Rutgers University: Algebra Written Qualifying Exam

January 2018: Problem 5 Solution

Exercise. Let I be a maximal ideal of $\mathbb{Z}[x]$. Prove that $\mathbb{Z}[x]/I$ is a finite field.

Solution.

$\mathbb{Z}[x]$ is a commutative ring with unity, and $\mathbb{Z}[x]/I$ is also a commutative ring with unity.
Also if I is a proper ideal of $\mathbb{Z}[x]$ then $\mathbb{Z}[x]/I$ is not the trivial ring.
Therefore, it suffices to prove that every nonzero element in $\mathbb{Z}[x]/I$ has a multiplicative inverse.

Let I be a maximal ideal of $\mathbb{Z}[x]$ and $a \notin I$.
Let J be the ideal $J = \{ab + x | b \in \mathbb{Z}[x], x \in I\}$.
Then, since I is a maximal ideal and $I \subsetneq J$, it follows that $J = \mathbb{Z}[x]$.
$\Longrightarrow$ $\exists b_0 \in \mathbb{Z}[x]$ and $x_0 \in I$ s.t. $1 = ab_0 + x_0$ and $1 - ab_0 = -x_0 \in I$
i.e. $\forall a \notin I$, $\exists b \in \mathbb{Z}[x]$ s.t.

$$1 - ab \in I.$$

$\Longrightarrow$ $\forall a \in \mathbb{Z}[x] - I$, $\exists b \in \mathbb{Z}[x]$ s.t.

$$(I + a)(I + b) = I + 1$$

$\Longrightarrow$ Every nonzero element of $\mathbb{Z}[x]/I$ has a multiplicative inverse.
Thus $\mathbb{Z}[x]/I$ is a field.