Rutgers University: Algebra Written Qualifying Exam

January 2018: Problem 3 Solution

Exercise.

(a) Prove that for any square matrices A and B of size n with coefficients in some field the characteristic polynomial of AB equals that of BA. -

Solution.

Similar matrices have the same characteristic polynomial.

Proof:
$$\begin{aligned} p_{M_1}(\lambda) &= \det(M_1 - \lambda I) \\ &= \det(P^{-1}M_2P - \lambda P^{-1}IP) \\ &= \det(P^{-1}(M_2 - \lambda I)P) \\ &= \det(P^{-1})\det(M_2 - \lambda I)\det(P) \\ &= \det(M_2 - \lambda I) \\ &= p_{M_2}(\lambda) \end{aligned} \qquad \square$$

If A or B is invertible then AB and BA are similar.

WLOG assume A is invertible, then $BA = A^{-1}(AB)A$

Thus AB and BA have the same characteristic polynomial.

If A and B are both non-invertible, we need to do a bit more:

By **Schur's formula:**

$$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(A)\det(D - CA^{-1}B), \text{ if } A \text{ is invertible}$$

$$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(D)\det(A - BD^{-1}C), \text{ if } D \text{ is invertible}$$

$$\begin{aligned} \det\begin{bmatrix} \lambda I_n & A \\ B & I_n \end{bmatrix} &= \det(\lambda I_n)\det(I_n - A\lambda^{-1}I_nB) \\ &= \det(\lambda I_n - \lambda\lambda^{-1}AB) \\ &= \det(\lambda I_n - AB) \\ &= p_{AB}(\lambda) \end{aligned}$$

$$\begin{aligned} \det\begin{bmatrix} \lambda I_n & A \\ B & I_n \end{bmatrix} &= \det(I_n)\det(\lambda I_n - BI_nA) \\ &= \det(\lambda I_n - BA) \\ &= p_{BA}(\lambda) \end{aligned}$$

$$\implies p_{AB}(\lambda) = p_{BA}(\lambda)$$

And thus the characteristic polynomial of AB equals that of BA

(b) Give an example of square matrices A and B such that the minimal polynomial of AB does not equal that of BA.

Solution.

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} && \text{and} && B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ \implies AB &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} && \text{and} && BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \implies q_{BA}(x) &= x && \text{BUT} && q_{AB} \neq x, \quad \text{since } AB \neq 0 \end{aligned}$$