Nature’s Monte Carlo Bakery: The Story of *Life as a Complex System*

GEK1530
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Kneading & Chaos

Lecture 3

Kneading is an essential step in making bread. It is also essential for understanding chaos. In this lecture we see how stretch and fold are at the heart of chaotic behavior.

The Bakery

Flour Water Yeast

Add Ingredients

Get some units

- ergo building blocks

mix n bake

Get something wonderful!

Process

Knead Wait

Bake

Eat & Live

Today's Lecture

Kneading

Kneading the process of stretching and folding.

Chaos

Chaos is a dynamical attribute where nearby points separate exponentially fast within a given phase space.

The Story

Many phenomena in nature can be explained phenomenologically by chaos theory. A key characteristic of chaos theory is that nearby points always separate exponentially fast leading to an effect often referred to as the butterfly effect.

How can points separate and stay close at the same time?

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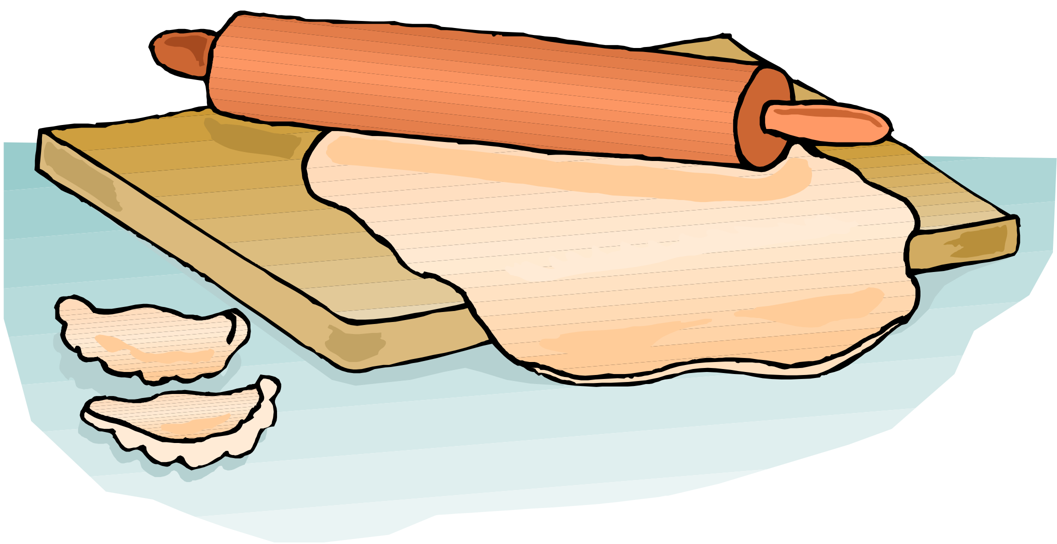
Life as a complex system

Chaos

Stretch and Fold

Kneading is the process of stretching and folding



Life as a complex system

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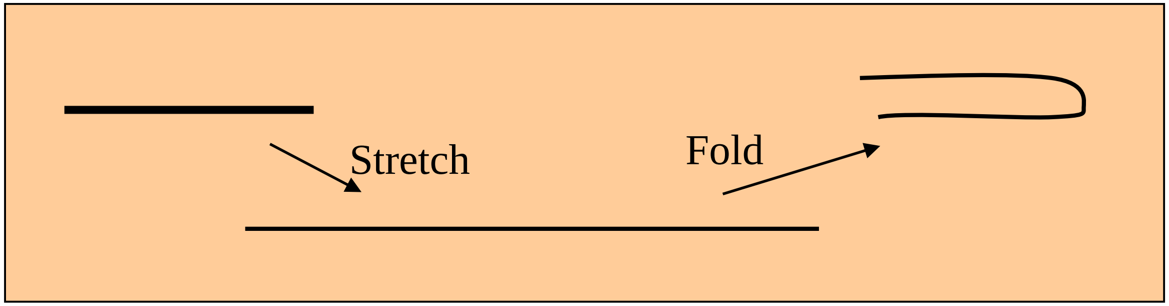
Chaos

What is Chaos?

The key to understanding Chaos is the concept of ***stretch and fold***.



Sensitive dependence on initial conditions

Two close-by points always separate yet stay in the same volume. Inside a layer, two points will separate, but, due the folding, when cutting through layers, they will also stay close.

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The Butterfly Effect

Sensitive dependence on initial conditions is what gave the world the butterfly effect.

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The Butterfly Effect

The *butterfly effect* is closely related to the notion of *sensitive dependence on initial conditions*.

The ***butterfly effect*** describes the notion that the flapping of the wings of a butterfly can ‘cause’ a typhoon at the other side of the world.

How? We saw with the stretch and fold example, where the distance between two points doubles each time, that a small distance/ difference can grow extremely quickly.

Due to the sensitive dependence on initial conditions in non-linear systems (of which the weather is one), the small disturbance caused by the butterfly (where we consider the disturbance to be the difference with the ‘no-butterfly’ situation) in a similar way can grow to become a storm.

Towards Chaos

At this point we should have a look at what ***simple*** and ***complex*** is.

Simple

6 : free from elaboration or figuration <*simple* harmony>
7 a (1) **:** not subdivided into branches or leaflets <a *simple* stem> <a *simple* leaf> (2) **:** consisting of a single carpel (3) **:** developing from a single ovary <a *simple* fruit> **b :** controlled by a single gene <*simple* inherited characters>
8 : not limited or restricted **:** UNCONDITIONAL <a *simple* obligation>
9 : readily understood or performed <*simple* directions> <the adjustment was *simple* to make>

From Merriam-Webster

Complex

1 : a whole made up of complicated or interrelated parts <a *complex* of university buildings> <a *complex* of welfare programs> <the military-industrial *complex*>
2 a : a group of culture traits relating to a single activity (as hunting), process (as use of flint), or culture unit **b** (1) **:** a group of repressed desires and memories that exerts a dominating influence upon the personality (2) **:** an exaggerated reaction to a subject or situation **c :** a group of obviously related units of which the degree and nature of the relationship is imperfectly known
3 : a chemical association of two or more species (as ions or molecules) joined usually by weak electrostatic bonds rather than covalent bonds

Simple? Complex?

Complex

The phenomena mentioned on the previous slides are very if not extremely complex. How can we ever understand them?

Try to write an equation for this.

Simple

Chaos and Fractals can be generated with what appear to be almost trivial mathematical formulas…

$$x_{n+1} = 1 - \alpha x_n^2$$

You could have done this in JC Right!??

Santa Fe

Many of the great scientists in the field of complex systems are linked to this institute.

“The Santa Fe Institute is a private, non-profit, multidisciplinary research and education center, founded in 1984. Since its founding SFI has devoted itself to creating a new kind of scientific research community, pursuing emerging science.”

“Operating as a small, visiting institution, SFI seeks to catalyze new collaborative, multidisciplinary projects that break down the barriers between the traditional disciplines, to spread its ideas and methodologies to other individuals and encourage the practical applications of its results.”

The Physical Aspect

Generally speaking a ***complex system*** is a system of interacting elements whose collective behavior cannot be described as the simple sum the elements' behavior

Hence many systems studied in physics are in that sense not complex. E.g. in Quantum Physics we can simply ADD the wave functions.

Five boys and five girls together on a deserted island though will likely behave quite differently from one boy or girl on the same deserted island.

Complex is not the same as complicated!

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Life as a complex system

The Physical Aspect

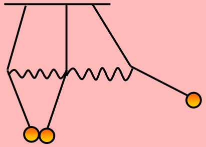
Complex is not the same as complicated!

Complicated:

Microprocessor

Function: difficult to design and understand
Behavior:
easy to understand

Complex:



Some double pendula linked by rubber bands

Function: easy to design and understand
Behavior: difficult to understand

The Physical Aspect

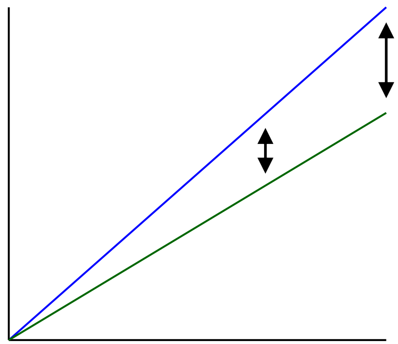
Examples:

Complicated:	**Complex:**
Car engine	Ant colony
	

The Physical Aspect

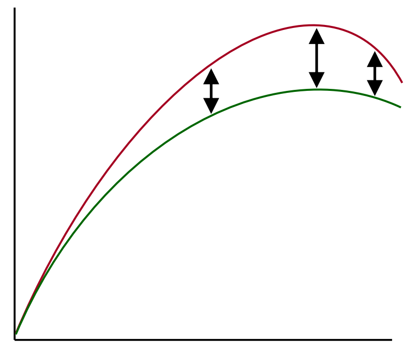
An essential ingredient of complex systems is their built-in non-linearity.

linear



A difference keeps on increasing

non-linear



A difference can increase
but also decrease

It's a bit similar to a winding mountain road. Take the wrong turn or make the wrong step and you could end up somewhere completely else!

The Physical Aspect

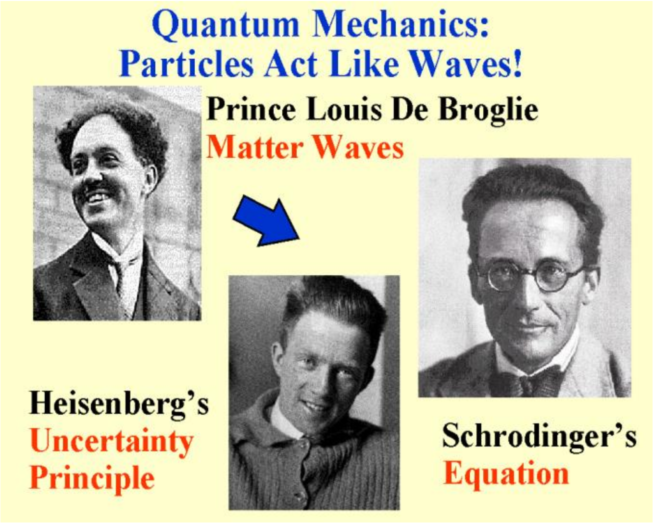
Example:

Linear:

Non-linear:

Quantum Mechanics

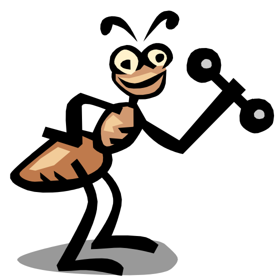
Turbulence



The Physical Aspect

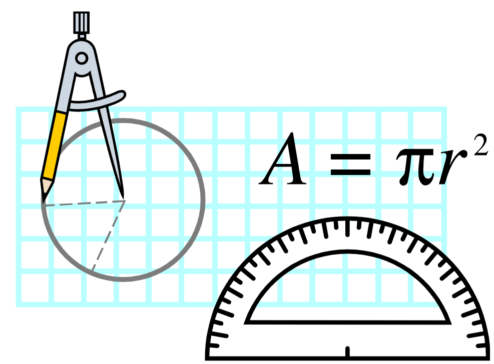
Complex systems often exhibit the following characteristics:

Robustness
Self-organization
Adaptability



Furthermore (in Physics):

Clear mathematical definition
Basically deterministic



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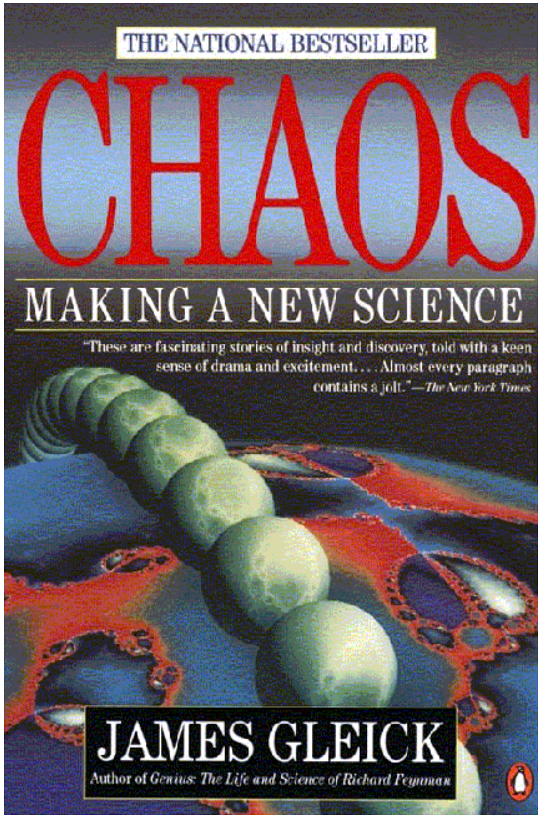
Understanding Chaos

In order to understand what's going on, let us have a somewhat more detailed look at what Chaos is.

Quite a nice book →



Poincaré & Lorenz

Poincaré, Henri (1854-1912)

He discovered chaos when he tried to win the Oscar price.

Edward N. Lorenz

Professor Emeritus at MIT

Modern discoverer of chaos

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Chaos

What is Chaos?

Chaos is often a more ‘catchy’ name for *non-linear dynamics*.

Non-linear = (roughly) the graph of the function is not a straight line.

Dynamics = (roughly) the time evolution of a system.

Are chaotic systems always chaotic?

No! Generally speaking, many researchers will call a system chaotic if it *can* be chaotic for certain parameters.

Parameter = (roughly) a constant in an equation. E.g. the slope of a line. This parameter can be adjusted.

Wait

If we wait long enough … which one wouldn't do of course when baking bread …

Yeast will start to multiply. Therefore, we're dealing with a population of yeast which can of course grow and shrink.

What can we say in general about populations?

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Verhulst

Pierre François Verhulst

Born: 28 Oct 1804 in Brussels, Belgium
Died: 15 Feb 1849 in Brussels, Belgium

He worked on the theory of numbers, and became interested in social statistics.

Verhulst's research on the law of population growth is important. The assumed belief before his work was that an increasing population followed a geometric progression. His contemporary Quetelet believed that there are forces which tend to prevent this population growth and that they increase with the square of the rate at which the population grows.
Verhulst showed in 1846 that forces which tend to prevent a population growth grow in proportion to the ratio of the excess population to the total population. The non-linear differential equation describing the growth of a biological population which he deduced and studied is now named after him.
Based on his theory Verhulst predicted the upper limit of the Belgium population would be 9,400,000. In fact the population in 1994 was 10,118,000 and, but for the affect of immigration, his prediction looks good. (MacTutor History of Mathematics)

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Logistic Map

The *logistic map* originates from population dynamics.

Consider a population from year to year. This can be described by:

$$N_{i+1} = p\, N_i$$

(the discrete version of the Malthus law)

Next year's population

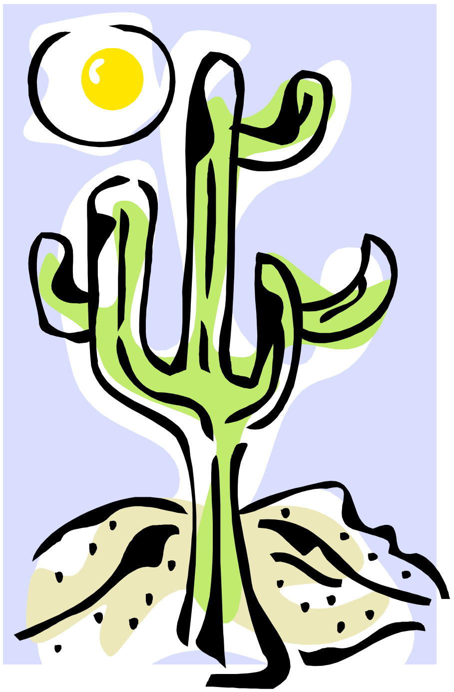
This year's Population

If p is larger than 1, this means that the population will grow ever bigger. This is of course not realistic. Hence Verhulst added the term $-bN^2$ to Malthus' law. We can do that here too and get:

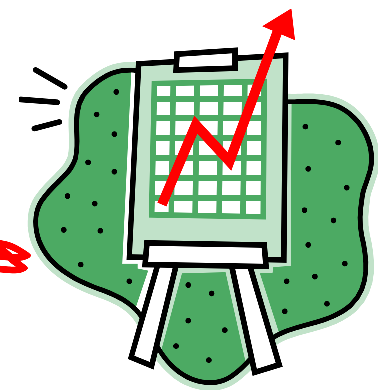
$$N_{i+1} = p\, N_i - bN_i^2$$

The logistic map (but written in a different way).

Logistic Map



Two opposing forces: tendency of population to grow exponentially and limitations by the environment.



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Life as a complex system

Logistic Map

The *logistic map* can be defined as:

$$x_{n+1} = 1 - \alpha x_n^2$$

Looks simple enough to me! What could be difficult about this?

In this equation, α is a so-called ***parameter***. This means, choose a value (e.g. 1.7) and start calculating. The little 'n' or 'n+1' below the variable x represents the time.

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Logistic Map

Let's have a closer look

So what is going on here?

$$x_{n+1} = 1 - \alpha x_n^2$$

Result:
Value of the variable 'x' at time 'n+1'.

Calculation:
Take the value of the variable 'x' at time 'n', square this value and multiply by the chosen value of α. Subtract this from 1 and you're done.

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Life as a complex system

Iteration

Iteration is just like stretch and fold

$$x_{n+1} = 1 - \alpha x_n^2$$

In math it means that you start with a certain value (given by you) calculate the result and then use this result as the starting value of a next calculation.

$$x_1 = 1 - \alpha x_0^2$$

given

$$x_2 = 1 - \alpha x_1^2$$

$$x_3 = 1 - \alpha x_2^2$$

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Life as a complex system

Iteration

Example:

Chose values for α and x_0 like $\alpha = 1.7$ and $x_0 = 0$.

$$x_1 = 1 - \alpha x_0^2 \qquad x_1 = 1 - 1.7 * 0^2 = \boxed{1}$$

$$x_2 = 1 - \alpha x_1^2 \qquad x_2 = 1 - 1.7 * \boxed{1}^2 = \boxed{-0.7}$$

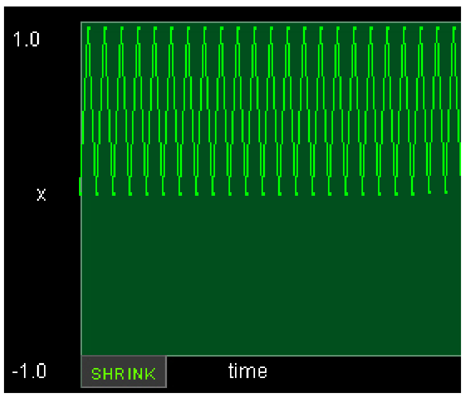
$$x_3 = 1 - \alpha x_2^2 \qquad x_3 = 1 - 1.7 * \boxed{-0.7}^2 = 0.167$$

Logistic Map

Time Series

A time series is a plot of the values of x versus the time n. E.g. we have:

value of x



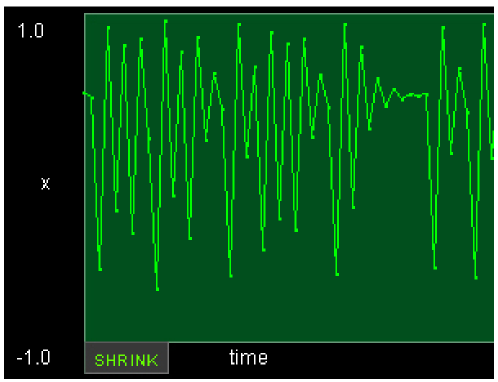
time n

$\alpha = 1.0$

Hey! Look at that, here, the values of x just seem to alternate between 2 values ….

value of x

$\alpha = 1.7$



time n

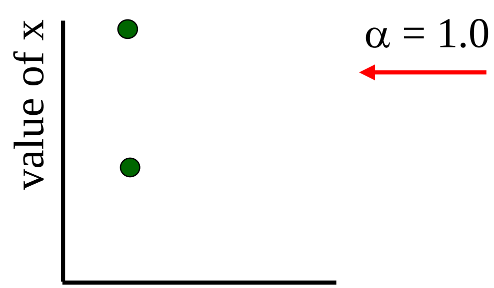
… and here, they jump all over the place.

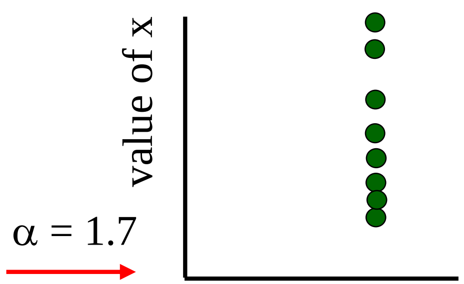
This must be related to α since that's the only difference between the two plots

Logistic Map

Towards …

Now if we are only interested in whether it alternates or kind of jumps around, then we do not need to actually draw the entire time series, we can just draw the points only.





Here we see the two values between which the time series alternates…

… and here, we see how the points almost join to form a line.

This naturally leads to the idea of the so-called ***bifurcation diagram***.

Feigenbaum

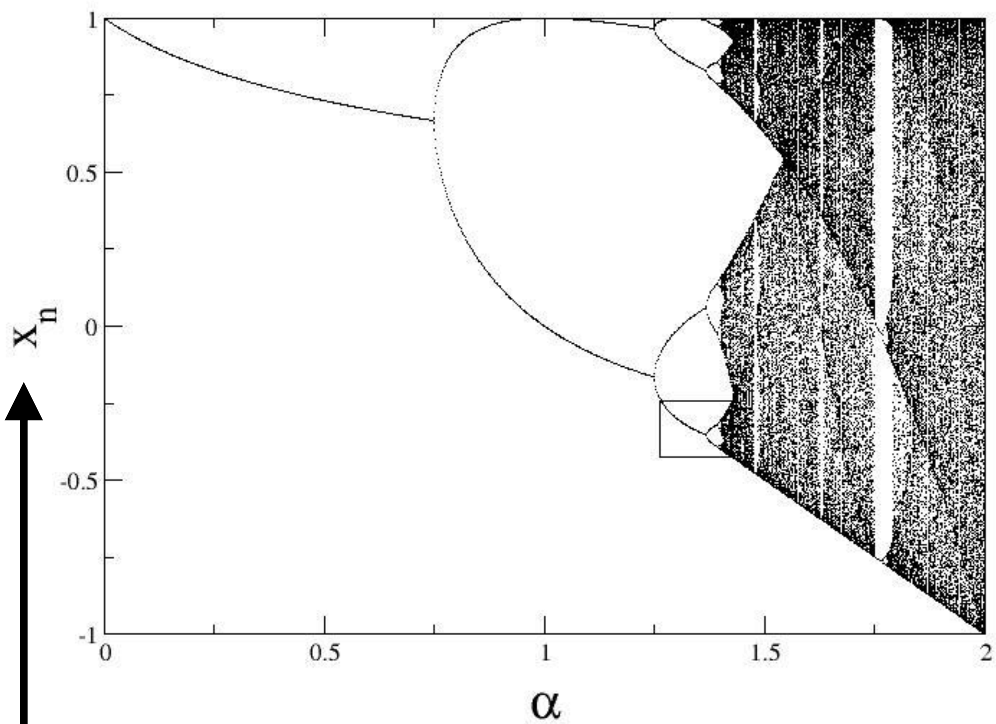
Born: 19 Dec 1944 in Philadelphia, USA

Discovered the universality of the bifurcation cascade.

Logistic Map

The so-called bifurcation diagram

Plot 200 successive values of x for every value of α

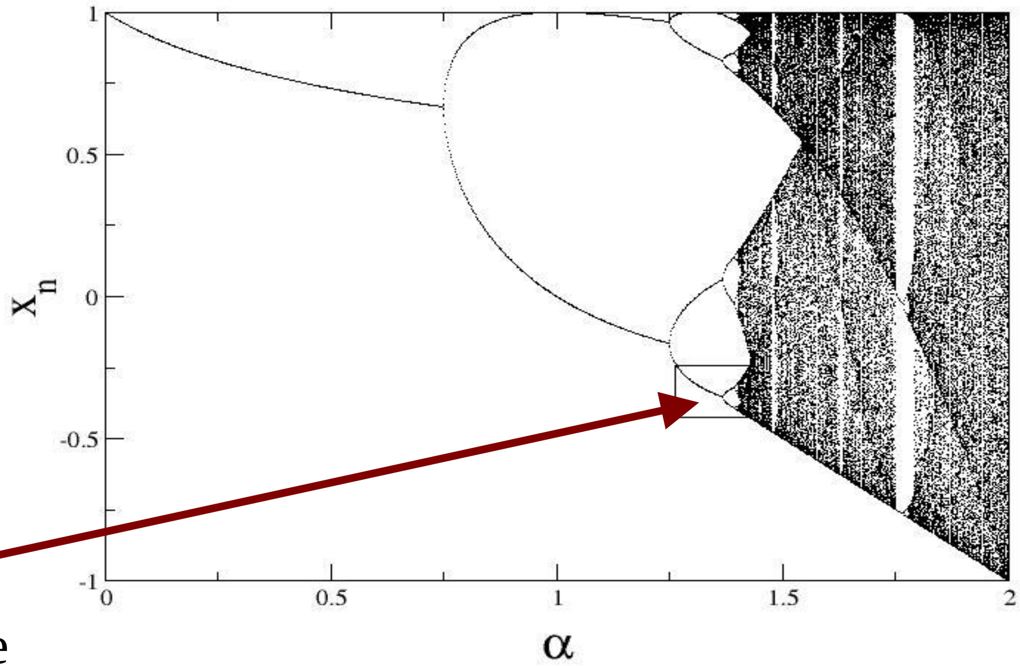


$$x_{n+1} = 1 - \alpha x_n^2$$

As the nonlinearity increases we sometimes encounter chaos

Logistic Map

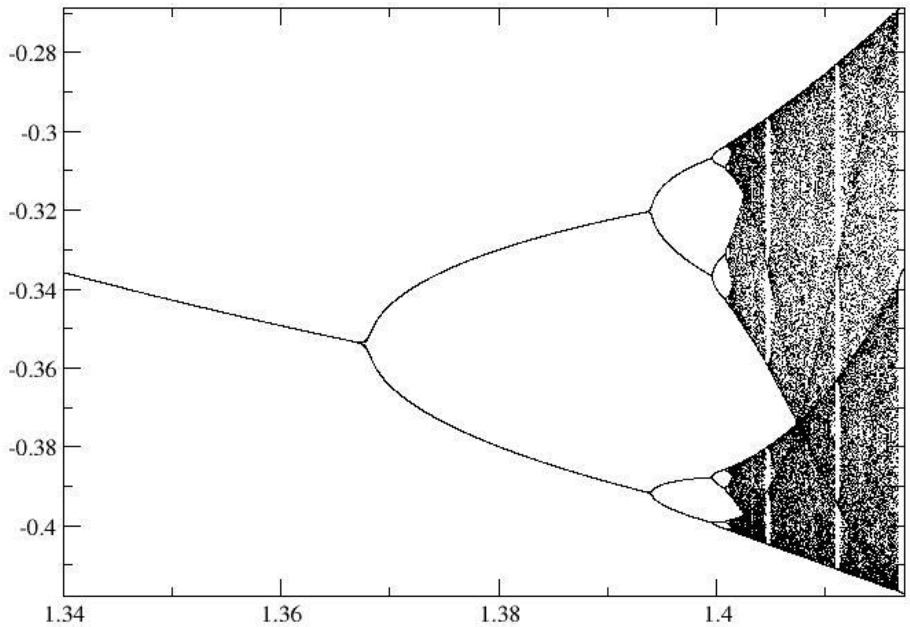
What's so special about this?
Let's have a closer look.



Let's enlarge
this area

Logistic Map

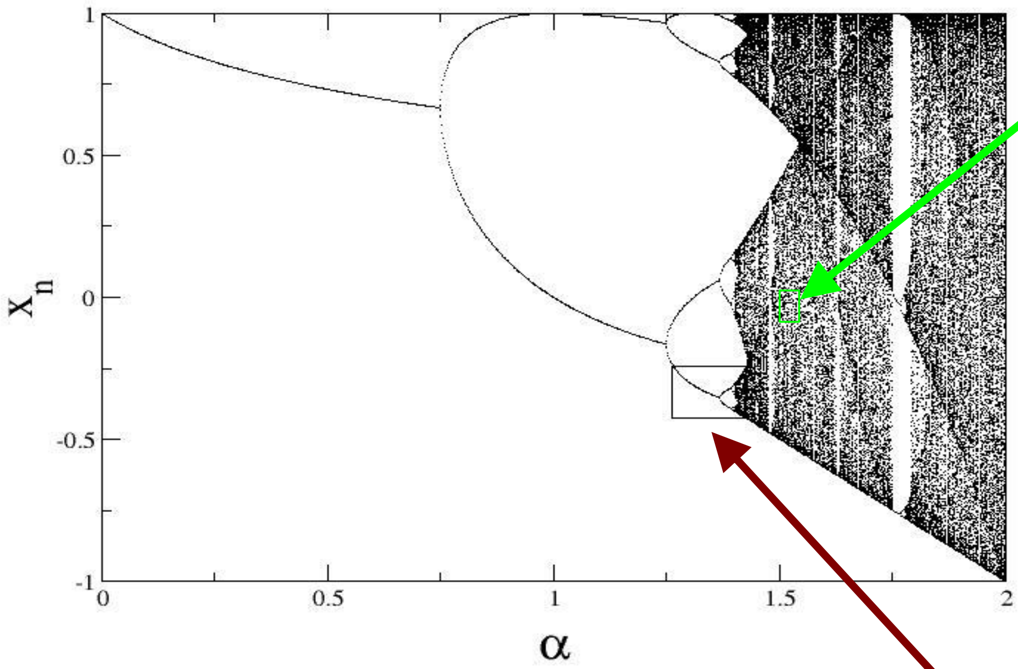
Hey! This looks almost the same!



Let's try this somewhere else...

Logistic Map

Let's enlarge a much smaller area!



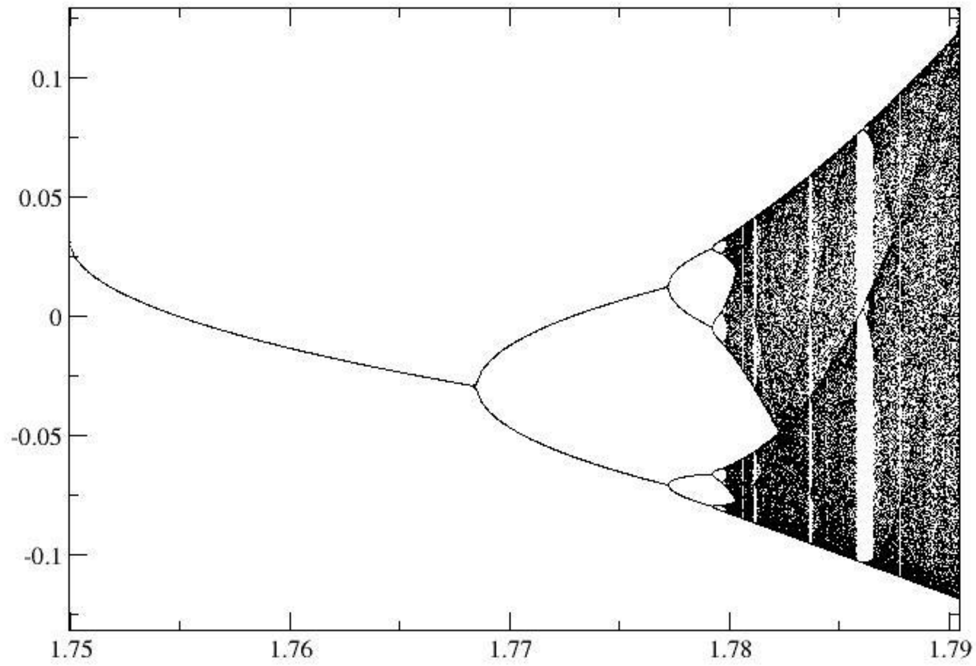
Now let's enlarge this area

Hard to see, isn't it?

The area we enlarged before.

Logistic Map

The same again!



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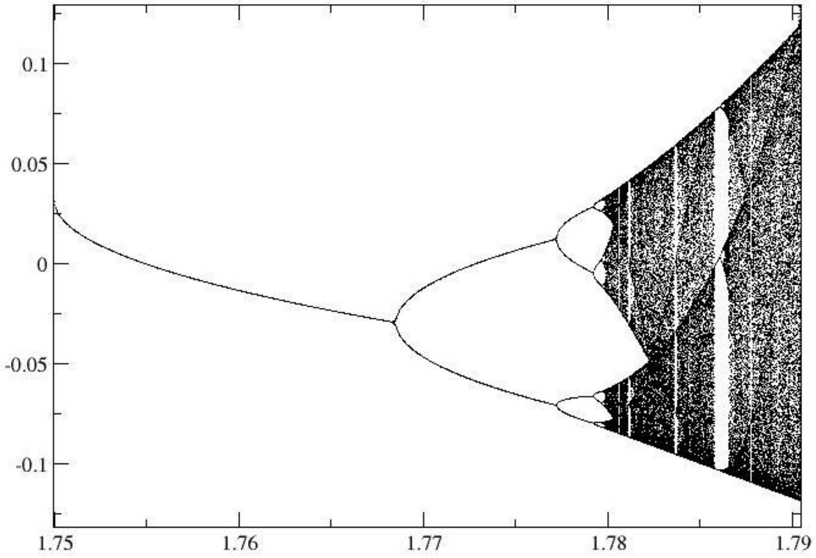
Life as a complex system

Logistic Map

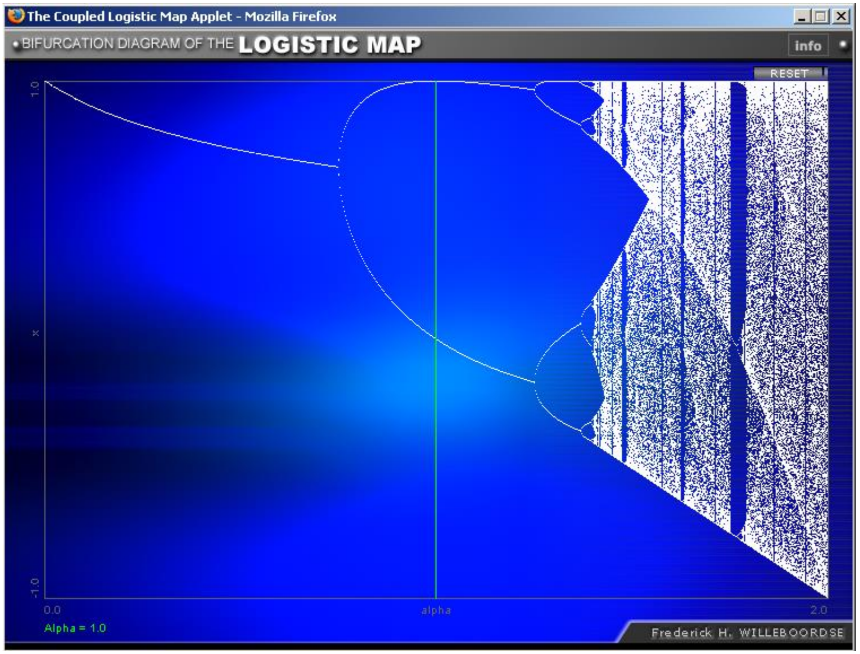
Indeed, the logistic map repeats itself over and over again at ever smaller scales

What's more, this behaviour was found to be universal!



Yes, there's a fractal hidden in here.

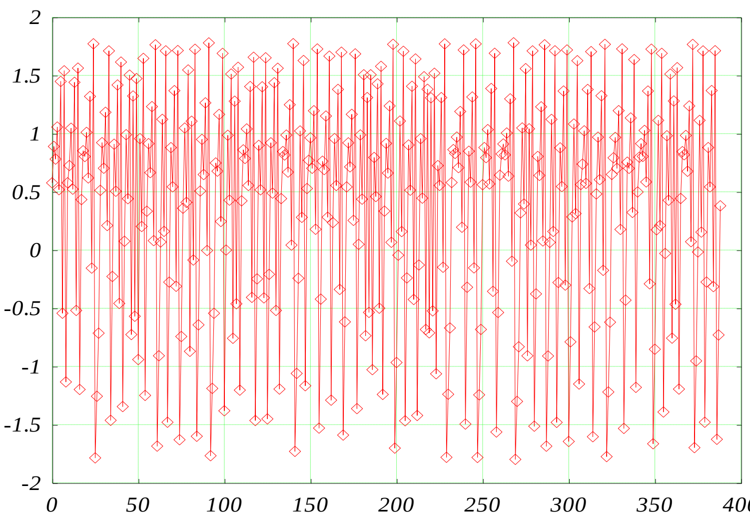
Applet

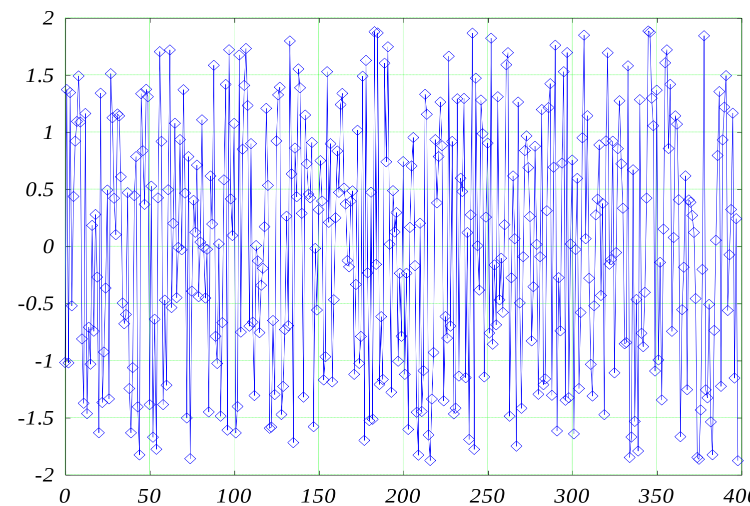

The Coupled Logistic Map Applet - Mozilla Firefox
BIFURCATION DIAGRAM OF THE LOGISTIC MAP
info
RESET
1.0
x
-1.0
0.0
alpha
2.0
Alpha = 1.0
Frederick H. WILLEBOORDSE

Chaos and Randomness

Chaos is NOT randomness though it can look pretty random.

Let us have a look at two time series:



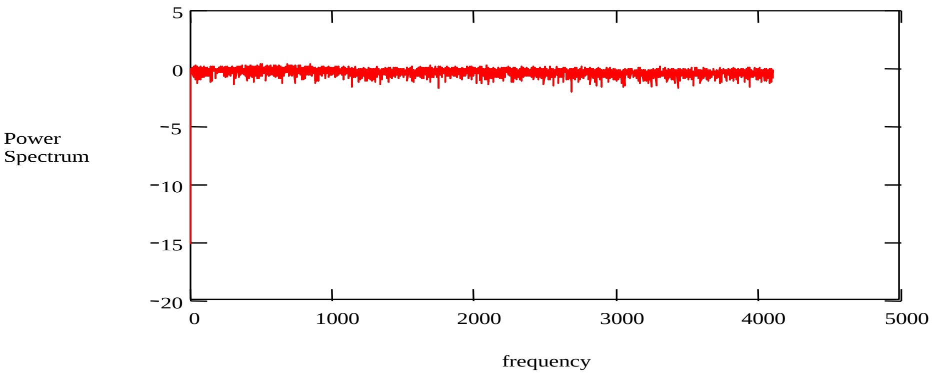


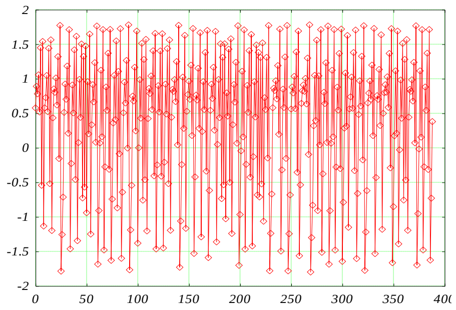
Data: Dr. C. Ting

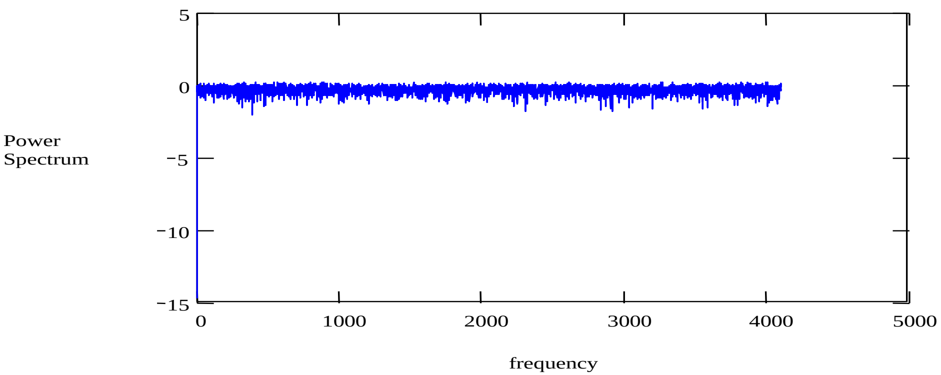
And analyze these with some standard methods

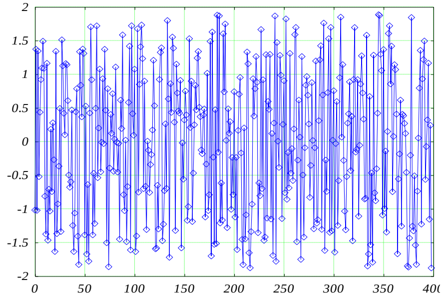
Chaos and Randomness

Power spectra





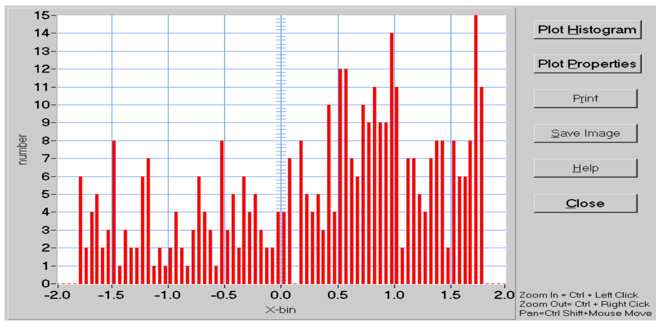


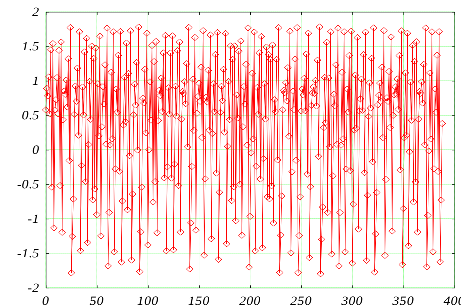


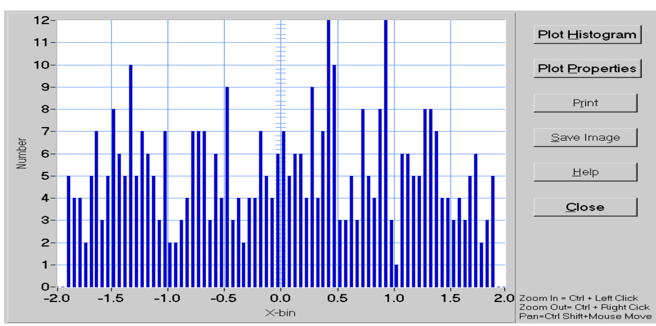
No qualitative differences!

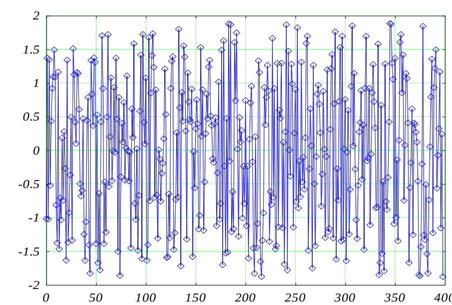
Chaos and Randomness

Histograms









No qualitative differences!

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Life as a complex system

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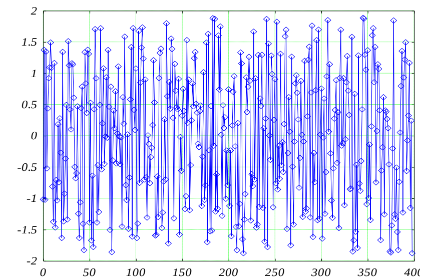
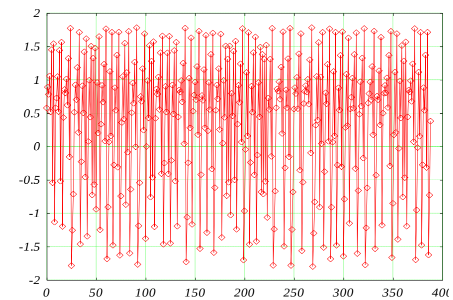
Chaos and Randomness

Well these two look pretty much the same.

Let us consider 4 options:

A: They are both Chaotic
B: Red is Chaotic and Blue is Random
C: Blue is Random and Red is Chaotic
D: They are both Random





What do you think?

Random???

Chaotic??

Chaotic?
?

Random???

Chaotic??

Random???

Chaotic??

Random???

Chaotic??

Chaos and Randomness

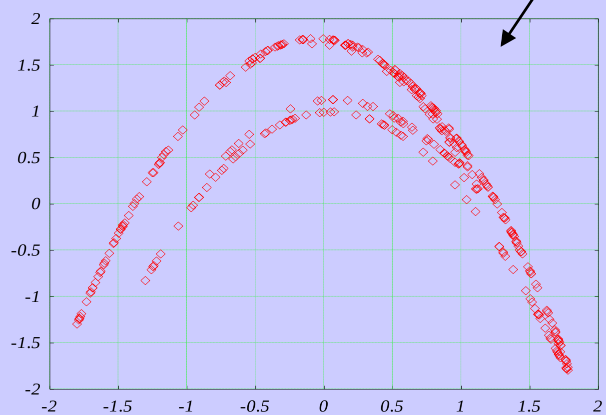
Red is Chaotic and Blue is Random!

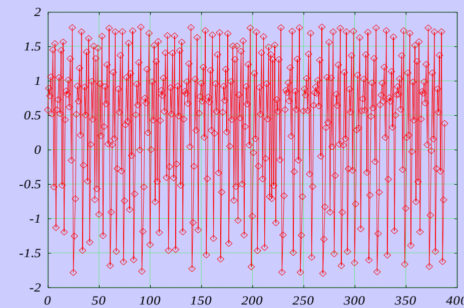
Return map
(plot x_{n+1} versus x_n)

Henon Map

Deterministic

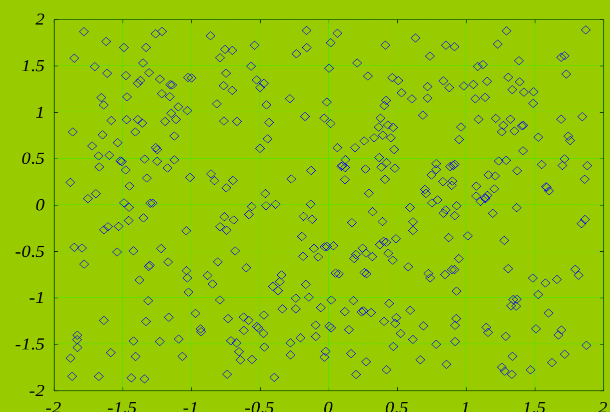
$$x_{n+1} = 1.4 - x^2_n + 0.3\, y_n$$
$$y_{n+1} = x_n$$

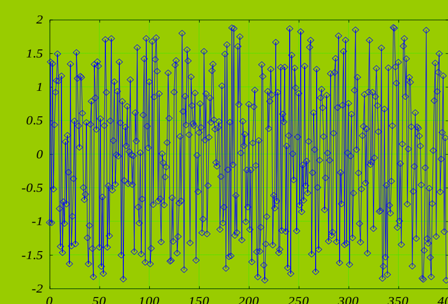




White Noise

Non-Deterministic





Wrapping up

Key Points of the Day

Populations
Complex/Simple
Chaos

Give it some thought

Is life chaotic?

References

WIKIPEDIA The Free Encyclopedia

http://www.wikipedia.org/