Limit Cycles from the Similarity Renormalization Group

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Background

- Schools previously attended:
  - UC Los Angeles (BS)
  - CSU Long Beach (MS)

- While at OSU:
  - Nuclear Theory Group under Dick Furnstahl
    - Joined in Spring 2019

- Elective courses taken:
  - Big Data Analytics in Physics
  - Quantum Field Theory I & II

- Relevant research experience:
  - Nuclear physics
    - Predict nuclei binding energies from unconverged no core shell model data using neural networks.
    - Apply eigenvector continuation to one-dimensional square well scattering problem to extrapolate phase shifts.

Outline

- Introduction
- Limit cycles
- Efimov physics
- Introduce the potential
- Regularization and renormalization of potential
- The similarity renormalization group (SRG)
- Detection of limit cycle using SRG
- Extraction of discrete scaling factor
- Eigenvector continuation (EC)
- Convolutional neural networks (CNN)
- Conclusion

P. Niemann · H.-W. Hammer

Limit Cycles from the Similarity Renormalization Group

arXiv:1504.04511v1

Introduction

- Effective field theories (EFT) provide a way of calculate observables at their appropriate energy scales.
- Pionless EFT is a low-energy theory that is obtained from chiral EFT when one over the pion mass is larger than the typical de Broglie wavelength of the nucleons.

- Regularizing and renormalizing the pionless EFT leads to the existence of a limit cycle.
- The potential $V(r) = \frac{c}{r^2}$ allows us to understand the behaviors of a limit cycle.

- Niemann and Hammer have provided a technique that can be used to analyze the limit cycle present in the pionless EFT.

What are limit cycles?

- Classical mechanics: closed trajectories in phase space as $t \to \pm\infty$.

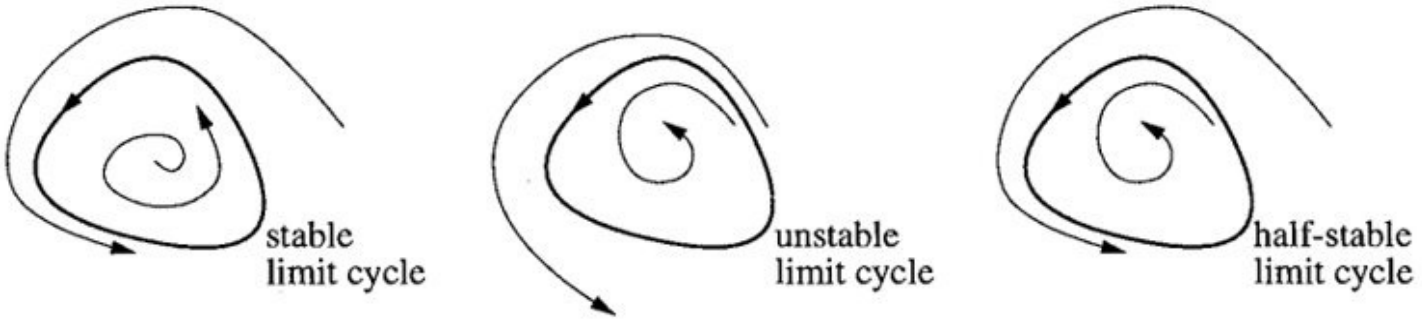


Figure source:
S. H. Strogatz. Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering.

- Quantum mechanics: arise from ill-defined potentials.
- How do we obtain unique solutions when solving the Schrödinger equation? Regularize and renormalize!
- What does renormalization have to do with limit cycles?
  - Two coupling constants will trace out trajectories as shown above as the cutoff is varied.
  - One coupling constant will lead to one-dimensional limit cycle $\longrightarrow$ will not trace out path as shown above.

The Efimov effect

- Occurs in a three-body system when each pair of particles are interacting enough to almost form a two-particle bound state.
- Three-body system develops a "tower" of three-body bound-state solutions, accumulating at zero.

$$\frac{E_3^{(n)}}{E_3^{(n+1)}} = \left(e^{\frac{\pi}{s_0}}\right)^2$$

Discrete scaling factor!

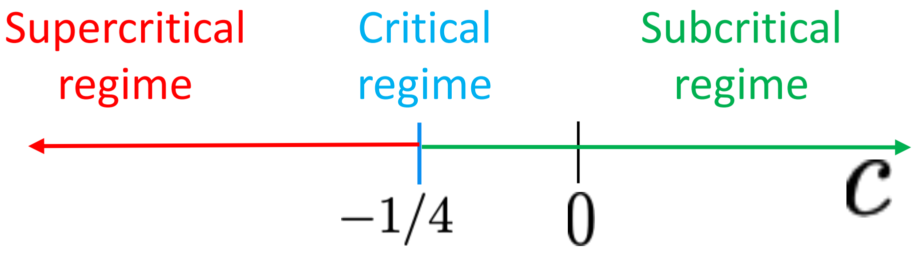
$s_0 = 1.00624...$ For three bosons

- Condition:
  - S-wave scattering length $>>$ range of interaction
- Why is this relevant?
  - Certain potentials with limit cycles display a similar spectrum!

Description of potential

$$V(r) = \frac{c}{r^2}$$

- Properties:
  - Singular potential with divergence at the origin.
  - Different solutions for different values of the potential strength c.



- Why is this potential of interest?
  - Has a limit cycle.
  - Bound-state spectrum analogous to Efimov effect.

Exploring the potential

3D Schrödinger equation

$$V(r) = \frac{c}{r^2}$$

Radial equation

$$\frac{d^2u}{dr^2} - \left[\frac{c}{r^2} + \frac{l(l+1)}{r^2} + E\right]u = 0$$

$$E = 0$$
$$\ell = 0$$
$$c < -1/4$$

Most general solution

$$R(r) \propto \frac{1}{\sqrt{r}} \cos\left(\nu \ln(kr) + \phi\right)$$

$$\nu = \sqrt{-c - \frac{1}{4}}$$

Does the solution have nodes?

$$r = \frac{1}{k}\exp\left\{\frac{1}{\nu}\left[\left(n + \frac{1}{2}\right)\pi - \phi\right]\right\}$$

We obtain an infinite number of nodes.

For the solutions to be orthogonal, the bound state spectrum must take the form

$$\frac{E_b}{E_a} = \left(e^{\frac{n\pi}{\nu}}\right)^2$$

Obtain a bound-state energy spectrum analogous to the Efimov effect!

Regularization and renormalization

- Deal with the singular potential using regularization and renormalization.

- Step 1: Impose a high-momentum cutoff to regulate physics at short-distances.
- Step 2: Make the observables independent of the imposed cutoff using renormalization by introducing a counterterm designed to describe the short-distance physics.

- Writing the potential in momentum space:

$$\tilde{V}(Q) = \frac{2\pi^2 c}{Q} \xrightarrow{\text{S-waves only!}} \tilde{V}(p,q) = 2\pi^2 c \left(\frac{\Theta(p-q)}{p} + \frac{\Theta(q-p)}{q} \right)$$

Q is momentum transfer

q (p) is the incoming (outgoing) momenta

$\Theta \rightarrow$ Step function

Regularize the Lippmann-Schwinger equation

- Calculate observables using the Lippmann-Schwinger equation:

$$t_E(p,k) = \tilde{V}(p,k) + \frac{1}{2\pi^2}\int_0^\infty \frac{dq\, q^2}{E - q^2 + i\epsilon}\, t_E(q,k)\tilde{V}(p,q). \Rightarrow k\cot\delta = ik - \frac{4\pi}{t_E(k,k)|_{E=k^2}}$$

- Introduce a counterterm in the potential:

$$V(q,\Lambda) = 2\pi^2 c\left(\frac{\Theta(p-q)}{p} + \frac{\Theta(q-p)}{q} + \frac{H(\Lambda)}{\Lambda}\right) \Rightarrow f(p,q) = \frac{\Theta(p-q)}{p} + \frac{\Theta(q-p)}{q}$$

- Regulate the integral with cutoff Λ:

$$t_E(p,k) = 2\pi^2 c\left[f(p,k) + \frac{H(\Lambda)}{\Lambda}\right] + c\int_0^\Lambda \frac{dq\, q^2}{E - q^2 + i\epsilon}\left[f(p,q) + \frac{H(\Lambda)}{\Lambda}\right] t_E(q,k)$$

- Coupling constant can be derived from the integral equation on the RHS of the Lippmann-Schwinger equation for the :

$$\phi_0(p) = -c\int_0^\Lambda dq\left[f(p,q) + \frac{H(\Lambda)}{\Lambda}\right]\phi_0(q)$$

Calculating the coupling constant H

- The cutoff cannot change the observables, which means the integral equation cannot vary with the cutoff Λ:

$$\frac{\partial}{\partial\Lambda}\phi_0(p) = 0$$

$$\nu > 0$$

$$\Downarrow$$

$$c < -1/4$$

- The coupling constant is:

$$H(\Lambda) = \frac{1 - 2\nu\tan(\nu\ln(\Lambda/\Lambda_*))}{1 + 2\nu\tan(\nu\ln(\Lambda/\Lambda_*))}$$

where Λ_* is a free parameter that fixes ϕ.

- Periodicity of the coupling constant:

$$\Lambda = \Lambda_* e^{n\pi/\nu}$$

Analogous to the Efimov bound-state energy!

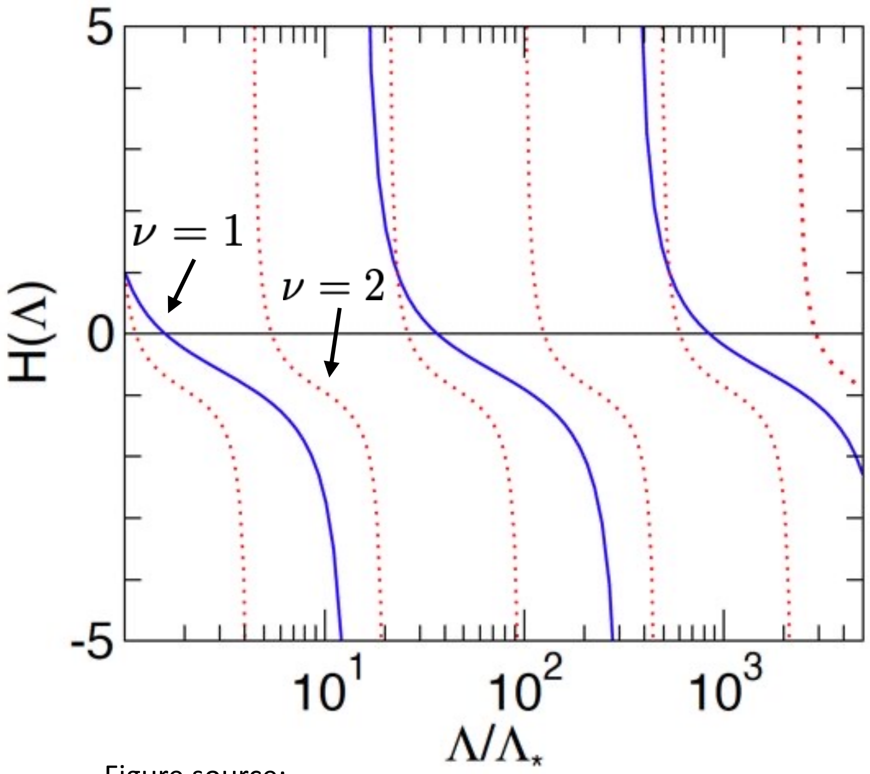


Figure source:
P. Niemann and H. W. Hammer. Limit Cycles from the Similarity Renormalization Group.

Allows us to study the properties of limit cycles by changing the bound-state spectrum. We can vary Λ using the SRG.

The similarity renormalization group (SRG)

- Used to decouple low- and high-momentum modes.
- Transformation is controlled by a flow parameter s that depends on a momentum cutoff λ.
- Generator drives the evolution of the potential. Usually kinetic energy.
- Drive Hamiltonian to band-diagonal form.

SRG flow equation:

$$\frac{dH(s)}{ds} = [[G, H(s)], H(s)]$$

Solution to flow equation:

$$\langle p| V_2(s) |q\rangle \approx \langle p| V_2(0) |q\rangle \, e^{-s(p^2-q^2)^2}$$

$$\lambda = s^{-1/4}$$
$$s = 0 \rightarrow \lambda = \infty$$

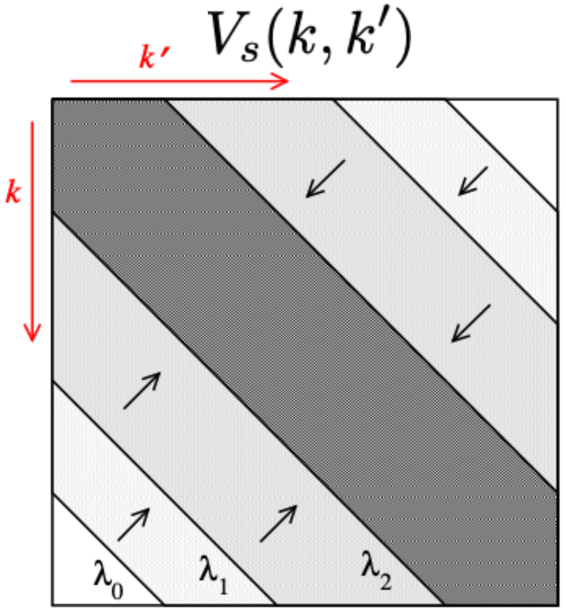


Figure source:
R. J. Furnstahl. The Renormalization Group in Nuclear Physics.

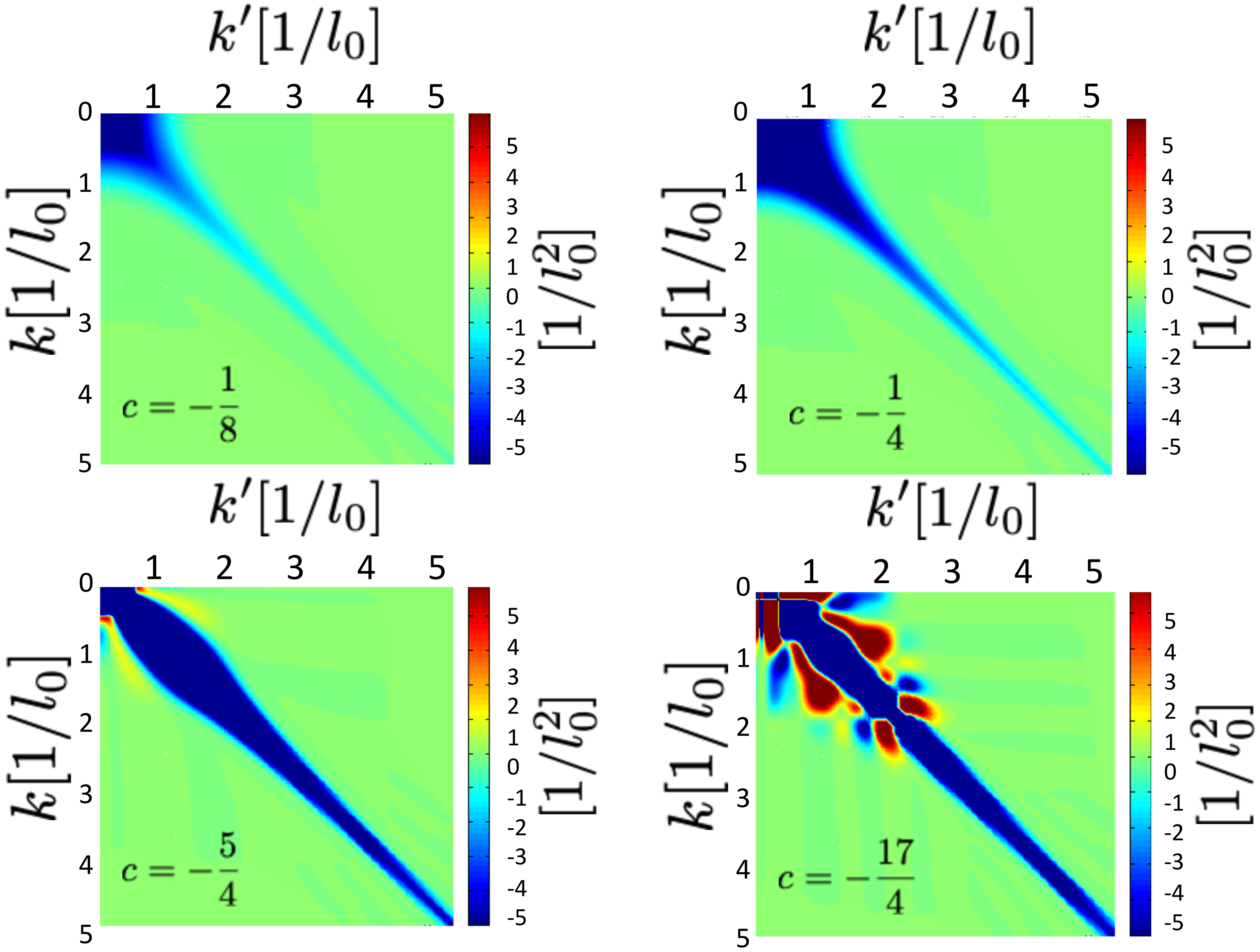
Alternate generators:

$$G_e = \sigma^2 \exp(-T/\sigma^2)$$

$$G_i = \frac{\sigma^2}{1 + T/\sigma^2},$$

- Alternate generators allows for decoupling at low energies.

Detecting the limit cycle with the SRG



Parameters:

$$\Lambda = 20\ l_0^{-1}$$
$$\lambda = 1\ l_0^{-1}$$
$$G = T$$

Figures source:

P. Niemann and H. W. Hammer. Limit Cycles from the Similarity Renormalization Group.

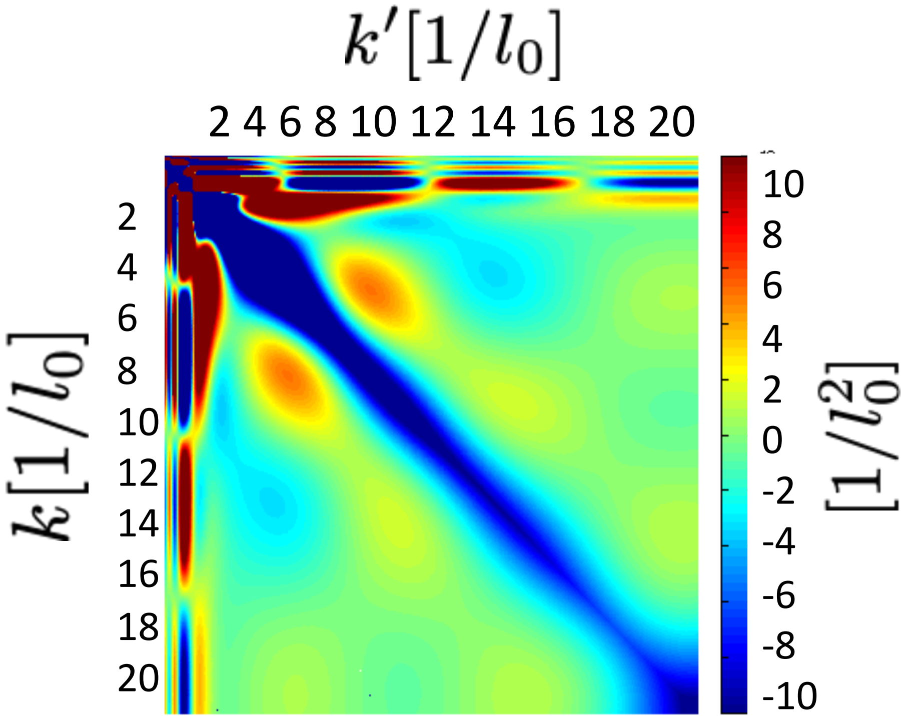
Detecting the limit cycle using alternate generators

Parameters:

$\Lambda = 20\ l_0^{-1}$

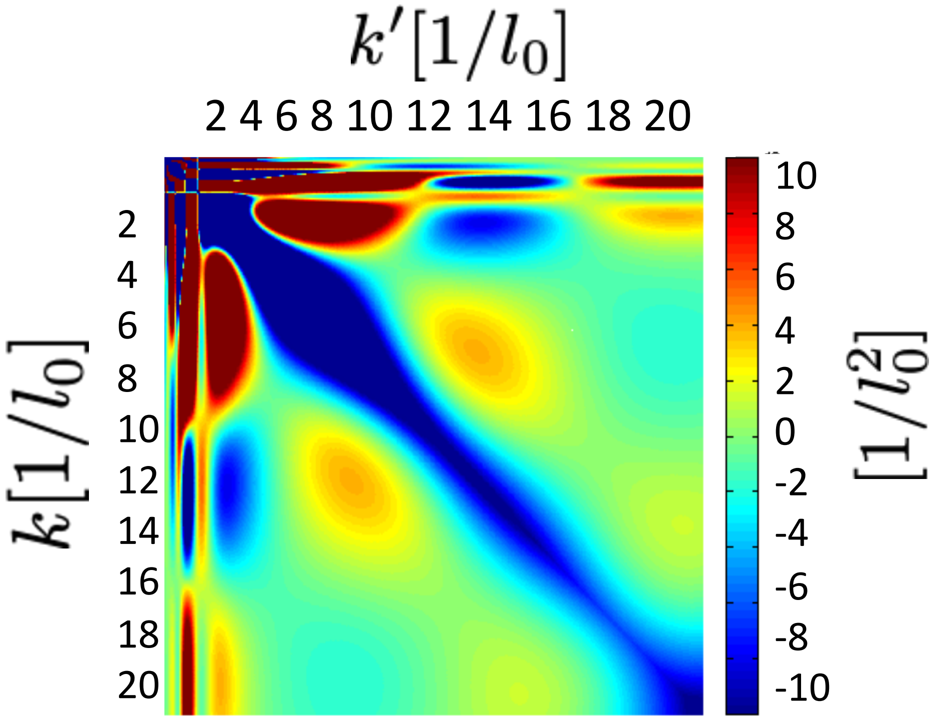
$\sigma = 2\ l_0^{-1}$

$\nu = 9$



$\lambda_a \approx 8.35\ l_0^{-1}$

$G_e = \sigma^2 \exp(-T/\sigma^2)$

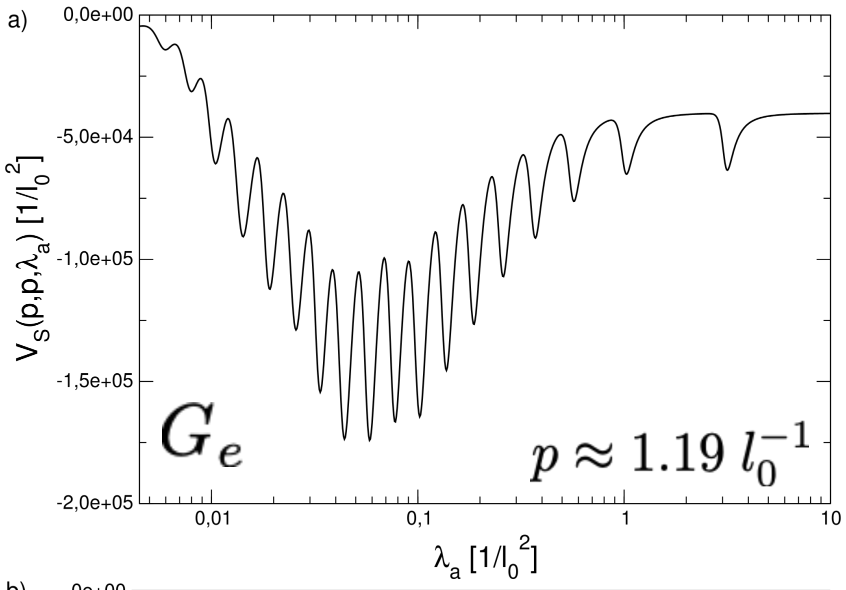


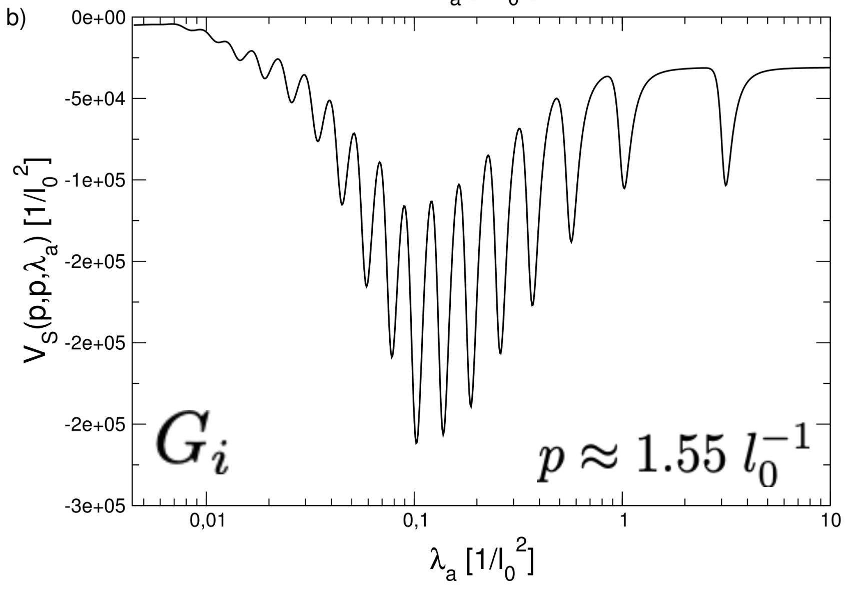
$\lambda_a \approx 10.75\ l_0^{-1}$

$$G_i = \frac{\sigma^2}{1 + T/\sigma^2}$$

Figures source:
P. Niemann and H. W. Hammer. Limit Cycles from the Similarity Renormalization Group.

Extraction of scaling factor of evolved potential





$$\Lambda = \Lambda_* e^{n\pi/\nu}$$

Analytical values:

$$\frac{\Lambda}{\Lambda_*} = e^{\pi/11} \approx 1.33 \quad \frac{\Lambda}{\Lambda_*} = e^{\pi/5} \approx 1.87$$

oscillation	$\nu = 11$		$\nu = 5$	
	maxima	minima	maxima	minima
1	2.94	3.08	3.26	3.36
2	1.76	1.80	2.11	2.11
3	1.52	1.54	1.91	1.91
4	1.42	1.43	1.87	1.85
5	1.38	1.39	1.83	1.81
6	1.36	1.36	1.85	1.80
7	1.35	1.34	1.84	1.79
8	1.31	1.32		1.78
9	1.33	1.33		
10	1.34	1.33		
11	1.30	1.31		
12	1.33	1.31		
13	1.34	1.34		
14	1.38	1.35		
15	1.36	1.36		
16	1.33	1.31		
17		1.34		

Parameters:

$$\nu = 11$$
$$\Lambda = 30\ l_0^{-1}$$
$$\sigma = 0.05\ l_0^{-1}$$
$$p \approx 1.19\sigma$$

Figures source:

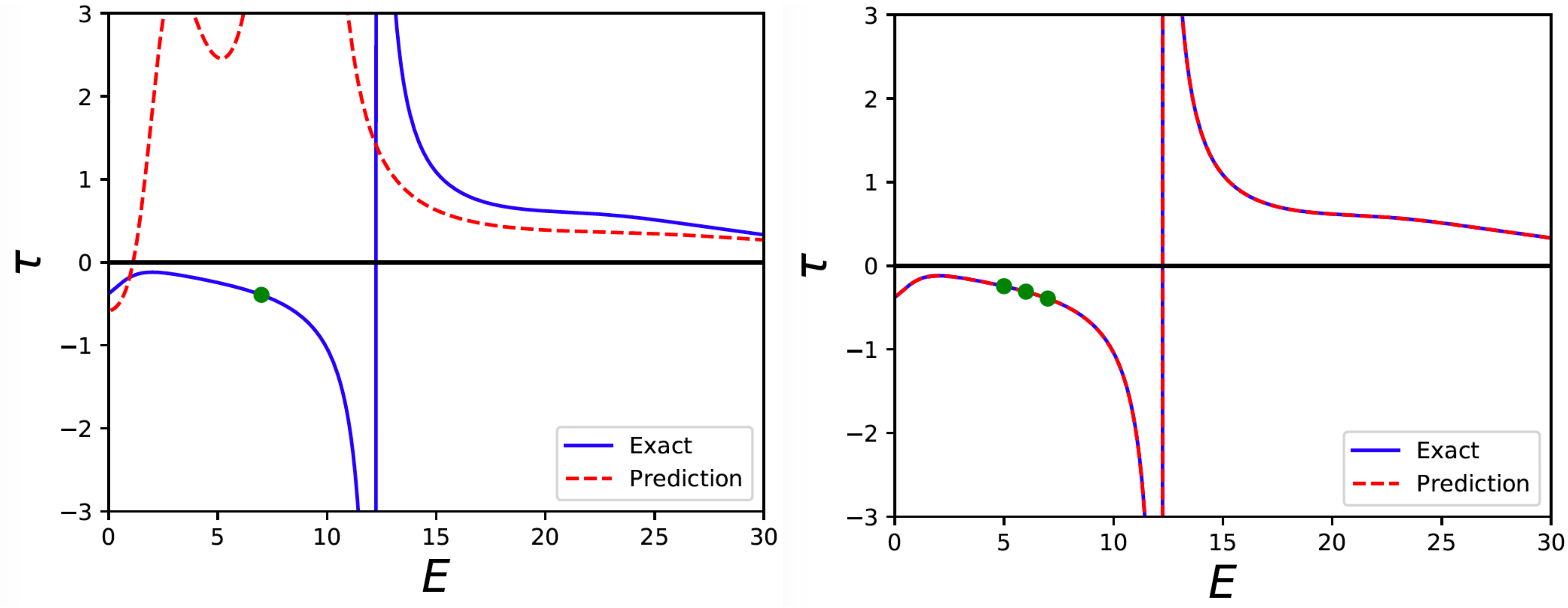
P. Niemann and H. W. Hammer. Limit Cycles from the Similarity Renormalization Group.

Eigenvector continuation (EC)

- A numerical technique
- Interpolate and extrapolate solutions of the Hamiltonian
- Example: scattering in a square well.

$$f = \frac{1}{k\cot\delta - ik}$$

$$\tau = \frac{1}{f}$$



- Application of EC to SRG:
  - Can be used to extrapolate to different values of the cutoff by taking "snapshots" of the potential at different cutoffs and using these to build a basis.

- Previously done by S. König for large cutoffs.

- Can be applied to chiral perturbation theory as a means of extrapolating down to pionless EFT.
- May be able to detect the pionless EFT limit cycle analogous to work done by Niemann for a three-body system.

Talk by König : http://www.int.washington.edu/talks/WorkShops/int_19_2a/People/Koenig_S/Koenig.pdf

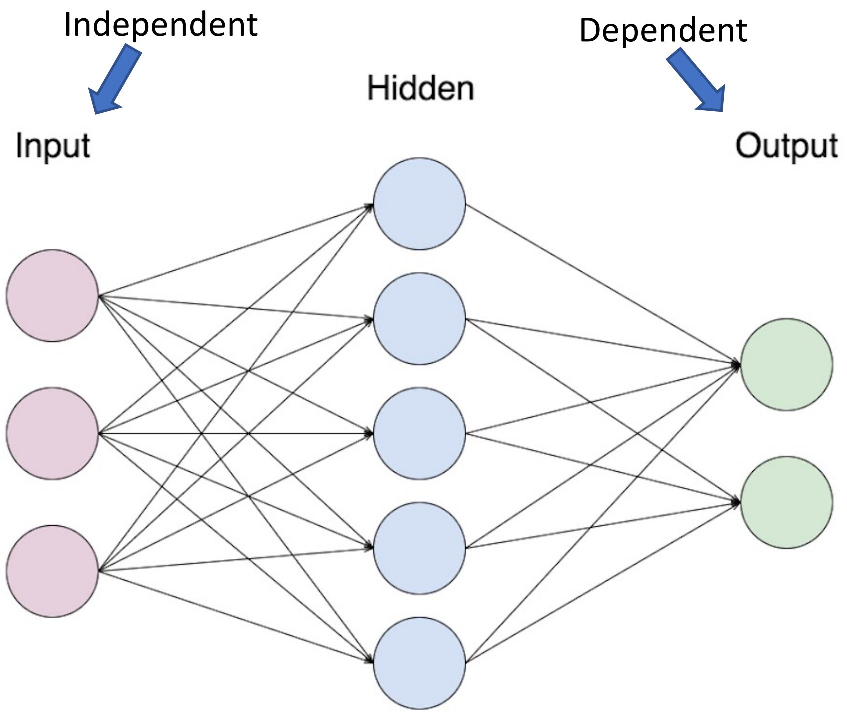
Convolutional neural networks

- Algorithms trained to recognize patterns and relationships between pictures in order to recognize and classify images.
- Assigns weights and biases to important features in images.
- Uses this information to identify characteristics of images.

$$X = (x_1, x_2, ..., x_n)$$

$$W = (w_1, w_2, ..., w_n)$$

$$Y = f(W \cdot X + b)$$



Basic feed-forward neural network

Figure source:
https://medium.com/@jamesdacombe/an-introduction-to-artificial-neural-networks-with-example-ad459bb6941b

- Applications of CNNs to limit cycles:
  - Can be used to extract the discrete scaling factor by training a CNN to recognize maximas and minimas from heat maps of an evolved potential.
  - By extension, CNNs can also be used to predict the discrete scaling factor from heats maps produced by a potential evolved using an unknown generator by training CNN to predict the scaling factor for known generators and extrapolating to the unknown case.

Conclusion

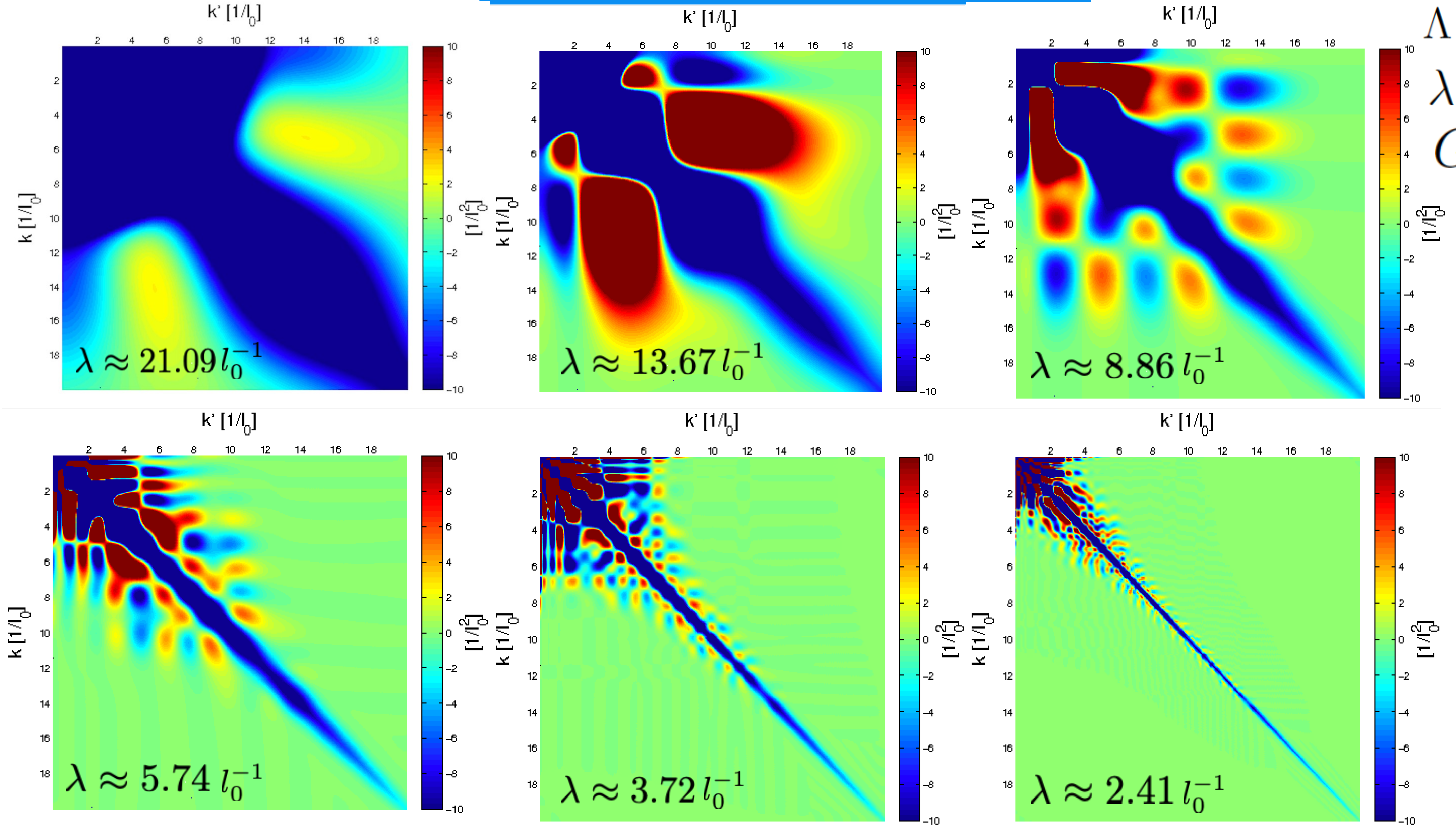
- The solutions of the Schrodinger equation for the potential $V(r) = \frac{c}{r^2}$ are not unique and have a bound-state energy spectrum similar to the Efimov effect when c is in the supercritical regime.
- Regularization and renormalization fixes this non-uniqueness by cutting off the short-distance physics.
- This introduces a log-periodic limit cycle for the coupling constant that varies as the cutoff is changed.
- The periodicity of the coupling constant is determined by a scaling factor related to the strength of the potential, which can be extracted using SRG.
- Extensions:
  - Use EC to detect limit cycles
  - Use CNNs to extract the discrete scaling parameter.

Thank you!

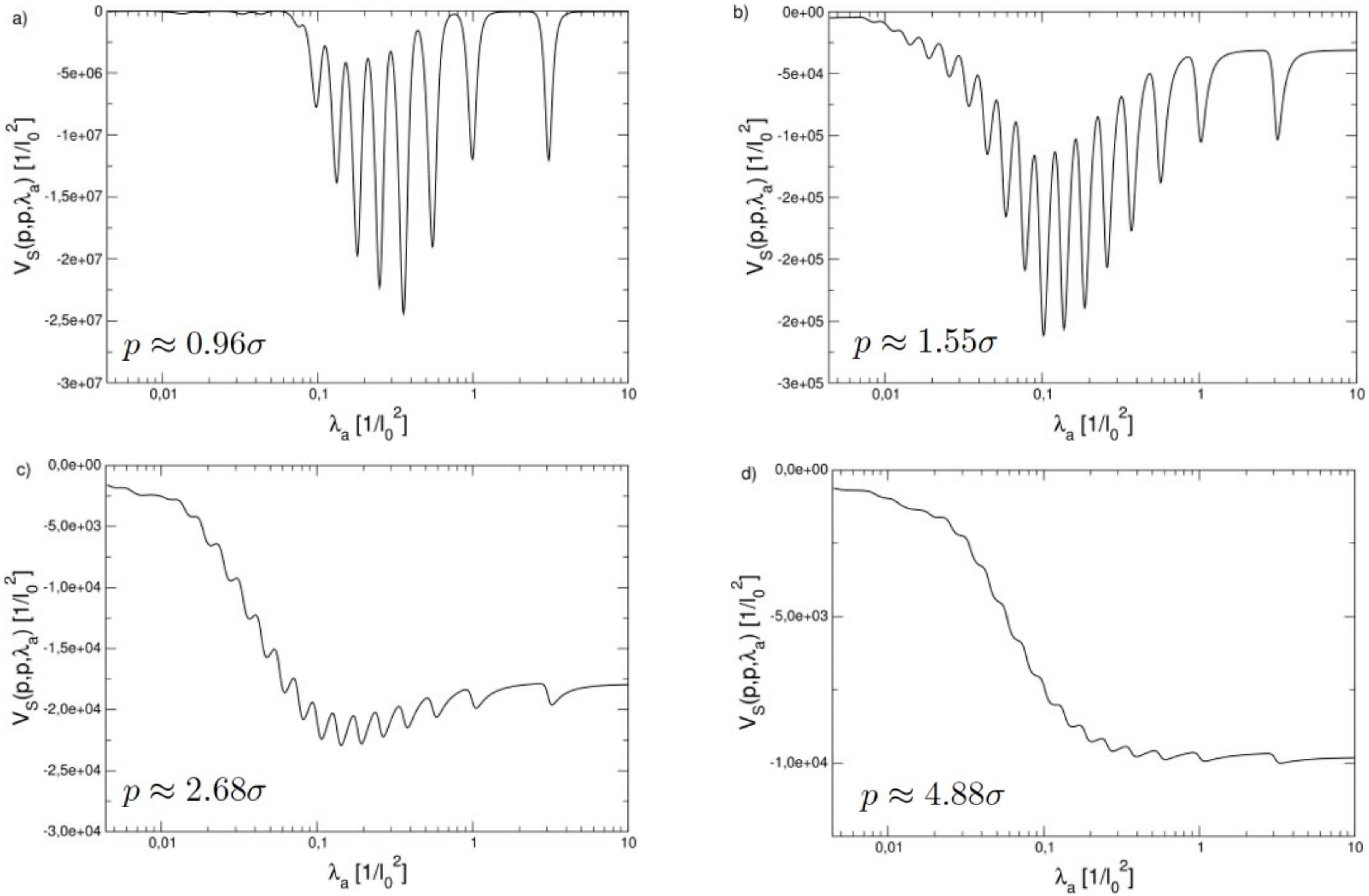
Example of SRG flow

Parameters:

$$\Lambda = 20\ l_0^{-1}$$
$$\lambda = 1\ l_0^{-1}$$
$$G = T$$



Dependence on σ: Inverse generator



Parameters:

$$\nu = 11$$

$$\Lambda = 30\ l_0^{-1}$$

$$\sigma = 0.05\ l_0^{-1}$$

Example of EC for scattering states