Describing the Motion of a Line Segment Moving Along a Vertical Line

AJM432

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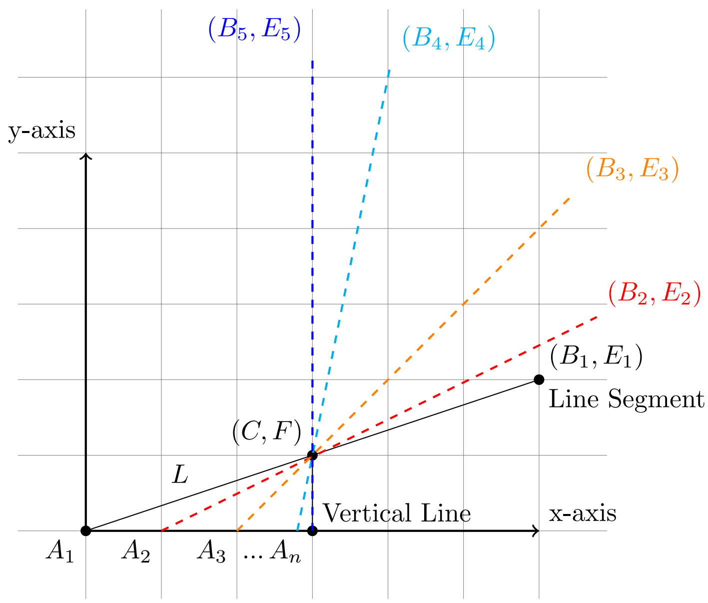
1 Introduction and Definitions

In this document we will derive the formulas responsible for the motion of a line segment moving along a vertical line. We will also attempt to explore the properties of such curves generated by the path of the line segment.



First we will define some variables involved in the motion of the line segment.

- let L denote the length of the line segment
- let A denote the distance of the tail of the line segment to the origin

- let B denote the distance of the head of the line segment to the origin
- let C denote the distance of the vertical line from the origin
- let D denote the height of the tail of the line segment (since the line slides across the x-axis $D = 0$ for all A
- let E denote the height of the head of the line segment
- let F denote the height of the vertical line

2 Finding the Equation of the Line Segment

let $g(x) = mx + h$ denote the function which describes the line segment
It follows that:

$g(A) = 0$
$g(C) = F$
$g(B) = E$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{F}{C - A}$$

$$mA + h = 0$$

$$h = -mA$$

$$h = \frac{-AF}{C - A}$$

$$g(x) = \frac{F}{C - A}x - \frac{AF}{C - A}$$

$$\therefore \boxed{g(x) = \frac{F(x - A)}{C - A}}$$

3 Finding E Using the Area Under the Line Segment

Now that we have $g(x)$ we can find the area under the curve $g(x)$ in two ways[1] since the points form ΔABE its area is $A_\Delta = \frac{1}{2}E(B-A)$. We can also find the equivalent expression $\int_A^B g(x)\,dx = A_\Delta$. Evaluating the integral we get:

$$\int_A^B g(x)\,dx = \int_A^B \frac{F(x-A)}{C-A}\,dx$$

$$= \frac{F}{C-A}\int_A^B x - A\,dx$$

$$= \frac{F}{C-A}(\frac{x^2}{2} - Ax)\Big|_A^B$$

$$\frac{1}{2}E(B-A) = \frac{F}{C-A}((\frac{B^2}{2} - AB) - (\frac{A^2}{2} - A^2))$$

$$E = \frac{2F}{(C-A)(B-A)}(\frac{1}{2}(B^2 - A^2) - A(B-A))$$

$$E = \frac{2F}{(C-A)(B-A)}(\frac{1}{2}(B+A)(B-A) - A(B-A))$$

$$E = \frac{2F}{C-A}(\frac{1}{2}B - \frac{1}{2}A)$$

$$\therefore \boxed{E = \frac{F(B-A)}{C-A}}$$

4 Finding B Using the Length of the Line Segment

The length of the line segment L is the hypotenuse of the triangle with sides $(B-A)$ and E. Thus:

$$L^2 = (B-A)^2 + E^2$$

$$L^2 = (B-A)^2 + \left(\frac{F(B-A)}{C-A}\right)^2$$

To simplify let $u = (B-A)$

[1] Note: we could have derived the equation for E by considering the area formed by $\Delta ACF + \Delta EF(B,F) +$ the rectangle $FCB(C,F)$

$$L^2 = u^2 + \frac{u^2F^2}{(C-A)^2}$$

$$L^2 = u^2\left(1 + \frac{F^2}{(C-A)^2}\right)$$

$$u^2 = \frac{L^2}{1 + \left(\frac{F}{C-A}\right)^2}$$

let $v = \frac{L^2}{1+\left(\frac{F}{C-A}\right)^2}$

$$(B-A)^2 = v$$

$$B^2 + B(-2A) + (A^2 - v) = 0$$

$$B = \frac{2A \pm \sqrt{4A^2 - 4(A^2 - v)}}{2}$$

$$B = \frac{2A \pm \sqrt{4v}}{2}$$

$A > 0$ thus:

$$B = A + \sqrt{v}$$

$$B = A + \sqrt{\frac{L^2}{1 + \left(\frac{F}{C-A}\right)^2}}$$

$$\therefore \boxed{B = A + \frac{L}{\sqrt{1 + \left(\frac{F}{C-A}\right)^2}}} \; L \geq 0$$

5 Plotting the Parametric Curve

Now that we have found E and B in terms of the defined constants we can plot them as a parametric curve since E depends on B. We can treat (B,E) as our (x,y) components since they represent the horizontal and vertical distance of the head of the line segment respectively.

Since the line segment and the vertical line form the right ΔACF we can assume that C can be simplified into $C = \sqrt{L^2 - F^2}$ as a special case for plotting the curve. Now we can plot the curve as $A \to C$ since A is the only changing quantity. In the final equation we have substituted $t = A, x = B$ and $y = E$.

$$\begin{cases} x = t + \frac{L}{\sqrt{1+\left(\frac{F}{C-t}\right)^2}} \\ y = \frac{F(x-t)}{C-t} \end{cases} \quad t \in \mathbb{R}, L \geq 0, C \neq t$$

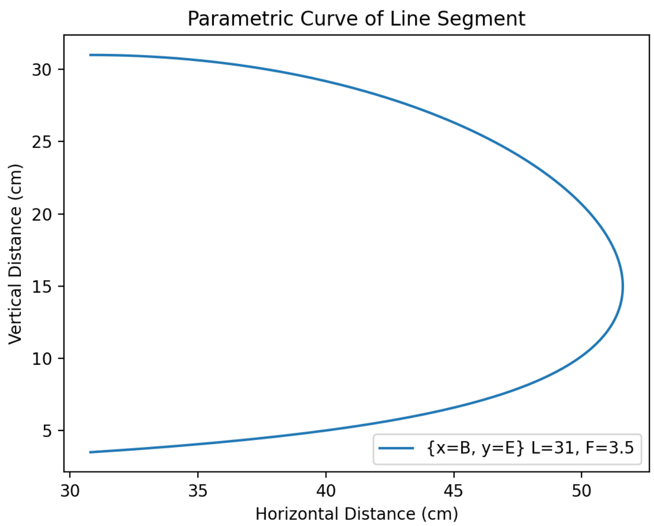


Figure 1: The curve with constants $L = 31$ and $F = 3.5$

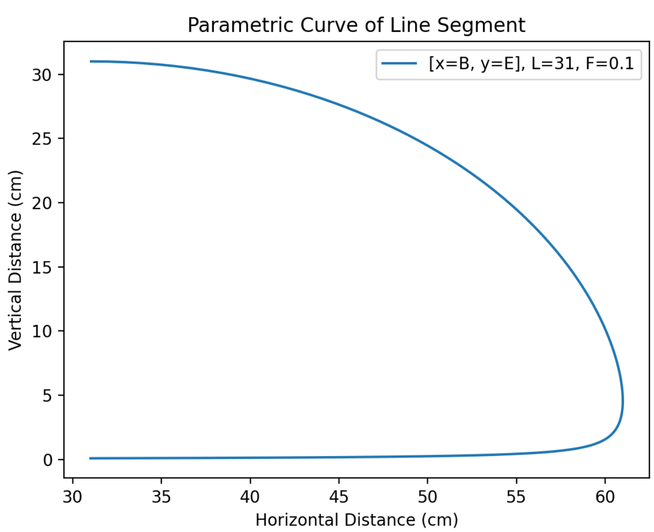


Figure 2: The curve with constants $L = 31$ and $F = 0.1$

6 Critical Point

Notice how the curve of the above figures always reach a critical point as $A \to C$ where B reaches a maxima. The critical point occurs precisely when $\frac{dB}{dA} = 0$ since B approaches an extrema at the critical point. Solving for the derivative of B with respect to A we get:

$$\frac{dB}{dA} = \frac{d}{dA}\left[A + \frac{L}{\sqrt{1+\left(\frac{F}{C-A}\right)^2}}\right]$$

$$\frac{dB}{dA} = 1 + L\frac{d}{dA}\left[\left(1+\left(\frac{F}{C-A}\right)^2\right)^{\frac{-1}{2}}\right]$$

$$\frac{dB}{dA} = 1 - \frac{L}{2}\frac{d}{dA}\left[1+\left(\frac{F}{C-A}\right)^2\right]\left(1+\left(\frac{F}{C-A}\right)^2\right)^{\frac{-3}{2}}$$

$$\frac{dB}{dA} = 1 - \frac{L}{2}\left(2F^2\left(C-A\right)^{-3}\right)\left(1+\left(\frac{F}{C-A}\right)^2\right)^{\frac{-3}{2}}$$

$$\therefore \boxed{\frac{dB}{dA} = 1 - \frac{LF^2}{\left(C-A\right)^3\left(1+\frac{F^2}{(C-A)^2}\right)^{\frac{3}{2}}}}$$

Now we can set the derivative of B with respect to A equal to 0 to solve for the critical point in terms of A.

$$\frac{dB}{dA} = 0$$

$$1 - \frac{LF^2}{\left(C-A\right)^3\left(1+\frac{F^2}{(C-A)^2}\right)^{\frac{3}{2}}} = 0$$

$$\left(C-A\right)^3\left(1+\left(\frac{F}{C-A}\right)^2\right)^{\frac{3}{2}} = LF^2$$

let $u = C - A$

$$u^3\left(1+\frac{F^2}{u^2}\right)^{\frac{3}{2}} = LF^2$$

$$u^3\sqrt{\frac{\left(u^2+F^2\right)^3}{u^6}} = LF^2$$

$$\sqrt{\left(u^2+F^2\right)^3} = LF^2$$

$$u^2 + F^2 = L^{\frac{2}{3}}F^{\frac{4}{3}}$$

$$(C-A)^2 = L^{\frac{2}{3}}F^{\frac{4}{3}} - F^2$$

$$A = C - \pm\sqrt{L^{\frac{2}{3}}F^{\frac{4}{3}} - F^2}$$

$A > 0$

$$\therefore \boxed{A = C - \sqrt{L^{\frac{2}{3}}F^{\frac{4}{3}} - F^2}}$$

We must evaluate B at $A = C - \sqrt{L^{\frac{2}{3}}F^{\frac{4}{3}} - F^2}$ to find the x-component of the critical point which we will denote P_x.

$$P_x = C - \sqrt{L^{\frac{2}{3}}F^{\frac{4}{3}} - F^2} + \frac{L}{\sqrt{1+\left(\frac{F}{C-C-\sqrt{L^{\frac{2}{3}}F^{\frac{4}{3}}-F^2}}\right)^2}}$$

$$P_x = C - \sqrt{L^{\frac{2}{3}}F^{\frac{4}{3}} - F^2} + \frac{L}{\sqrt{1+\frac{F^2}{L^{\frac{2}{3}}F^{\frac{4}{3}}-F^2}}}$$

$$\therefore \boxed{P_x = C - \sqrt{L^{\frac{2}{3}}F^{\frac{4}{3}} - F^2} + \frac{L}{\sqrt{1+\frac{1}{\left(\frac{L}{F}\right)^{\frac{2}{3}}-1}}}}$$

We can now graph the vertical tangent $x = P_x$ along with our parametric equation.

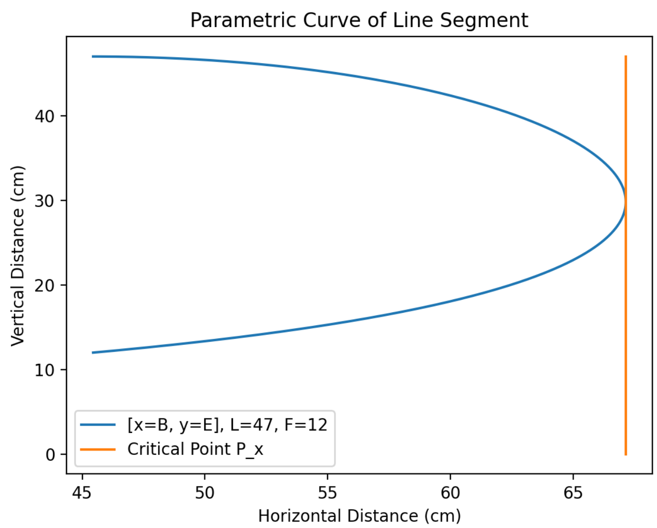


Figure 3: The curve with constants $L = 47$ and $F = 12$

7 Taking Limits

As we decrease the height of the vertical line F, intuitively the critical point should approach $L + C$ since at the limit we are essentially sliding a line horizontally. Taking the limit as F approaches 0 of P_x we get:

$$\lim_{F\to 0} P_x = \lim_{F\to 0}\left[C - \sqrt{L^{\frac{2}{3}}F^{\frac{4}{3}} - F^2} + \frac{L}{\sqrt{1 + \frac{1}{\left(\frac{L}{F}\right)^{\frac{2}{3}} - 1}}}\right]$$

$$= \lim_{F\to 0} C - \lim_{F\to 0}\left[\sqrt{L^{\frac{2}{3}}F^{\frac{4}{3}} - F^2}\right] + \lim_{F\to 0}\left[\frac{L}{\sqrt{1 + \frac{1}{\left(\frac{L}{F}\right)^{\frac{2}{3}} - 1}}}\right]$$

$$= C - 0 + \frac{L}{\sqrt{1 + \lim_{F\to 0}\left[\frac{1}{\left(\frac{L}{F}\right)^{\frac{2}{3}} - 1}\right]}}$$

$$= C + \frac{L}{\sqrt{1 + \frac{1}{\infty}}}$$

$$\therefore \boxed{\lim_{F\to 0} P_x = C + L}$$