Susana Gomes, Kamran Pentland 22/10/2021

Stochastic Modelling and Random Processes

Example sheet 2

1 Visualisation of the Gershgorin disk theorem and lazy Markov chains

Recall, from handout 1, that given a matrix $A \in \mathbb{R}^{n \times n}$, the Gershgorin disk theorem states that all eigenvalues lie in a least one Gershogorin disk, D_i, where D_i is centered on $a_{i,i}$ with radius

$$R_i = \sum_{j \neq i} a_{i,j}.$$

(a) Visualise this for a couple of examples (e.g. the weather DTMC from last week, or a simple random walk with periodic boundary conditions and state space $\{1, 2, 3, 4, 5\}$).

(b) Let $(X_n : n \in \mathbb{N}_0)$ be a DTMC with transition matrix $p(x, y)$ (e.g. the simple random walk from (a)). The DTMC with transition matrix

$$p^\epsilon(x, y) = \epsilon \delta_{x,y} + (1 - \epsilon)\, p(x, y) \ , \quad \epsilon \in (0, 1)$$

is called a **lazy version** of the original chain.

i. Check that P^ϵ has the same eigenvectors as P with eigenvalues $\lambda_i^\epsilon = \lambda_i(1-\epsilon) + \epsilon$.

ii. In particular, check that this implies $|\lambda_i^\epsilon| < |\lambda_i| \leq 1$ unless $\lambda_i = 1$.
Since all diagonal elements are bounded below by $\epsilon > 0$, the Gershgorin theorem now also implies for the eigenvalues of P^ϵ

$$|\lambda_i| = 1 \quad \Rightarrow \quad \lambda_i = 1.$$

Such a matrix P^ϵ is called **aperiodic**, and there are no persistent oscillations (because there are not any eigenvalues $\mathbb{C} \ni \lambda \neq 1$ with $|\lambda| = 1$). Visualise this for the lazy versions of the DTMCs in (a) for a couple of values of ϵ.

(c) Consider a Markov chain with state space $S = \{1, 2, 3, 4, 5, 6, 7\}$ and the following transition matrix:

$$P = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

i. Draw a graph representation of this chain.

ii. Compute its eigenvalues (and visualise them) and conclude that there are persistent oscillations. What is the stationary distribution?

iii. Simulate 100 realisations of this DTMC up to $T = 1000$ to get an idea of its behaviour.

(d) Come up with an example of a DTMC which is not irreducible and find its stationary distributions.

2 Geometric random walk

Let $X_1, X_2, \dots$ be a sequence of iidrv's with $X_i \sim \mathcal{N}(\mu, \sigma^2)$ where $\mu \in \mathbb{R}$ and $\sigma^2 > 0$. Consider the discrete-time random walk (DTRW) on state space $\mathbb{R}$

$$(Y_n : n \geq 0) \quad \text{with} \quad Y_{n+1} = Y_n + X_{n+1} \quad \text{and} \quad Y_0 = 0\,.$$

(a) State the weak law of large numbers and the central limit theorem for Y_n.

(b) Using that sums of Gaussian random variables are again Gaussian, what is the distribution of Y_n for any arbitrary $n \geq 0$?

Now consider the discrete-time process $(Z_n : n \geq 0)$ on the state space $[0, \infty)$ with $Z_n = \exp(Y_n)$, which is called a **geometric random walk**.

(c) Give a recursive definition of $(Z_n : n \geq 0)$ analogous to the above.
Show that Z_n has a log-normal distribution for all $n \geq 1$ by deriving the PDF.
Give the mean, variance and median of Z_n (you can look this up on the web).

(d) Simulate $M = 500$ realizations of Z_n for $n = 0, \dots, 100$ with $\mu = 0$ and $\sigma = 0.2$.
Plot the **empirical average** $\hat{\mu}_n^M := \frac{1}{M}\sum_{i=1}^{M} Z_n^i$ as a function of time n, with error bars indicating the standard deviation.
At times $n = 10$ and 100 produce boxplots, plot the empirical PDF using a kernel density estimation, and compare it to the theoretical prediction.
For a single realization, plot the **ergodic average** $\bar{\mu}_N := \frac{1}{N}\sum_{n=1}^{N} Z_n$ as a function of N up to $N = 100$.

We will see more about this when we look at the Geometric Brownian motion in a week or so