Complex Quals

Contour Integration

Jan 2015 2. $\int_0^{2\pi} \frac{d\theta}{a+\sin\theta}$, where $a>1$

$z = e^{i\theta}$ $dz = ie^{i\theta}d\theta \Rightarrow d\theta = \frac{1}{ie^{i\theta}}dz = \frac{1}{iz}dz$ $C: |z|=1$ $e^{i\theta}$ $0 \le \theta \le 2\pi$

$\sin\theta = \frac{z-z^{-1}}{2i}$

$\int_0^{2\pi} \frac{d\theta}{a+\sin\theta} = \int_C \frac{1}{a+\frac{z-z^{-1}}{2i}}\left(\frac{1}{iz}\right)dz = \int_C \frac{1}{iaz+\frac{z^2-1}{2}}dz = 2\int_C \frac{1}{z^2+2aiz-1}dz$

$z = \frac{-2ai \pm \sqrt{-4a^2-4(-1)}}{2} = -ai \pm i\sqrt{a^2-1}$

$\Rightarrow f(z) = \frac{1}{z^2+2aiz-1}$ has simple poles at $z = (-a+\sqrt{a^2-1})i$ and $z = (-a-\sqrt{a^2-1})i$

but $|(-a-\sqrt{a^2-1})i| = a+\sqrt{a^2-1} > 1$ since $a>1$

So only $(-a+\sqrt{a^2-1})i$ lies inside contour $|z|=1$ $|(-a+\sqrt{a^2-1})i| = -a+\sqrt{a^2-1}$

$\int_C f(z)dz = 2\pi i \operatorname{Res}(f, (-a+\sqrt{a^2-1})i)$

$= 2\pi i\left(\frac{1}{(-a+\sqrt{a^2-1}+a+\sqrt{a^2-1})i}\right) = \frac{2\pi}{2\sqrt{a^2-1}} = \frac{\pi}{\sqrt{a^2-1}}$

$\int_0^{2\pi} \frac{d\theta}{a+\sin\theta} = 2\int_C f(z)\,dz = 2\left(\frac{\pi}{\sqrt{a^2-1}}\right) = \boxed{\frac{2\pi}{\sqrt{a^2-1}}}$

Aug 2016 1. $\int_0^{2\pi} \frac{d\theta}{(2+\cos\theta)^2}$

$z = e^{i\theta}$ $dz = ie^{i\theta}d\theta \Rightarrow d\theta = \frac{1}{ie^{i\theta}}dz = \frac{1}{iz}dz$ $C: |z|=1$ $e^{i\theta}$ $0 \le \theta \le 2\pi$

$\cos\theta = \frac{z+z^{-1}}{2}$

$I = \int_{|z|=1} \frac{1}{(2+\frac{z+z^{-1}}{2})^2}\left(\frac{1}{iz}\right)dz = \frac{4}{i}\int_{|z|=1} \frac{1}{z(4+z+z^{-1})^2}dz = \frac{4}{i}\int_{|z|=1} \frac{z}{(4z+z^2+1)^2}dz$

$z = \frac{-4 \pm \sqrt{16-4}}{2} = -2 \pm \sqrt{3}$

$-2-\sqrt{3}$ is NOT in the unit circle but $-2+\sqrt{3}$ is. degree 2

$I = \frac{4}{i}(2\pi i \operatorname{Res}(f, -2+\sqrt{3})) = 8\pi\left[\lim_{z\to -2+\sqrt{3}} \frac{d}{dz}\left(\frac{z}{(z+2+\sqrt{3})^2}\right)\right] = 8\pi \lim_{z\to -2+\sqrt{3}}\left(\frac{(z+2+\sqrt{3})^2 - z(2)(z+2+\sqrt{3})}{(z+2+\sqrt{3})^4}\right)$

$= 8\pi \frac{(-2+\sqrt{3}+2+\sqrt{3})^2 - 2(-2+\sqrt{3})(-2+\sqrt{3}+2+\sqrt{3})}{(-2+\sqrt{3}+2+\sqrt{3})^4} = 8\pi \frac{(2\sqrt{3})^2+(4-2\sqrt{3})(2\sqrt{3})}{(2\sqrt{3})^4} = 8\pi \frac{12+8\sqrt{3}-12}{16\cdot 9} = \boxed{\pi\left(\frac{4\sqrt{3}}{9}\right)}$

$\int_C \frac{3z+e^{z^2}}{z^3}dz$ $z:0$

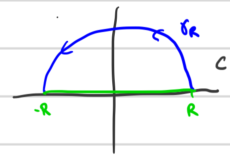
$\lim_{z\to 0} \frac{1}{2}\frac{d^2}{dz^2}(3z+e^{z^2}) = 0+2ze^{z^2} \to 2e^{z^2}+4z^2e^{z^2} \to \frac{1}{2}(2) = 1$

Contour Integration

Jan 2011

2. $\int_0^\infty \frac{1}{(1+x^2)^2}dx = \int_0^\infty \frac{1}{(x+i)^2(x-i)^2}dx = \frac{1}{2}\int_{-\infty}^\infty \frac{1}{(x+i)^2(x-i)^2}dx = \frac{1}{2}\lim_{R\to\infty}\int_{-R}^R \frac{1}{(x+i)^2(x-i)^2}dx$

poles: $\pm i$ with degree 2



$\int_C f(z)dz = \int_{-R}^R f(x)dx + \int_{\gamma_R} f(x)dx$

$\Rightarrow \int_{-R}^R f(x)dx = \int_C f(z)dz - \int_{\gamma_R} f(x)dx$

$I = \frac{1}{2}\lim_{R\to\infty}\int_{-R}^R \frac{1}{(1+x^2)^2}dx = \frac{1}{2}\lim_{R\to\infty}\left(\int_C f(z)dz - \int_{\gamma_R} f(x)dx\right)$

$\int_C f(z)dz$: for $R>1$, i is the only pole in C

$\int_C f(z)dz = 2\pi i\, Res(f,i) = 2\pi i\left[\lim_{z\to i}\frac{d}{dz}\left(\frac{1}{(z+i)^2}\right)\right] = 2\pi i \lim_{z\to i}\frac{-2}{(z+i)^3} = \frac{-4\pi i}{(2i)^3} = \frac{-4\pi i}{-8i} = \frac{\pi}{2}$

$\int_{\gamma_R} f(x)dx$: $\left|\int_{\gamma_R} f(x)dx\right| \le \int_{\gamma_R}|f(x)dx| \le \ell(\gamma_R)\cdot \max|f|$

$\ell(\gamma_R) = \pi R$, $\max|f| \to ?$

$|f| = \left|\frac{1}{(1+x^2)^2}\right|$ $\quad |(1+z^2)|^2 = |1+z^2||1+z^2| > (|z^2|-1)(|z^2|-1) = (R^2-1)^2$ $\quad \int_{-R}^R f(x)dx = \int_C f(z) - \int_{\gamma_R} f(x)dx$

$\left|\frac{1}{1+z^2}\right| < \frac{1}{(R^2-1)^2} \to 0$ $\quad R^4-2R^2+1$

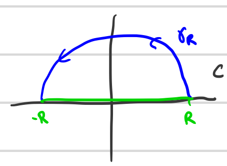
$\Rightarrow \lim_{R\to\infty}\int_{\gamma_R} f(x)dx = 0$

$I = \frac{1}{2}\lim_{R\to\infty}\left(\int_C f(z)dz - \int_{\gamma_R} f(x)dx\right) = \frac{1}{2}\left(\frac{\pi}{2} - 0\right) = \boxed{\frac{\pi}{4}}$

Aug 2014

1. $\int_{-\infty}^\infty \frac{\cos ax}{1+x^2} = Re\left(\int_{-\infty}^\infty \frac{e^{iax}}{1+x^2}dx\right)$ since $e^{iax} = \cos(ax) + i\sin ax$

$= Re\left(\lim_{R\to\infty}\int_{-R}^R \frac{e^{iax}}{1+x^2}dx\right)$



$\int_C f(z)dz = \int_{-R}^R f(x)dx + \int_{\gamma_R} f(x)dx$

$\Rightarrow \int_{-R}^R f(x)dx = \int_C f(z)dz - \int_{\gamma_R} f(x)dx$

poles: $i, -i$

$\int_C f(z)dz$: for $R>1$, i is the only pole in the contour

$\int_C f(z)dz = 2\pi i\, Res(f,i) = 2\pi i\left(\lim_{z\to i}\frac{e^{iaz}}{(z+i)}\right) = 2\pi i\left(\frac{e^{-a}}{2i}\right) = \frac{\pi}{e^a}$

$\left|\int_{\gamma_R} f(x)dx\right| \le \ell(\gamma_R)\cdot\max|f| = \pi R\cdot \max|f|$

$|f| = \left|\frac{e^{iaz}}{1+z^2}\right| = \frac{1}{|1+z^2|} < \frac{1}{|z|^2-1} = \frac{1}{R^2-1} \to 0$ as $R\to\infty$

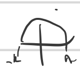
for $|z|>1$ because $|1+z^2| > |z|^2-1$

$\Rightarrow \lim_{R\to\infty}\int_{\gamma_R} f(x)dx = 0$

$\Rightarrow I = Re\left(\lim_{R\to\infty}\int_{-R}^R \frac{e^{iax}}{1+x^2}dx\right) = Re\left(\int_C f(z)dz - \lim_{R\to\infty}\int_{\gamma_R} f(x)dx\right) = Re\left(\frac{\pi}{e^a} - 0\right)$

$= \boxed{\frac{\pi}{e^a}}$

$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} = \frac{1}{2}\int_0^\infty \left(\frac{\sin x}{x}\right)^2$



$\left(\frac{e^{ix}-e^{-ix}}{2i}\right)^2 = \frac{e^{2ix} - 2 - e^{-2ix}}{-4}$ $\quad \int \frac{1-e^{2ix}}{4x}$ $\quad \int \frac{1-e^{2ix}}{x^2}$

$1-\cos^2\theta = 1 - e^{i\theta}(\cos\theta)$ $\quad \cos 2x = \cos x$

$\sin^2$ $\quad e^{i\theta} = \cos\theta + i\sin\theta$ $\quad \sin^2 x = 1-\cos x = \sqrt{\frac{1-\cos 2x}{2}}$

$\int_\gamma r(z)f(z) = 0$ $\quad \frac{\sin^2 x}{x}$ $\quad 2\pi i\left(\frac{1}{4(i)}\right)$

$x = e^{iz}$

$x = \ln(z)$

$\frac{1}{x^2}$

$Im\left(\int \frac{e^{ix}\sin x}{x^2}\right)$

$\pm\left(\frac{1-\cos 2x}{2}\right)$

$2\pi i(0)$

$\frac{1-\cos 2x}{2x^2}$

$\lim_{R\to\infty} \frac{-1}{R}\Big|_{-R}^{R} = 0$

$\int_{-\infty}^\infty \frac{1}{x^2} - \int \frac{\cos^2}{x^2} =$

$2\pi i\left(\frac{d}{dz}\left(1\right)\right) = 0$ $\quad \int 1$

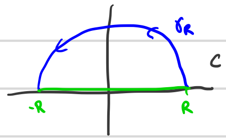
Contour Integration

Jan 2014

3. $\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)(x^2+9)}\,dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{\cos x}{(x+i)(x-i)(x+3i)(x-3i)} = \mathrm{Re}\left(\lim_{R\to\infty} \int_{-R}^{R} \frac{e^{ix}}{(x+i)(x-i)(x+3i)(x-3i)}\,dx \right)$

poles: $i, -i, -3i, 3i$ all have degree 1

$-i$ and $-3i$ not in contour, but i and $3i$ are for $R \geq 3$



$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{\gamma_R} f(x)\,dx$

$\Rightarrow \int_{-R}^{R} f(x)\,dx = \int_C f(z)\,dz - \int_{\gamma_R} f(x)\,dx$

$$\int_C f(z)\,dz = 2\pi i\left[\mathrm{Res}(f,i) + \mathrm{Res}(f,3i)\right] = 2\pi i\left[\left(\lim_{z\to i} \frac{e^{iz}}{(z+i)(z+3i)(z-3i)}\right)\left(\lim_{z\to 3i} \frac{e^{iz}}{(z+i)(z-i)(z+3i)}\right)\right]$$

$$= 2\pi i\left(\frac{e^{-1}}{(2i)(4i)(-2i)} + \frac{e^{-3}}{(4i)(2i)(6i)}\right) = \pi\left(\frac{e^{-1}}{8} - \frac{e^{-3}}{24}\right)$$

$$= \pi\left(\frac{3e^2-1}{24e^3}\right)$$

$\left|\int_{\gamma_R} f(z)\,dz\right| < \int_{\gamma_R} |f(z)|\,dz$ and $|f(z)| = \left|\frac{e^{iz}}{(z^2+1)(z^2+9)}\right| = \frac{1}{|z^2+1||z^2+9|} < \frac{1}{(|z^2|-1)(|z|^2-9)} = \frac{1}{(R^2-1)(R^2-9)}$ on γ_R

For $|z|>3$: $|z^2+1| > |z|^2-1$ $\Rightarrow |z^2+1||z^2+3| > (|z|^2-1)(|z|^2-3)$ for $|z|^2>3$

$|z^2+3| > |z|^2-3$

$$\left|\int_{\gamma_R} f(x)\,dx\right| < (\pi R)\left(\frac{1}{R^4-10R^2-1}\right) \to 0 \text{ as } R\to\infty$$

$$I = \mathrm{Re}\left(\frac{(3e^2-1)\pi}{24e^3} - 0\right) = \frac{(3e^2-1)\pi}{24e^3}$$

Aug 2012

1. $\int_0^{\infty} \frac{dx}{1+x^4} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R} \frac{1}{1+x^4}\,dx$

$(x^2+i)(x^2-i) = (x+\sqrt{i})(x-\sqrt{i})$

$\pm\sqrt{i}$ $\pm i\sqrt{i}$

$\lim_{z\to i\sqrt{i}} \frac{1}{(\sqrt{i}+\sqrt{i})(\sqrt{i}+i\sqrt{i})(\sqrt{i}-i\sqrt{i})} + \frac{1}{(\sqrt{i}+i\sqrt{i})(-\sqrt{i}+i\sqrt{i})(2i\sqrt{i})}$

$\frac{1}{2i(1+i)(\sqrt{i}+i\sqrt{i})}$ $\frac{1}{2i(\sqrt{i})(i)(1-i)(-1+i)} = \frac{1}{2(1-i)^2\sqrt{i}}$

$\frac{3z+e^{z^2}dz}{z^3} = \int \frac{3}{z^2} + \int \frac{e^{z^2}}{z^3}$

$z=0$ $z=0$

$\int_{C(a,r)} \frac{f(z)}{z-z_0} = \frac{1}{i}\int$ $2\pi r$

$u(0,0) = \frac{1}{\pi}\int_R u(x,y)\,dx\,dy$ $\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}$

$\int_r \frac{\ln x}{x^2+1} = \frac{1}{i}\int_{-\infty}^{\infty} \frac{\ln x}{x^2+1}$ $x = 1$ and i

$h, (i)$

$2\pi i \left(\frac{\ln(z)}{}\right)$

$1.\ f(z) = \frac{\ln z}{}$ $g(z) = \frac{}{z+1}$ $\frac{\ln z}{z+1}$

$\frac{\ln(z)}{z+i}$ $\int \frac{f(w)}{w-z}\,dw$ $z=i$ $\frac{\ln(i)}{2i}$ $2\pi i$ $0 = \frac{1}{2\pi i}$

0

γ_R

$\left|\frac{\ln x}{x^2+1}\right| < \frac{1}{x^2+1} < \frac{1}{R^2-1}$

$\to 0$

$f(z) = \frac{\ln z}{z+1}$ $g(z) = \frac{1}{z-1}$

$1 \cdot \frac{\ln(1)}{2} = 0$ $\frac{1}{2\pi i}\int \frac{f(z)}{z-}$

$\int \ln(1)$ $2\pi i\,(0$

$$e^{h(u)} = e^{\int_0^u \frac{\gamma'(t)}{\gamma(t)-a}\,dt} = e^{\int_{\gamma(0)}^{\gamma(u)} \frac{1}{z-a}\,dz} = e^{\ln(z-a)\Big|_{\gamma(0)}^{\gamma(u)}} = e^{\ln(\gamma(u)-a)-\ln(\gamma(0)-a)} = \gamma(u)-a\,/\,\gamma(0)-a$$

$z=\gamma(t) \quad dz=\gamma'(t)\,dt$

$$\frac{d}{du}\left[e^{-\int_0^u \frac{\gamma'(t)}{\gamma(t)-a}dt}\right](\gamma(u)-a) = e^{-h(u)}(-h'(u))(\gamma(u)-a) + e^{-h(u)}(\gamma'(u)) = e^{-h(u)}\left(\frac{\gamma'(u)}{\gamma(u)-a}\cdot\gamma(u)-a+\gamma'(u)\right) = e^{-h(u)}(2\gamma'(u))$$

$e^{-h(u)}(2\gamma'(u))$

$\frac{\gamma(u)-\gamma(u)}{\gamma(t)-a}(\gamma'(u))$

$\int \frac{1}{(z-a)}^{-1}$ D amultr Gt LN

$2(1)$ $= \frac{1}{2\pi i}\int \frac{1}{z-a}$

lim

pole at a $\lim \int \frac{f(z)}{z-a} = 2\pi i\, f(a)$

$\lim \frac{1}{2\pi i} \int \frac{f(z)}{z-a} = \lim f(a) \left(\frac{1}{z}\right)$

Tak $U(0,0) = \frac{1}{2\pi i} \int_B \frac{U(z)}{(z-0)}\, dz$

$= \frac{1}{2\pi i} \int_B \frac{U(x,y)}{(x+iy)}\, dx\, dy$

$\int_B \frac{u(z)}{z}$ pole at z.

$B(v,v) = (v,v)$

$v, v \neq 0$

$\vec{v} = (x, y, z) \in \mathbb{R}^3$, let $f(\vec{v}) = x^2 + y^4 + z^6 - 3$ and

$g(\vec{v}) = x + y + z$

$S = \{v \in \mathbb{R}^3 : f(v) = g(v) = 0\}$ prove for every $\vec{v} \in S$, $\exists$

a) An open subset $\Omega \subset \mathbb{R}^3$ containing $\vec{v}$

f, g continuow

Counting zeros

Aug 2009 $f(z)=6z^5-10z^3-2z-1$. Find the number of zeros (with multiplicity) of f in annulus $\{z\in\mathbb{C}: 1<|z|<2\}$

Rouche's thm: f, g analytic in open set U and γ a simple closed path in U, with its interior contained in U and with parameter interval I. If f has no zero on $\gamma(I)$ and $|f(z)-g(z)|\leq|g(z)|$ on $\gamma(I)$, then f and g have the same number of zeros, counting order, inside γ

$\{z\in\mathbb{C}: 1<|z|<2\} = D_2(0)\setminus D_1(0)$

so # of zeros in annulus = # of zeros in $D_2(0)$ − # of zeros in $D_1(0)$.

$D_2(0)$: $g(z)=6z^5$ and $\gamma: \partial D_2(0)$

the zeros of $g(z)$ are: 0 with order 5 all in $D_2(0)$

On $\partial D_2(0)$, $|g(z)|=|6z^5|=6|z|^5=6\cdot 32=192$

$|f(z)-g(z)|=|-10z^3-2z-1|\leq 10|z|^3+2|z|+1=80+4+1=85<192=|g(z)|$

By Rouche's thm, f and g have the same # of zeros in $D_2(0)$.

$\Rightarrow f$ has 5 zeros in $D_2(0)$.

$D_1(0)$: $g_2(x)=-10z^3-2z=-2z(5z^2-1)$ and $\gamma_2: \partial D_1(0)$

the zeros of g_2 are $0, \sqrt{\frac{1}{5}}, -\sqrt{1/5}$ all in $D_1(0)$

On $\partial D_1(0)$: $|g(z)|=|-10z^3-2z|\geq 10|z|^3-2|z|=10-2=8$

$|f(z)-g(z)|=|6z^5-1|\leq 6|z|^5-1=7<8\leq|g(z)|$

By Rouche's, f has 3 zero in $D_1(0)$.

f has $5-3=2$ zeros in the annulus.

Aug 2017

4 Schwarz lemma: $f: \Delta \to \Delta$ holic, $f(0)=0$ $\Big\} \Rightarrow$ (1) $|f(z)| \le |z|$
(2) $|f'(0)| \le 1$
(3) If $|f(z)| = |z|$ for some $z \in \Delta$ on $|f'(0)|=1$, then $f(z) = e^{i\theta} z$

candidate $p_n(z) = \sum_{j=0}^{n} a_j z^j$

$f(z) - p_n(z) = \sum_{j=n+1}^{\infty} a_n z^j$

$|f(z) - p_n(z)| \le |f(z)| + |p_n(z)| \underset{\text{assump}}{\le} 1 + |p_n(z)|$ $\le 1 + \sum |a_j| |z^j|$ $\le 1 + \sum_0^n |z|^j \le n+2$ $\le (n+2)|z|^{n+1}$

claim $|a_j| \le 1 \; \forall j$

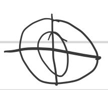
Have $|f(z) - p_n(z)| \le n+2$

$f(z) - p_n(z) = (z^{n+1}) \sum_{j=0} a_{j+n+1} z^j$

$\dfrac{f(z) - p_n(z)}{z^{n+1}} = g_n(z)$ holic, max value on $\partial\Delta$ (*)

$|g_n(z)| \le \dfrac{n+2}{|z|^{n+1}} \xrightarrow{} n+2$ as $|z| \to 1$

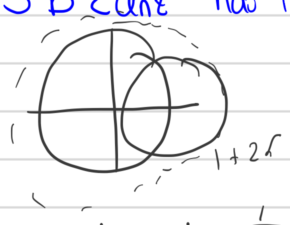
Let $\Delta_\varepsilon = \{ |z| < 1 - \varepsilon$



on $|g_n(z)| \le \max_{\partial D_\varepsilon} |g_n(z)|$ ← max mod principle

send $\varepsilon \to 0$ on Δ, $|g_n(z)| \le n+2$

5 b $\sum a_n z^n$ has radius of conv > 1



claim roc $\ge 1 + 2\delta$
pf $|z| < 1 + 2\delta$.

Hadarmard: $\dfrac{1}{roc} = \limsup_{n \to \infty} |a_n|^{1/n}$

$\dfrac{1}{roc} = \limsup_{n \to \infty} |a_n|^{1/n}$

$f(z) = \sum b_n (z-1)^n$

obs: Binom thm

$(1+\delta)^n = \sum_{k=0}^{n} \binom{n}{k} \delta^k = \sum_{k=0}^{n} \frac{n!}{(n-k)!\,k!} \delta^k = \sum_{k=0}^{n} \frac{n!}{(n-k)!\,k!} \delta^k$

$b_n = \dfrac{f^{(n)}(1)}{n!}$

$f(1+\delta) = \sum b_n \delta^n \le c$

$= \sum \frac{f^{(k)}(1)}{k!} \delta^k = \sum_{m \ge k} \sum_{k \ge 0} \frac{m!}{(m-k)!\,k!} \delta^k a_m$

a. Prove $f^{(k)}(1) \in \mathbb{R}$ $\forall k \geq 0$ and $f^{(k)}(1) \geq \frac{m!}{(m-k)!} a_m$ for all $0 \leq k \leq m$

$f^{(k)}(z) = \sum_{n \geq k} \frac{n!}{(k-1)!} a_n z^{n-k}$

$f^{(k)}(1) = \lim_{r \to 1^-} f^{(k)}(r) = \lim_{r \to 1^-} \sum_{n \geq k} \frac{n!}{(n-k)!} a_n r^n$

For $m \geq k$ $\quad \geq \lim_{r \to 1^-} \frac{m!}{(m-k)!} a_m r^m$

$= \frac{m!}{(m-k)!} a_m$

$f(x)$ continuously diff'ble real-valued function over $(-\infty,\infty)$ with $f(0)=0$. Suppose $|f'(x)| \leq |f(x)|$ $\forall x \in (-\infty,\infty)$.

Show $f(x)=0$ $\forall x \in (-\varepsilon,\varepsilon)$, for some $\varepsilon>0$.

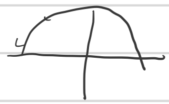
f <u>ctsly diff'ble</u>: f' is cts.

So f' is continuous at $x=0$.

Thus, $\forall \varepsilon>0$, $\exists \delta>0$ s.t. $|0-x|<\delta \Rightarrow |f'(x)| = |f'(x)-f'(0)| < |f'(x)| + |f'(0)| < (|f'(x)| + |f(0)|) < \varepsilon$

Show that $f(x)=0$ $\forall x \in (-\infty,\infty)$

$$\int_0^\infty \frac{1}{(1+x^2)^2}\,dx = \int_0^\infty \frac{1}{(x+i)^2(x-i)^2}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{1}{(x+i)^2(x-i)^2}\,dx$$



$\mathrm{Res}(f, i) =$

1. Show that if f is an entire function s.t. $f(\mathbb{C}) \cap \{x \in \mathbb{R} : x > 0\} = \emptyset$, then f is constant.

f bounded $f(\mathbb{C}) = \{x$

$u_x = v_y$ and $u_y = -v_x$

Aug 2016 3. $u(x,y)$ harmonic on $D=\{z:|z|<1\}$, u twice differentiable on D and $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ where $z=x+iy$.

a. Show $f(z)=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}$ is a holomorphic function on D.

$f(z)=g(z)+ih(z)$ where $g(z)=u_x(z)$ and $h(z)=-u_y(z)$

$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \Rightarrow \frac{\partial^2 u}{\partial x^2}=-\frac{\partial^2 u}{\partial y^2}$

$\Rightarrow g_x = h_y$

since order of differentiation doesn't matter, $u_{xy}=u_{yx}$

$\Rightarrow (u_x)\frac{\partial}{\partial y} = -(-u_y)\frac{\partial}{\partial x}$

$\Rightarrow g_y = -h_x$

$\Rightarrow f$ satisfies the Cauchy-Riemann equation on D

$\Rightarrow f$ is holomorphic on D

b. For any piecewise smooth curve $\gamma\subset D$ connecting 0 to $z\in D$, define $F(z)=\int_\gamma f(z)\,dz$. Prove that F is a well defined holomorphic function on D.

$F(z)=\int_\gamma f(z)\,dz=\int_{\gamma_1} f(z)\,dz+\dots+\int_{\gamma_n} f(z)\,dz$

$=\int_{a_1}^{b_1} f(\gamma_1(t))\gamma_1'(t)\,dt+\dots+\int$

8/2015

5. f holomorphic on open disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Assume $f(0) = \frac{1}{3}$, and that $|f(z)| \leq 1 \quad \forall z \in \mathbb{D}$. Prove that $|f(\frac{1}{20})| \leq \frac{2}{5}$.

$f(x,y) = u(x,y) + iv(x,y)$ then $\gamma(x) = x - \frac{1}{3}$

$u_x = v_y$ and $u_y = -v_x$

$\gamma(0) = 0$

$\gamma \circ f(0) = 0$

$|\gamma \circ f(\frac{1}{20})| < \frac{1}{20}$ $\frac{1}{20} + \frac{7}{20} = \frac{8}{20}$ [illegible]

$-\frac{1}{20} < \gamma \circ f(\frac{1}{20}) < \frac{1}{20}$

$-\frac{1}{20} + \frac{1}{3} < f(\frac{1}{20}) < \frac{1}{20} + \frac{1}{3}$

$f(\frac{1}{20}) < \frac{23}{60} < \frac{24}{60} = \frac{2}{5}$

1/2009 $f:\mathbb{R}\to\mathbb{R}$ twice diff'ble s.t. $f(0)=0$, $f'(0)>0$, and $f''(x)\geq f(x)$ $\forall x\geq 0$. Prove that $f(x)>0$ $\forall x>0$.

If $f(b)\leq 0$ for $b>0$, then by the mean value thm $\exists c\in(0,b)$ s.t.

$$f'(c)=\frac{f(b)-f(0)}{b-0}\leq 0$$

And $\exists d\in(0,c)$ s.t.

$$f''(d)=\frac{f'(c)-f'(0)}{c-0}<\frac{f'(c)}{c}<0$$

$f'(0)>0 \Rightarrow f$ increasing

$f'(b)=0$

$f'(x)=0$

$$f'(c)=\frac{f'(b)-f'(a)}{b-0}\leq 0$$

If $f(b)=0$

$f''(c)<0$

$f(c)>0$

f increasing at 0. $\lim_{x\to 0^+}\frac{f(x+h)-f(x)}{h}>0$

1/2015

5. $u:\mathbb{C}\to\mathbb{R}$ harmonic which is bound below (ie $\exists C\in\mathbb{R}$ s.t. $u(z)\geq C$ $\forall z\in\mathbb{C}$), then u must be constant.

u harmonic $\Rightarrow \frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}=0$

$\Rightarrow \frac{d^2u}{dx^2}=-\frac{d^2u}{dy^2}$

Questions on Entire Functions

1/2016 4. f entire $\iint_{\mathbb{C}} |f(z)|^2\,dx\,dy < \infty$. Show that $f(z) = 0\ \forall z \in \mathbb{C}$.

$\iint_{\mathbb{C}} |f(z)|^2 dx\,dy < \infty \Rightarrow \iint_{\mathbb{C}} |f(z)|^2 dx\,dy$ exists

$\Rightarrow |f(z)|^2$ is bounded

$\Rightarrow |f(z)|$ bounded $\Rightarrow f(z)$ bounded

So f is a bounded, entire function $\Rightarrow f$ is constant.

If $f(z) \neq 0$ then $|f(z)|^2 = c \in \mathbb{R}^+$

$\Rightarrow \iint_{\mathbb{C}} |f(z)|^2 dx\,dy = c \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 1\,dx\,dy = \infty$

So $f(z) = 0$ //

Aug 2015 1. Show if entire $f(\mathbb{C}) \cap (x \in \mathbb{R} : x > 0) = \emptyset$ then f constant

$f(\mathbb{C}) =$ ___ + ___

e entire $x \in \mathbb{R}$ $x > 0$

$e^{f(z)}$ -

$g \circ f$

1/2016

2. $\mathbb{D}=\{z:|z|<1\}\subset\mathbb{C}$ and $f:\mathbb{D}\to\mathbb{D}$ holomorphic. Prove that $\dfrac{|f(0)|-|z|}{1+|f(0)||z|}\leq|f(z)|\leq\dfrac{|f(0)|+|z|}{1-|f(0)||z|}$

$f(x,y)=u(x,y)+iv(x,y)$ $\sqrt{u(x,y)^2+v(x,y)^2}<1$ $|f(0)|+|z|\geq 1-|f(0)||z|$ obvious.

$|f(z)|<1$ $|f(0)|+|z|<1-f(0)z$

for $|z|\geq|f(0)|$ obvious $|f(0)|(1+|z|)+|z|<1$ $\sqrt{u^2x^2+u^2y^2+v^2x^2+v^2y^2}<1$

$|z|<|f(0)|$ $\dfrac{|f(0)|-|z|}{1+|f(0)||z|}$ $|z|(1+|f(0)|)\leq 1$ $\sqrt{u^2+v^2}\cdot\sqrt{x^2+y^2}$

$u_x=v_y$ $v_x=u_y$

$\dfrac{|f(0)|-|z|}{1+}=$ $u_x=v_y$ and $u_y=v_x$ $|z|<|f(0)|$ $|f(0)|(1+|z|)+|z|<1$

$u_x=v_y$ $f(1)$ $\leq|f(0)|+|f(0)||z|+|z|$ <3

and $u_y=-v_x$ $|f(z)|>\dfrac{|f(0)|+|z|}{1-|f(0)||z|}>\dfrac{|f(0)|+|z|}{1}$

$|f'(z)|=1$

1/2018

1. $\lambda > 1$. Show $ze^{\lambda - z} = 1$ has a real solution in the unit disk, and that there are no other solutions in the unit disc.

$z = x + iy \Rightarrow ze^{\lambda - z} = (x+iy)\frac{e^{\lambda}}{e^{x}e^{iy}} = \frac{e^{\lambda}}{e^{x}}\left(\frac{x+iy}{\cos y + i\sin y}\right) = \left(\frac{e^{\lambda}}{e^{x}}\right)\left((x\cos y + y\sin y) + i(y\cos y - x\sin y)\right) \overset{!}{=} 1$

$\Rightarrow y\cos y = x\sin y \Rightarrow y = 0$ or $x = y\frac{\cos y}{\sin y}$

$\Rightarrow |z| = \sqrt{y^2\frac{\cos^2 y}{\sin^2 y} + y^2} = |y|\sqrt{\frac{\cos^2 y}{\sin^2 y} + 1} = \left|\frac{y}{\sin y}\right| > 1$ when $0 < |y| \leq 1$ (because $y = \sin y$ at $y = 0$ and $\frac{d}{dy}(y) = 1 > \cos y = \frac{d}{dy}(\sin y)$ for $0 < y \leq 1$)

So $y = 0 \Rightarrow z = x \in \mathbb{R}$

If $x \leq 0$, then $xe^{\lambda - x} = x\frac{e^{\lambda}}{e^{x}} \leq 0 \Rightarrow x\frac{e^{\lambda}}{e^{x}} \neq 1$

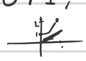
Therefore, $x > 0$

$x\frac{e^{\lambda}}{e^{x}}$ is continuous since x, e^{λ}, and $\frac{1}{e^{x}}$ are all continuous

$x = 0$: $0\frac{e^{\lambda}}{1} = 0$ $\qquad$ $x = 1$ $1\frac{e^{\lambda}}{e^{x}} > 1$

By IVT

If $x \leq 0$ then $xe^{\lambda - x} = 0 \neq 1$, so $x > 1$

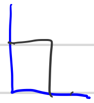
$\frac{e^{x}}{x} = e^{\lambda}$ $\qquad$ $\frac{e^{x}}{x}$ is  $\qquad$ $\frac{d}{dx}\frac{e^{x}}{x} = \frac{e^{x} - xe^{x}}{x^2} > 0$ $\quad$ $\frac{e^{x}(1-x)}{x^2}$ increasing $\Rightarrow$ Only 1 Solution

e^{0} $x = 1$: $e^{\lambda - 1} = 1 \Leftrightarrow \lambda = 1$ contradiction

$\frac{e^{x}}{x} = e^{\lambda}$ $\quad$ $\lambda = \ln(e^{x}/x) = x - \ln x$ $\quad$ e

$x = 1$ $\quad$ $e^{1} \geq e > 1$ $\quad$ $x = 0$ $e^{0} = 0$ $\quad$ $1\frac{e^{\lambda}}{e^{x}} > 1$ $\quad$ $0 \Rightarrow$ IVT

$f(z+1) = f(z) = f(z+i)$ $+L$



$f(0) = f(i) = f(1) = f(1+i)$

$f(x) \neq 0$ $x+y+z = 0$ $x^2+y^4+z^6 - 3 = 0$

Power Series Questions

1/2017 4. $f(z)=\sum_{n=0}^{\infty} a_n z^n$ converges and $|f(z)|<1$ $\forall z\in\mathbb{C}$ with $|z|<1$. Prove $|a_0|^2+|a_1|\le 1$. In addition, determine all f that satisfies $|a_0|^2+|a_1|=1$.

$|a_0|^2 < |a_0| < |f(0)|$

$\frac{\sqrt{n+1}\ z^{n+1}}{\sqrt{n}\ z^n} = \frac{\sqrt{n+1}}{\sqrt{n}}\cdot z < 1$ for $|z|<1$

$|n^{\frac{1}{n}} z$ converge $\Rightarrow$ holomorphic

at $z=1$ $\sum\sqrt{n} = \sum n^{1/2}$

$\sqrt{n} > \frac{1}{n}$

diverg

at $z=1$ $f(1)=\sum\sqrt{n}$

$\frac{\sqrt{n}}{n^2} = n^{3/2}$

Aug 2013 $f(z)=\sum_{n=1}^{\infty}\sqrt{n}\, z^n$

a. Show $f(z)$ holomorphic in open unit disc $\{z\in\mathbb{C} : |z|<1\}$

$\lim_{n\to\infty}\left|\frac{\sqrt{n+1}\ z^{n+1}}{\sqrt{n}\ z^n}\right| = |z|\lim_{n\to\infty}\frac{\sqrt{n+1}}{\sqrt{n}} = |z|\cdot 1 < 1$ for $|z|<1$

Radius of convergence is $r=1$

$\Rightarrow f(z)$ is holomorphic on $D_1(0)$ since power series are holomorphic on their disc of convergence

*not sure b. Show f does not extend to a continuous complex valued function on the closed unit disc.

If $z=1$: $f(1)=\sum_{n=1}^{\infty}\sqrt{n}$

$\frac{1}{n} < \sqrt{n}$ $\forall n\in\mathbb{N}$

$\sum\frac{1}{n}$ diverges by the p-test $\Rightarrow \sum\sqrt{n}$ diverges.

$\Rightarrow f$ diverges at $z=1$

$\Rightarrow$ it does not extend to a cts complex valued function on $\overline{D_1(0)}$

1/2015 3. Find a conformal (ie biholomorphic) map from the quarter circle $Q = \{z \in \mathbb{C} : Re(z) > 0, Im(z) > 0, \text{ and } |z| < 1\}$ onto the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$

$f_1(z) = z^2$ maps the quarter circle to semicircle

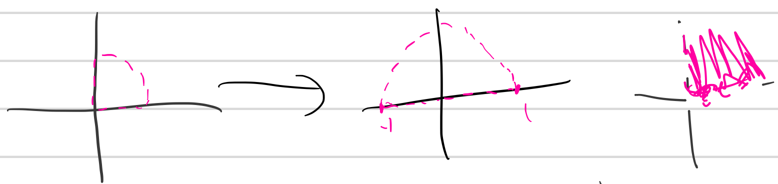
$f_2(z) = \frac{z-1}{z+1}$ $\quad f(0) = -1 \quad f(i) = \frac{i-1}{i+1} = \frac{-1-2i+1}{-1-1} = i$

$f_2(z) = -i \frac{z-1}{z+1}$

maps semicircle to 1^{st} quad

$f_3(z) = z^2$ map 1^{st} quad to upper

$f_4(z)$ map upper to unit disc $\frac{z-i}{z+i}$



$\frac{1-i}{1+i} \quad \frac{1-2i+1}{1+1} = -i$

$i \to 0$ $\quad 1 \to$

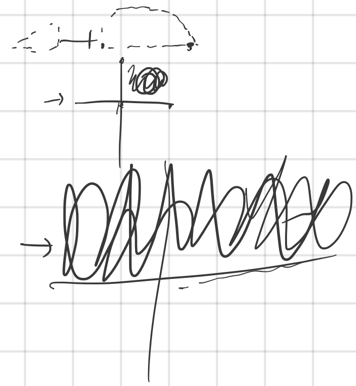
$0 = -1$ $\quad 1 \to i$ $\quad \frac{z-i}{z+i} \quad \left|\frac{z-i}{z+i}\right| \quad \frac{z^2-2i-1}{z^2+1}$

$z=0$

$f_1(z) = z^3$

$f_2(z) = \frac{z-1}{z+1}$ $\quad 0 \to -1 \quad i \to \frac{i-1}{i+1} = \frac{-1-2i+1}{-1-1} = i$

$-i \frac{z-1}{z+1}$



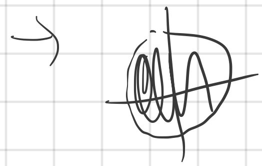
$f_3(z) = -\frac{(z-1)}{(z+1)} i$

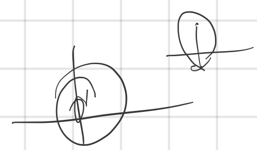
$f_4(z) = \frac{z-i}{z+i}$

$-(x+iy) \quad -x-iy \quad \to c$

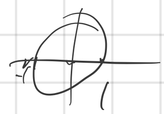
$x<0$ any $y>0$

$-i(x+iy) = -ix + y$

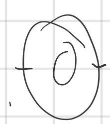




$z \mapsto z + bi$



$z \mapsto \sqrt{z+bi}$



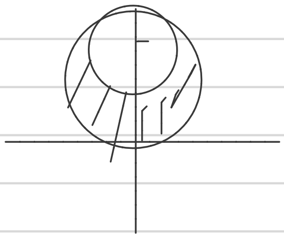
$\frac{i(1+z)}{1-z}$

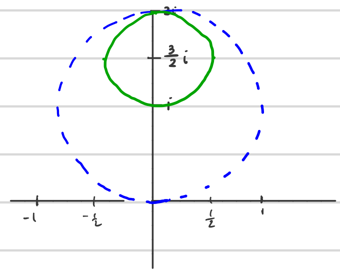
$i\left(\frac{1+z}{1-z}\right)$

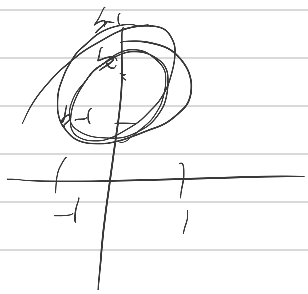
$i\left(\frac{b-z}{a+z}\right)$

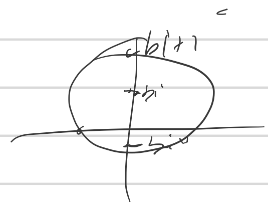
$f(0) = 0 \quad f'(0) = 1$

$f(z) =$

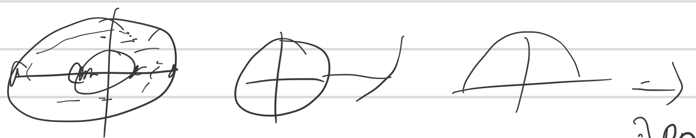






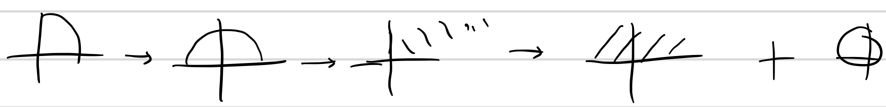


biholom



$\exists\, p_n$

$|f(z) - p_n(z)| < ((n+2)/3)^{n+1}$



1/2009 Let $f: \mathbb{R} \to \mathbb{R}$ be a twice differentiable function that satisfies $f(0)=0$ and $f'(0)=0$ and $f''(x) \geq f(x)$ $\forall x \geq 0$. Prove that $f(x)>0$ $\forall x > 0$.

If $f(x)<0$ for some $x>0$, then $f'(x)<0$ for some $x>0$

$f''(x)<0$

$f''(c) = \frac{f(a)-f(b)}{a-b}$

$x^2 + y^4 + z^6 - 3 = 0$ $\qquad$ $y^2 - 2yz + z^2 + y^4 + z^6 - 3 = 0$

$x + y + z = 0$

h harmonic $\Rightarrow h_{xx} + h_{yy} = 0$

$f = u + iv$ holomorphic $\Rightarrow u_x = v_y \qquad u_y = -v_x$

$$\frac{\partial}{\partial x}(h \circ f) = (h_x \circ f)(u_x + iv_x) + (h_y \circ f)(u_x + iv_x)$$

$$\frac{\partial^2}{\partial x^2}(h \circ f) = (h_{xx} \circ f)(u_x + iv_x)^2 + (h_{xy} \circ f)(u_x + iv_x)^2 + (h_x \circ f)(u_{xx} + iv_{xx}) + (h_{yx} \circ f)(u_x + iv_x)^2 + (h_{yy} \circ f)(u_x + iv_x)^2$$
$$+ (h_y \circ f)(u_{xx} + iv_{xx})$$

$$\frac{\partial}{\partial y}(h \circ f) = (h_y \circ f)(u_y + iv_y) + (h_x \circ f)(u_y + iv_y)$$

$$\frac{\partial^2}{\partial y^2}(h \circ f) = (h_{yy} \circ f)(u_y + iv_y)^2 + (h_{yx})(u_y + iv_y)^2 + (h_y \circ f)(u_{yy} + iv_{yy}) + (h_{xy} \circ f)(u_y + iv_y)^2 + (h_{xx} \circ f)(u_y + iv_y)^2 + (h_x \circ f)(u_{yy} + iv_{yy})$$

$= -v_x + iu_x \quad (i(u_x + iv_y))^2 = -(u_x + iv_y)^2$

$= h$

$$(h \circ f)_x = h_x(f) f_x$$
$$h_{xx}(f) f_x^2 + h_x(f) f_{xx}$$
$$= h_{xx}(f) u_x + i h_{xx}(f) v_x + h_x(f) u_{xx} + i h_x(f) v_{xx}$$

$$(h \circ f)_{yy} = h_{yy}(f) f_y^2 + h_y(f) f_{yy}$$
$$= -h_{xx}(f) f_y + h_y(f) f_{yy}$$
$$= h_{xx}(f) v_x + i h_{xx}(f) u_x + h_x(f) u_x$$

$h(z)$ real valued harmonic $h(z)^2$ harm.. $\Rightarrow h(z)$ constant

$h_{xx} + h_{yy} = 0$

$(x+iy)^2 = x^2 + 2xyi - y^2$

$h \cdot h \qquad h_x h + h h_x$

$(h^2)_x = (2h(x)) h_x$

$2hh_x \qquad 2h_x^2 + 2hh_{xx}$

$2h_x^2 + 2hh_{xx} + 2h_y^2 + 2hh_{yy} = 2h_x^2 + 2h_y^2 \; != 0$

$h_x^2 = -h_y^2$

$2h_y^2 + 2hh_{yy} \qquad 2h_x^2 = -2h_y^2 \, ?$

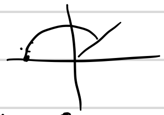
$\Rightarrow h_x = 0$ and $h_y = 0$

$\Rightarrow h_x^2 = -h_y^2 = 0 \, ?$

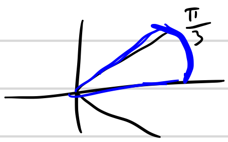
$\Rightarrow h$ is constant

(

$\int_0^\infty \frac{1}{1+x^3}\,dx$



$1+x^3=0$ at $-1,\ e^{\frac{\pi}{3}i},\ e^{\frac{5\pi}{3}i}$



$\int_0^\infty = \int_{C_R} + \int_{\gamma_1} + \int_{\gamma_2}$

$\int_0^\infty \frac{1}{1+x^3} = \int_0^\infty \frac{1}{(x+1)(x-e^{\frac{\pi}{3}i})(x-e^{\frac{5\pi}{3}i})}$

$\mathrm{Res}(f, e^{\frac{\pi}{3}i}) = \frac{1}{(e^{\frac{\pi}{3}i}+1)(e^{\frac{\pi}{3}i}-e^{\frac{5\pi}{3}i})} = \frac{1}{(\frac{3}{2}+\frac{\sqrt{3}}{2}i)(\frac{1}{2}+\frac{\sqrt{3}}{2}i-\frac{1}{2}+\frac{\sqrt{3}}{2}i)} = \frac{1}{(\frac{3}{2}+\frac{\sqrt{3}}{2}i)(\sqrt{3}i)} = \frac{2}{-3+3\sqrt{3}i}$

$\int_0^\infty \frac{1}{1+x^3}\,dx = \int_C f(z)\,dz - \int_{\gamma_1} f(z)\,dz - \int_{\gamma_2} f(z)\,dz$

$\gamma_2 = t e^{\frac{2\pi}{3}i}$ from R to 0

$\int_0^R \frac{1}{1+t^3}(e^{\frac{2\pi}{3}i})\,dt = (-\frac{1}{2}+\frac{\sqrt{3}}{2}i)\int_0^R \frac{1}{1+t^3}$

$(\frac{3}{2}-\frac{\sqrt{3}}{2})i \int_0^\infty \frac{1}{1+x^3}\,dx = \frac{2\pi}{3}\left(\frac{i\,2}{-3+3\sqrt{3}i}\right)$

$\int_0^\infty = \frac{2\pi}{3}\left(\frac{4}{(3\sqrt{3}+3i)(3-\sqrt{3}i)}\right) = \frac{2\pi}{3}\left(\frac{4}{9\sqrt{3}+9i+3\sqrt{3}-9i}\right) = \frac{2\pi}{3}\left(\frac{4}{12\sqrt{3}}\right) = \frac{2\pi}{(3\sqrt{3})}$ ✓