1) = 1^2 = 1$$

This is true. Moving on to the inductive step, we assume

$$P_k : \sum_{i=1}^{k}(2i-1) = k^2$$

For all $k \in \mathbb{N}$. Proving $P_{k+1} : \sum_{i=1}^{k+1}(2i-1) = (k+1)^2$, it follows that

$$\sum_{i=1}^{k+1}(2i-1) = \sum_{i=1}^{k}(2i-1) + 2(k+1) - 1$$

$$= k^2 + 2k + 1$$

$$= (k+1)^2$$

2. Our base case is the smallest n for which we can tile the board with the L-shape. This is for $n = 1$ since we need at least 3 tiles for the L-shape. Clearly, this works since a 2×2 grid with a square missing is exactly the L-shape.

   For the inductive hypothesis, assume that this is the case for all natural $n \geq 1$ on a $2^n \times 2^n$ board. We need to show that this is also the case for $n+1$ on a $2^{n+1} \times 2^{n+1}$ board. We can divide the board into four $2^n \times 2^n$ sub-boards where each removed tile is adjacent to the next. By the induction hypothesis, we can tile all of these sub-boards and therefore tile the entire $2^{n+1} \times 2^{n+1}$ board.

3. Our base case is for $n = 0, n = 1$. We get

$$a_0 = 3^0 = 1, a_1 = 3^1 = 3$$

We assume for the inductive hypothesis that

$$a_k = 3^k$$

for all $k \geq 2 \in \mathbb{N}$. We want to show that $a_{k+1} = 2a_k + 3a_{k-1}$. To do this, substitute 3^k for a_k due to the inductive hypothesis, and accordingly 3^{k-1} for a_{k-1}.

$$a_{k+1} = 2(3^k) + 3a_{k-1}$$

$$= 2(3^k) + 3(3^{k-1})$$

$$= 2(3^k) + 3^k$$

$$= 3(3^k) = 3^{k+1}$$

4. Our base is case is for the smallest number of blocks. For 1 block, we have $1(1-1)/2 = 0$ points, and are told we get 0 points for stacks of size 1, so this holds.

   Our strong inductive hypothesis is that for any $k \in \mathbb{N}$, $1 \leq k \leq n$, k blocks will yield $\frac{k(k-1)}{2}$ points. We need to show that $n+1$ blocks yields $\frac{n(n+1)}{2}$ points.

   To do this, we go through three iterations of the game. In the first, for $n+1$ blocks, we have stacks of size a and $n+1-a$ making $a(n+1-a)$ points. In the second, we apply the inductive hypothesis and use a blocks for k (since $a \leq n-1$ and therefore also $a \leq k$) to make $\frac{a(a-1)}{2}$ points. In the third, we can do this again, except using $n-a$ for k (since a can be at most $n-1$, in which case $n+1-a$ becomes 1) to make $\frac{(n-a)(n+1-a)}{2}$ points. Adding these up for some point total P,

$$P = a(n+1-a) + \frac{a(a-1)}{2} + \frac{(n-a)(n+1-a)}{2}$$

$$\begin{aligned} 2P &= 2an + 2a - 2a^2 + a^2 - a + n^2 + n - an - an - a + a^2 \\ &= n^2 + n \end{aligned}$$

   So,

$$P = \frac{n(n+1)}{2}$$

5. a)

$$\begin{aligned} \sum_{n=1}^{k} 4n + 1 &= 4\sum_{n=1}^{k} + \sum_{n=1}^{k} 1 \\ &= 2k(k+1) + k + 1 \end{aligned}$$

   b)

$$\begin{aligned} \sum_{n=1}^{k} (4n-3) - (4n-1) &= \sum_{n=1}^{k} 4n - 4n - 2 \\ &= \sum_{n=1}^{k} -2 = -2k \end{aligned}$$