AMCS 394E Homework 1

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Abstract

Homework regarding the first week. The goal is to work with basic numerical approximation of PDE's' and functions.

1 Using the method of lines

Consider the one-dimensional advection diffusion equation:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \mu \frac{\partial^2 u}{\partial x^2} = 0 \qquad \forall x \in \Omega = [0,1] \quad \text{and} \quad t > 0 \tag{1}$$

where $\mu > 0$ is a coefficient. Consider periodic boundary conditions and the following initial condition:

$$u(x,0) = \sin(2\pi x) \tag{2}$$

What do we expect the exact solution to do? Due to the advective part, the initial condition travels at constant speed to the right. At the same time, due to the diffusive term, the initial condition is dissipated at a rate that depends on μ.

Consider the following discretization. Use second-order central finite differences to approximate u_x and u_{xx}. Use forward and backward Euler to obtain full discretization (write down the schemes). Consider a fixed mesh with $\Delta x = 10^{-2}$.

Exersize 1

Consider a final time of $t = 1$ and $\mu = 0.01$. For each full discretization proceed as follows:

- Experiment using the following time step sizes: $\Delta t = 10^{-4}$, 10^{-3}, 10^{-2} and 10^{-1} .
- How do the explicit and implicit methods behave for these time steps?

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x y^2} + c\frac{\partial^2 u}{\partial y^2} = 0 \tag{3}$$

$$\Delta = b^2 - 2ac \tag{4}$$

Exersize 2

Consider $\mu = 0$ and solve eq. (1) using the explicit and the implicit methods. Use $\Delta t = 10^{-4}$ and solve the problem for the following final times: $t = 1$, 5, 10, 15 and 20. Comment on the behaviour of each full discretization as the final time increases.

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2 Approximation of functions

Consider the function

$$f(x) = \sin^4(2\pi x) \qquad \forall x \in \Omega = [0,1] \tag{5}$$

for which we have to find multiple global and local approximations. Let $f_h(x)$ be such an approximation for a given grid. We consider the following errors:

$$E_1 := \int_\Omega |f(x) - f_h(x)|\,\mathrm{d}x \quad \text{and} \quad E_2 := \int_\Omega \left(f(x) - f_h(x)\right)^2 \mathrm{d}x$$

Exersize 3: Global approximations

Consider the following approximations all with N terms:

a) the Taylor series around $x = 0.5$,
b) the Fourier series,
c) a global polynomial interpolation given by

$$f_h(x) = a_0 + a_1 x + a_2 x^2 + \ldots$$

with $f_h(x_i) = f(x_i)$ for an evenly spaced set of N points on the closed interval.

Consider different levels of refinement, $N = 4, 5, 6, \ldots, 10$ and for each approximation report both E_1 and E_2

Exercize 4: Local approximations

Split the domain Ω into N cells. For each cell K, compute linear and quadratic approximations $f_K(x)$ with $f_K(x_i) = f(x_i)$ where x_i are evenly spaced grid points (including the boundaries of the cell) within cell K. Compute and report the errors E_1 and E_2 for different number of cells; e.g., $N = 4, 5, 6, \ldots, 10$.