Finding The Equation Of A Curved Surface

AJM432

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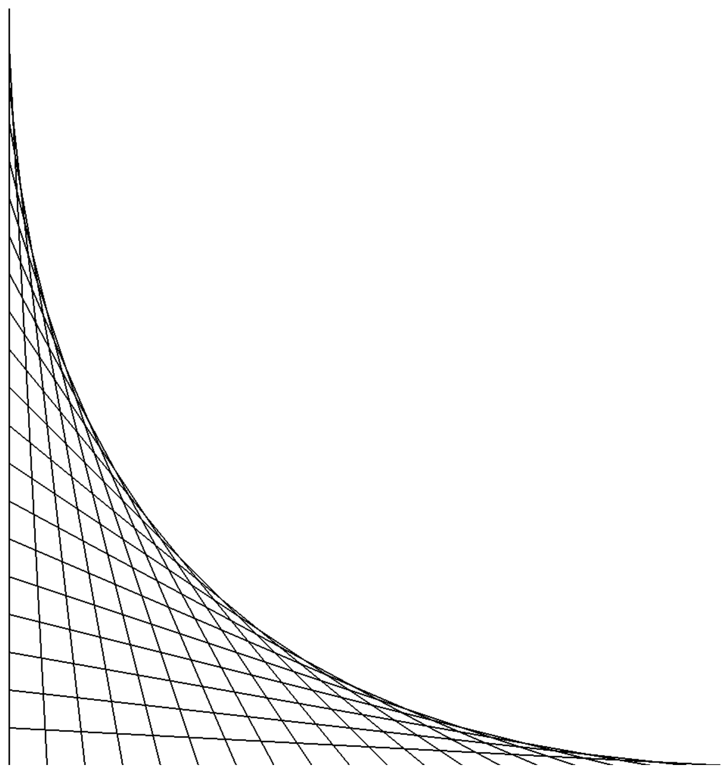

Figure 1: A curved surface composed of intersecting lines

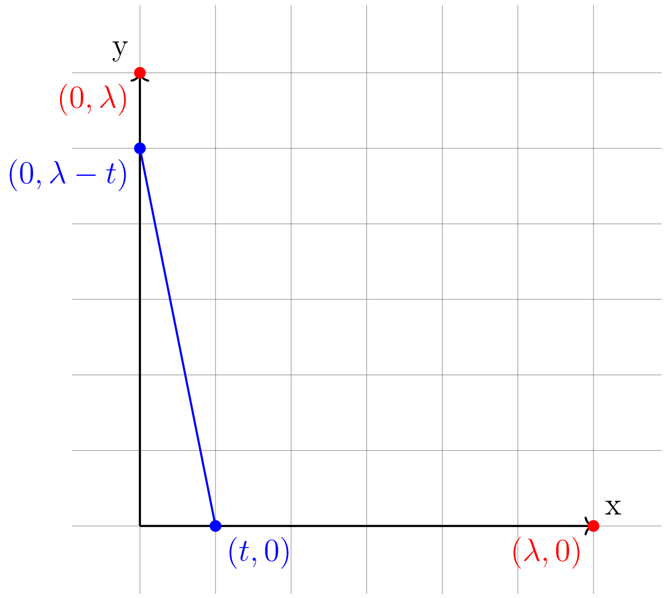

Figure 2: Labelled diagram of a single line

1 Purpose

The purpose of this paper will be to find and analyze the equation of the curve in Figure 1.

2 Finding a General Equation for all Line Segments

Let $g(x)$ represent the general equation of each line where $g(x) = mx + b$. The lines begin from $(0, \lambda - t)$ and ends at $(t, 0)$ where λ represents the bounding region of the curve.

$$g(0) = \lambda - t$$
$$\implies g(x) = mx + \lambda - t$$
$$g(t) = 0$$
$$0 = mt + \lambda - t$$
$$\implies m = \frac{t - \lambda}{t}$$
$$\boxed{g(x) = \frac{t - \lambda}{t} x + \lambda - t}$$

3 Finding the Equation of the Bounding Curve $f(x)$

3.1 Deriving the Tangent of a Function at any Point

Let us recall how to find the equation of a tangent line of function at any point. Let $T(x)$ denote the tangent line of a function $f(x)$ at $x = a$. The slope of the tangent is given by $f'(x)$. Therefore the tangent function is of the form $T(x) = f'(a)x + b$.

$$T(a) = f(a)$$
$$T(a) = af'(a) + b$$
$$f(a) = af'(a) + b$$
$$\implies b = f(a) - af'(a)$$
$$\boxed{T(x) = f'(a)x + f(a) - af'(a)}$$

3.2 The Differential Relationship Between $g(x)$ and $f(x)$

Now we must reverse this process and find $f(x)$. Let $f(x)$ represent the curve created by the tangents of the intersecting lines at the curved boundary in Figure 1. In the previously derived equation $g(x)$ we see that it is the output of mapping $f(x)$ to $T(x)$.

$$T : f \mapsto g$$

Essentially, we are solving a differential equation which satisfies the following equation where a is the point of tangency and λ represents the bounding region of the curve.

$$g(x) = f'(a)x + f(a) - af'(a)$$

3.3 Finding $f(x)$ Through a Limiting Process of Adjoining Tangents

Let us consider the points that compose $f(x)$. These points are the result of the intersection of the lines created by altering the value of t in $g(x)$ between the interval $(0, \lambda)$. To find $f(x)$ we need to find the intersection points for any value of t in the defined interval. This intersection is our a value from the tangent function definition.

Let ϵ represent a small change in x that will be used to find the intersection of two consecutive lines $g(x,t)$ and $g(x,t+\epsilon)$ where $g(x,t) := \frac{t-\lambda}{t}x + \lambda - t$. To find the intersection of these lines we will set them equal to each other.

$$g(x,t) = g(x,t+\epsilon)$$
$$\frac{t-\lambda}{t}x + \lambda - t = \frac{t+\epsilon-\lambda}{t+\epsilon}x + \lambda - t - \epsilon$$

$$x\left(\frac{t-\lambda}{t}-\frac{t+\epsilon-\lambda}{t+\epsilon}\right)=\lambda-t-\epsilon-\lambda+t$$

$$x=\frac{t(t+\epsilon)}{\lambda}$$

We have now found the x-value of the intersection of two lines whose points of tangency to $f(x)$ are separated by a distance of ϵ. Now we may consider what occurs when ϵ approaches zero. We will get the exact value of x at which a point of tangency occurs on $f(x)$.

$$x=\lim_{\epsilon\to 0^+}\frac{t(t+\epsilon)}{\lambda}$$

$$\boxed{x=\frac{t^2}{\lambda}}$$

Since we have found the exact value of x at which the tangency occurs in terms of t we can plug this result back into $g(x,t)$. However, since $f(x)$ does not depend on t we must solve for t in terms of x to solve for $f(x)$.

$$t=\sqrt{x\lambda}$$

We can now transform $g(x)$ into the non-linear function $f(x)$ by a change of variables with t and x.

$$f(x)=g\left(x,t\right)\ \text{ at the point of tangency } x=a$$

$$f(x)=\frac{t-\lambda}{t}x+\lambda-t$$

We will now substitute $t=\sqrt{x\lambda}$ to find $f(x)$.

$$f(x)=\frac{\sqrt{x\lambda}-\lambda}{\sqrt{x\lambda}}x+\lambda-\sqrt{x\lambda}$$

$$\boxed{f(x)=x+\lambda-2\sqrt{x\lambda}}\,x\in(0,\lambda)$$

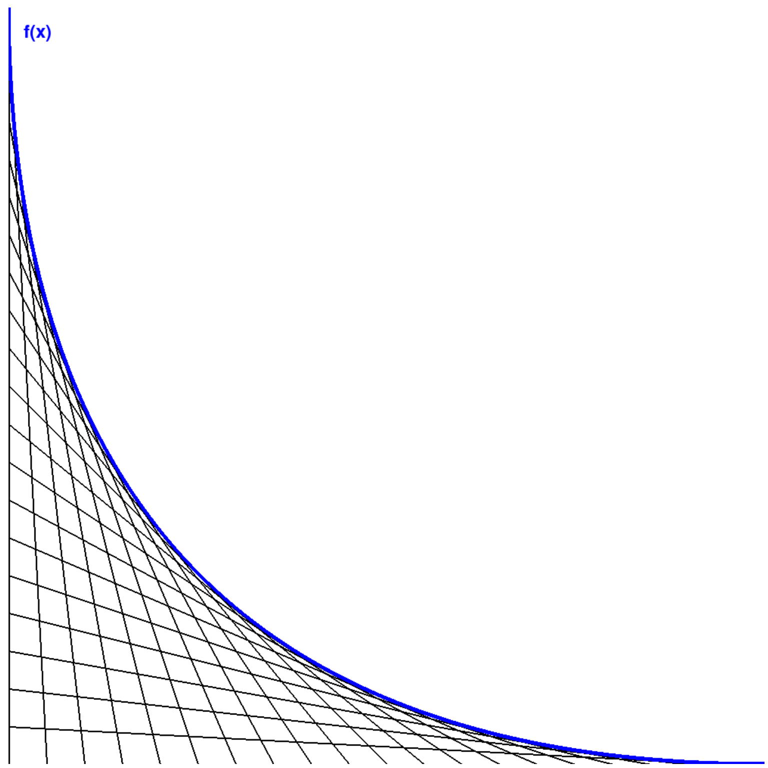

Figure 3: Solution Curve of $f(x)$