Rutgers University: Complex Variables and Advanced Calculus Written Qualifying Exam

January 2015: Problem 2 Solution

Exercise. Use contour integration to evaluate the integral

$$\int_0^{2\pi} \frac{d\theta}{a+\sin\theta}, \qquad \text{for } a > 1.$$

Solution.

$$z = e^{i\theta} \qquad dz = ie^{i\theta}d\theta \implies d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz} \qquad C : |z| = 1 \qquad e^{i\theta} \qquad 0 \le \theta \le 2\pi$$

$$\sin\theta = \frac{z - z^{-1}}{2i}$$

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{a+\sin\theta} &= \int_C \frac{1}{a + \frac{z-z^{-1}}{2i}} \left(\frac{1}{iz}\right) dz \\
&= \int_C \frac{1}{iaz + \frac{z^2-1}{2}} dz \\
&= 2\int_C \frac{1}{z^2 + 2aiz - 1} dz \\
z - \frac{-2ai \pm \sqrt{-4a^2 - 4(-1)}}{2} & \\
&= -ai \pm i\sqrt{a^2-1}
\end{aligned}$$

$\implies f(z) = \dfrac{1}{x^2 + 2aiz - 1}$ has simple poles at $z = (-a + \sqrt{a^2-1})i$ and $z = (-a - \sqrt{a^2-1})i$
But $|(-a - \sqrt{a^2-1})i| = a + \sqrt{a^2+1} > 1$ since $a > 1$.
So only $(-a + \sqrt{a^2-1})i$ lies instead contour $|z| = 1$ and $|(-a + \sqrt{a^2-1})i| = -a + \sqrt{a^2-1}$

$$\begin{aligned}
\int_C f(z)dz &= 2\pi i Res\Big(f, (-a + \sqrt{a^2-1})i\Big) \\
&= 2\pi i \cdot \frac{1}{\left(-a + \sqrt{a^2-1} + a + \sqrt{a^2-1}\right) i} \\
&= \frac{2\pi}{2\sqrt{a^2-1}} \\
&= \frac{\pi}{\sqrt{a^2-1}} \\
\int_0^{2\pi} \frac{d\theta}{a+\sin\theta} &= 2\int_C f(z)dz \\
&= 2\left(\frac{\pi}{\sqrt{a^2-1}}\right) \\
&= \boxed{\frac{2\pi}{\sqrt{a^2-1}}}
\end{aligned}$$