Practice

From Math 20630

HW 2: More Induction and Functions

1. We start with the two base cases $n = 1$ and $n = 2$. For both cases, a_n is odd by definition.

   Our inductive hypothesis is that a_n and a_{n-1} are odd. We want to show that a_{n+1} is consequently also odd. To do this, we note that $a_{n+1} = 2a_n + 3_{n-1}$ is the sum of an even number (any $2n \in \mathbb{N}$ is even by definition) and an odd number (by the inductive hypothesis). An even number plus an odd number is always odd, so a_{n+1} must be odd.

2. One way to accomplish this is without induction. A power set is constructed by taking all possible subsets of a set. In other words, for every element in the set, the set of every combination of sets given a cut size is a part of the power set. We can write this as

$$\sum_{k=1}^{n} \binom{n}{k}$$

   Where n is the number of elements in the set, and k is the number of elements we can choose per iteration of the combination. We sum from 1 to n since we add every combination of singleton, couplet, triplet, etc. of elements into the power set. To compute this, we can use the binomial theorem evaluated at 1:

$$\sum_{k=1}^{n} \binom{n}{k} x^k = (1+x)^n$$

$$\sum_{k=1}^{n} \binom{n}{k} 1^k = (1+1)^n$$

$$\sum_{k=1}^{n} \binom{n}{k} = 2^n$$

   Alternatively, we can use induction. Our base case is for $n = 0$. An set with 0 elements is simply $\emptyset$, which indeed has $2^0 = 1$ subset, itself.

   Our inductive hypothesis is that for any set A with n elements, $|\mathcal{P}(\mathcal{A})| = 2^n$. We want to show that for a set A^* with $n+1$ elements, $|\mathcal{P}(\mathcal{A}^*)| = 2^{n+1}$.

   Say $A^* = A \cup \ell$. We can divide $\mathcal{P}(\mathcal{A}^*)$ into sets X and Y where X is the set of all subsets that do not contain ℓ and Y is the set of all subsets that do. Since no subset of X contains ℓ, then X contains all subsets of A, and so $|X| = 2^n$ by the inductive hypothesis. Now, notice that for any subset $E \subset Y$, $E \setminus \ell$ is also in X. Further, for any subset $F \subset X$, $F \cup \ell$ is in Y. This means $|X| = |Y| = 2^n$. Since $\mathcal{P}(\mathcal{A}^*) = X \cup Y$, then $\mathcal{P}(\mathcal{A}^*) = |X| + |Y| = 2(2^n) = 2^{n+1}$ as desired.

3. Let $F : A \to B$ be defined by $F(S) =$

$$\begin{cases} S \setminus \{n\} & \text{if } n \in S \\ S \cup \{n\} & \text{if } n \notin S \end{cases}$$

   To show this is a bijection, we need to show it is both an injection and surjection.

   To show this is an injection, take sets M and $N \in A$ which both contain n and so $F(M) = F(N) = P$. If $n \in P$, then $M = P \setminus n$ and $N = P \setminus n$, so $M = N$. Similarly, if $n \notin P$, then $M = P \cup n$ and $N = P \cup n$, so $M = N$.

To show this a surjection, take a set $U \subseteq B$. Since U is in B, then U must be have an odd number of elements. If $n \in U$, we can remove n and get an even number of elements, which is the $K \subset A$ which can map to U. If $n \notin U$, we can add it and get an even number of elements again, which again is the $K \subset A$ which can map to U.