Taming Chaos

GEM2505M

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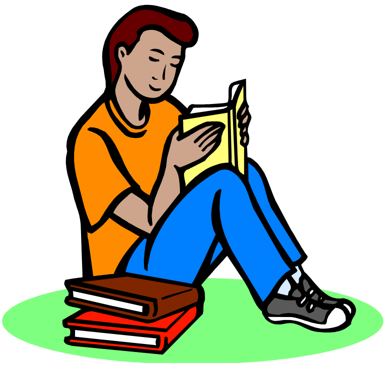
The Logistic Map

Lecture 7

Today’s Lecture

- Population Dynamics
- The Logistic Map
- Cobwebs
- Bifurcation Diagram



Population Dynamics

Consider a population ***N*** from year to year. This can be described by:

$$N_{i+1} = p\, N_i$$

(the discrete version of Malthus' Law)

Next year's population

This year's Population

?

What do you think this means?

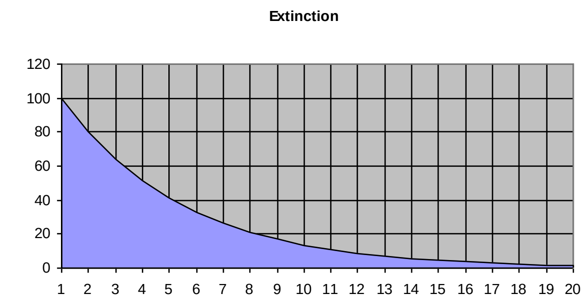
1. The population grows until it reaches equilibrium
2. Population Explosion
3. Population Extinction
4. It depends on ***p***

Population Dynamics

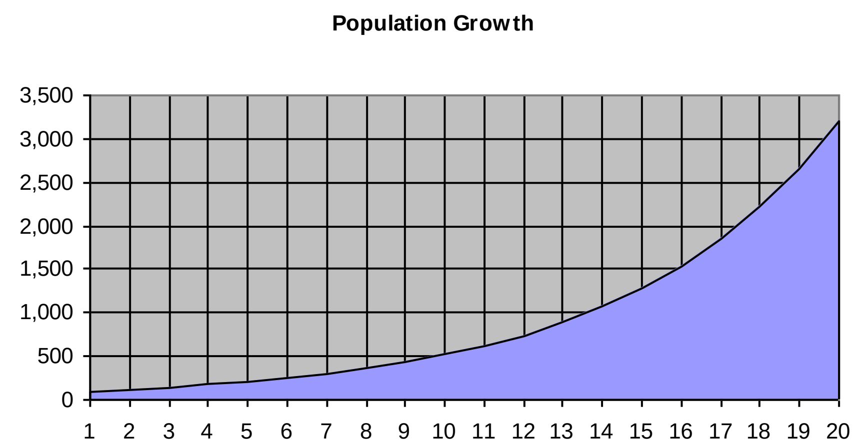
Answer: It depends on ***p***.

$$N_{i+1} = p\, N_i$$

p = 1 nothing happens
p > 1 population explosion
p < 1 extinction



If p is positive, this means that the population will grow ever bigger.



Population Dynamics

Of course an ever growing population is not realistic. Hence Verhulst added the term $-bN^2$ to Malthus' Law. We can do that here too and obtain:

$$\boxed{N_{i+1} = p\,N_i - bN_i^{\,2}}$$

This is called the logistic map (though usually it's written in a different way).

With some math it can be expressed as:

$$\boxed{x_{n+1} = 1 - \alpha x_n^2}$$

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Population Dynamics

Not in exam

One may feel a bit uncomfortable with the discrete approach and consider a more continuous description.

$$\frac{\Delta N}{\Delta T} = (b - d)N$$

Where:

Δ This Greek D called delta means "change in"

N Size of the population

T Time

d Death rate

b Birth rate

$$r = (b - d)$$

Population Dynamics

Not in exam

Growth

Regardless of the exact value of r, if it is larger than 0, exponential growth will inevitable lead to the exhaustion of all available resources not matter how small the organism!

As a differential equation we have:

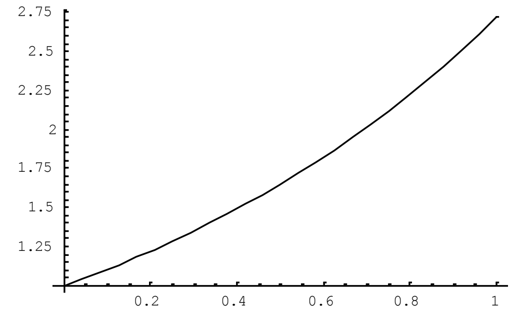
$$\frac{dn(t)}{dt} = (b - d)n(t)$$

Population Dynamics

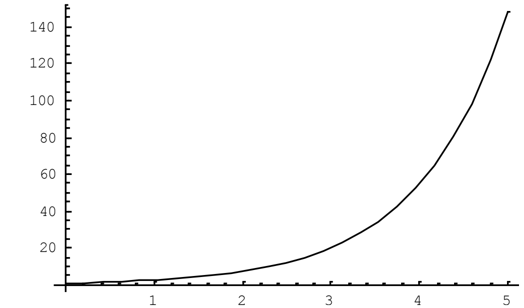
Not in exam

Exponential Growth

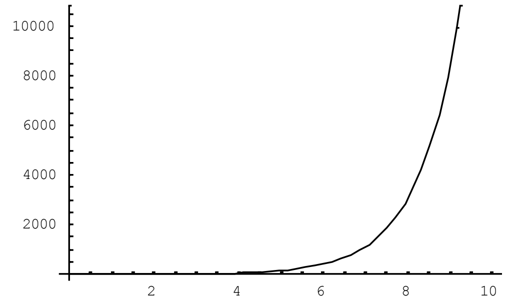
```
DSolve[{n'[t] == r n[t], n[0] == 1}, n[t], t]
Plot[ (n[t] /. %)[[1]] /. {r -> 1}, {t, 0, 1}]
{{n[t] → e^(r t)}}
```




```
DSolve[{n'[t] == r n[t], n[0] == 1}, n[t], t]
Plot[ (n[t] /. %)[[1]] /. {r -> 1}, {t, 0, 5}]
{{n[t] → e^(r t)}}
```




```
DSolve[{n'[t] == r n[t], n[0] == 1}, n[t], t]
Plot[ (n[t] /. %)[[1]] /. {r -> 1}, {t, 0, 10}]
{{n[t] → e^(r t)}}
```



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Population Dynamics

Not in exam

Limiting growth

In the most simple case, growth will stop accelerating, start decelerating and approach the carrying capacity of the environment.

When it has reached the carrying capacity, the growth and death rates should roughly be equal.

This can be incorporated into the equation by adding the term

$$\frac{k-N}{k}$$

k = carrying capacity

on the right hand side.

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Population Dynamics

Not in exam

Limiting growth

Thus we obtain:

$$\frac{\Delta N}{\Delta T} = (b-d)\frac{k-N}{k}N$$

Or as a differential equation:

$$\frac{dn(t)}{dt} = (b-d)\frac{k-n(t)}{k}n(t)$$

Population Dynamics

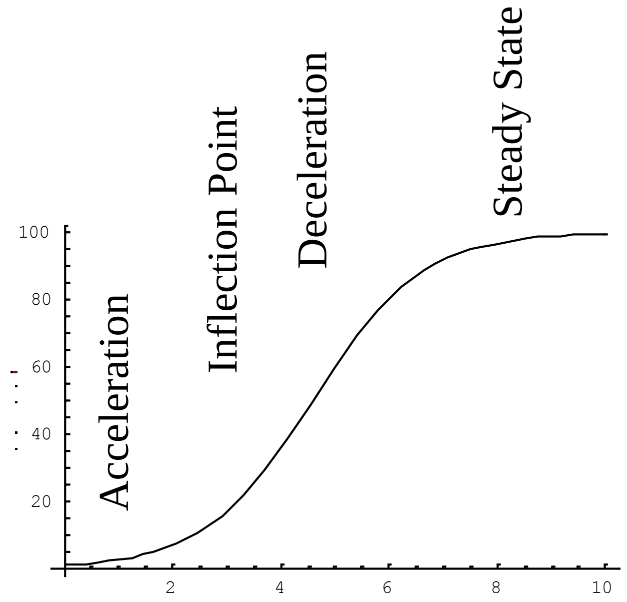
Not in exam

Limiting growth

The result is an s-shape like growth curve.

```
DSolve[{n'[t] == ((k - n[t]) / k) r n[t], n[0] == 1}, n[t], t]
Plot[ (n[t] /. %)[[1]] /. {r -> 1, k→100}, {t, 0, 10}]
```

$$\left\{\left\{n[t] \to \frac{e^{r t} k}{-1 + e^{r t} + k}\right\}\right\}$$



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Population Dynamics

Not in exam

Back to the discrete case

We'll now show that the logistic map is just a discrete version of: $\frac{dn(t)}{dt} = (b-d)\frac{k-n(t)}{k}n(t)$

$$N_{t+1} - N_t = (b-d)\frac{k-N_t}{k}N_t$$

$$N_{t+1} = (b-d)N_t - \frac{(b-d)}{k}N_t^2 + N_t$$

This can easily be transformed by considering the right hand side as a square.

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Population Dynamics

Not in exam

Towards the logistic map

First, lets write $N_{t+1} = (b-d)N_t - \frac{(b-d)}{k}N_t^2 + N_t$ a bit simpler

$$N_{t+1} = aN_t - cN_t^2$$

Then, it is clear that the right hand side can be written as a square:

$$N_{t+1} = \frac{a^2}{4c} - \Bigg(\underbrace{\sqrt{c}N_t - \frac{a}{2\sqrt{c}}}_{\equiv\, y_t}\Bigg)^2$$

Population Dynamics

Not in exam

Towards the logistic map

$$y_t = \sqrt{c}N_t - \frac{a}{2\sqrt{c}}$$

$$N_t = \frac{y_t}{\sqrt{c}} + \frac{a}{2c}$$

$$N_{t+1} = \frac{y_{t+1}}{\sqrt{c}} + \frac{a}{2c}$$

$$N_{t+1} = \frac{a^2}{4c} - \left(\sqrt{c}N_t - \frac{a}{2\sqrt{c}}\right)^2$$

$$N_{t+1} = \frac{y_{t+1}}{\sqrt{c}} + \frac{a}{2c} = \frac{a^2}{4c} - y_t^2$$

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Population Dynamics

Not in exam

Towards the logistic map

$$N_{t+1} = \frac{y_{t+1}}{\sqrt{c}} + \frac{a}{2c} = \frac{a^2}{4c} - y_t^2$$

$$\frac{y_{t+1}}{\sqrt{c}} = \frac{a^2-2a}{4c} - y_t^2$$

$$\underbrace{y_{t+1}\frac{4\sqrt{c}}{a^2-2a}}_{\equiv x_{t+1}} = 1 - \frac{4c}{a^2-2a}y_t^2$$

Population Dynamics

Not in exam

Towards the logistic map

$$\underbrace{y_{t+1}\frac{4\sqrt{c}}{a^2-2a}}_{\equiv\, x_{t+1}} = 1 - \frac{4c}{a^2-2a}y_t^2$$

← From previous slide.

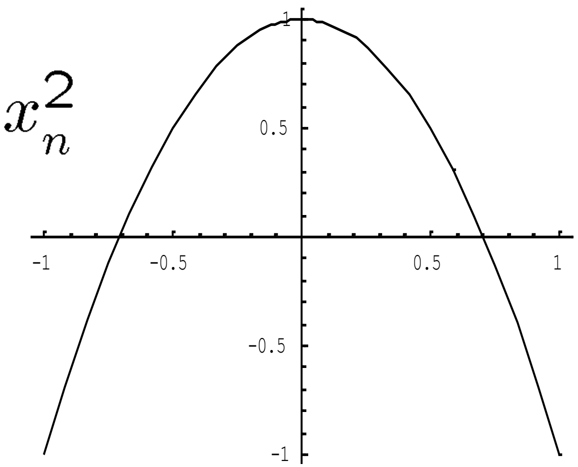
$$x_{t+1} = 1 - \frac{4c}{a^2-2a}\left(\frac{a^2-2a}{4\sqrt{c}}\right)^2 x_t^2$$

$$x_{t+1} = 1 - \underbrace{\frac{a^2-2a}{4}}_{\equiv\, \alpha} x_t^2$$

Population Dynamics

The logistic map $x_{n+1} = 1 - \alpha x_n^2$

If we make a function plot of the logistic map we obtain:



`Plot[1 - 2 x^2, {x, -1, 1}]`

?

What do you think this means?

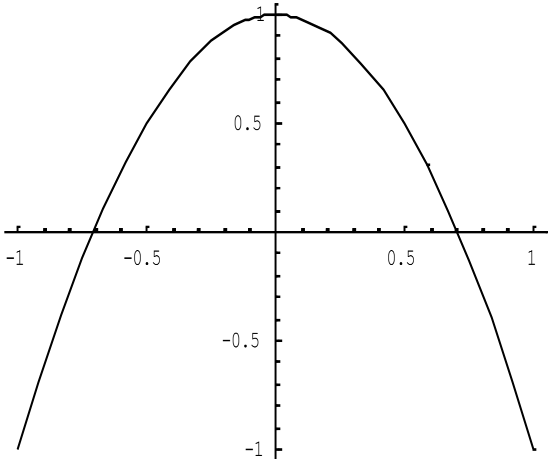
1. Nothing
2. We can read of x_n in this graph
3. We can read of x_{n+1} in this graph
4. It's upside down

Population Dynamics

The logistic map $$x_{n+1} = 1 - \alpha x_n^2$$

Answer:

With the help of such a plot, we can graphically determine the value of x_{n+1}.



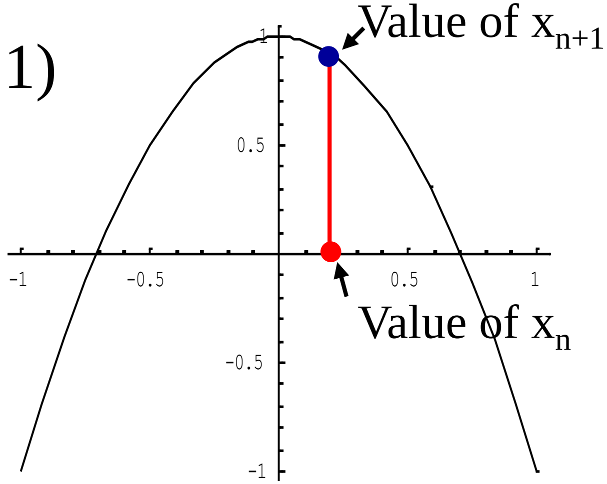
Plot[1 - 2 x^2, {x, -1, 1}]

Cobwebs

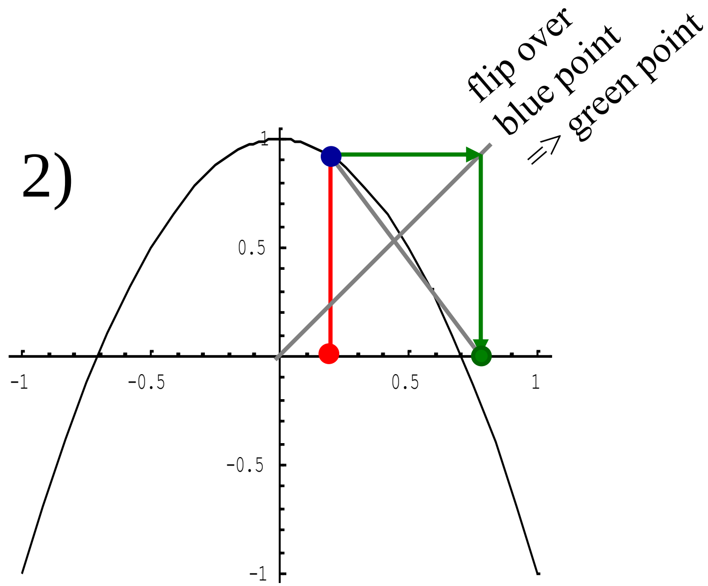
Let’s take a = 2.0. I.e.

$$x_{n+1} = 1 - 2x_n^2$$

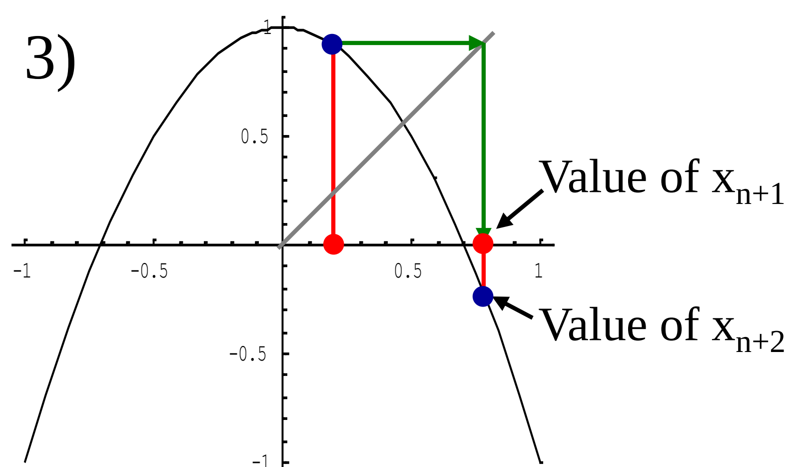
1)



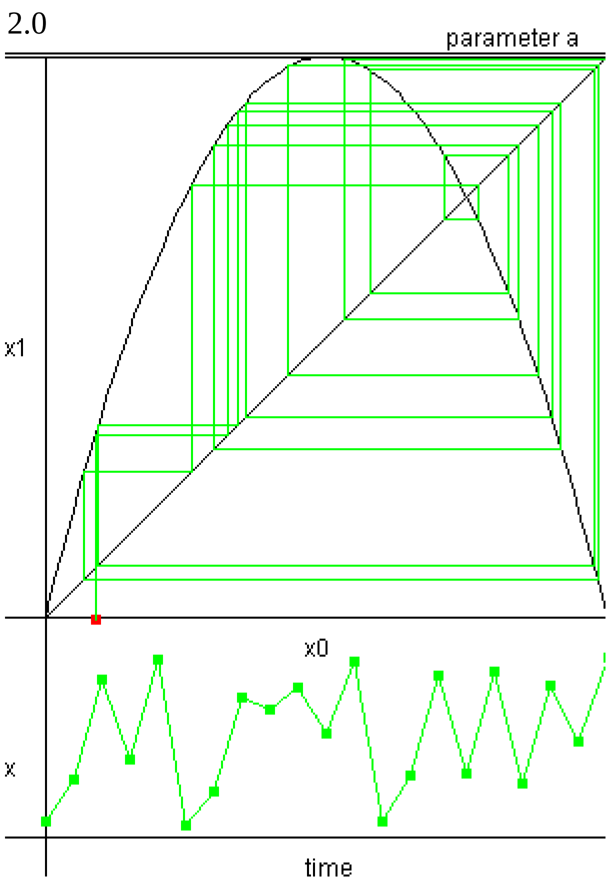
2)

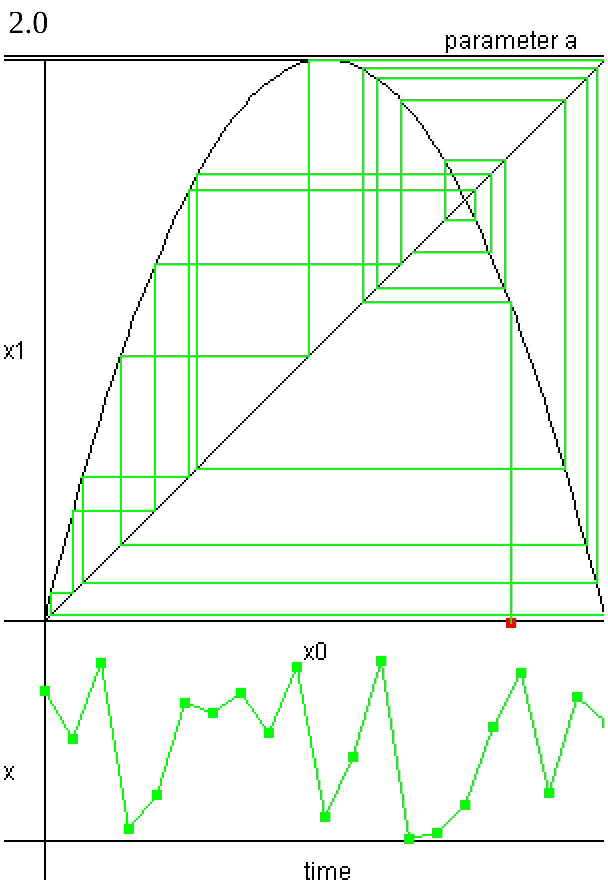


3)

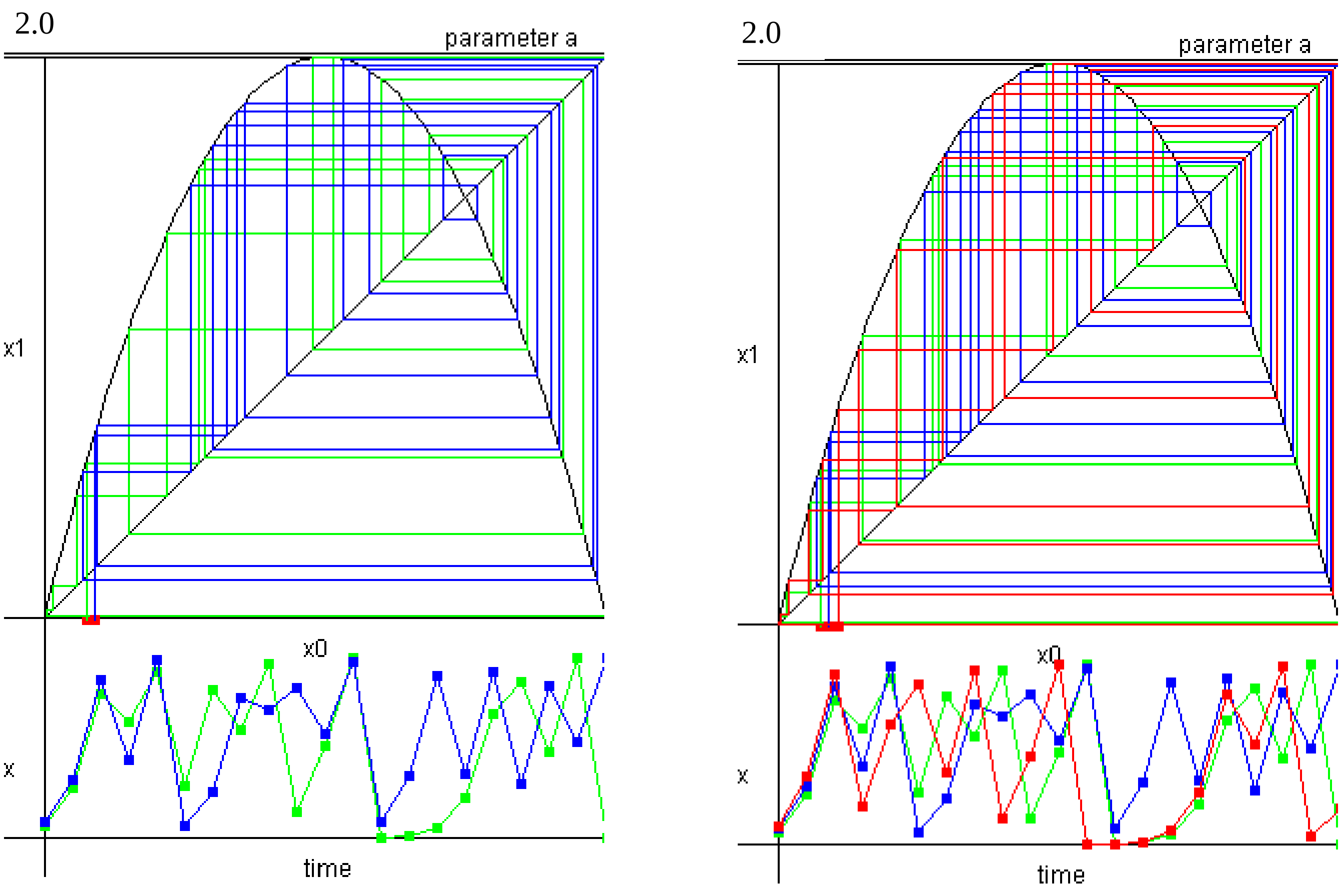


Cobwebs





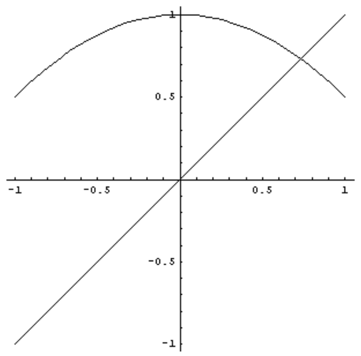
Sensitive Dependence

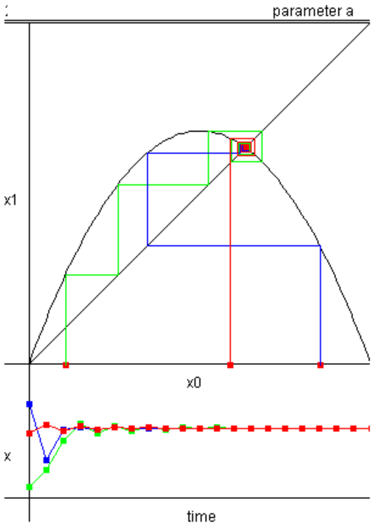


Parameter dependence

$\alpha = 0.5$

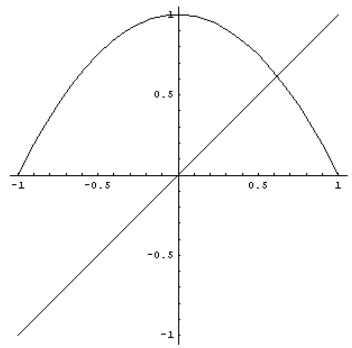
Plot[{1 - 0.5 x^2, x}, {x, -1, 1}, AspectRatio → 1]

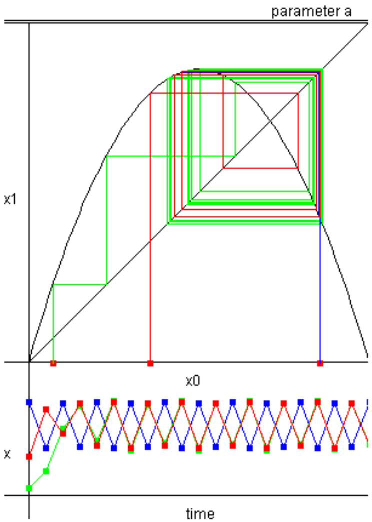




$\alpha = 1.0$

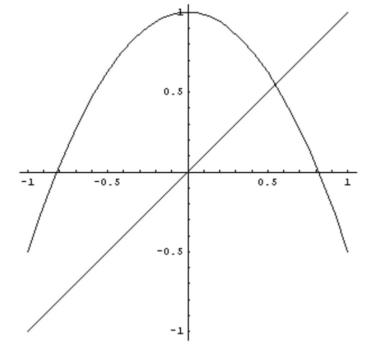
Plot[{1 - 1 x^2, x}, {x, -1, 1}, AspectRatio → 1]

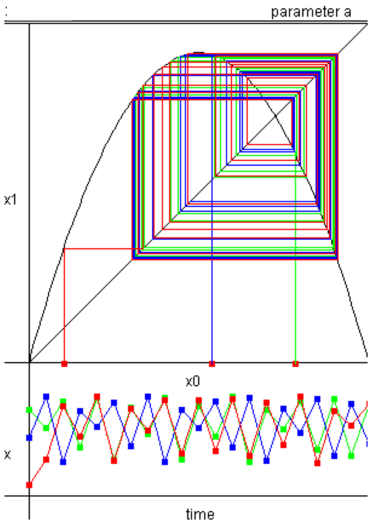




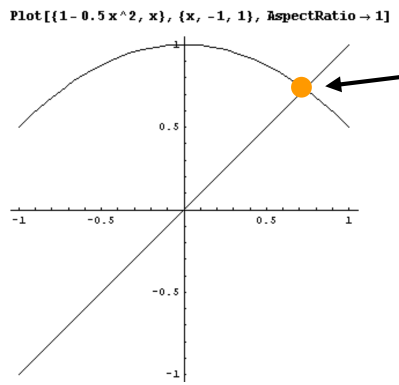
$\alpha = 1.5$

Plot[{1 - 1.5 x^2, x}, {x, -1, 1}, AspectRatio → 1]

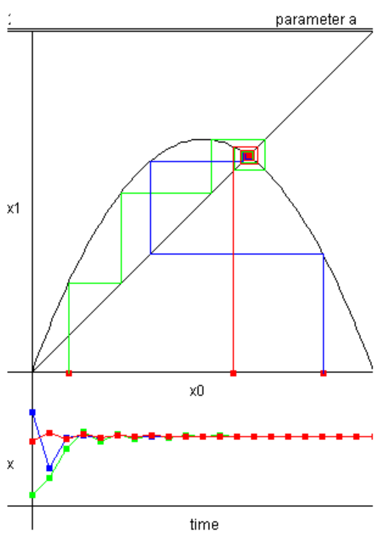




Fixed Points



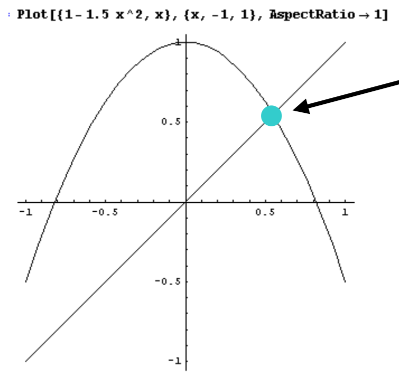
This point will not change when applying the function map.



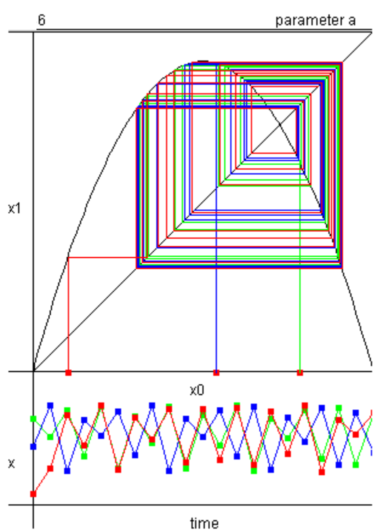
As we can see from the cobweb, wherever we start, we'll eventually end up at this point for this value of α.

This kind of a fixed point is called an ***attracting*** fixed point.

Fixed Points



This point will not change when applying the function map.



As we can see from the cobweb, wherever we start, we'll basically never end up at this point for this value of α.

This kind of a fixed point is called a ***repelling*** fixed point.

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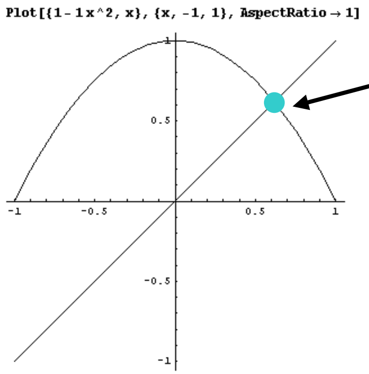
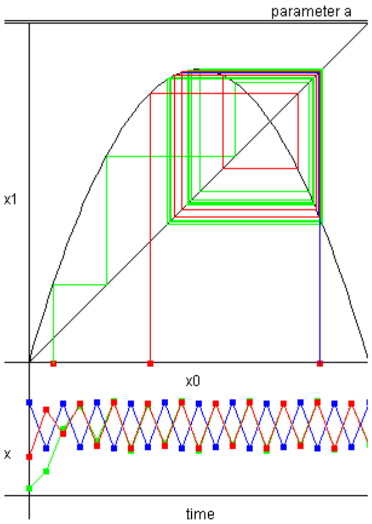
Fixed Points



This point will not change when applying the function map.



Again the fixed point is repelling, but this time we see a regular period 2 orbit.

How does this work?

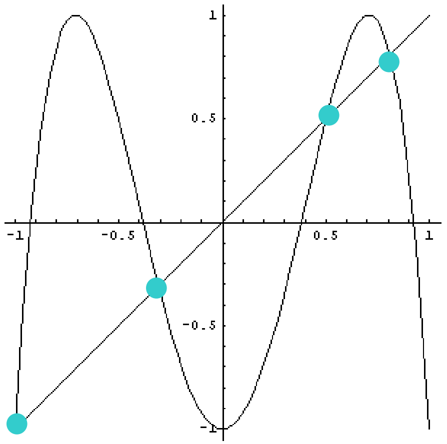
Taming Chaos

Fixed Points

Compositions

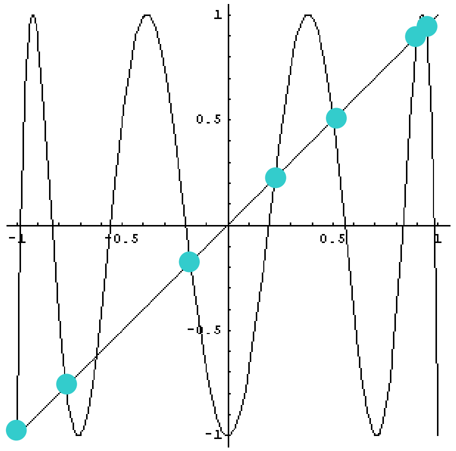
Just like for the original function, we can plot higher compositions.

```
Plot[{1-2 (1-2 x^2)^2, x}, {x, -1, 1}, AspectRatio→1]
```



$$x_{n+2} = 1 - 2(1 - 2x_n^2)^2$$

```
Plot[{1-2 (1-2 (1-2x^2)^2)^2, x}, {x, -1, 1}, AspectRatio→1]
```



$$x_{n+3} = 1 - 2(1 - 2(1 - 2x_n^2)^2)^2$$

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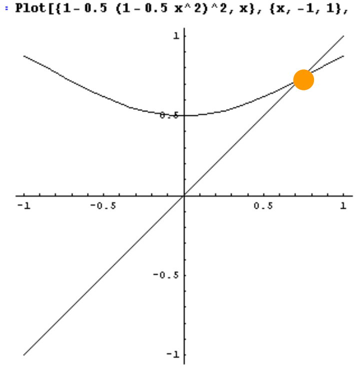
Compositions – Parameter Dep.

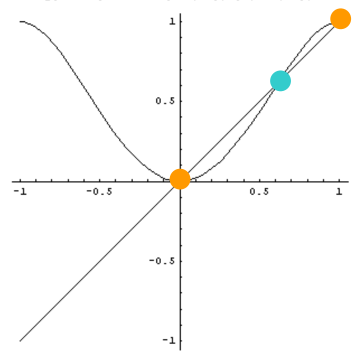
2nd Composition

Plot[{1 - 0.5 (1 - 0.5 x^2)^2, x}, {x, -1, 1},



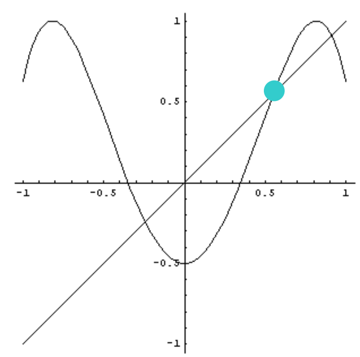
2nd Composition

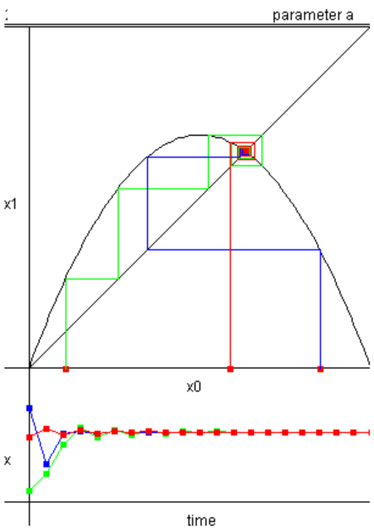
Plot[{1 - 1 (1 - 1 x^2)^2, x}, {x, -1, 1},

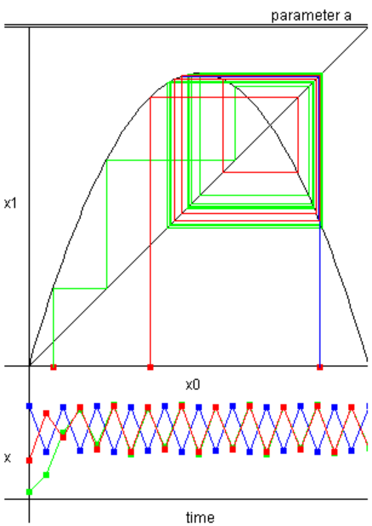


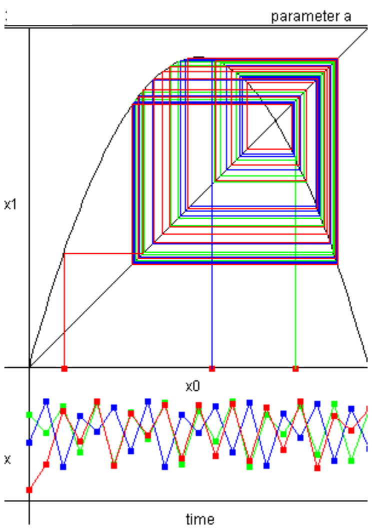
2nd Composition

Plot[{1 - 1.5 (1 - 1.5 x^2)^2, x}, {x, -1, 1},





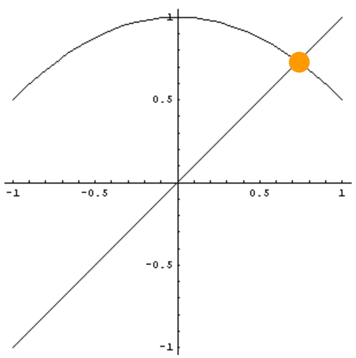




Compositions – Parameter Dep.

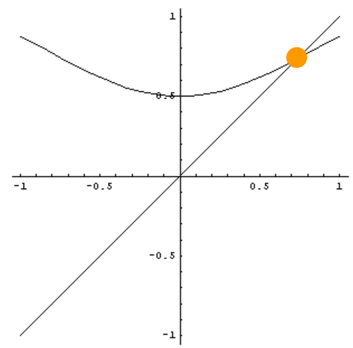
1st Composition

```
Plot[{1-0.5x^2, x}, {x, -1, 1}, AspectRatio→1]
```



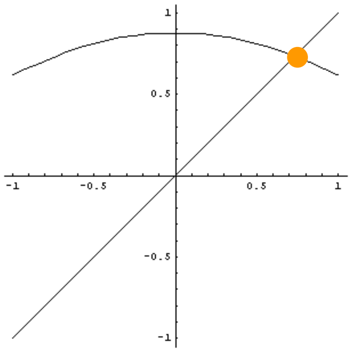
2nd Composition

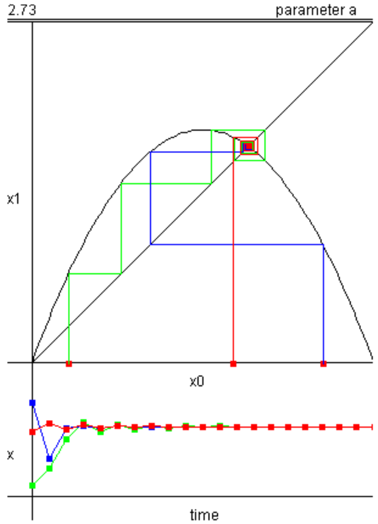
```
Plot[{1-0.5 (1-0.5 x^2)^2, x}, {x, -1, 1},
```



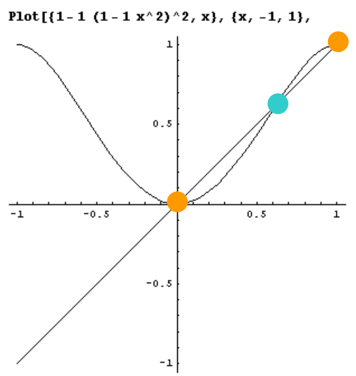
3rd Composition

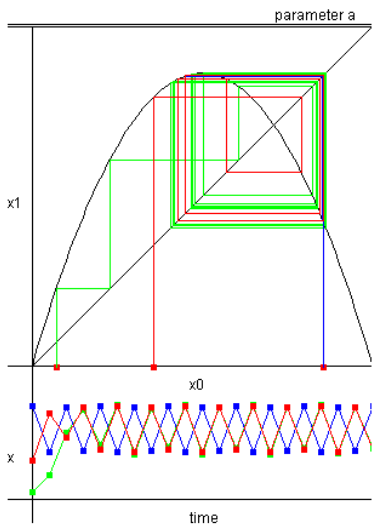
```
Plot[{1-0.5 (1-0.5 (1-0.5x^2)^2)^2, x}, {x,
```





Compositions – Parameter Dep.



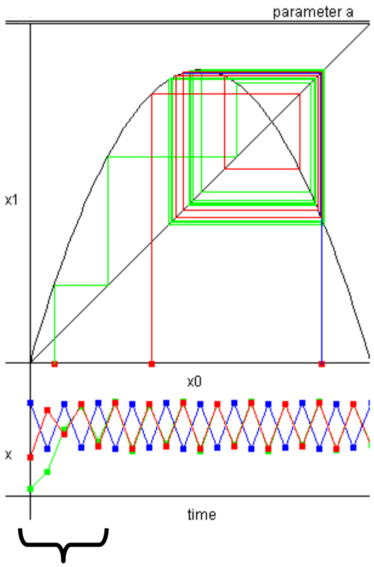


The period 1 fixed point is also a fixed point of the second composition.

But something special happened. It changed from being attracting to being repelling.

Also there are two new fixed points. They are attracting and the time series alternates between them.

Transients



Transient.

As can be seen from the cobweb, it usually takes a few time steps until the sequence settles down to a some kind of a stable pattern..

This time is called the ***transient time*** and the part of the sequence that falls into this time the ***transient***.

Note: exactly where the transient ends is somewhat arbitrary.

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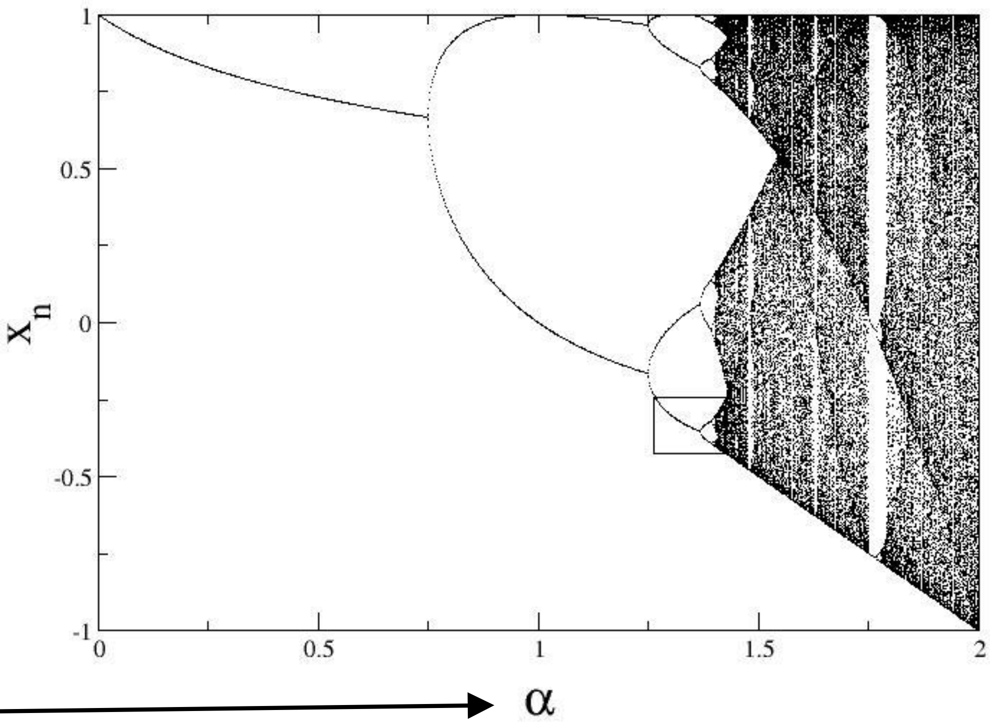
Bifurcation Diagram

In a bifurcation diagram, the possible values of x are plotted versus the parameter.



Plot 200 successive values of x for every value of α

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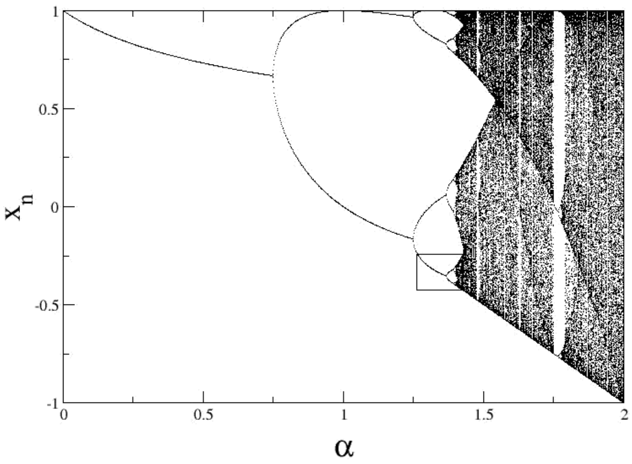
Bifurcation Diagram

At the end of the bifurcation cascade, we have the maximum amount of chaos



?

How many fixed points are there for $\alpha=2$?

1. 16
2. Infinitely many
3. Zero
4. 1

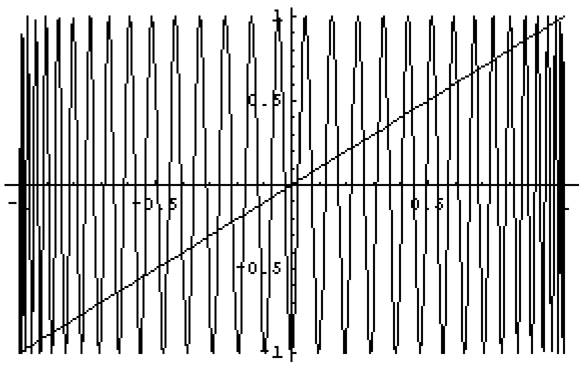
Bifurcation Diagram

Answer:

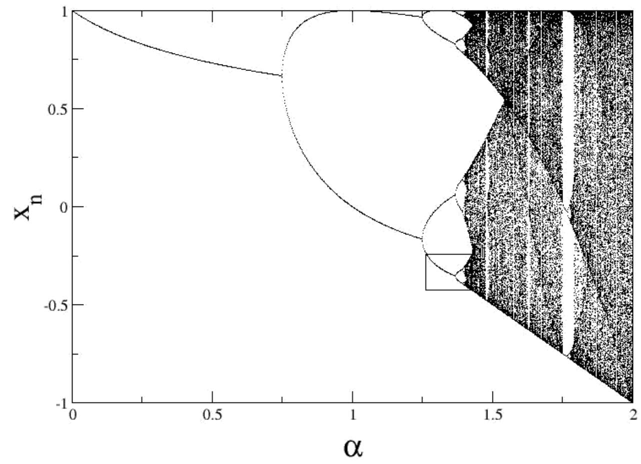
Infinitely many!

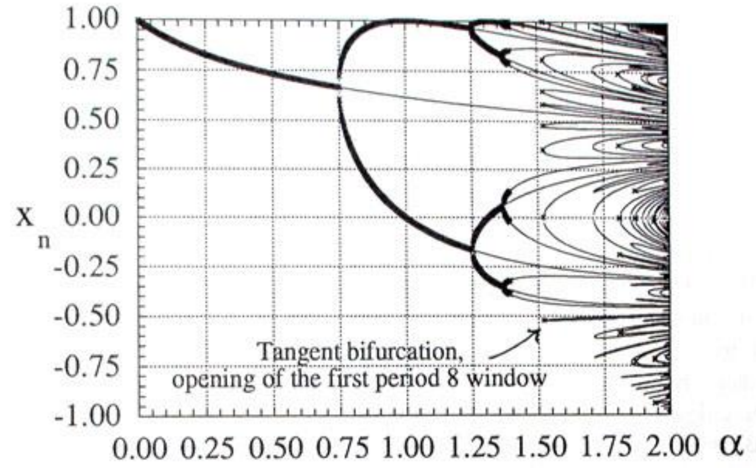
```
f[x_] := 1 - a x x
g[x_] = f[f[f[f[f[f[x]]]]]] /. a → 2
Plot[{ g[y], y} , {y, -1, 1}]
```

$$1-2\left(1-2\left(1-2\left(1-2\left(1-2\left(1-2x^2\right)^2\right)^2\right)^2\right)^2\right)^2$$



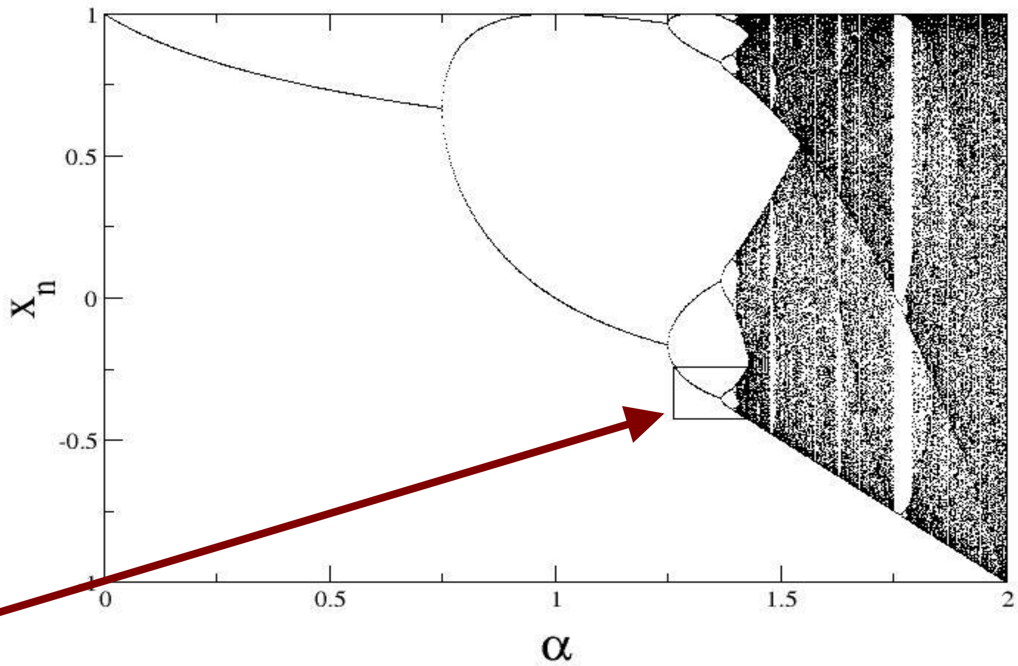
Sixth iterate





Bifurcation Diagram

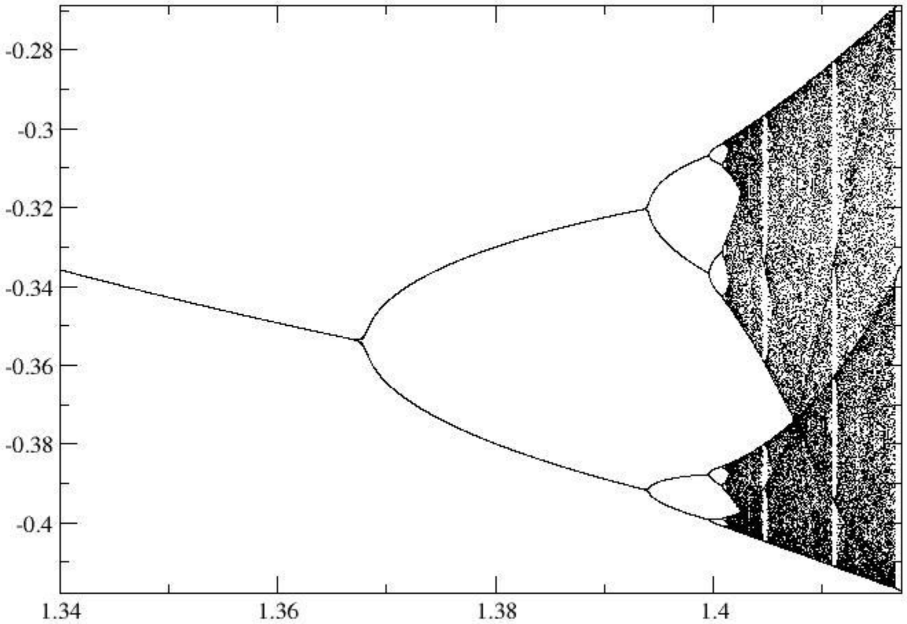
What's so special about this?
Let's have a closer look.



Let's enlarge this area

Bifurcation Diagram

Hey! This looks almost the same!



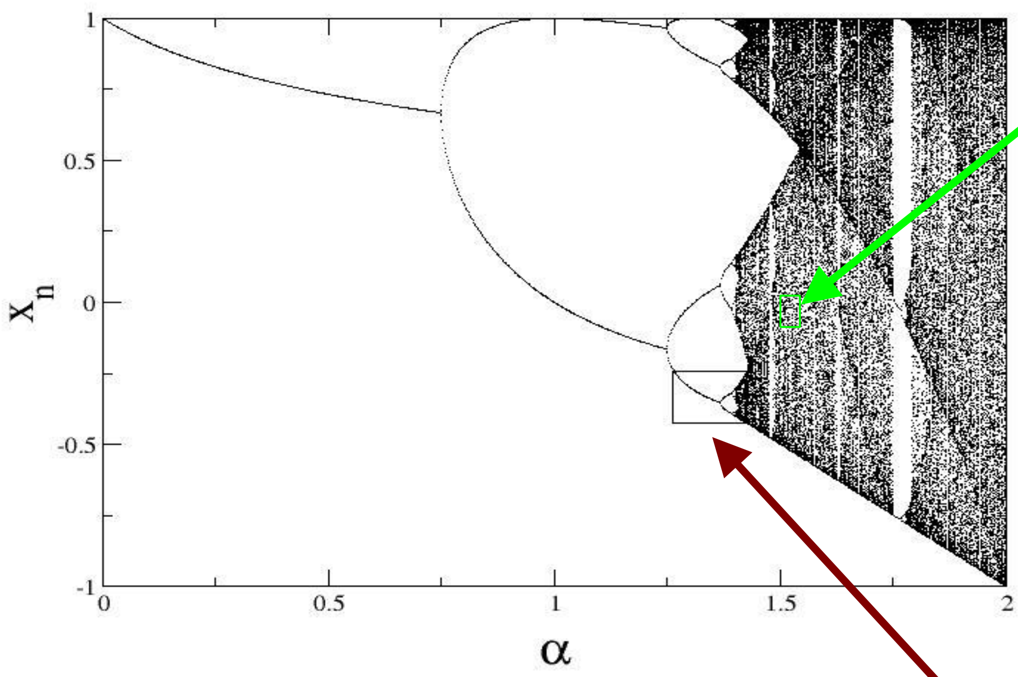
Let's try this somewhere else...

Taming Chaos

Bifurcation Diagram

Let's enlarge a much smaller area!

Now let's enlarge this area

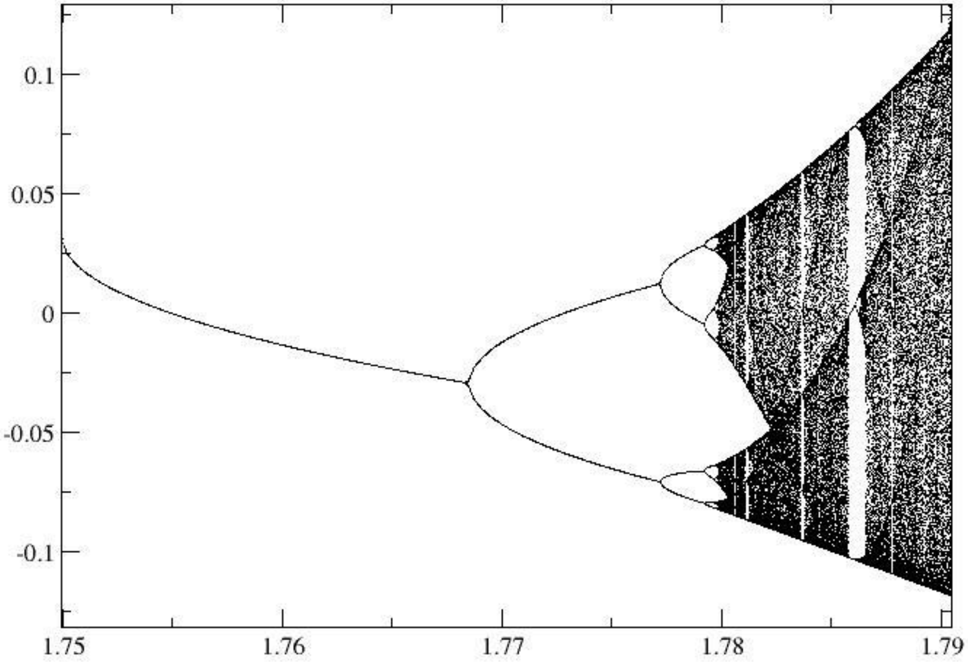


Hard to see, isn't it?

The area we enlarged before.

Bifurcation Diagram

The same again!

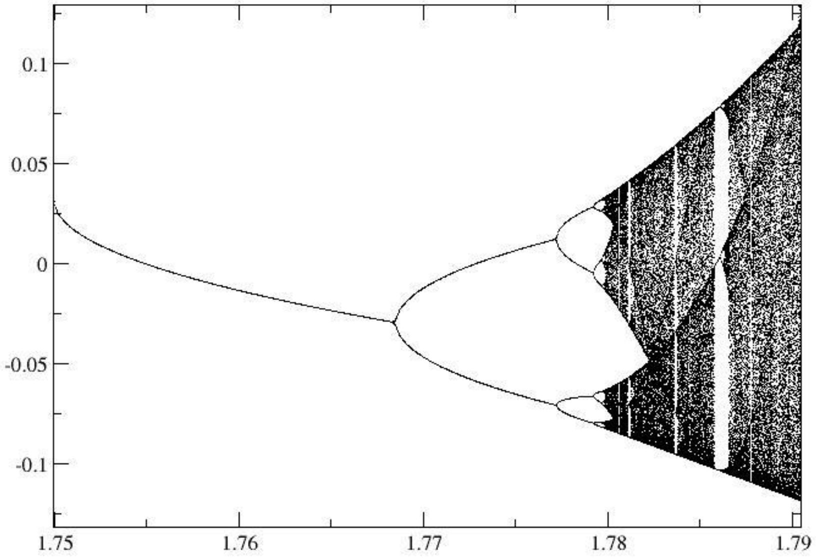


Taming Chaos

Bifurcation Diagram

Indeed, the logistic map repeats itself over and over again at ever smaller scales

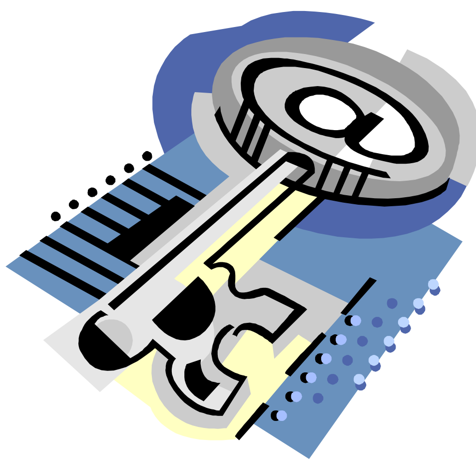
What's more, this behaviour was found to be universal!



Yes, there's a fractal hidden in here.

Key Points of the Day

- Simple Map.
- Amazing Properties!



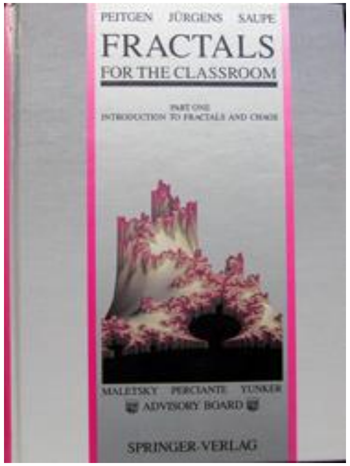
Taming Chaos

Think about it!

- What is the map of nature?

Map,
Directions,
Lost,
Chaos!

References



http://www.cmp.caltech.edu/~mcc/Chaos_Course/Lesson4/Demo1.html

http://www.expm.t.u-tokyo.ac.jp/~kanamaru/Chaos/e/