Derivation Stress Response in Plane-stress Viscoelasticity

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The derivation of a numerical formula for calculating stress from measured in-plane strains in plane stress linear viscoelasticity begins with the process laid out in Mun 2006 "Numerical Computation of Convolution Integral for the Linear Viscoelasticity of Asphalt Concrete." We start with equations (16) and (17) from the paper represented with equations (1) and (2) respectively:

$$\varepsilon_{kl_m}(t_n) = \varepsilon_{kl}(t_{n-1}) + exp(\frac{-\Delta t}{\tau_m})[\varepsilon_{kl_m}(t_{n-1}) - \varepsilon_{kl}(t_{n-1})] + \frac{\varepsilon_{kl}(t_n) - \varepsilon_{kl}(t_{n-1})}{\Delta t}[\Delta t - \tau_m(1 - exp(\frac{-\Delta t}{\tau_m}))] \tag{1}$$

$$\begin{aligned}\sigma_{ij}(t_{n+1}) = C_{ijkl_0}\varepsilon_{kl}(t_{n+1}) - \sum_{m=1}^{M} C_{ijkl_m}\Bigg[\varepsilon_{kl}t_n + exp(\frac{-\Delta t}{\tau_m})\left[\varepsilon_{kl_m}(t_n) - \varepsilon_{kl}(t_n)\right]\ldots \\ + \Big(\varepsilon_{kl}(t_{n+1}) - \varepsilon_{kl}(t_n)\Big)\left[1 - \frac{\tau_m}{\Delta t}\left(1 - exp(\frac{-\Delta t}{\tau_m})\right)\right]\Bigg]\end{aligned} \tag{2}$$

where:

- ε_{kl_m} is a state variable representing the viscous strain tensor in Maxwell element m
- t is time with subscripts $n-1$,n, and $n+1$ representing the previous, current, and subsequent time increment respectively
- ε_{kl} is the total strain tensor
- Δt represents the time step
- C_{ijkl_m} is the stiffness tensor of each spring in the model, and $m = 0$ is the instantaneous stiffness tensor. The instantaneous stiffness tensor is the sum of all the stiffness tensors in the model.
- τ_m defines the time constant and $\tau_m = C_{1111_m}/\eta - 1111_m$, where eta is the damping constant of the dashpot. Note in this derivation, we are assuming the time constants for all components of the stiffness tensor are the same within any single Maxwell element. If different parameters have different rate sensitivities they should be represented by another Maxwell element with a different time constant.
- σ_{ij} is the Cauchy stress tensor

Next, because we are solving a plane stress problem, we know $\sigma_{i3} = 0$ for $i = 1,2,3$ and $\varepsilon_{23} = \varepsilon_{13} = 0$. However, ε_{33} is not necessarily 0 during loading, and as is clear from (1) and (2) is necessary for calculating in-plane stresses. Unfortunately, in image based experiments with a single camera, we typically cannot measure ε_{33} directly and must calculate it from measured in-plane strains. To do this we set $\sigma_{33} = 0$ and expand the indicial notation

$$\begin{aligned}\sigma_{33}(t_{n+1}) = 0 = C_{3311_0}\varepsilon_{11}(t_{n+1}) + C_{3322_0}\varepsilon_{22}(t_{n+1}) + C_{3333_0}\varepsilon_{33}(t_{n+1})\ldots \\ - \sum_{m=1}^{M}\left[C_{3311_m}B_{11_m} + C_{3322_m}B_{22_m} + C_{3333_m}B_{33_m}\right]\end{aligned} \tag{3a}$$

$$B_{11_m} = \varepsilon_{11}(t_n) + exp(\frac{-\Delta t}{\tau_m})\left[\varepsilon_{11_m}(t_n) - \varepsilon_{11}(t_n)\right] + \left[\varepsilon_{11}(t_{n+1}) - \varepsilon_{11}(t_n)\right]\left[1 - \frac{\tau_m}{\Delta t}\left(1 - exp(\frac{-\Delta t}{\tau_m})\right)\right] \tag{3b}$$

$$B_{22_m} = \varepsilon_{22}(t_n) + exp(\frac{-\Delta t}{\tau_m})\left[\varepsilon_{22_m}(t_n) - \varepsilon_{22}(t_n)\right] + \left[\varepsilon_{22}(t_{n+1}) - \varepsilon_{22}(t_n)\right]\left[1 - \frac{\tau_m}{\Delta t}\left(1 - exp(\frac{-\Delta t}{\tau_m})\right)\right] \tag{3c}$$

$$
\begin{aligned}
B_{33_m} &= \varepsilon_{33}(t_n) + exp(\frac{-\Delta t}{\tau_m})\Big[\varepsilon_{33_m}(t_n) - \varepsilon_{33}(t_n)\Big] + \Big[\varepsilon_{33}(t_{n+1}) - \varepsilon_{33}(t_n)\Big]\Big[1 - \frac{\tau_m}{\Delta t}\Big(1 - exp(\frac{-\Delta t}{\tau_m})\Big)\Big] \\
&= \varepsilon_{33}(t_{n+1})\Big[1 - \frac{\tau_m}{\Delta t}\Big(1 - exp(\frac{-\Delta t}{\tau_m})\Big)\Big] + \varepsilon_{33}(t_n) + exp(\frac{-\Delta t}{\tau_m})\Big[\varepsilon_{33_m}(t_n) - \varepsilon_{33}(t_n)\Big]\ldots \\
&- \varepsilon_{33}(t_n)\Big[1 - \frac{\tau_m}{\Delta t}\Big(1 - exp(\frac{-\Delta t}{\tau_m})\Big)\Big]
\end{aligned} \tag{3d}
$$

Now we gather the $\varepsilon_{33}(t_{n+1})$ terms and solve for $\varepsilon_{33}(t_{n+1})$

$$
\begin{aligned}
&\varepsilon_{33}(t_{n+1}) = A/D \\
&\quad A = \sum_{m=1}^{M}\Bigg[C_{3311_m}B_{11_m} + C_{3322_m}B_{22_m} + C_{3333_m}\Big[\varepsilon_{33}(t_n) + exp(\frac{-\Delta t}{\tau_m})\Big[\varepsilon_{33_m}(t_n) - \varepsilon_{33}(t_n)\Big]\ldots \\
&\qquad - \varepsilon_{33}(t_n)\Big[1 - \frac{\tau_m}{\Delta t}\Big(1 - exp(\frac{-\Delta t}{\tau_m})\Big)\Big]\Big]\Bigg] - C_{3311_0}\varepsilon_{11}(t_{n+1}) - C_{3322_0}\varepsilon_{22}(t_{n+1}) \\
&\quad D = C_{3333_0} - \sum_{m=1}^{M} C_{3333_m}\Big[1 - \frac{\tau_m}{\Delta t}\Big(1 - exp(\frac{-\Delta t}{\tau_m})\Big)\Big]
\end{aligned} \tag{4}
$$

Now that we have an expression for ε_{33} we can derive expressions for the in-plane stresses

$$
\sigma_{11}(t_{n+1}) = C_{1111_0}\varepsilon_{11}(t_{n+1}) + C_{1122_0}\varepsilon_{22}(t_{n+1}) + C_{1133_0}\varepsilon_{3}3(t_{n+1}) + \sum_{m=1}^{M}[C_{1111_m}B_{11_m} + C_{1122_m}B_{22_m} + C_{1133_m}B_{33_m}] \tag{5}
$$

$$
\sigma_{22}(t_{n+1}) = C_{2211_0}\varepsilon_{11}(t_{n+1}) + C_{2222_0}\varepsilon_{22}(t_{n+1}) + C_{2233_0}\varepsilon_{3}3(t_{n+1}) + \sum_{m=1}^{M}[C_{2211_m}B_{11_m} + C_{2222_m}B_{22_m} + C_{2233_m}B_{33_m}] \tag{6}
$$

$$
\sigma_{12}(t_{n+1}) = C_{1212_0}\varepsilon_{12}(t_{n+1}) + \sum_{m=1}^{M} C_{1212_m}B_{12_m} \tag{7a}
$$

$$
B_{12_m} = \varepsilon_{12}(t_n) + exp(\frac{-\Delta t}{\tau_m})\Big[\varepsilon_{12_m}(t_n) - \varepsilon_{12}(t_n)\Big] + \Big[\varepsilon_{12}(t_{n+1}) - \varepsilon_{12}(t_n)\Big]\Big[1 - \frac{\tau_m}{\Delta t}\Big(1 - exp(\frac{-\Delta t}{\tau_m})\Big)\Big] \tag{7b}
$$

$$
\varepsilon_{12_m} = \varepsilon_{12}(t_{n-1}) + exp(\frac{-\Delta t}{\tau_m})[\varepsilon_{12_m}(t_{n-1}) - \varepsilon_{12}(t_{n-1})] + \frac{\varepsilon_{12}(t_n) - \varepsilon_{12}(t_{n-1})}{\Delta t}[\Delta t - \tau_m(1 - exp(\frac{-\Delta t}{\tau_m}))] \tag{7c}
$$