Appendix 12: Magnetic Precession in Static and Oscillating Magnetic Fields:
The Rotating Coordinate System

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Before learning the full quantum mechanical treatment of a spin 1/2 particle in a magnetic field, it is useful to consider the more familiar and intuitive problem of the evolution of a classical particle with magnetic moment $\vec{M}$. The quantum mechanical problem will turn out to be analogous.

The unperturbed system treated here is a single object or particle with magnetic moment $\vec{M}$ in the presence of a strong static magnetic field in the z direction. The perturbation will be a weak rotating magnetic field in the x-y plane (which is largely equivalent to an oscillating field in the x direction).

Unperturbed System

The energy of the unperturbed system depends on the angle θ between the magnetic moment and the magnetic field.

$$\vec{B} = B_0 \hat{z}$$

$$U = -\vec{M} \cdot \vec{B} = -\left|\vec{M}\right| B_0 \cos\theta$$

For a negatively charged particle

$$\vec{M} = -g\vec{L}$$

where g is called the gyromagnetic ratio, and $\vec{L}$ is the angular momentum of the particle (which for an electron is also the spin).

The torque on the magnetic moment is given by

$$\vec{\tau} = \frac{d\vec{L}}{dt} = \vec{M} \times \vec{B}$$

Hence

$$\frac{d\vec{M}}{dt} = -g\frac{d\vec{L}}{dt} = -g\vec{M} \times \vec{B} = g^2\vec{L} \times \vec{B}.$$

Note that the magnitude of the magnetic moment does not change since

$$\frac{d\left|\vec{L}\right|^2}{dt} = \frac{d\left(\vec{L}\cdot\vec{L}\right)}{dt} = 2\vec{L}\cdot\frac{d\vec{L}}{dt} = 2\vec{L}\cdot\left(\vec{M}\times\vec{B}\right) = 2\vec{L}\cdot\left(-g\vec{L}\times\vec{B}\right) = 0$$

Similarly the angle between $\vec{M}$ and $\vec{B}$ (i.e., the projection of $\vec{M}$ along the z axis) does not change, since

$$\frac{d\left(\vec{B}\cdot\vec{L}\right)}{dt} = 0 + \vec{B}\cdot\frac{d\vec{L}}{dt} = \vec{B}\cdot\left(\vec{M}\times\vec{B}\right) = -g\vec{B}\cdot\left(\vec{L}\times\vec{B}\right) = 0$$

In the quantum mechanical problem of a particle with spin there will be only two possible orientations of the spin, up or down. Both spin up and spin down will be eigenstates, just as the z component of the spin is a constant in the classical case.

The equations of motion for the magnetic moment become

$$\frac{dM_x}{dt} = -\omega_0 M_y$$

$$\frac{dM_y}{dt} = \omega_0 M_x$$

$$\frac{dM_z}{dt} = 0$$

where $\omega_0 = gB_0$ is known as the Larmor frequency.

With the initial condition

$$\vec{M}(0) = M_x(0)\hat{x} + M_z(0)\hat{z}$$

i.e., $\theta \neq 0$, The solutions are

$$M_x(t) = M_x(0)\cos\omega_0 t$$
$$M_y(t) = M_x(0)\sin\omega_0 t$$

That is, the magnetic moment precesses around the z-axis at the Larmor frequency.

Rotating Frame

Now consider the problem in the reference frame rotating at the Larmor frequency ω_0. In this reference frame, $\vec{M}$ is a constant, which makes the time evolution trivial. The time evolution in the lab frame can then be obtained by transforming from the rotating frame back to the lab frame.

Consider in general the transformation to a reference frame rotating at a frequency ω. The rotation matrix for the transformation between the two frames is

$$R = \begin{pmatrix} \cos\omega t & \sin\omega t & 0 \\ -\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The magnetic moment in the new reference frame is given by

$$M_{x'}(t) = M_x \cos\omega t + M_y \sin\omega t$$
$$M_{y'}(t) = -M_x \sin\omega t + M_y \cos\omega t$$
$$M_{z'}(t) = M_z$$

The equations or motion in the rotating frame are

$$\frac{dM_{x'}}{dt} = (\omega - \omega_0)M_{y'}$$
$$\frac{dM_{y'}}{dt} = -(\omega - \omega_0)M_{x'}$$
$$\frac{dM_{z'}}{dt} = 0$$

These equations can be recombined into a vector equation

$$\frac{d\vec{M}'}{dt} = -g\vec{M}' \times \vec{B}_{eff}$$

where

$$\vec{B}_{eff} = \left(B_0 - \frac{\omega}{g}\right)\hat{z} = -\frac{\Delta}{g}\hat{z}$$

and $\Delta = \omega - \omega_0$.

Note that at resonance, one has $\omega = \omega_0$ and thus $\vec{B}_{eff} = 0$, so there is no precession of the magnetic moment in the rotating frame. For any other frequency there is a precession because the rotating frame is out of synch with the Larmor frequency. The equations above can be transformed back into the lab frame:

$$\vec{M} \rightarrow \vec{M}'$$

$$\frac{d\vec{M}}{dt} \rightarrow \frac{d\vec{M}'}{dt} + \vec{\omega} \times \vec{M}'$$

$$\vec{B}_{eff} \rightarrow \vec{B} - \frac{\vec{\omega}}{g}$$

Perturbation by a Rotating Magnetic Field

Next, consider an applied B field, which precesses in the x-y plane at frequency ω:

$$\vec{B}_{pert} = B_1 \left(\cos\omega t \ \hat{x} \ + \ \sin\omega t \ \hat{y} \right)$$

[Note: in the laboratory it is often impractical to create a rotating magnetic field, so one commonly employs a field that oscillates along a particular direction, e.g., the x axis. This oscillating field is equivalent to the sum of two counter-rotating fields: one field rotates with the same helicity as the magnetic moment's rotation, while the other field rotates with opposite helicity and can usually be ignored because it is effectively very far off resonance (at $\omega \cong 2\omega_0$).]

With the addition of $\vec{B}_{pert}$ the equations of motion become

$$\frac{d\vec{M}}{dt} = -g\vec{M} \times \vec{B} = -g\vec{M} \times \left[B_0\hat{z} + B_1 \left(\cos\omega t \ \hat{x} \ + \ \sin\omega t \ \hat{y} \right) \right]$$

In the frame rotating at ω this becomes

$$\frac{d\vec{M}'}{dt} = -g\vec{M}' \times \vec{B}_{eff} = -g\vec{M}' \times \left[\left(B_0 - \frac{\omega}{g} \right)\hat{z} + B_1\hat{x} \right]$$

In component form one has

$$\frac{dM'_x}{dt} = \Delta M_y$$

$$\frac{dM'_y}{dt} = -\Delta M_x - gB_1 M_z$$

$$\frac{dM'_z}{dt} = gB_1 M_y$$

On resonance in the rotating frame, i.e., at $\omega = \omega_0$, the magnetic field lies ONLY along the x direction, so the magnetic moment simply precesses around x. This means that M_x is constant and the component of $\vec{M}$ perpendicular to x rotates in the y-z plane. Thus the component of $\vec{M}$ which is along the z-axis at time t=0 will be along the -z axis at $t = \pi/gB_1$. In other words, all of the projection of $\vec{M}$ along the positive z-axis will be transferred to a

projection along the negative z-axis. Classically and geometrically, this is a consequence of the axis of rotation being perpendicular to the y-z plane. Quantum mechanically, the applied rotating field allows a coupling between the spin up and spin down states. In either picture, this inversion effect is known as resonant Rabi oscillation.

Mathematically, the solutions are

$$M'_x(t) = M'_x(0)$$
$$M'_y(t) = -M'_z(0)\sin gB_1 t + M'_y(0)\cos gB_1 t$$
$$M'_z(t) = -M'_y(0)\sin gB_1 t + M'_z(0)\cos gB_1 t$$

Thus the magnetic moment rotates around the x-axis at the Rabi frequency $\Omega = gB_1$.

If $\vec{B}_{pert}$ is tuned off resonance, such that $\omega \cong \omega_0$, then the effective magnetic field will have some z component as well as an x component. Thus the component of $\vec{M}$ in the positive z direction will NEVER rotate completely into the negative z direction (and visa versa). As the detuning of ω from ω_0 gets larger and larger, the effective field approaches the z direction and the rotations between the positive and negative components of the magnetic moment approaches zero.

Mathematically, in the rotating frame $\vec{M}'$ precesses about $\vec{B}$ at a frequency

$$\Omega_{eff} = g\left|\vec{B}_{eff}\right|^2 = \sqrt{\Delta^2 + (gB_1)^2} = \sqrt{\Delta^2 + \Omega^2}$$

This is the off-resonant Rabi oscillation frequency.

The effect of the field is thus to set up a correlation between the spin up and spin down states since something, which is in spin up at one time, will rotate into spin down at another time. In the stationary frame, the moment precesses around the moving magnetic field vector.

Mathematically, in the rotating frame, the time evolution of the components becomes

$$M'_x(t) = M'_x(0)\frac{(gB_1)^2 + \Delta^2 \cos\Omega_{eff}t}{\Omega_{eff}^2} + M'_y(0)\frac{\Delta}{\Omega_{eff}}\sin\Omega_{eff}t + M'_z(0)\frac{\Delta\Omega}{\Omega_{eff}^2}\left(\cos\Omega_{eff}t - 1\right)$$

$$M'_y(t) = -M'_x(0)\frac{\Delta}{\Omega_{eff}}\sin\Omega_{eff}t + M'_y(0)\cos\Omega_{eff}t - M'_z(0)\frac{\Omega}{\Omega_{eff}^2}\sin\Omega_{eff}t$$

$$M'_z(t) = M'_x(0)\frac{\Delta\Omega}{\Omega_{eff}^2}\left(\cos\Omega_{eff}t - 1\right) + M'_y(0)\frac{\Omega}{\Omega_{eff}^2}\sin\Omega_{eff}t + M'_z(0)\left[1 + \left(\frac{\Omega}{\Omega_{eff}}\right)^2\left(\cos\Omega_{eff}t - 1\right)\right]$$