Evaluation of Numerical Schemes for
Stochastic Neural Field Equations

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1 Notes

The classical neural field equation was introduced by Amari in [1] and has the form

$$\frac{\partial u(x,t)}{\partial t} = -u(x,t) + \int_\Omega w(x-y) f(u(y,t)-h) \mathrm{d}y + S(x,t), \tag{1}$$

where $u(x,t)$ represents the activity at time t of a neuron at position x, $w(x-y)$ denotes the strength of connections between neurons separated by a distance y, $S(x,t)$ denotes a time-dependent localized input at site x, $f(x)$ gives the firing rate function with threshold h.
In the following we consider a stochastic version of Amari equation driven by Q-Wiener process W

$$\mathrm{d}U(x,t) = \left(-U(x,t) + \int_\Omega w(x-y) f(U(y,t)-h) + S(x,t)\right) \mathrm{d}t + \epsilon \mathrm{d}W(x,t) \tag{2}$$

where $t \in [0,T]$, $x \in \Omega \subset \mathbb{R}^n$ and W is a Q-Wiener process.
We assume $f(x)$ is a Heaviside function, that is

$$f(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases} \tag{3}$$

We choose $w(x)$ as periodicaly modulated connectivity function

$$w(x) = Ae^{-b|x|}(b \sin|\alpha x| + \cos(\alpha x)), \tag{4}$$

with $A = 2$, $b = 0.08$ and $\alpha = \pi/10$.
External input $S(x)$ is given by

$$S(x) = -0.5 + 8\exp(-\frac{x^2}{18}). \tag{5}$$

2 Objectives

1. Implement, using the language of your choice (preferably Matlab, alternatively Julia or Python) the stochastic neural field equation (2) using:
   - The Euler-Maruyama method,
   - The Milstein and/or the order 1 Runge-Kutta method (see [5] and the codes in [3]).

   For an example of the Euler-Maruyama method in the context of neural fields you may refer to [4].

2. Simulate the deterministic case ($\epsilon = 0$).

3. Present the simulation results (with 100 paths) for each method, for different noise levels. Plot the example bump solutions and save the maximum and minimum value of solution in each timestep. For the maximums, plot the evolution of the highest and the lowest value, as well as the expected value in time. Then do the same for the minimums (see for example Fig. 3 in [4]).

4. Compare of performance of methods for the same field parameters. For presenting the results see for example [2].

5. For each method, investigate how the performance is affected by:

   – discretization of the field,

   – the way of computing the integral (e.g. directly with a trapezoidal rule vs. FFTs).

References

[1] S. Amari. Dynamics of pattern formation in lateral-inhibition type neural fields. *Biological Cybernetics*, 27(2):77–87, 1977.

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[3] D. J. Higham. An algorithmic introduction to numerical simulation of stochastic differential equations. *SIAM review*, 43(3):525–546, 2001.

[4] P. M. Lima and E. Buckwar. Numerical Solution of Stochastic Neural Fields with Delays. *ArXiv e-prints*, January 2017.

[5] T. Sauer. *Numerical Solution of Stochastic Differential Equations in Finance*, pages 529–550. Springer Berlin Heidelberg, 2012.