Rutgers University: Complex Variables and Advanced Calculus Written Qualifying Exam

August 2009 Day 1: Problem 4 Solution

Exercise. Let $f(z) = 3z^5 - 5z^3 - z - \frac{1}{2}$. How many zeros (counted with multiplicity) does f have in the annulus $\{z \in \mathbb{C}, 1 < |z| < 2\}$? Prove your statement.

Solution.

Rouche's Theorem: f, g analytic in open set U and γ a simple path in U, with its interior contained in U and with parameter interval I. If f has no zeros on $\gamma(I)$ and $|f(z) - g(z)| \leq |g(z)|$ on $\gamma(I)$ then f and g have the same number of zeros, counting order, inside γ

$$\{z \in \mathbb{C} : 1 < |z| < 2\} = D_2(0) \backslash D_1(0)$$

So the number of zeros in annulus = [number of zeros in $D_2(0)$] − [number of zeros in $D_1(0)$]

$D_2(0)$: $g(z) = 6z^5$ and $\gamma : \delta D_2(0)$

The zeros of $g(z)$ are: 0 with order 5 all in $D_2(0)$

On $\delta D_2(0)$, $|g(z)| = |6z^5| = 6|z^5| = 6 \cdot 2^5 = 192$

$|f(z) - g(z)| = |-10z^3 - 2z - 1| \leq 10|z|^3 + 2|z| + 1 = 80 + 4 + 1 = 85 < 192 = |g(z)|$

By Rouche's Theorem, f and g have the same number of zeros in $D_2(0)$

$\implies$ f has 5 zeros in $D_2(0)$

$D_1(0)$: $g_2(z) = -10z^3 - 2z = -2z(5z^2 + 1)$ and $\gamma_2 : \delta D_1(0)$

The zeros of $g_2(z)$ are: $0, \frac{1}{\sqrt{5}}, -\frac{1}{\sqrt{5}}$ with order 1 and all in $D_1(0)$

On $\delta D_1(0)$, $|g_2(z)| = |-10z^3 - 2z| \geq 10|z|^3 - 2|z| = 10 - 2 = 8$

$|f(z) - g(z)| = |6z^5 - 1| \leq 6|z|^5 - 1 = 7 < 9 \leq |g(z)|$

By Rouche's Theorem, f and g have the same number of zeros in $D_1(0)$

$\implies$ f has 3 zeros in $D_1(0)$

Thus, f has $5 - 3 = 2$ zeros in the annulus.