Simulation of the Trajectory of a Charged Particle in Two Superpositioned Magnetic Fields

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Abstract

Cosmic rays—highly energised charged particles—are one of today's great mysteries in astrophysics. There is an observed spectra of particles with energies up to 10^{20} eV. The origin of these have been described by theories about magnetic fields, such as magnetic mirrors, that accelerate charged particles. Recent studies show that it may also be possible for charged particles to be accelerated even in a simple magnetic field, if the particle moves in a chaotic path. This study aimed to model the trajectory of a charged particle in a system of a magnetic dipole field in superposition with a uniform magentic field, to see if the path could be chaotic. A simulation was done to compute the path of the particle. The parameters that were changed was the strength of each magnetic field. Results showed no chaotic paths but the conclusion was made that this needs to be investigated further.

Contents

1 Introduction

Highly energised charged particles, called *cosmic rays*, are one of today's great mysteries in physics and have been ever since they were first discovered in 1912 by Victor Hess. Cosmic rays consist mainly of protons. It was previuosly thought that the cosmic rays came only from the Sun. However, this was disproved when the radiation showed to be the same both day and night. Supernovae remnants are a well investigated source of the radiation. Although, particles with energy of nearly 10^{20} eV has been detected and the origin of these particles cannot be explained by supernovae. Hence, it was concluded that particles must be accelerated somewhere else in the Universe. What makes life on Earth possible despite the large amount of the cosmic radiation is the magnetic field caused by our planet [1–3].

There are theories that charged particles are accelerated by magnetic fields. In 1949 the physicist Enrico Fermi published a seminal paper where he presented a theory of a phenomenon called magnetic mirrors. The theory proposes that charged particles can gain energy from a system of two converging magnetic fields moving towards each other. Theories on other types of magnetic fields, such as helical fields, have also been proposed but none of the fields have been observed [4, 5].

Previous studies have shown that an assumed simple symmetric magnetic field can make charged particles move in chaotic paths [4]. This study aims to examine the trajectory of a charged particle in a system of a magnetic dipole field in superposition with an external uniform field. If the trajectory shows to be chaotic, it may be possible for a charged particle to get accelerated in such a system of magnetic fields, in the case the magnetic fields are time dependent [4]. In this paper the magnetic fields are constant.

The path of the particles in this study was modelled by a code called Pyoden written in Python and was plotted with Matplotlib (see Appendix A). The parametres varied were the strength of the external uniform field and the strength of the dipole field.

2 Theory

2.1 External Uniform Magnetic Field

A uniform magnetic field can be described by

$$\boldsymbol{B}_{\text{ext}} = \hat{\boldsymbol{y}} \cdot B_0 \sin\theta + \hat{\boldsymbol{z}} \cdot B_0 \cos\theta \tag{1}$$

where θ is the angle of the field in the yz-plane, B_0 is the strength of the field and $\hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{z}}$ are unit vectors. The $\hat{\boldsymbol{x}}$-component of this field is zero.

In a magnetic field, $\boldsymbol{B}$, a moving particle with charge q and velocity $\boldsymbol{v}$ will be acted on by a force

$$\boldsymbol{F} = q \cdot (\boldsymbol{v} \times \boldsymbol{B}) \tag{2}$$

called the Lorentz force (see Fig. 1 and 2). Note that the force is independent of the mass of the particle and that $\boldsymbol{F}$ is always perpendicular to $\boldsymbol{v}$ and $\boldsymbol{B}$ [6]. This force makes the particle move in a circle, with the force always pointing at the centre of it.

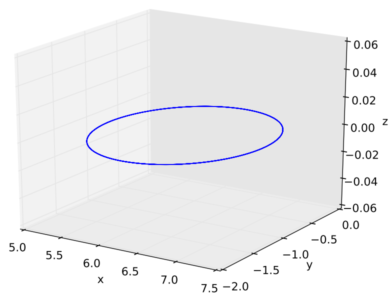


Figure 1: Uniform magnetic field in the z-plane. Initial velocity in x direction. Lorentz force acting in xy-plane making the particle move in a circle. Units are arbitrary.

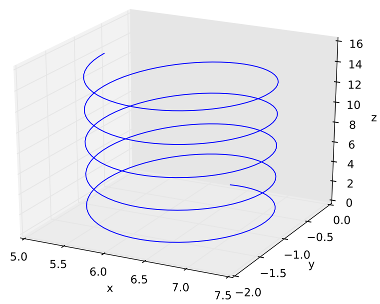


Figure 2: Uniform magnetic field in the z-plane. Initial velocity in x and z direction. Lorentz force acting in xy-plane making the particle move in a circle. Velocity in z direction remains constant and makes the particle move in a helical path. Units are arbitrary.

2.2 Magnetic Dipole Field

A magnetic dipole field can be described with a current loop. *Magnetic moment* is here defined as $\boldsymbol{\mu} = I \cdot \boldsymbol{S}$, where I is the current and $\boldsymbol{S}$ is the area of the supposed current loop.

Astronomical bodies with a magnetic moment $\boldsymbol{\mu}$ creates an external magnetic dipole field according to

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\left(\frac{3\boldsymbol{r}(\boldsymbol{\mu}\cdot\boldsymbol{r})}{r^5} - \frac{\boldsymbol{\mu}}{r^3}\right) \tag{3}$$

where $\boldsymbol{r}$ is the position $\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$ from the centre of the dipole [7]. The constant μ_0 is called the magnetic permeability and has the value $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{T\,m\,A^{-1}}$ [8]. Note that μ_0 is not the same as $\boldsymbol{\mu}$. See Fig. 3 for a visualisation of the dipole field.

The magnetic moment is a vector $\boldsymbol{\mu} = \mu \cdot \hat{\boldsymbol{z}}$ in the coordinate system xyz. From this follows that $\boldsymbol{\mu}\cdot\boldsymbol{r} = \mu z$ since the x and y components in $\boldsymbol{\mu}\cdot\boldsymbol{r}$ are perpendicular to z and therefore are zero. Eq. 3 can now be divided into the three components

$$\begin{cases} B_x = \dfrac{3x\mu z}{r^5} \\[2ex] B_y = \dfrac{3y\mu z}{r^5} \\[2ex] B_z = \dfrac{3z^2\mu}{r^5} - \dfrac{\mu}{r^3}. \end{cases} \tag{4}$$

The constant $\frac{\mu_0}{4\pi}$ is not important since the units are arbitrary in the simulation.

2.3 Superposition of Magnetic Fields

The total magnetic field is a superposition of the external field and the dipole field. Moreover, adding these two fields

$$\boldsymbol{B}_{\text{tot}} = \boldsymbol{B} + \boldsymbol{B}_{\text{ext}} \tag{5}$$

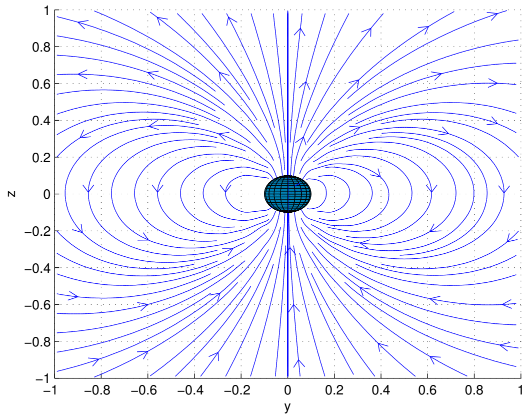


Figure 3: Magnetic dipole field in the yz-plane of a xyz coordinate system. In the centre is an atronomical body. Note that the lines only represent the direction of the magnetic field in each point, not the strength of the field. Units are arbitrary.

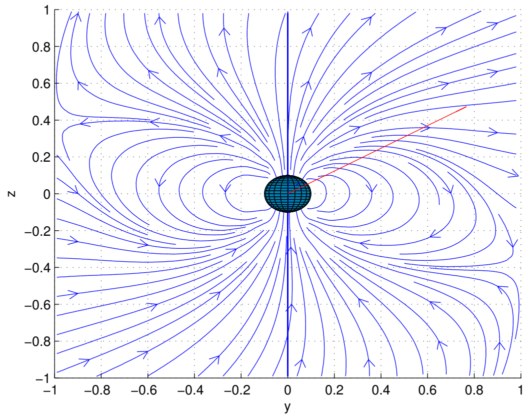


Figure 4: Magnetic dipole field in superposition with a uniform external magnetic field along the direction of the red line. Note that the lines only represent the direction of the magnetic field in each point, not the strength of the field. Units are arbitrary.

gives a system where the dipole field in the x-plane remains constant and is disordered in the yz-plane (see Fig. 4).

3 Simulation

Newton's second law of motion is

$$\boldsymbol{F} = m \cdot \boldsymbol{a} = m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = m\frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}t^2} \tag{6}$$

and combined with Eq. 2

$$q \cdot (\boldsymbol{v} \times \boldsymbol{B}) = m \frac{\mathrm{d}^2 \boldsymbol{x}}{\mathrm{d}t^2} \tag{7}$$

is obtained, which is a second order differential equation. If $\boldsymbol{B}$ was constant the acceleration $\boldsymbol{a}$ would vary upon the velocity $\boldsymbol{v}$ which varies upon the position $\boldsymbol{x}$. Now $\boldsymbol{B}$ is not constant but a superposition of two magnetic fields, according to Eq. 5, that in turn also vary upon the position $\boldsymbol{x}$. This makes the equation more complex to solve.

Using the fourth order Runge-Kutta method Eq. 7 was divided into the first order equations

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{v} \tag{8}$$

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{q}{m}(\boldsymbol{v} \times \boldsymbol{B}) \tag{9}$$

since the code only solves first order differential equations. By solving these equations the position of the particle was determined.

In the simulation one particle was modelled and origo was defined in the centre of the dipole field. Two parameters were varied, while the rest remained constant. All parameters were arbitrarily chosen to observe qualitative changes, not quantitative. Therefore the following values were selected.

Initially, the strength of the external magnetic field, B_0, was varied between 0.5000 and 0.5249. In this simulation the angle, θ, was set to $\pi/4$, the ratio charge by mass to 1, the step time, $\mathrm{d}t$, to 0.1 and the final time, t_{max}, to 2000. Initial velocity was zero in y direction and 0.1 in both x and z direction. Initial position was set to zero in the y and z direction and 2π in the x direction.

Thereafter, the strength of the dipole field, $\boldsymbol{\mu}$, was varied. Apart from $t_{\mathrm{max}} = 700$ and $B_0 = 0.524$ initial conditions were the same as in the first simulation.

4 Results

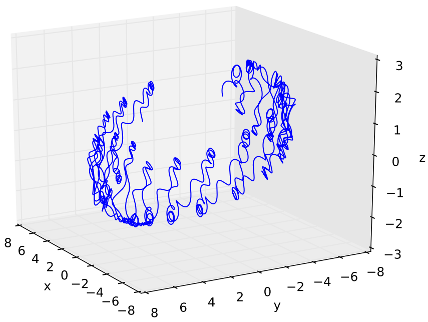
The results is presented in Fig. 5 and Fig. 6. Each subfigure depicts the trajectory of one particle in the magnetic field described in section 2 and 3. It spirals back and forth and in some conditions the particle leaves the dipole field and is only affected by the external uniform field. Fig. 5 shows the results of the first simulation, where the strength of the external uniform field was varied, while Fig. 6 shows the results of the second simulation, where the strength of the dipole field was varied.

5 Discussion

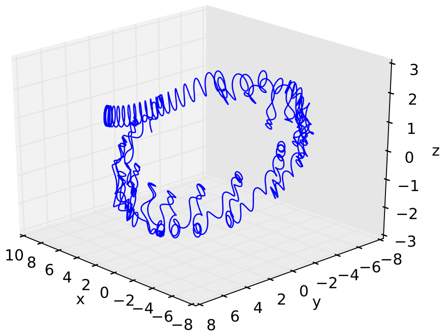
In Fig. 5 the path of the particle is mostly symmetric. In Fig. 5e the particle reaches the escape velocity at a point and leaves the dipole field. There is also a tendency to this in Fig. 5b. Moreover it can be seen that small changes in the initial conditions cause rather big changes in the trajectory. See Fig. 5c and Fig. 5d, where although the difference in the strength of the magnetic field is 0.001, the figures vary more than Fig. 5a and Fig. 5b where the difference in the strength of the magnetic field is 0.01. This indicates that there might be a dynamic nature of the system.

The trajectories within the second simulation (Fig. 6) are different from the ones in the first simulation (Fig. 5), partly because the final time was 700 instead of 2000. The strength of the external field, B_0, was the same in the second simulation as in Fig. 5d. Once again it is shown that the path of the particle is sensitive for small changes in initial conditions, in this case the strength of the dipole field. For instance, see Fig. 6c and Fig. 6f, two situations where the particle may seem to reach an escape velocity—at least it is leaving the current symmetry—but does not move in the same direction. As in the previous paragraph this may indicate a system of dynamic nature.

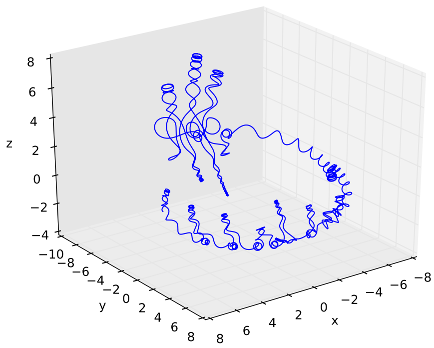
None of the trajectories in Fig. 5 can be seen as chaotic, since the particle moves in a helical path symmetric with the dipole field. Although, there are tendences that the particle may move in unexpected paths. The same goes for the trajectories in Fig. 6.



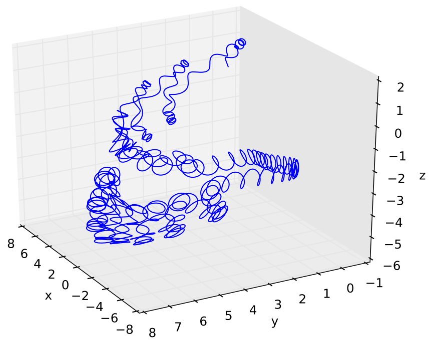
(a) $B_0 = 0.5000$



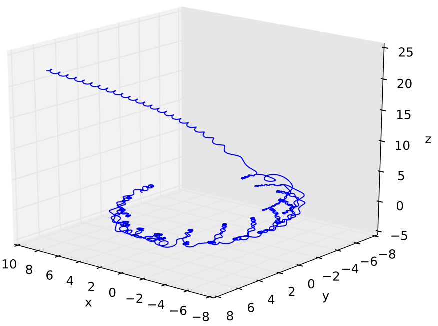
(b) $B_0 = 0.5200$



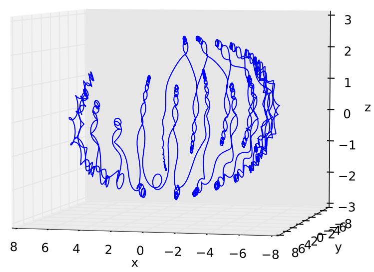
(c) $B_0 = 0.5230$



(d) $B_0 = 0.5240$

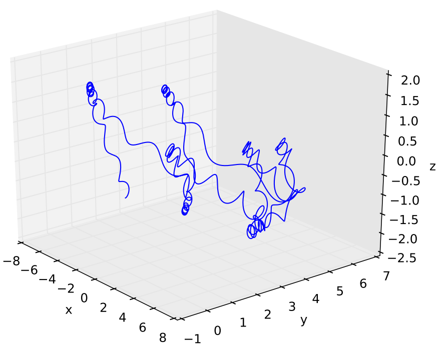


(e) $B_0 = 0.5247$

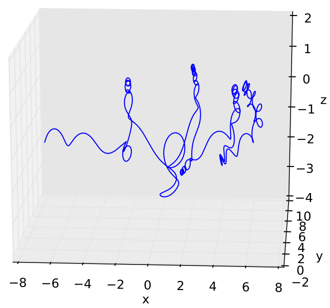


(f) $B_0 = 0.5249$

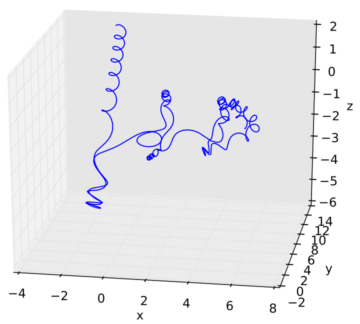
Figure 5: The trajectory of one particle in two magnetic fields in superposition. The only parameter changing is the strength of the uniform field, B_0, according to what is said in the subfigures. See section 3 for details of other parameters.



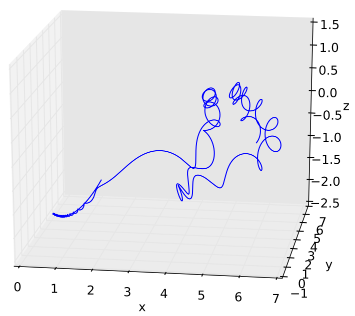
(a) $\boldsymbol{\mu} = 191$



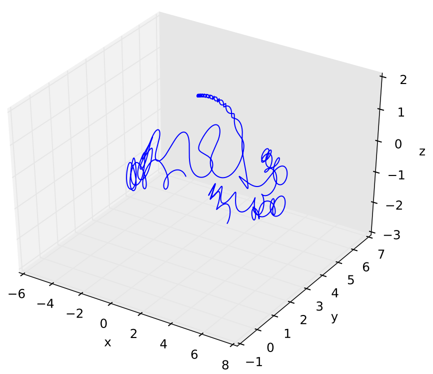
(b) $\boldsymbol{\mu} = 190$



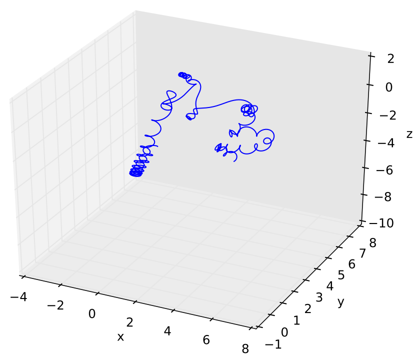
(c) $\boldsymbol{\mu} = 189$



(d) $\boldsymbol{\mu} = 187$



(e) $\boldsymbol{\mu} = 186$



(f) $\boldsymbol{\mu} = 185.8$

Figure 6: The trajectory of one particle in two magnetic fields in superposition. The only parameter changing is the strength of the dipole field, $\boldsymbol{\mu}$, according to what is said in the subfigures. See section 3 for details of the other parameters.

When computing there might be mistakes in the integrating due to the time steps being too large. This could be the case in both simulations that would need to be investigated further on.

In this simulation only two magnetic fields were considered. Although, in reality, if there are a bundle of charged particles with velocities they will create magnetic fields around them. This needs to be taken into consideration when computing the trajectory of the charged particle, even though this effect may be insignificant.

Since all magnetic fields are dipole fields, a uniform field only occurs when looking at it at a sufficient distance. Therefore, it might also be interesting to look at two dipole fields in superposition at a distance R from each other. Furthermore adding several fields interacting could be interesting, since it mirrors the real world better.

In conclusion, it is not accurate to interpret the paths as chaotic from the results in this paper. More simulations need to be done with other initial conditions varying other variables, such as the angle θ. If the trajectory then appears to be chaotic under some conditions it will be relevant to rotate the field and make it time dependent to investigate if the particle actually can gain energy in a system of magnetic fields like this.

6 Acknowledgements

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References

[1] Hillas, A. M. *The Origin of Ultra-High-Energy Cosmic Rays*, Annual Review of Astronomy and Astrophysics, 1984.

[2] Erlykin, A. D., Wolfendale, A. W. *Cosmic rays: the centenary of their discovery*, Europhysics News, **43**(2), 2012, P. 26-28.

[3] Bhattacharjee, P., Sigl, G. *Origin and Propagation of Extremely High-Energy Cosmic Rays*, Physics Reports, **327**, 2000.

[4] Mitra, D., Brandenburg, A., Dasgrupta, B., Niklasson, E., Ram, A. *Particle energization through time-periodic helical magnetic fields*, Physical Review E, **89**, 2014.

[5] Fermi, E. *On the Origin of the Cosmic Radiation*, Physical Review, **75**(8), 1949.

[6] Young, H. D., Freedman, R. A., Ford, A. L. *University Physics*, 12th ed., Pearson Addison-Wesley, San Fransisco, 2008.

[7] Griffiths, D. J., *Introduction to Electrodynamics*, 4th ed., Pearson Education, Harlow, 2014.

[8] Knight, R., Jones, B., Field, S., *College Physics – A Strategic Approach*, Pearson Addison-Wesley, San Fransisco, 2007.

A Python codes

A.1 Particle in Magnetic Field

```
from pylab import *
import numpy as np
import noise as noise
from mpl_toolkits.mplot3d import Axes3D
# ---------------------
Bfield='dipole';
Nparticle=1;
m=200;
kk=32.
qbym = 1
dim=3;
l_lyapunov=0;
mag_BB = 0.524;
mag_BBc = 0.
pi=3.141592653589793238462643383279502884197169399
angle=0;
if l_lyapunov==1:
  ddim = 4*dim
  snumber=0.1
else:
  ddim = 2*dim
pstride=ddim;
N=Nparticle*(ddim);
f=np.zeros(N);
y0=np.zeros(N);
# ---------------------
def model(y,t,dt,istep):
# y[0:2] is position, y[3:5] is momentum
    for ip in range(0,Nparticle):
        x=y[ip*pstride+0*dim:ip*pstride+1*dim];
        v=y[ip*pstride+1*dim:ip*pstride+2*dim];
        if l_lyapunov==1:
            dx = y[ip*pstride+2*dim:ip*pstride+3*dim]
            dv = y[ip*pstride+3*dim:ip*pstride+4*dim]
            dBBdx = magfield(x,1)
        BB = magfield(x,0);
        f[ip*pstride+0*dim:ip*pstride+1*dim]=v;
        f[ip*pstride+1*dim:ip*pstride+2*dim] = qbym * np.cross(v,BB);
```

```
        if l_lyapunov==1:
          f[ip*pstride+2*dim:ip*pstride+3*dim] = dv;
          f[ip*pstride+3*dim:ip*pstride+4*dim] = qbym * ( np.cross(dv,BB)+np.cross(v-dv,np.dot(dBBdx,dx)) )
    #print f
    return f;
# ---------------------
def dimension():
    return N;
# ---------------------
def modsqr(A):
    size=len(A);
    xx=0;
    for ix in range(0,size):
      xx=xx+A[ix]*A[ix];
    return xx;
# ---------------------
def magfield(x,der):
# Create array for the magnetic field
    if Bfield == 'default':
        print 'no default magnetic field set';
        technical.end_program();
    elif Bfield == 'dipole':
        magfield,dmagfield=dipole(x)
    else:
        print 'No such magnetic field';
        technical.end_program();
    if der == 1:
        return dmagfield
    else :
        return magfield
# ---------------------
def dipole(x):
    BBx=np.zeros(dim)
    BB=np.zeros(dim)
    dBB=np.zeros([dim,dim])
    BBx[1]=mag_BB*sin(angle)
    BBx[2]=mag_BB*cos(angle)
    r=np.sqrt(modsqr(x))
    BB[0]=(3 * x[0] * m * x[2]) / (r**5)
    BB[1]=(3 * x[1] * m * x[2]) / (r**5)
    BB[2]=((3 * (x[2]**2) * m) / (r**5)) - (m / (r**3))
    BB=BB + BBx
```

```
    return BB, dBB
# ---------------------
def iniconf():
    for ip in range(0,Nparticle):
        y0[ip*pstride+0]=2*3.141592; #*np.random.ranf();
        y0[ip*pstride+1]=0;
        y0[ip*pstride+2]=0;
        y0[ip*pstride+3]=0.1; #np.random.ranf()
        y0[ip*pstride+4]=0;
        y0[ip*pstride+5]=0.1;
        if l_lyapunov==1:
            y0[ip*pstride+6]=snumber*(1./kk)*noise.uniform(-1,1);
    return y0;
# ---------------------
def diagnostic(y,t,counter,fname,ldiag2file):
    energy=0;
    rsqr=0
    mFTLE=0
    for ip in range(0,Nparticle):
        x=y[ip*pstride+0:ip*pstride+dim];
        x0=y0[ip*pstride+0:ip*pstride+dim];
        rsqr=rsqr+modsqr(x-x0);
        v=y[ip*pstride+dim:ip*pstride+(2*dim)];
        energy=energy+modsqr(v);
        if l_lyapunov == 1 :
          dx=y[ip*pstride+(2*dim):ip*pstride+(3*dim)];
          dxsqrt= np.sqrt(modsqr(dx));
          dx0=y0[ip*pstride+(2*dim):ip*pstride+(3*dim)];
          dx0sqrt= np.sqrt(modsqr(dx0));
          if t == 0:
              FTLE=0
          else:
              FTLE=(1./t)*np.log(dxsqrt/dx0sqrt)
          mFTLE=mFTLE+FTLE
    #print t,energy/Nparticle,rsqr/Nparticle,mFTLE/Nparticle;
    if ldiag2file == 1:
      fname.write(str(t)+"\t"+str(x[0])+"\t"+str(x[1])+"\t"+str(x[2])+"\t"+str(rsqr)+"\t"+str(mFTLE)+"\n")
      vel = open("vel.txt", "a")
      vel.write(str(v[0])+" "+str(v[1]) + " " + str(v[2]) + "\n")
      vel.close()
      print "current tstep: " + str(t)
    else:
```

```
        print t, x[0],x[1],x[2], rsqr,mFTLE
# ---------------------
```

A.2 Other Files

For the access to the rest of the code, see `http://code.google.com/p/pyoden/source/browse/#svn%2Ftrunk`. It might be particularly useful to look at odeN.py, where the step time and the final time is changed.