Rutgers University: Algebra Written Qualifying Exam
August 2018: Problem 2 Solution

Exercise. Let p and q be distinct primes. Let $\overline{q} \in \mathbb{Z}/p\mathbb{Z}$ denote the class q modulo p and let k denote the order of $\overline{q}$ as ab ekenebt if $(\mathbb{Z}/p\mathbb{Z})^*$. Prove that no group of order pq^ℓ with $1 \le \ell \le k$ is simple.

Solution.

Let $|G| = pq^\ell$ such that $1 \le \ell \le k$.
Since k is the order of q in $\mathbb{Z}_p$,

$$\begin{aligned} q^k &\equiv 1 \mod p, \text{ and} \\ q^j &\not\equiv 1 \mod p \text{ for } 1 \le j < k \end{aligned}$$

$n_p \cong 1 \mod p$ and $n_p \mid q^\ell$
$\implies n_p = 1$ or $n_p = q^k$ and $\ell = k$

If $n_p = 1$, there is a unique Sylow p-subgroup
$\implies$ the Sylow $p-$subgroup is normal in G
$\implies$ G is not simple.

If $n_p = q^k$, then $\ell = k$ and $|G| = pq^k$
Since the Sylow p-subgroups have prime order, they are cyclic
If P and Q are two Sylow $p-$subgroups
$P \cap Q = \{e\}$ or P.
This is because $P \cap Q$ is a subgroup of both P and Q and its order must divide p.
If $n_p = q^k$ then there are q^k Sylow $p-$subgroups that pairwise only intersect with the identity element.
So, the q^k Sylow $p-$subgroups contain a total of $(p-1)q^k = pq^k - q^k$ elements of order p.

The remaining q^k elements must be elements of the Sylow $q-$ subgroup:
$\implies n_q = 1$
$\implies$ The Sylow $q-$subgroup is normal
$\implies$ G is not simple