Lectures on Differential Equations[1]

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Preface

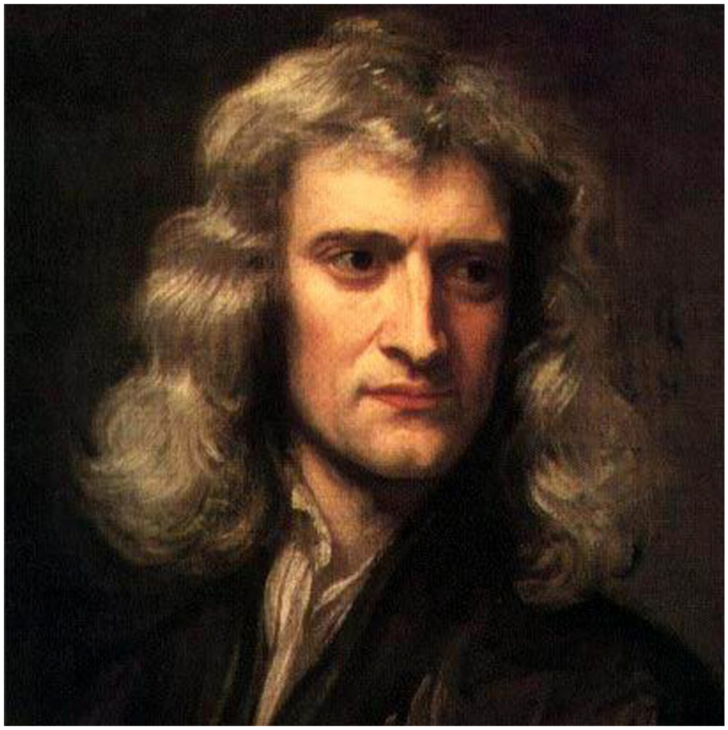


Figure 1: Sir Isaac Newton, December 25, 1642–March 20, 1727 (Julian Calendar).

These notes are for a one-quarter course in differential equations. The approach is to tie the study of differential equations to specific applications in physics with an emphasis on oscillatory systems.

> Mathematics is a part of physics. Physics is an experimental science, a part of natural science. Mathematics is the part of physics where experiments are cheap.
>
> In the middle of the twentieth century it was attempted to divide physics and mathematics. The consequences turned out to be catastrophic. Whole generations of mathematicians grew up without knowing half of their science and, of course, in total ignorance of any other sciences. They first began teaching their ugly scholastic pseudo-mathematics to their students, then to schoolchildren (forgetting Hardy's warning that ugly mathematics has no permanent place under the Sun).
>
> Since scholastic mathematics that is cut off from physics is fit neither for teaching nor for application in any other science, the result was the universal hate towards mathematicians—both on the part of the poor schoolchildren (some of whom in the meantime became ministers) and of the users.
>
> V. I. Arnold, *On Teaching Mathematics*

> Newton's fundamental discovery, the one which he considered necessary to keep secret and published only in the form of an anagram, consists of the following: *Data aequatione quotcunque fluentes quantitae involvente fluxions invenire et vice versa*. In contemporary mathematical language, this means: "It is useful to solve differential equations".
>
> V. I. Arnold, *Geometrical Methods in the Theory of Ordinary Differential Equations.*

I thank Eunghyun (Hyun) Lee for his help with these notes during the 2008–09 academic year. Also thanks to Andrew Waldron for his comments on the notes.

Craig Tracy, Sonoma, California

Notation

Symbol	**Definition of Symbol**
$\mathbb{R}$	field of real numbers
$\mathbb{R}^n$	the n-dimensional vector space with each component a real number
$\mathbb{C}$	field of complex numbers
$\dot{x}$	the derivative dx/dt, t is interpreted as time
$\ddot{x}$	the second derivative d^2x/dt^2, t is interpreted as time
$:=$	equals by definition
$\Psi = \Psi(x,t)$	wave function in quantum mechanics
ODE	ordinary differential equation
PDE	partial differential equation
KE	kinetic energy
PE	potential energy
det	determinant
δ_{ij}	the Kronecker delta, equal to 1 if $i = j$ and 0 otherwise
$\mathcal{L}$	the Laplace transform operator
$\binom{n}{k}$	The binomial coefficient n choose k.
MAPLE	is a registered trademark of Maplesoft.
MATHEMATICA	is a registered trademark of Wolfram Research.
MATLAB	is a registered trademark of the MathWorks, Inc.

Chapter 1

Introduction

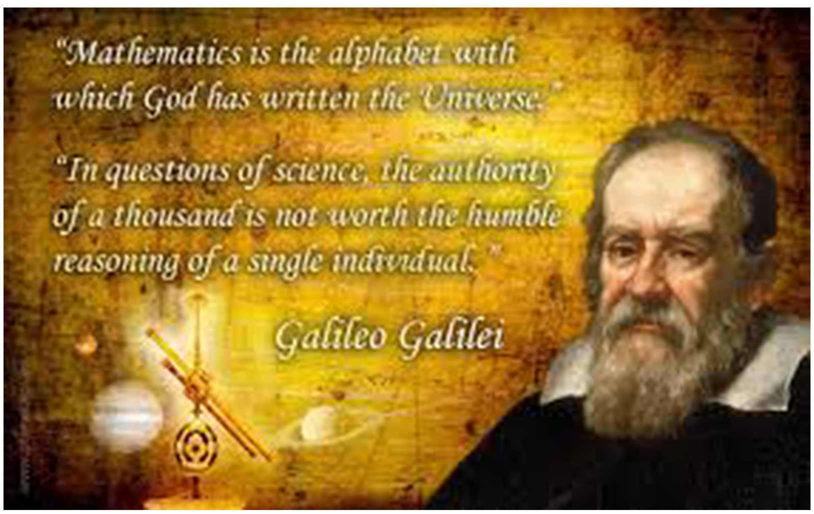


Figure 1.1: Galileo Galilei, 1564–1642. From *The Galileo Project*: "Galileo's discovery was that the period of swing of a pendulum is independent of its amplitude–the arc of the swing–the isochronism of the pendulum. Now this discovery had important implications for the measurement of time intervals. In 1602 he explained the isochronism of long pendulums in a letter to a friend, and a year later another friend, Santorio Santorio, a physician in Venice, began using a short pendulum, which he called "pulsilogium," to measure the pulse of his patients. The study of the pendulum, the first harmonic oscillator, date from this period."

See the You Tube video `http://youtu.be/MpzaCCbX-z4`.

1.1 What is a differential equation?

From Birkhoff and Rota [3]

> A *differential equation* is an equation between specified derivative on an unknown function, its values, and known quantities and functions. Many physical laws are most simply and naturally formulated as differential equations (or DEs, as we will write for short). For this reason, DEs have been studied by the greatest mathematicians and mathematical physicists since the time of Newton.
>
> *Ordinary* differential equations are DEs whose unknowns are functions of a single variable; they arise most commonly in the study of dynamical systems and electrical networks. They are much easier to treat that *partial* differential equations, whose unknown functions depend on two or more independent variables.
>
> Ordinary DEs are classified according to their order. The *order* of a DE is defined as the largest positive integer, n, for which an nth derivative occurs in the equation. Thus, an equation of the form
>
> $$\phi(x, y, y') = 0$$
>
> is said to be of the *first order*.

From Wikipedia

> A *differential equation* is a mathematical equation that relates some function of one or more variables with its derivatives. Differential equations arise whenever a deterministic relation involving some continuously varying quantities (modeled by functions) and their rates of change in space and/or time (expressed as derivatives) is known or postulated. Because such relations are extremely common, differential equations play a prominent role in many disciplines including engineering, physics, economics, and biology.
>
> Differential equations are mathematically studied from several different perspectives, mostly concerned with their solutions the set of functions that satisfy the equation. Only the simplest differential equations admit solutions given by explicit formulas; however, some properties of solutions of a given differential equation may be determined without finding their exact form. If a self-contained formula for the solution is not available, the solution may be numerically approximated using computers. The theory of dynamical systems puts emphasis on qualitative analysis of systems described by differential equations, while many numerical methods have been developed to determine solutions with a given degree of accuracy.
>
> Many fundamental laws of physics and chemistry can be formulated as differential equations. In biology and economics, differential equations are used to model the behavior of complex systems. The mathematical theory of differential equations first developed together with the sciences where the equations had originated and where the results found application. However, diverse problems, sometimes originating in quite distinct scientific fields, may give rise to identical differential equations. Whenever this happens, mathematical theory behind the equations

can be viewed as a unifying principle behind diverse phenomena. As an example, consider propagation of light and sound in the atmosphere, and of waves on the surface of a pond. All of them may be described by the same second-order partial differential equation, the wave equation, which allows us to think of light and sound as forms of waves, much like familiar waves in the water. Conduction of heat, the theory of which was developed by Joseph Fourier, is governed by another second-order partial differential equation, the heat equation. It turns out that many diffusion processes, while seemingly different, are described by the same equation; the Black–Scholes equation in finance is, for instance, related to the heat equation.

1.1.1 Examples

1. A simple example of a differential equation (DE) is

$$\frac{dy}{dx} = \lambda y$$

where λ is a constant. The unknown is y and the independent variable is x. The equation involves both the unknown y as well as the unknown dy/dx; and for this reason is called a *differential* equation. We know from calculus that

$$y(x) = c\,\mathrm{e}^{\lambda x},\ \ c = \text{constant},$$

satisfies this equation since

$$\frac{dy}{dx} = \frac{d}{dx} c\,\mathrm{e}^{\lambda x} = c\,\lambda \mathrm{e}^{\lambda x} = \lambda y(x).$$

The constant c is uniquely specified once we give the *initial condition* which in this case would be to give the value of $y(x)$ at a particular point x_0. For example, if we impose the initial condition $y(0) = 3$, then the constant c is now determined, i.e. $c = 3$.

2. Consider the DE

$$\frac{dy}{dx} = y^2$$

subject to the initial condition $y(0) = 1$. This DE was solved in your calculus courses using the method of separation of variables:

- First rewrite DE in differential form:

$$dy = y^2 dx$$

- Now separate variables (all x's on one side and all y's on the other side):

$$\frac{dy}{y^2} = dx$$

- Now integrate both sides

$$-\frac{1}{y} = x + c$$

where c is a constant to be determined.

- Solve for $y = y(x)$

$$y(x) = -\frac{1}{x+c}$$

- Now require $y(0) = -1/c$ to equal the given initial condition:

$$-\frac{1}{c} = 1$$

Solving this gives $c = -1$ and hence the solution we want is

$$y(x) = -\frac{1}{x-1}$$

3. An example of a *second order* ODE is

$$F(x) = m\frac{d^2x}{dt^2} \tag{1.1}$$

where $F = F(x)$ is a given function of x and m is a positive constant. Now the unknown is x and the independent variable is t. The problem is to find functions $x = x(t)$ such that when substituted into the above equation it becomes an identity. Here is an example; choose $F(x) = -kx$ where $k > 0$ is a positive number. Then (1.1) reads

$$-kx = m\frac{d^2x}{dt^2}$$

We rewrite this ODE as

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0. \tag{1.2}$$

You can check that

$$x(t) = \sin\left(\sqrt{\frac{k}{m}}\, t\right)$$

satisfies (1.2). Can you find other functions that satisfy this same equation? One of the problems in differential equations is to find *all* solutions $x(t)$ to the given differential equation. We shall soon prove that all solutions to (1.2) are of the form

$$x(t) = c_1 \sin\left(\sqrt{\frac{k}{m}}\, t\right) + c_2 \cos\left(\sqrt{\frac{k}{m}}\, t\right) \tag{1.3}$$

where c_1 and c_2 are arbitrary constants. Using differential calculus[1] one can verify that (1.3) when substituted into (1.2) satisfies the differential equation (show this!). It is another matter to show that *all* solutions to (1.2) are of the form (1.3). This is a problem we will solve in this class.

[1] Recall the differentiation formulas

$$\frac{d}{dt}\sin(\omega t) = \omega\cos(\omega t), \quad \frac{d}{dt}\cos(\omega t) = -\omega\sin(\omega t)$$

where ω is a constant. In the above the constant $\omega = \sqrt{k/m}$.

1.2 Differential equation for the pendulum

Newton's principle of determinacy

> The initial state of a mechanical system (the totality of positions and velocities of its points at some moment of time) uniquely determines all of its motion.
>
> It is hard to doubt this fact, since we learn it very early. One can imagine a world in which to determine the future of a system one must also know the acceleration at the initial moment, but experience shows us that our world is not like this.

V. I. Arnold, *Mathematical Methods of Classical Mechanics*[1]

Many interesting ordinary differential equations (ODEs) arise from applications. One reason for understanding these applications in a mathematics class is that you can combine your physical intuition with your mathematical intuition in the same problem. Usually the result is an improvement of both. One such application is the motion of pendulum, i.e. a ball of mass m suspended from an ideal rigid rod that is fixed at one end. The problem is to describe the motion of the mass point in a constant gravitational field. Since this is a mathematics class we will not normally be interested in deriving the ODE from physical principles; rather, we will simply write down various differential equations and claim that they are "interesting." However, to give you the flavor of such derivations (which you will see repeatedly in your science and engineering courses), we will derive from Newton's equations the differential equation that describes the time evolution of the angle of deflection of the pendulum.

Let

$$\begin{aligned} \ell &= \text{length of the rod measured, say, in meters,} \\ m &= \text{mass of the ball measured, say, in kilograms,} \\ g &= \text{acceleration due to gravity} = 9.8070\, m/s^2. \end{aligned}$$

The motion of the pendulum is confined to a plane (this is an assumption on how the rod is attached to the pivot point), which we take to be the xy-plane (see Figure 1.2). We treat the ball as a "mass point" and observe there are two forces acting on this ball: the force due to gravity, mg, which acts vertically downward and the tension $\vec{T}$ in the rod (acting in the direction indicated in figure). Newton's equations for the motion of a point $\vec{x}$ in a plane are vector equations[2]

$$\vec{F} = m\vec{a}$$

where $\vec{F}$ is the sum of the forces acting on the the point and $\vec{a}$ is the acceleration of the point, i.e.

$$\vec{a} = \frac{d^2\vec{x}}{dt^2}.$$

Since acceleration is a *second derivative* with respect to time t of the position vector, $\vec{x}$, Newton's equation is a second-order ODE for the position $\vec{x}$. In x and y coordinates Newton's

[2]In your applied courses vectors are usually denoted with arrows above them. We adopt this notation when discussing certain applications; but in later chapters we will drop the arrows and state where the quantity lives, e.g. $x \in \mathbb{R}^2$.

equations become two equations

$$F_x = m\frac{d^2x}{dt^2}\,,\ \ F_y = m\frac{d^2y}{dt^2},$$

where F_x and F_y are the x and y components, respectively, of the force $\vec{F}$. From the figure (note definition of the angle θ) we see, upon resolving $\vec{T}$ into its x and y components, that

$$F_x = -T\sin\theta,\ \ F_y = T\cos\theta - mg.$$

(T is the magnitude of the vector $\vec{T}$.)

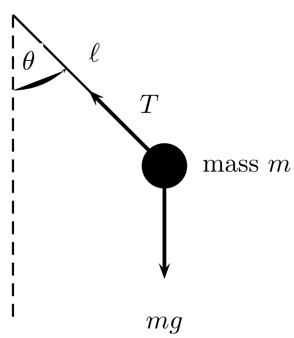


Figure 1.2: Simple pendulum

Substituting these expressions for the forces into Newton's equations, we obtain the differential equations

$$\begin{aligned} -T\sin\theta &= m\frac{d^2x}{dt^2}, && (1.4)\\ T\cos\theta - mg &= m\frac{d^2y}{dt^2}. && (1.5) \end{aligned}$$

From the figure we see that

$$x = \ell\sin\theta,\ y = \ell - \ell\cos\theta. \tag{1.6}$$

(The origin of the xy-plane is chosen so that at $x = y = 0$, the pendulum is at the bottom.) Differentiating[3] (1.6) with respect to t, and then again, gives

$$\begin{aligned} \dot{x} &= \ell\cos\theta\,\dot{\theta},\\ \ddot{x} &= \ell\cos\theta\,\ddot{\theta} - \ell\sin\theta\,(\dot{\theta})^2, && (1.7)\\ \dot{y} &= \ell\sin\theta\,\dot{\theta},\\ \ddot{y} &= \ell\sin\theta\,\ddot{\theta} + \ell\cos\theta\,(\dot{\theta})^2. && (1.8) \end{aligned}$$

Substitute (1.7) in (1.4) and (1.8) in (1.5) to obtain

$$\begin{aligned} -T\sin\theta &= m\ell\cos\theta\,\ddot{\theta} - m\ell\sin\theta\,(\dot{\theta})^2, && (1.9)\\ T\cos\theta - mg &= m\ell\sin\theta\,\ddot{\theta} + m\ell\cos\theta\,(\dot{\theta})^2. && (1.10) \end{aligned}$$

[3] We use the dot notation for time derivatives, e.g. $\dot{x} = dx/dt$, $\ddot{x} = d^2x/dt^2$.

Now multiply (1.9) by $\cos\theta$, (1.10) by $\sin\theta$, and add the two resulting equations to obtain

$$-mg\sin\theta = m\ell\ddot{\theta},$$

or

$$\boxed{\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0.} \tag{1.11}$$

Remarks

- The ODE (1.11) is called a second-order equation because the highest derivative appearing in the equation is a second derivative.
- The ODE is nonlinear because of the term $\sin\theta$ (this is not a linear function of the unknown quantity θ).
- A solution to this ODE is a function $\theta = \theta(t)$ such that when it is substituted into the ODE, the ODE is satisfied for all t.
- Observe that the mass m dropped out of the final equation. This says the motion will be independent of the mass of the ball. If an experiment is performed, will we observe this to be the case; namely, the motion is independent of the mass m? If not, perhaps in our model we have left out some forces acting in the real world experiment. Can you think of any?
- The derivation was constructed so that the tension, $\vec{T}$, was eliminated from the equations. We could do this because we started with two unknowns, T and θ, and two equations. We manipulated the equations so that in the end we had one equation for the unknown $\theta = \theta(t)$.
- We have not discussed how the pendulum is initially started. This is very important and such conditions are called the *initial conditions.*

We will return to this ODE later in the course. At this point we note that if we were interested in only small deflections from the origin (this means we would have to start out near the origin), there is an obvious approximation to make. Recall from calculus the Taylor expansion of $\sin\theta$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots .$$

For small θ this leads to the approximation $\sin\theta \approx \theta$. Using this small deflection approximation in (1.11) leads to the ODE

$$\boxed{\ddot{\theta} + \frac{g}{\ell}\theta = 0.} \tag{1.12}$$

We will see that (1.12) is mathematically simpler than (1.11). The reason for this is that (1.12) is a linear ODE. It is linear because the unknown quantity, θ, and its derivatives appear only to the first or zeroth power. Compare (1.12) with (1.2).

1.3 Introduction to MatLab, Mathematica and Maple

In this class we may use the computer software packages MatLab, Mathematica or Maple to do routine calculations. It is strongly recommended that you learn to use at least one of these software packages. These software packages take the drudgery out of routine calculations in calculus and linear algebra. Engineers will find that MatLab is used extenstively in their upper division classes. Both MatLab and Maple are superior for symbolic computations (though MatLab can call Maple from the MatLab interface).

1.3.1 MatLab

What is MatLab ? "MatLab is a powerful computing system for handling the calculations involved in scientific and engineering problems."[4] MatLab can be used either interactively or as a programming language. For most applications in Math 22B it suffices to use MatLab interactively. Typing `matlab` at the command level is the command for most systems to start MatLab . Once it loads you are presented with a prompt sign `>>`. For example if I enter

```
>> 2+22
```

and then press the enter key it responds with

```
ans=24
```

Multiplication is denoted by `*` and division by `/` . Thus, for example, to compute

$$\frac{(139.8)(123.5-44.5)}{125}$$

we enter

```
>> 139.8*(123.5-44.5)/125
```

gives

```
ans=88.3536
```

MatLab also has a Symbolic Math Toolbox which is quite useful for routine calculus computations. For example, suppose you forgot the Taylor expansion of $\sin x$ that was used in the notes just before (1.12). To use the Symbolic Math Toolbox you have to tell MatLab that x is a symbol (and not assigned a numerical value). Thus in MatLab

```
>> syms x
>> taylor(sin(x))
```

[4] Brian D. Hahn, *Essential* MatLab *for Scientists and Engineers.*

gives

```
ans = x -1/6*x^3+1/120*x^5
```

Now why did `taylor` expand about the point $x = 0$ and keep only through x^5? By default the Taylor series about 0 up to terms of order 5 is produced. To learn more about `taylor` enter

```
>> help taylor
```

from which we learn if we had wanted terms up to order 10 we would have entered

```
>> taylor(sin(x),10)
```

If we want the Taylor expansion of $\sin x$ about the point $x = \pi$ up to order 8 we enter

```
>> taylor(sin(x),8,pi)
```

A good reference for MatLab is *MatLab Guide* by Desmond Higham and Nicholas Higham.

1.3.2 Mathematica

There are alternatives to the software package MatLab. Two widely used packages are Mathematica and Maple. Here we restrict the discussion to Mathematica . Here are some typical commands in Mathematica .

1. To define, say, the function $f(x) = x^2 e^{-2x}$ one writes in Mathematica

```
f[x_]:=x^2*Exp[-2*x]
```

2. One can now use f in other Mathematica commands. For example, suppose we want $\int_0^\infty f(x)\,dx$ where as above $f(x) = x^2 e^{-2x}$. The Mathematica command is

```
Integrate[f[x],{x,0,Infinity}]
```

Mathematica returns the answer 1/4.

3. In Mathematica to find the Taylor series of $\sin x$ about the point $x = 0$ to fifth order you would type

```
Series[Sin[x],{x,0,5}]
```

4. Suppose we want to create the 10×10 matrix

$$M = \left(\frac{1}{i+j+1}\right)_{1 \le i,j \le 10}.$$

In Mathematica the command is

```
M=Table[1/(i+j+1),{i,1,10},{j,1,10}];
```

(The semicolon tells MATHEMATICA not to write out the result.) Suppose we then want the determinant of M. The command is

```
Det[M]
```

MATHEMATICA returns the answer

$$1/273739709893086064093902013446617579389091964235284480000000000$$

If we want this number in scientific notation, we would use the command `N[· ]` (where the number would be put in place of $\cdot$). The answer MATHEMATICA returns is 3.65311×10^{-63}. The (numerical) eigenvalues of M are obtained by the command

```
N[Eigenvalues[M]]
```

MATHEMATICA returns the list of 10 distinct eigenvalues. (Which we won't reproduce here.) The reason for the `N[·]` is that MATHEMATICA cannot find an exact form for the eigenvalues, so we simply ask for it to find approximate numerical values. To find the (numerical) eigenvectors of M, the command is

```
N[Eigenvectors[M]]
```

5. MATHEMATICA has nice graphics capabilities. Suppose we wish to graph the function $f(x) = 3e^{-x/10}\sin(x)$ in the interval $0 \le x \le 50$. The command is

```
Plot[3*Exp[-x/10]*Sin[x],{x,0,50},PlotRange->All,
AxesLabel->{x},PlotLabel->3*Exp[-x/10]*Sin[x]]
```

The result is the graph shown in Figure 1.3.

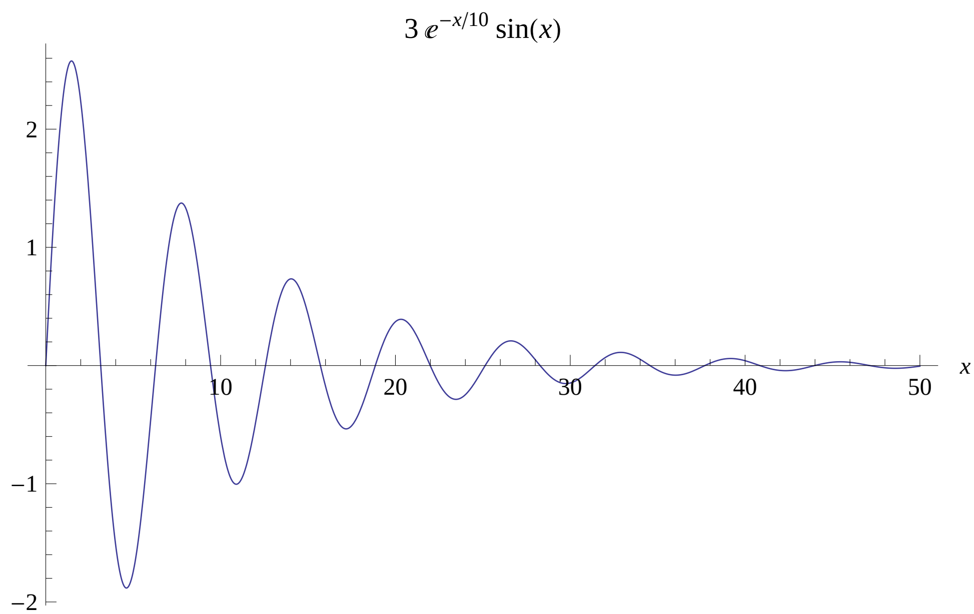


Figure 1.3:

1.4 Exercises

#1. MatLab and/or Mathematica Exercises

1. Use MatLab or Mathematica to get an estimate (in scientific notation) of 99^{99}. Now use

```
>> help format
```

to learn how to get more decimal places. (All MatLab computations are done to a relative precision of about 16 decimal places. MatLab defaults to printing out the first 5 digits.) Thus entering

```
>> format long e
```

on a command line and then re-entering the above computation will give the 16 digit answer.

In Mathematica to get 16 digits accuracy the command is

```
N[99^(99),16]
```

Ans.: $3.697296376497268 \times 10^{197}$.

2. Use MatLab to compute $\sqrt{\sin(\pi/7)}$. (Note that MatLab has the special symbol `pi`; that is `pi` $\approx \pi = 3.14159\ldots$ to 16 digits accuracy.)

   In Mathematica the command is

```
N[Sqrt[Sin[Pi/7]],16]
```

3. Use MatLab or Mathematica to find the determinant, eigenvalues and eigenvectors of the 4×4 matrix

$$A = \begin{pmatrix} 1 & -1 & 2 & 0 \\ \sqrt{2} & 1 & 0 & -2 \\ 0 & 1 & \sqrt{2} & -1 \\ 1 & 2 & 2 & 0 \end{pmatrix}$$

   Hint: In MatLab you enter the matrix A by

```
>> A=[1 -1 2 0; sqrt(2) 1 0 -2;0 1 sqrt(2) -1; 1 2 2 0]
```

   To find the determinant

```
>> det(A)
```

   and to find the eigenvalues

```
>> eig(A)
```

   If you also want the eigenvectors you enter

```
>> [V,D]=eig(A)
```

   In this case the columns of V are the eigenvectors of A and the diagonal elements of D are the corresponding eigenvalues. Try this now to find the eigenvectors. For the determinant you should get the result `16.9706`. One may also calculate the determinant symbolically. First we tell MatLab that A is to be treated as a symbol (we are assuming you have already entered A as above):

```
>> A=sym(A)
```

   and then re-enter the command for the determinant

```
det(A)
```

   and this time MatLab returns

```
ans =
12*2^(1/2)
```

   that is, $12\sqrt{2}$ which is approximately equal to 16.9706.

4. Use MatLab or Mathematica to plot $\sin\theta$ and compare this with the approximation $\sin\theta \approx \theta$. For $0 \le \theta \le \pi/2$, plot both on the same graph.

#2. Inverted pendulum

This exercise derives the small angle approximation to (1.11) when the pendulum is nearly inverted, i.e. $\theta \approx \pi$. Introduce

$$\phi = \theta - \pi$$

and derive a small ϕ-angle approximation to (1.11). How does the result differ from (1.12)?

Chapter 2

First Order Equations & Conservative Systems

2.1 Linear first order equations

2.1.1 Introduction

The simplest differential equation is one you already know from calculus; namely,

$$\frac{dy}{dx} = f(x). \tag{2.1}$$

To find a solution to this equation means one finds a function $y = y(x)$ such that its derivative, dy/dx, is equal to $f(x)$. The fundamental theorem of calculus tells us that all solutions to this equation are of the form

$$y(x) = y_0 + \int_{x_0}^{x} f(s)\, ds. \tag{2.2}$$

Remarks:

- $y(x_0) = y_0$ and y_0 is arbitrary. That is, there is a one-parameter family of solutions; $y = y(x; y_0)$ to (2.1). The solution is unique once we specify the initial condition $y(x_0) = y_0$. This is the solution to the initial value problem. That is, we have found a function that satisfies both the ODE and the initial value condition.

- Every calculus student knows that differentiation is easier than integration. Observe that solving a differential equation is like integration—you must find a function such that when it and its derivatives are substituted into the equation the equation is identically satisfied. Thus we sometimes say we "integrate" a differential equation. In the above case it is exactly integration as you understand it from calculus. This also suggests that solving differential equations can be expected to be difficult.

- For the integral to exist in (2.2) we must place some restrictions on the function f appearing in (2.1); here it is enough to assume f is continuous on the interval $[a, b]$. It was implicitly assumed in (2.1) that x was given on some interval—say $[a, b]$.

A simple generalization of (2.1) is to replace the right-hand side by a function that depends upon both x and y

$$\frac{dy}{dx} = f(x,y).$$

Some examples are $f(x,y) = xy^2$, $f(x,y) = y$, and the case (2.1). The simplest choice in terms of the y dependence is for $f(x,y)$ to depend linearly on y. Thus we are led to study

$$\frac{dy}{dx} = g(x) - p(x)y,$$

where $g(x)$ and $p(x)$ are functions of x. We leave them unspecified. (We have put the minus sign into our equation to conform with the standard notation.) The conventional way to write this equation is

$$\boxed{\frac{dy}{dx} + p(x)y = g(x).} \tag{2.3}$$

It's possible to give an algorithm to solve this ODE for more or less general choices of $p(x)$ and $g(x)$. We say more or less since one has to put some restrictions on p and g—that they are continuous will suffice. It should be stressed at the outset that this ability to find an explicit algorithm to solve an ODE is the exception—most ODEs encountered will not be so easily solved.

But before we give the general solution to (2.3), let's examine the special case $p(x) = -1$ and $g(x) = 0$ with initial condition $y(0) = 1$. In this case the ODE becomes

$$\frac{dy}{dx} = y \tag{2.4}$$

and the solution we know from calculus

$$y(x) = \mathrm{e}^x.$$

In calculus one typically *defines* e^x as the limit

$$\mathrm{e}^x := \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$$

or less frequently as the solution $y = y(x)$ to the equation

$$x = \int_1^y \frac{dt}{t}.$$

In calculus courses one then proves from either of these starting points that the derivative of e^x equals itself. One could also take the point of view that $y(x) = \mathrm{e}^x$ is defined to be the (unique) solution to (2.4) satisfying the initial condition $y(0) = 1$. Taking this last point of view, can you explain why the Taylor expansion of e^x,

$$\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

follows almost immediately?

2.1.2 Method of integrating factors

If (2.3) were of the form (2.1), then we could immediately write down a solution in terms of integrals. For (2.3) to be of the form (2.1) means the left-hand side is expressed as the derivative of our unknown quantity. We have some freedom in making this happen—for instance, we can multiply (2.3) by a function, call it $\mu(x)$, and ask whether the resulting equation can be put in form (2.1). Namely, is

$$\mu(x)\frac{dy}{dx} + \mu(x)p(x)y = \frac{d}{dx}\left(\mu(x)y\right)\ ? \tag{2.5}$$

Taking derivatives we ask can μ be chosen so that

$$\mu(x)\frac{dy}{dx} + \mu(x)p(x)y = \mu(x)\frac{dy}{dx} + \frac{d\mu}{dx}y$$

holds? This immediately simplifies to[1]

$$\mu(x)p(x) = \frac{d\mu}{dx},$$

or

$$\frac{d}{dx}\log\mu(x) = p(x).$$

Integrating this last equation gives

$$\log\mu(x) = \int p(s)\,ds + c.$$

Taking the exponential of both sides (one can check later that there is no loss in generality if we set $c = 0$) gives[2]

$$\mu(x) = \exp\left(\int^x p(s)\,ds\right). \tag{2.6}$$

Defining $\mu(x)$ by (2.6), the differential equation (2.5) is transformed to

$$\frac{d}{dx}\left(\mu(x)y\right) = \mu(x)g(x).$$

This last equation is precisely of the form (2.1), so we can immediately conclude

$$\mu(x)y(x) = \int^x \mu(s)g(s)\,ds + c,$$

and solving this for y gives our final formula

$$\boxed{y(x) = \frac{1}{\mu(x)}\int^x \mu(s)g(s)\,ds \;+\; \frac{c}{\mu(x)}\,,} \tag{2.7}$$

where $\mu(x)$, called the *integrating factor*, is defined by (2.6). The constant c will be determined from the initial condition $y(x_0) = y_0$.

[1]Notice y and its first derivative drop out. This is a good thing since we wouldn't want to express μ in terms of the unknown quantity y.

[2]By the symbol $\int^x f(s)\,ds$ we mean the indefinite integral of f in the variable x.

An example

Suppose we are given the DE

$$\frac{dy}{dx} + \frac{1}{x}\,y = x^2, \;\; x > 0$$

with initial condition

$$y(1) = 2.$$

This is of form (2.3) with $p(x) = 1/x$ and $g(x) = x^2$. We apply formula (2.7):

- First calculate the integrating factor $\mu(x)$:

$$\mu(x) = \exp\left(\int p(x)\,dx\right) = \exp\left(\int \frac{1}{x}\,dx\right) = \exp(\log x) = x.$$

- Now substitute into (2.3)

$$y(x) = \frac{1}{x}\int x \cdot x^2\,dx + \frac{c}{x} = \frac{1}{x}\cdot\frac{x^4}{4} + \frac{c}{x} = \frac{x^3}{4} + \frac{c}{x}.$$

- Impose the initial condition $y(1) = 2$:

$$\frac{1}{4} + c = 2, \;\text{ solve for }\; c, \;\; c = \frac{7}{4}.$$

- Solution to DE is

$$y(x) = \frac{x^3}{4} + \frac{7}{4x}.$$

2.1.3 Application to mortgage payments

Suppose an amount P, called the principal, is borrowed at an interest I (100I%) for a period of N years. One is to make monthly payments in the amount $D/12$ (D equals the amount paid in one year). The problem is to find D in terms of P, I and N. Let

$$y(t) = \text{amount owed at time } t \text{ (measured in years)}.$$

We have the initial condition

$$y(0) = P \;\text{ (at time 0 the amount owed is } P).$$

We are given the additional information that the loan is to be paid off at the end of N years,

$$y(N) = 0.$$

We want to derive an ODE satisfied by y. Let Δt denote a small interval of time and Δy the change in the amount owed during the time interval Δt. This change is determined by

- Δy is increased by compounding at interest I; that is, Δy is increased by the amount $Iy(t)\Delta t$.

- Δy is decreased by the amount paid back in the time interval Δt. If D denotes this constant rate of payback, then $D\Delta t$ is the amount paid back in the time interval Δt.

Thus we have

$$\Delta y = Iy\Delta t - D\Delta t,$$

or

$$\frac{\Delta y}{\Delta t} = Iy - D.$$

Letting $\Delta t \to 0$ we obtain the sought after ODE,

$$\frac{dy}{dt} = Iy - D. \tag{2.8}$$

This ODE is of form (2.3) with $p = -I$ and $g = -D$. One immediately observes that this ODE is not exactly what we assumed above, i.e. D is not known to us. Let us go ahead and solve this equation for any constant D by the method of integrating factors. So we choose μ according to (2.6),

$$\begin{aligned}\mu(t) &:= \exp\left(\int^t p(s)\,ds\right)\\ &= \exp\left(-\int^t I\,ds\right)\\ &= \exp(-It).\end{aligned}$$

Applying (2.7) gives

$$\begin{aligned}y(t) &= \frac{1}{\mu(t)}\int^t \mu(s)g(s)\,ds + \frac{c}{\mu(t)}\\ &= e^{It}\int^t e^{-Is}(-D)\,ds + ce^{It}\\ &= -De^{It}\left(-\frac{1}{I}e^{-It}\right) + ce^{It}\\ &= \frac{D}{I} + ce^{It}.\end{aligned}$$

The constant c is fixed by requiring

$$y(0) = P,$$

that is

$$\frac{D}{I} + c = P.$$

Solving this for c gives $c = P - D/I$. Substituting this expression for c back into our solution $y(t)$ gives

$$y(t) = \frac{D}{I} - \left(\frac{D}{I} - P\right)e^{It}.$$

First observe that $y(t)$ *grows* if $D/I < P$. (This might be a good definition of loan sharking!) We have not yet determined D. To do so we use the condition that the loan is to be paid off at the end of N years, $y(N) = 0$. Substituting $t = N$ into our solution $y(t)$ and using this condition gives

$$0 = \frac{D}{I} - \left(\frac{D}{I} - P\right) e^{NI}.$$

Solving for D,

$$D = PI \, \frac{e^{NI}}{e^{NI} - 1}, \tag{2.9}$$

gives the sought after relation between D, P, I and N. For example, if $P = \$100,000$, $I = 0.06$ (6% interest) and the loan is for $N = 30$ years, then $D = \$7,188.20$ so the monthly payment is $D/12 = \$599.02$. Some years ago the mortgage rate was 12%. A quick calculation shows that the monthly payment on the same loan at this interest would have been \$1028.09.

We remark that this model is a continuous model—the rate of payback is at the continuous rate D. In fact, normally one pays back only monthly. Banks, therefore, might want to take this into account in their calculations. I've found from personal experience that the above model predicts the bank's calculations to within a few dollars.

Suppose we increase our monthly payments by, say, \$50. (We assume no prepayment penalty.) This \$50 goes then to paying off the principal. The problem then is how long does it take to pay off the loan? It is an exercise to show that the number of years is (D is the total payment in one year)

$$-\frac{1}{I} \log\left(1 - \frac{PI}{D}\right). \tag{2.10}$$

Another questions asks on a loan of N years at interest I how long does it take to pay off one-half of the principal? That is, we are asking for the time T when

$$y(T) = \frac{P}{2}.$$

It is an exercise to show that

$$T = \frac{1}{I} \log\left(\frac{1}{2}(e^{NI} + 1)\right). \tag{2.11}$$

For example, a 30 year loan at 9% is half paid off in the 23rd year. Notice that T does not depend upon the principal P.

2.2 Conservative systems

2.2.1 Energy conservation

Consider the motion of a particle of mass m in one dimension, i.e. the motion is along a line. We suppose that the force acting at a point x, $F(x)$, is *conservative.* This means there exists a function $V(x)$, called the *potential energy*, such that

$$F(x) = -\frac{dV}{dx}.$$

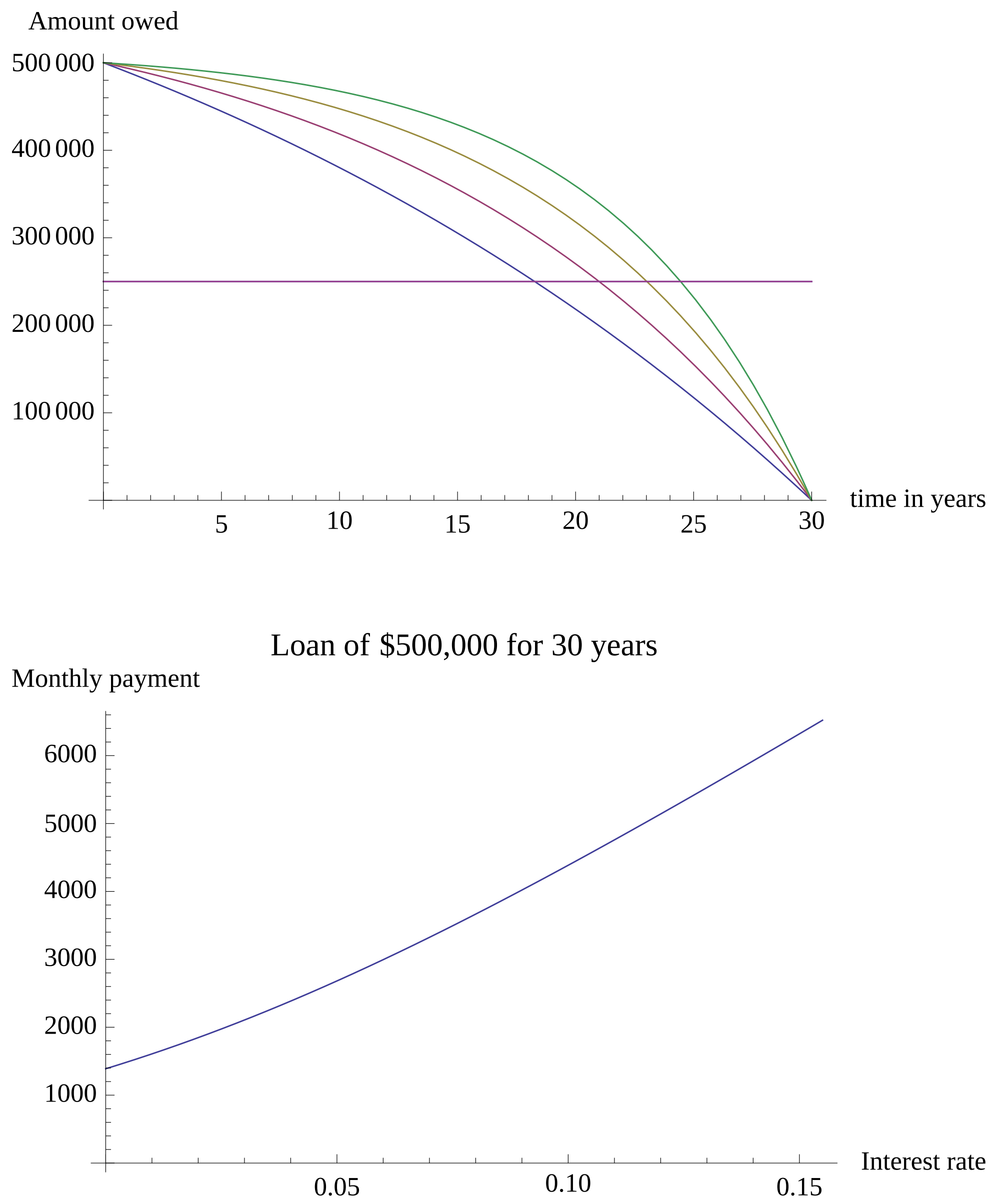


Figure 2.1: The top figure is the graph of the amount owned, $y(t)$, as a function of time t for a 30-year loan of \$500,000 at interest rates 3%, 6%, 9% and 12%. The horizontal line in the top figure is the line $y = \$250,000$; and hence, its intersection with the $y(t)$-curves gives the time when the loan is half paid off. The lower the interest rate the lower the $y(t)$-curve. The bottom figure gives the monthly payment on a 30-year loan of \$500,000 as a function of the interest rate I.

(Tradition has it we put in a minus sign.) In one dimension this requires that F is only a function of x and not $\dot{x}$ $(= dx/dt)$ which physically means there is no friction. In higher spatial dimensions the requirement that $\vec{F}$ is conservative is more stringent. The concept of *conservation of energy* is that

$$E = \text{Kinetic energy } + \text{ Potential energy}$$

does not change with time as the particle's position and velocity evolves according to Newton's equations. We now prove this fundamental fact. We recall from elementary physics that the kinetic energy (KE) is given by

$$\text{KE} = \frac{1}{2}mv^2, \;\; v = \text{velocity} = \dot{x}.$$

Thus the energy is

$$E = E(x, \dot{x}) = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + V(x).$$

To show that $E = E(x, \dot{x})$ does not change with t when $x = x(t)$ satisfies Newton's equations, we differentiate E with respect to t and show the result is zero:

$$\begin{aligned} \frac{dE}{dt} &= m\frac{dx}{dt}\frac{d^2x}{dt^2} + \frac{dV}{dx}\frac{dx}{dt} \text{ (by the chain rule)} \\ &= \frac{dx}{dt}\left(m\frac{d^2x}{dt^2} + \frac{dV(x)}{dx}\right) \\ &= \frac{dx}{dt}\left(m\frac{d^2x}{dt^2} - F(x)\right). \end{aligned}$$

Now not any function $x = x(t)$ describes the motion of the particle—$x(t)$ must satisfy

$$F = m\frac{d^2x}{dt^2},$$

and we now get the desired result

$$\frac{dE}{dt} = 0.$$

This implies that E is constant on solutions to Newton's equations.

We now use energy conservation and what we know about separation of variables to solve the problem of the motion of a point particle in a potential $V(x)$. Now

$$E = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + V(x) \tag{2.12}$$

is a nonlinear first order differential equation. (We know it is nonlinear since the first derivative is squared.) We rewrite the above equation as

$$\left(\frac{dx}{dt}\right)^2 = \frac{2}{m}\left(E - V(x)\right),$$

or

$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}\left(E - V(x)\right)}.$$

(In what follows we take the + sign, but in specific applications one must keep in mind the possibility that the − sign is the correct choice of the square root.) This last equation is of the form in which we can separate variables. We do this to obtain

$$\frac{dx}{\sqrt{\frac{2}{m}\left(E - V(x)\right)}} = dt.$$

This can be integrated to

$$\boxed{\pm \int \frac{1}{\sqrt{\frac{2}{m}\left(E - V(x)\right)}}\, dx \;=\; t - t_0.} \tag{2.13}$$

2.2.2 Hooke's Law

Figure 2.2: Robert Hooke, 1635–1703.

Consider a particle of mass m subject to the force

$$F = -kx,\; k > 0,\;\; \text{(Hooke's Law)}. \tag{2.14}$$

The minus sign (with $k > 0$) means the force is a restoring force—as in a spring. Indeed, to a good approximation the force a spring exerts on a particle is given by Hooke's Law. In

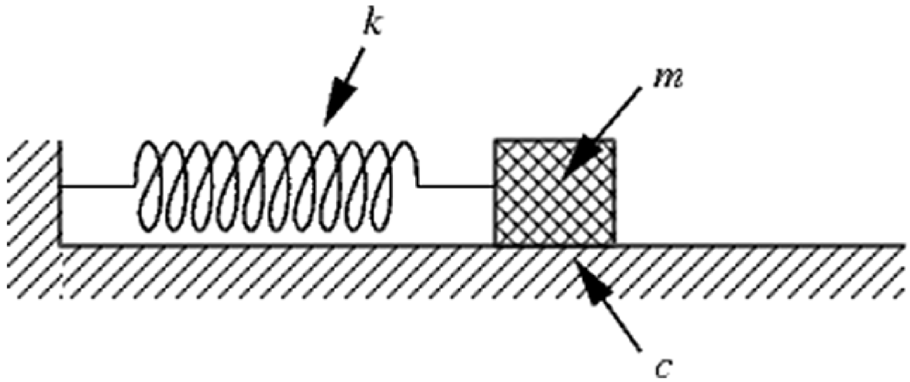


Figure 2.3: The mass-spring system: k is the spring constant in Hook's Law, m is the mass of the object and c represents a frictional force between the mass and floor. We neglect this frictional force. (Later we'll consider the effect of friction on the mass-spring system.)

this case $x = x(t)$ *measures the displacement from the equilibrium position at time* t; and the constant k is called the spring constant. Larger values of k correspond to a stiffer spring.

Newton's equations are in this case

$$m\frac{d^2x}{dt^2} + kx = 0. \tag{2.15}$$

This is a second order linear differential equation, the subject of the next chapter. However, we can use the energy conservation principle to derive an associated nonlinear first order equation as we discussed above. To do this, we first determine the potential corresponding to Hooke's force law.

One easily checks that the potential equals

$$V(x) = \frac{1}{2}\,k\,x^2.$$

(This potential is called the *harmonic potential*.) Let's substitute this particular V into (2.13):

$$\int \frac{1}{\sqrt{2E/m - kx^2/m}}\,dx = t - t_0. \tag{2.16}$$

Recall the indefinite integral

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{|a|}\right) + c.$$

Using this in (2.16) we obtain

$$\begin{aligned}\int \frac{1}{\sqrt{2E/m - kx^2/m}}\,dx &= \frac{1}{\sqrt{k/m}}\int \frac{dx}{\sqrt{2E/k - x^2}} \\ &= \frac{1}{\sqrt{k/m}}\,\arcsin\left(\frac{x}{\sqrt{2E/k}}\right) + c.\end{aligned}$$

Thus (2.16) becomes[3]

$$\arcsin\left(\frac{x}{\sqrt{2E/k}}\right) = \sqrt{\frac{k}{m}}\, t + c.$$

Taking the sine of both sides of this equation gives

$$\frac{x}{\sqrt{2E/k}} = \sin\left(\sqrt{\frac{k}{m}}\, t + c\right),$$

or

$$x(t) = \sqrt{\frac{2E}{k}}\, \sin\left(\sqrt{\frac{k}{m}}\, t + c\right). \tag{2.17}$$

Observe that there are two constants appearing in (2.17), E and c. Suppose one initial condition is

$$x(0) = x_0.$$

Evaluating (2.17) at $t = 0$ gives

$$x_0 = \sqrt{\frac{2E}{k}}\, \sin(c). \tag{2.18}$$

Now use the sine addition formula,

$$\sin(\theta_1 + \theta_2) = \sin\theta_1\, \cos\theta_2 + \sin\theta_2\, \cos\theta_1,$$

in (2.17):

$$\begin{aligned} x(t) &= \sqrt{\frac{2E}{k}} \left\{ \sin\left(\sqrt{\frac{k}{m}}\, t\right) \cos c + \cos\left(\sqrt{\frac{k}{m}}\, t\right) \sin c \right\} \\ &= \sqrt{\frac{2E}{k}}\, \sin\left(\sqrt{\frac{k}{m}}\, t\right) \cos c + x_0 \cos\left(\sqrt{\frac{k}{m}}\, t\right) \end{aligned} \tag{2.19}$$

where we use (2.18) to get the last equality.

Now substitute $t = 0$ into the energy conservation equation,

$$E = \frac{1}{2}\, m v_0^2 + V(x_0) = \frac{1}{2}\, m v_0^2 + \frac{1}{2}\, k\, x_0^2.$$

(v_0 equals the velocity of the particle at time $t = 0$.) Substituting (2.18) in the right hand side of this equation gives

$$E = \frac{1}{2}\, m v_0^2 + \frac{1}{2}\, k\, \frac{2E}{k} \sin^2 c$$

or

$$E(1 - \sin^2 c) = \frac{1}{2}\, m v_0^2.$$

Recalling the trig identity $\sin^2\theta + \cos^2\theta = 1$, this last equation can be written as

$$E \cos^2 c = \frac{1}{2}\, m v_0^2.$$

[3] We use the same symbol c for yet another unknown constant.

Solve this for v_0 to obtain the identity

$$v_0 = \sqrt{\frac{2E}{m}}\,\cos c.$$

We now use this in (2.19)

$$x(t) = v_0\sqrt{\frac{m}{k}}\,\sin\left(\sqrt{\frac{k}{m}}\,t\right) + x_0\cos\left(\sqrt{\frac{k}{m}}\,t\right).$$

To summarize, we have eliminated the two constants E and c in favor of the constants x_0 and v_0. As it must be, $x(0) = x_0$ and $\dot{x}(0) = v_0$. The last equation is more easily interpreted if we define

$$\omega_0 = \sqrt{\frac{k}{m}}. \tag{2.20}$$

Observe that ω_0 has the units of 1/time, i.e. frequency. Thus our final expression for the position $x = x(t)$ of a particle of mass m subject to Hooke's Law is

$$\boxed{x(t) = x_0\cos(\omega_0 t) + \frac{v_0}{\omega_0}\,\sin(\omega_0 t).} \tag{2.21}$$

Observe that this solution depends upon two arbitrary constants, x_0 and v_0.[4] In (2.7), the general solution depended only upon one constant. It is a general fact that the number of independent constants appearing in the general solution of a nth order[5] ODE is n.

Period of mass-spring system satisfying Hooke's Law

The sine and cosine are periodic functions of period 2π, i.e.

$$\sin(\theta + 2\pi) = \sin\theta, \ \ \cos(\theta + 2\pi) = \cos\theta.$$

This implies that our solution $x = x(t)$ is periodic in time,

$$x(t + T) = x(t),$$

where the period T is

$$\boxed{T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}.} \tag{2.23}$$

Observe that the period T, for the mass-spring system following Hooke's law, depends on the mass m and the spring constant k but *not* on the initial conditions .

[4] ω_0 is a constant too, but it is a parameter appearing in the differential equation that is fixed by the mass m and the spring constant k. Observe that we can rewrite (2.15) as

$$\ddot{x} + \omega_0^2 x = 0. \tag{2.22}$$

Dimensionally this equation is pleasing: $\ddot{x}$ has the dimensions of d/t^2 (d is distance and t is time) and so does $\omega_0^2\,x$ since ω_0 is a frequency. It is instructive to substitute (2.21) into (2.22) and verify directly that it is a solution. Please do so!

[5] The order of a scalar differential equation is equal to the order of the highest derivative appearing in the equation. Thus (2.3) is first order whereas (2.15) is second order.

2.2.3 Period of the nonlinear pendulum

In this section we use the method of separation of variables to derive an exact formula for the period of the pendulum. Recall that the ODE describing the time evolution of the angle of deflection, θ, is (1.11). This ODE is a second order equation and so the method of separation of variables does not apply to this equation. However, we will use energy conservation in a manner similar to the previous section on Hooke's Law.

To get some idea of what we should expect, first recall the approximation we derived for small deflection angles, (1.12). Comparing this differential equation with (2.15), we see that under the identification $x \to \theta$ and $\frac{k}{m} \to \frac{g}{\ell}$, the two equations are identical. Thus using the period derived in the last section, (2.23), we get as an approximation to the period of the pendulum

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{\ell}{g}}. \tag{2.24}$$

An important feature of T_0 is that it does not depend upon the amplitude of the oscillation.[6] That is, suppose we have the initial conditions[7]

$$\theta(0) = \theta_0, \;\; \dot{\theta}(0) = 0, \tag{2.25}$$

then T_0 does not depend upon θ_0. We now proceed to derive our formula for the period, T, of the pendulum.

We claim that the energy of the pendulum is given by

$$E = E(\theta, \dot{\theta}) = \frac{1}{2} m\ell^2 \dot{\theta}^2 + mg\ell(1 - \cos\theta). \tag{2.26}$$

Proof of (2.26)

We begin with

$$\begin{aligned} E &= \text{Kinetic energy } + \text{ Potential energy} \\ &= \frac{1}{2}mv^2 + mgy. \end{aligned} \tag{2.27}$$

(This last equality uses the fact that the potential at height h in a constant gravitational force field is mgh. In the pendulum problem with our choice of coordinates $h = y$.) The x and y coordinates of the pendulum ball are, in terms of the angle of deflection θ, given by

$$x = \ell\sin\theta, \;\; y = \ell(1 - \cos\theta).$$

Differentiating with respect to t gives

$$\dot{x} = \ell\cos\theta\,\dot{\theta}, \;\; \dot{y} = \ell\sin\theta\,\dot{\theta},$$

[6] Of course, its validity is only for small oscillations.

[7] For simplicity we assume the initial angular velocity is zero, $\dot{\theta}(0) = 0$. This is the usual initial condition for a pendulum.

from which it follows that the velocity is given by

$$\begin{aligned} v^2 &= \dot{x}^2 + \dot{y}^2 \\ &= \ell^2\,\dot{\theta}^2. \end{aligned}$$

Substituting these in (2.27) gives (2.26).

The energy conservation theorem states that for solutions $\theta(t)$ of (1.11), $E(\theta(t), \dot{\theta}(t))$ is independent of t. Thus we can evaluate E at $t = 0$ using the initial conditions (2.25) and know that for subsequent t the value of E remains unchanged,

$$\begin{aligned} E &= \frac{1}{2}\,m\ell^2\,\dot{\theta}(0)^2 + mg\ell\,(1 - \cos\theta(0)) \\ &= mg\ell(1 - \cos\theta_0). \end{aligned}$$

Using this (2.26) becomes

$$mg\ell(1 - \cos\theta_0) = \frac{1}{2}\,m\ell^2\,\dot{\theta}^2 + mg\ell(1 - \cos\theta),$$

which can be rewritten as

$$\frac{1}{2}\,m\ell^2\dot{\theta}^2 = mg\ell(\cos\theta - \cos\theta_0).$$

Solving for $\dot{\theta}$,

$$\dot{\theta} = \sqrt{\frac{2g}{\ell}\,(\cos\theta - \cos\theta_0)}\,,$$

followed by separating variables gives

$$\frac{d\theta}{\sqrt{\frac{2g}{\ell}\,(\cos\theta - \cos\theta_0)}} = dt. \tag{2.28}$$

We now integrate (2.28). The next step is a bit tricky—to choose the limits of integration in such a way that the integral on the right hand side of (2.28) is related to the period T. By the definition of the period, T is the time elapsed from $t = 0$ when $\theta = \theta_0$ to the time T when θ first returns to the point θ_0. By symmetry, $T/2$ is the time it takes the pendulum to go from θ_0 to $-\theta_0$. Thus if we integrate the left hand side of (2.28) from $-\theta_0$ to θ_0 the time elapsed is $T/2$. That is,

$$\frac{1}{2}\,T = \int_{-\theta_0}^{\theta_0} \frac{d\theta}{\sqrt{\frac{2g}{\ell}\,(\cos\theta - \cos\theta_0)}}\,.$$

Since the integrand is an even function of θ,

$$\boxed{T = 4\int_0^{\theta_0} \frac{d\theta}{\sqrt{\frac{2g}{\ell}\,(\cos\theta - \cos\theta_0)}}\,.} \tag{2.29}$$

This is the sought after formula for the period of the pendulum. For small θ_0 we expect that T, as given by (2.29), should be approximately equal to T_0 (see (2.24)). It is instructive to see this precisely.

We now assume $|\theta_0| \ll 1$ so that the approximation

$$\cos\theta \approx 1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4$$

is accurate for $|\theta| < \theta_0$. Using this approximation we see that

$$\begin{aligned} \cos\theta - \cos\theta_0 &\approx \frac{1}{2!}(\theta_0^2 - \theta^2) - \frac{1}{4!}(\theta_0^4 - \theta^4) \\ &= \frac{1}{2}(\theta_0^2 - \theta^2)\left(1 - \frac{1}{12}(\theta_0^2 + \theta^2)\right). \end{aligned}$$

From Taylor's formula[8] we get the approximation, valid for $|x| \ll 1$,

$$\frac{1}{\sqrt{1-x}} \approx 1 + \frac{1}{2}\,x.$$

Thus

$$\begin{aligned} \frac{1}{\sqrt{\frac{2g}{\ell}\left(\cos\theta - \cos\theta_0\right)}} &\approx \sqrt{\frac{\ell}{g}}\,\frac{1}{\sqrt{\theta_0^2 - \theta^2}}\,\frac{1}{\sqrt{1 - \frac{1}{12}\left(\theta_0^2 + \theta^2\right)}} \\ &\approx \sqrt{\frac{\ell}{g}}\,\frac{1}{\sqrt{\theta_0^2 - \theta^2}}\left(1 + \frac{1}{24}\left(\theta_0^2 + \theta^2\right)\right). \end{aligned}$$

Now substitute this approximate expression for the integrand appearing in (2.29) to find

$$\frac{T}{4} = \sqrt{\frac{\ell}{g}}\int_0^{\theta_0} \frac{1}{\sqrt{\theta_0^2 - \theta^2}}\left(1 + \frac{1}{24}\left(\theta_0^2 + \theta^2\right)\right) + \text{higher order corrections.}$$

Make the change of variables $\theta = \theta_0 x$, then

$$\begin{aligned} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\theta_0^2 - \theta^2}} &= \int_0^1 \frac{dx}{\sqrt{1 - x^2}} = \frac{\pi}{2}, \\ \int_0^{\theta_0} \frac{\theta^2\, d\theta}{\sqrt{\theta_0^2 - \theta^2}} &= \theta_0^2 \int_0^1 \frac{x^2\, dx}{\sqrt{1 - x^2}} = \theta_0^2\frac{\pi}{4}. \end{aligned}$$

Using these definite integrals we obtain

$$\begin{aligned} \frac{T}{4} &= \sqrt{\frac{\ell}{g}}\left(\frac{\pi}{2} + \frac{1}{24}(\theta_0^2\frac{\pi}{2} + \theta_0^2\frac{\pi}{4})\right) \\ &= \sqrt{\frac{\ell}{g}}\,\frac{\pi}{2}\left(1 + \frac{\theta_0^2}{16}\right) + \text{higher order terms.} \end{aligned}$$

[8] You should be able to do this without resorting to MatLab. But if you wanted higher order terms MatLab would be helpful. Recall to do this we would enter

```
>> syms x
>> taylor(1/sqrt(1-x))
```

Recalling (2.24), we conclude

$$T = T_0 \left(1 + \frac{\theta_0^2}{16} + \cdots\right) \tag{2.30}$$

where the $\cdots$ represent the higher order correction terms coming from higher order terms in the expansion of the cosines. These higher order terms will involve higher powers of θ_0. It now follows from this last expression that

$$\lim_{\theta_0 \to 0} T = T_0.$$

Observe that the first correction term to the linear result, T_0, depends upon the initial amplitude of oscillation θ_0.

In Figure 2.4 shows the graph of the ratio $T(\theta_0)/T_0$ as a function of the initial displacement angle θ_0.

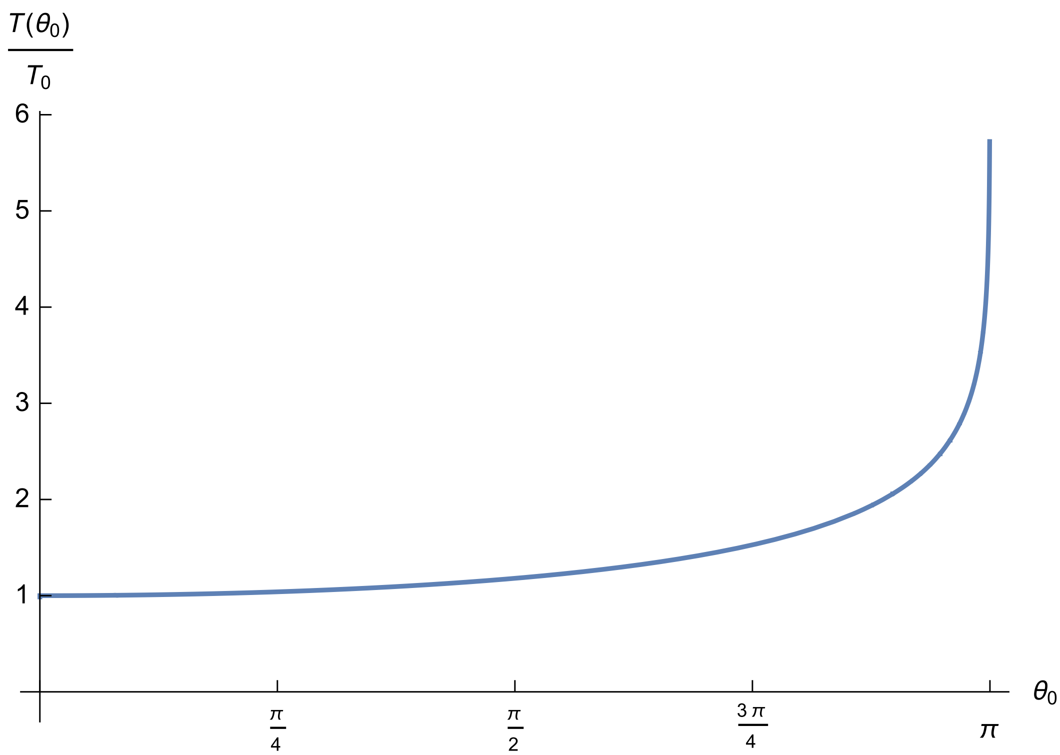


Figure 2.4: Graph of the the exact period $T(\theta_0)$ of the pendulum divided by the linear approximation $T_0 = 2\pi\sqrt{\frac{\ell}{g}}$ as a function of the initial deflection angle θ_0. It can be proved that as $\theta_0 \to \pi$, the period $T(\theta_0)$ diverges to $+\infty$. Even so, the linear approximation is quite good for moderate values of θ_0. For example at 45° ($\theta_0 = \pi/4$) the ratio is 1.03997. At 20° ($\theta_0 = \pi/9$) the ratio is 1.00767. The approximation (2.30) predicts for $\theta_0 = \pi/9$ the ratio 1.007632.

Remark: To use MatLab to evaluate symbolically these definite integrals you enter (note the use of ')

```
>> int('1/sqrt(1-x^2)',0,1)
```

and similarly for the second integral

```
>> int('x^2/sqrt(1-x^2)',0,1)
```

Numerical example

Suppose we have a pendulum of length $\ell = 1$ meter. The linear theory says that the period of the oscillation for such a pendulum is

$$T_0 = 2\pi\sqrt{\frac{\ell}{g}} = 2\pi\sqrt{\frac{1}{9.8}} = 2.0071 \text{ sec.}$$

If the amplitude of oscillation of the of the pendulum is $\theta_0 \approx 0.2$ (this corresponds to roughly a 20 cm deflection for the one meter pendulum), then (2.30) gives

$$T = T_0\left(1 + \frac{1}{16}\,(.2)^2\right) = 2.0121076 \text{ sec.}$$

One might think that these are so close that the correction is not needed. This might well be true if we were interested in only a few oscillations. What would be the difference in one week (1 week=604,800 sec)?

One might well ask how good an approximation is (2.30) to the exact result (2.29)? To answer this we have to evaluate numerically the integral appearing in (2.29). Evaluating (2.29) numerically (using say Mathematica's `NIntegrate`) is a bit tricky because the end-point θ_0 is singular—an integrable singularity but it causes numerical integration routines some difficulty. Here's how you get around this problem. One isolates where the problem occurs—near θ_0—and takes care of this analytically. For $\varepsilon > 0$ and $\varepsilon \ll 1$ we decompose the integral into two integrals: one over the interval $(0, \theta_0 - \varepsilon)$ and the other one over the interval $(\theta_0 - \varepsilon, \theta_0)$. It's the integral over this second interval that we estimate analytically. Expanding the cosine function about the point θ_0, Taylor's formula gives

$$\cos\theta = \cos\theta_0 - \sin\theta_0\,(\theta - \theta_0) - \frac{\cos\theta_0}{2}\,(\theta - \theta_0)^2 + \cdots .$$

Thus

$$\cos\theta - \cos\theta_0 = \sin\theta_0\,(\theta - \theta_0)\left(1 - \frac{1}{2}\cot\theta_0\,(\theta - \theta_0)\right) + \cdots .$$

So

$$\begin{aligned}
\frac{1}{\sqrt{\cos\theta - \cos\theta_0}} &= \frac{1}{\sqrt{\sin\theta_0\,(\theta - \theta_0)}}\,\frac{1}{\sqrt{1 - \frac{1}{2}\cot\theta_0(\theta_0 - \theta)}} + \cdots \\
&= \frac{1}{\sqrt{\sin\theta_0\,(\theta_0 - \theta)}}\left(1 + \frac{1}{4}\cot\theta_0\,(\theta_0 - \theta)\right) + \cdots
\end{aligned}$$

Thus

$$\begin{aligned}
\int_{\theta_0-\varepsilon}^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta-\cos\theta_0}} &= \int_{\theta_0-\varepsilon}^{\theta_0} \frac{d\theta}{\sqrt{\sin\theta_0\,(\theta_0-\theta)}}\left(1+\frac{1}{4}\cot\theta_0\,(\theta-\theta_0)\right)d\theta+\cdots \\
&= \frac{1}{\sqrt{\sin\theta_0}}\left(\int_0^{\varepsilon} u^{-1/2}\,du+\frac{1}{4}\cot\theta_0\int_0^{\varepsilon} u^{1/2}\,du+\cdots\right) \quad (u:=\theta_0-\theta) \\
&= \frac{1}{\sqrt{\sin\theta_0}}\left(2\varepsilon^{1/2}+\frac{1}{6}\cot\theta_0\,\varepsilon^{3/2}\right)+\cdots.
\end{aligned}$$

Choosing $\varepsilon = 10^{-2}$, the error we make in using the above expression is of order $\varepsilon^{5/2} = 10^{-5}$. Substituting $\theta_0 = 0.2$ and $\varepsilon = 10^{-2}$ into the above expression, we get the approximation

$$\int_{\theta_0-\varepsilon}^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta-\cos\theta_0}} \approx 0.4506$$

where we estimate the error lies in fifth decimal place. Now the numerical integration routine in MatLab quickly evaluates this integral:

$$\int_0^{\theta_0-\varepsilon} \frac{d\theta}{\sqrt{\cos\theta-\cos\theta_0}} \approx 1.7764$$

for $\theta_0 = 0.2$ and $\varepsilon = 10^{-2}$. Specifically, one enters

```
>> quad('1./sqrt(cos(x)-cos(0.2))',0,0.2-1/100)
```

Hence for $\theta_0 = 0.2$ we have

$$\int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta-\cos\theta_0}} \approx 0.4506+1.77664 = 2.2270$$

This implies

$$T \approx 2.0121.$$

Thus the first order approximation (2.30) is accurate to some four decimal places when $\theta_0 \leq 0.2$. (The reason for such good accuracy is that the correction term to (2.30) is of order θ_0^4.)

Remark: If you use MatLab to do the integral from 0 to θ_0 directly, i.e.

```
>> quad('1./sqrt(cos(x)-cos(0.2))',0,0.2)
```

what happens? This is an *excellent* example of what may go wrong if one uses software packages without *thinking first*! Use `help quad` to find out more about numerical integration in MatLab .

The attentive reader may have wondered how we produced the graph in Figure 2.4. It turns out that the integral (2.29) can be expressed in terms of a special function called "elliptic integral of the first kind". The software Mathematica has this special function and hence graphing it is easy to do: Just enter

```
Integrate[1/Sqrt[Cos[x]-Cos[x0]],{x,0,x0},Assumptions->{0<x0<Pi}]
```

to get the integral in terms of this special function. You can now ask Mathematica to plot the result.

2.3 Level curves of the energy

For the mass-spring system (Hooke's Law) the energy is

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \tag{2.31}$$

which we can rewrite as

$$\left(\frac{x}{a}\right)^2 + \left(\frac{v}{b}\right)^2 = 1$$

where $a = \sqrt{2E/k}$ and $b = \sqrt{2E/m}$. We recognize this last equation as the equation of an ellipse. Assuming k and m are fixed, we see that for various values of the energy E we get different ellipses in the (x, v)-plane. Thus the values of $x = x(t)$ and $v = v(t)$ are fixed to lie on various ellipses. The ellipse is fixed once we specify the energy E of the mass-spring system.

For the pendulum the energy is

$$E = \frac{1}{2}\, m\ell^2\omega^2 + mg\ell(1 - \cos\theta) \tag{2.32}$$

where $\omega = d\theta/dt$. What do the contour curves of (2.32) look like? That is we want the curves in the (θ, ω)-plane that obey (2.32).

To make things simpler, we set $\frac{1}{2}\, m\ell^2 = 1$ and $mg\ell = 1$ so that (2.32) becomes

$$E = \omega^2 + (1 - \cos\theta) \tag{2.33}$$

We now use Mathematica to plot the contour lines of (2.33) in the (θ, ω)-plane (see Figure 2.5). For small E the contour lines look roughly like ellipses but as E gets larger the ellipses become more deformed. At $E = 2$ there is a curve that separates the deformed elliptical curves from curves that are completely different (those contour lines corresponding to $E > 2$). In terms of the pendulum what do you think happens when $E > 2$?

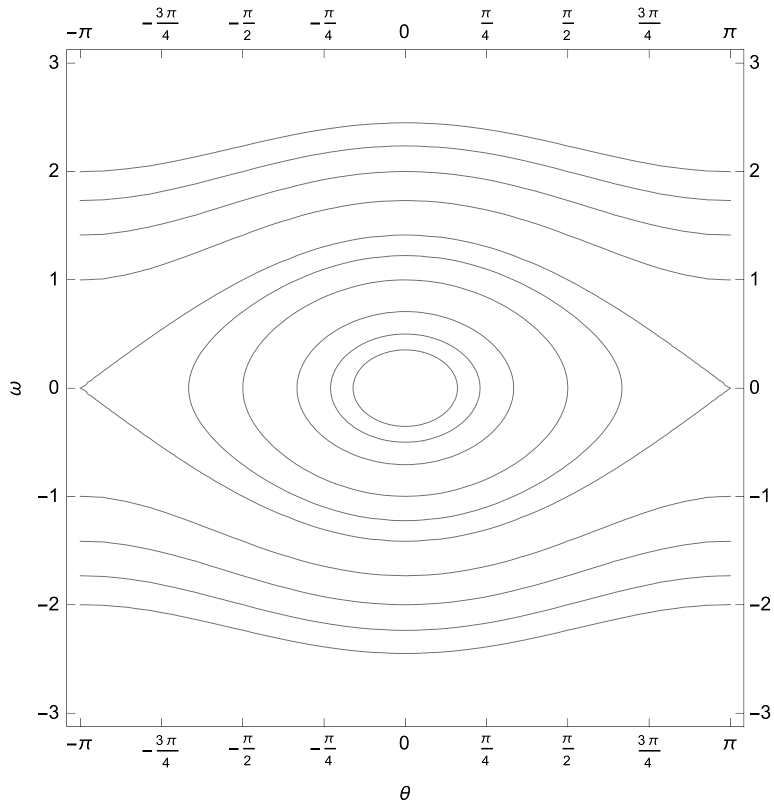


Figure 2.5: Contour lines for (2.33) for various values of the energy E.

2.4 Exercises for Chapter 2

#1. Radioactive decay

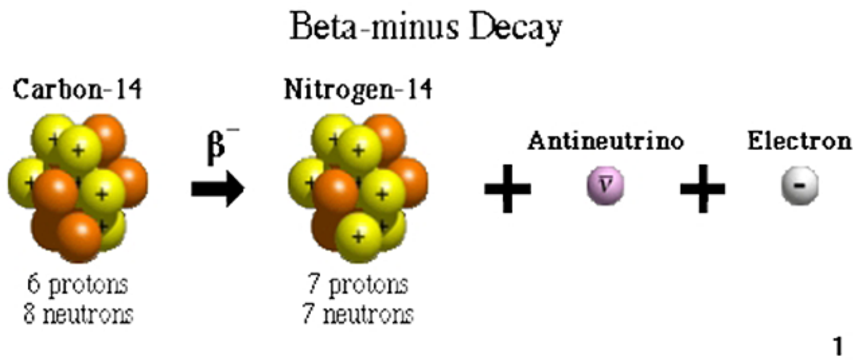


Figure 2.6:

Carbon 14 is an unstable (radioactive) isotope of stable Carbon 12. If $Q(t)$ represents the amount of C14 at time t, then Q is known to satisfy the ODE

$$\frac{dQ}{dt} = -\lambda Q$$

where λ is a constant. If $T_{1/2}$ denotes the half-life of C14 show that

$$T_{1/2} = \frac{\log 2}{\lambda}.$$

Recall that the half-life $T_{1/2}$ is the time $T_{1/2}$ such that $Q(T_{1/2}) = Q(0)/2$. It is known for C14 that $T_{1/2} \approx 5730$ years. In Carbon 14 dating[9] it becomes difficult to measure the levels of C14 in a substance when it is of order 0.1% of that found in currently living material. How many years must have passed for a sample of C14 to have decayed to 0.1% of its original value? The technique of Carbon 14 dating is not so useful after this number of years.

[9]From Wikipedia: The Earth's atmosphere contains various isotopes of carbon, roughly in constant proportions. These include the main stable isotope C12 and an unstable isotope C14. Through photosynthesis, plants absorb both forms from carbon dioxide in the atmosphere. When an organism dies, it contains the standard ratio of C14 to C12, but as the C14 decays with no possibility of replenishment, the proportion of carbon 14 decreases at a known constant rate. The time taken for it to reduce by half is known as the half-life of C14. The measurement of the remaining proportion of C14 in organic matter thus gives an estimate of its age (a raw radiocarbon age). However, over time there are small fluctuations in the ratio of C14 to C12 in the atmosphere, fluctuations that have been noted in natural records of the past, such as sequences of tree rings and cave deposits. These records allow fine-tuning, or "calibration", of the raw radiocarbon age, to give a more accurate estimate of the calendar date of the material. One of the most frequent uses of radiocarbon dating is to estimate the age of organic remains from archaeological sites. The concentration of C14 in the atmosphere might be expected to reduce over thousands of years. However, C14 is constantly being produced in the lower stratosphere and upper troposphere by cosmic rays, which generate neutrons that in turn create C14 when they strike nitrogen–14 atoms. Once produced, the C14 quickly combines with the oxygen in the atmosphere to form carbon dioxide. Carbon dioxide produced in this way diffuses in the atmosphere, is dissolved in the ocean, and is taken up by plants via photosynthesis. Animals eat the plants, and ultimately the radiocarbon is distributed throughout the biosphere.

#2: Mortgage payment problem

In the problem dealing with mortgage rates, prove (2.10) and (2.11). Using either a hand calculator or some computer software, create a table of monthly payments on a loan of \$200,000 for 30 years for interest rates from 1% to 15% in increments of 1%.

#3: Discontinuous forcing term

Solve

$$y' + 2y = g(t),\ y(0) = 0,$$

where

$$g(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & t > 1 \end{cases}$$

We make the additional assumption that the solution $y = y(t)$ should be a continuous function of t. Hint: First solve the differential equation on the interval $[0, 1]$ and then on the interval $[1, \infty)$. You are given the initial value at $t = 0$ and after you solve the equation on $[0, 1]$ you will then know $y(1)$.[10] Plot the solution $y = y(t)$ for $0 \le t \le 4$. (You can use any computer algebra program or just graph the $y(t)$ by hand.)

#4. Application to population dynamics

In biological applications the population P of certain organisms at time t is sometimes assumed to obey the equation

$$\frac{dP}{dt} = aP\left(1 - \frac{P}{E}\right) \tag{2.34}$$

where a and E are positive constants. This model is sometimes called the *logistic growth model*.

1. Find the equilibrium solutions. (That is solutions that don't change with t.)
2. From (2.34) determine the regions of P where P is increasing (decreasing) as a function of t. Again using (2.34) find an expression for d^2P/dt^2 in terms of P and the constants a and E. From this expression find the regions of P where P is convex ($d^2P/dt^2 > 0$) and the regions where P is concave ($d^2P/dt^2 < 0$).
3. Using the method of separation of variables solve (2.34) for $P = P(t)$ assuming that at $t = 0$, $P = P_0 > 0$. Find
$$\lim_{t\to\infty} P(t)$$
Hint: To do the integration first use the identity
$$\frac{1}{P(1 - P/E)} = \frac{1}{P} + \frac{1}{E - P}$$
4. Sketch P as a function of t for $0 < P_0 < E$ and for $E < P_0 < \infty$.

[10] This is problem #32, pg. 74 (7th edition) of the Boyce & DiPrima [4].

#5: Mass-spring system with friction

We reconsider the mass-spring system but now assume there is a frictional force present and this frictional force is proportional to the velocity of the particle. Thus the force acting on the particle comes from two terms: one due to the force exerted by the spring and the other due to the frictional force. Thus Newton's equations become

$$-kx - \beta \dot{x} = m\ddot{x} \tag{2.35}$$

where as before $x = x(t)$ is the displacement from the equilibrium position at time t. β and k are positive constants. Introduce the energy function

$$E = E(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2, \tag{2.36}$$

and show that if $x = x(t)$ satisfies (2.35), then

$$\frac{dE}{dt} < 0.$$

What is the physical meaning of this last inequality?

#6: Nonlinear mass-spring system

Consider a mass-spring system where $x = x(t)$ denotes the displacement of the mass m from its equilibrium position at time t. The linear spring (Hooke's Law) assumes the force exerted by the spring on the mass is given by (2.14). Suppose instead that the force F is given by

$$F = F(x) = -kx - \varepsilon\, x^3 \tag{2.37}$$

where ε is a small positive number.[11] The second term represents a nonlinear correction to Hooke's Law. Why is it reasonable to assume that the first correction term to Hooke's Law is of order x^3 and not x^2? (Hint: Why is it reasonable to assume $F(x)$ is an *odd* function of x?) Using the solution for the period of the pendulum as a guide, find an *exact* integral expression for the period T of this nonlinear mass-spring system assuming the initial conditions

$$x(0) = x_0, \;\; \frac{dx}{dt}(0) = 0.$$

Define

$$z = \frac{\varepsilon x_0^2}{2k}.$$

Show that z is dimensionless and that your expression for the period T can be written as

$$T = \frac{4}{\omega_0}\int_0^1 \frac{1}{\sqrt{1-u^2+z-zu^4}}\, du \tag{2.38}$$

where $\omega_0 = \sqrt{k/m}$. We now assume that $z \ll 1$. (This is the precise meaning of the parameter ε being small.) Taylor expand the function

$$\frac{1}{\sqrt{1-u^2+z-zu^4}}$$

[11] One could also consider $\varepsilon < 0$. The case $\varepsilon > 0$ is a called a *hard* spring and $\varepsilon < 0$ a *soft* spring.

in the variable z to first order. You should find

$$\frac{1}{\sqrt{1-u^2+z-zu^4}} = \frac{1}{\sqrt{1-u^2}} - \frac{1+u^2}{2\sqrt{1-u^2}}\, z + \mathrm{O}(z^2).$$

Now use this approximate expression in the integrand of (2.38), evaluate the definite integrals that arise, and show that the period T has the Taylor expansion

$$T = \frac{2\pi}{\omega_0}\left(1 - \frac{3}{4}\, z + \mathrm{O}(z^2)\right).$$

#7: Motion in a central field

A (three-dimensional) force $\vec{F}$ is called a *central force*[12] if the direction of $\vec{F}$ lies along the the direction of the position vector $\vec{r}$. This problem asks you to show that the motion of a particle in a central force, satisfying

$$\vec{F} = m\frac{d^2\vec{r}}{dt^2}, \tag{2.39}$$

lies in a plane.

1. Show that
$$\vec{M} := \vec{r} \times \vec{p} \;\text{ with }\; \vec{p} := m\vec{v} \tag{2.40}$$
is *constant* in t for $\vec{r} = \vec{r}(t)$ satisfying (2.39). (Here $\vec{v} = d\vec{r}/dt$ is the velocity vector and $\vec{p}$ is the *momentum vector*. In words, the momentum vector is mass times the velocity vector.) The $\times$ in (2.40) is the vector cross product. Recall (and you may assume this result) from vector calculus that
$$\frac{d}{dt}(\vec{a} \times \vec{b}) = \frac{d\vec{a}}{dt} \times \vec{b} + \vec{a} \times \frac{d\vec{b}}{dt}.$$
The vector $\vec{M}$ is called the *angular momentum* vector.

2. From the fact that $\vec{M}$ is a constant vector, show that the vector $\vec{r}(t)$ lies in a plane perpendicular to $\vec{M}$. Hint: Look at $\vec{r} \cdot \vec{M}$. Also you may find helpful the vector identity
$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b}).$$

#8: Motion in a central field (cont)

From the preceding problem we learned that the position vector $\vec{r}(t)$ for a particle moving in a central force lies in a plane. In this plane, let (r, θ) be the polar coordinates of the point $\vec{r}$, i.e.

$$x(t) = r(t)\cos\theta(t), \;\; y(t) = r(t)\sin\theta(t) \tag{2.41}$$

[12] For an in depth treatment of motion in a central field, see [1], Chapter 2, §8.

1. In components, Newton's equations can be written (why?)
$$F_x = f(r)\frac{x}{r} = m\ddot{x},\;\; F_y = f(r)\frac{y}{r} = m\ddot{y} \tag{2.42}$$
where $f(r)$ is the magnitude of the force $\vec{F}$. By twice differentiating (2.41) with respect to t, derive formulas for $\ddot{x}$ and $\ddot{y}$ in terms of r, θ and their derivatives. Use these formulas in (2.42) to show that Newton's equations in polar coordinates (and for a central force) become
$$\frac{1}{m}\,f(r)\cos\theta \;=\; \ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta - r\ddot{\theta}\sin\theta, \tag{2.43}$$
$$\frac{1}{m}\,f(r)\sin\theta \;=\; \ddot{r}\sin\theta + 2\dot{r}\dot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta + r\ddot{\theta}\cos\theta. \tag{2.44}$$
Multiply (2.43) by $\cos\theta$, (2.44) by $\sin\theta$, and add the resulting two equations to show that
$$\ddot{r} - r\dot{\theta}^2 = \frac{1}{m}\,f(r). \tag{2.45}$$
Now multiply (2.43) by $\sin\theta$, (2.44) by $\cos\theta$, and substract the resulting two equations to show that
$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0. \tag{2.46}$$
Observe that the left hand side of (2.46) is equal to
$$\frac{1}{r}\,\frac{d}{dt}(r^2\dot{\theta}).$$
Using this observation we then conclude (why?)
$$r^2\dot{\theta} = H \tag{2.47}$$
for some constant H. Use (2.47) to solve for $\dot{\theta}$, eliminate $\dot{\theta}$ in (2.45) to conclude that the polar coordinate function $r = r(t)$ satisfies
$$\ddot{r} = \frac{1}{m}\,f(r) + \frac{H^2}{r^3}\,. \tag{2.48}$$

2. Equation (2.48) is of the form that a second derivative of the unknown r is equal to some function of r. We can thus apply our general *energy method* to this equation. Let Φ be a function of r satisfying
$$\frac{1}{m}\,f(r) = -\frac{d\Phi}{dr},$$
and find an *effective potential* $V = V(r)$ such that (2.48) can be written as
$$\ddot{r} = -\frac{dV}{dr} \tag{2.49}$$
(Ans: $V(r) = \Phi(r) + \frac{H^2}{2r^2}$). Remark: The most famous choice for $f(r)$ is the inverse square law
$$f(r) = -\frac{mMG_0}{r^2}$$
which describes the gravitational attraction of two particles of masses m and M. (G_0 is the universal gravitational constant.) In your physics courses, this case will be analyzed in great detail. The starting point is what we have done here.

3. With the choice
$$f(r) = -\frac{mMG_0}{r^2}$$
the equation (2.48) gives a DE that is satisfied by r as a function of t:
$$\ddot{r} = -\frac{G}{r^2} + \frac{H^2}{r^3} \tag{2.50}$$
where $G = MG_0$. We now use (2.50) to obtain a DE that is satisfied by r as a function of θ. This is the quantity of interest if one wants the *orbit* of the planet. Assume that $H \neq 0$, $r \neq 0$, and set $r = r(\theta)$. First, show that by chain rule
$$\ddot{r} = r''\dot{\theta}^2 + r'\ddot{\theta}. \tag{2.51}$$
(Here, $'$ implies the differentiation with respect to θ, and as usual, the dot refers to differentiation with respect to time.) Then use (2.47) and (2.51) to obtain
$$\ddot{r} = r''\frac{H^2}{r^4} - (r')^2\frac{2H^2}{r^5} \tag{2.52}$$
Now, obtain a second order DE of r as a function of θ from (2.50) and (2.52). Finally, by letting $u(\theta) = 1/r(\theta)$, obtain a simple linear constant coefficient DE
$$u'' + u = \frac{G}{H^2} \tag{2.53}$$
which is known as *Binet's equation.*[13]

#9: Euler's equations for a rigid body with no torque

In mechanics one studies the motion of a rigid body[14] around a stationary point in the absence of outside forces. Euler's equations are differential equations for the angular velocity vector $\mathbf{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$. If I_i denotes the moment of inertia of the body with respect to the ith principal axis, then Euler's equations are
$$\begin{aligned} I_1\frac{d\Omega_1}{dt} &= (I_2 - I_3)\Omega_2\Omega_3 \\ I_2\frac{d\Omega_2}{dt} &= (I_3 - I_1)\Omega_3\Omega_1 \\ I_3\frac{d\Omega_3}{dt} &= (I_1 - I_2)\Omega_1\Omega_2 \end{aligned}$$
Prove that
$$\mathcal{M} = I_1^2\Omega_1^2 + I_2^2\Omega_2^2 + I_3^2\Omega_3^2$$
and
$$\mathcal{E} = \frac{1}{2}I_1\Omega_1^2 + \frac{1}{2}I_2\Omega_2^2 + \frac{1}{2}I_3\Omega_3^2$$
are both first integrals of the motion. (That is, if the Ω_j evolve according to Euler's equations, then $\mathcal{M}$ and $\mathcal{E}$ are independent of t.)

[13] For further discussion of Binet's equation see [8].
[14] For an in-depth discussion of rigid body motion see Chapter 6 of [1].

#10. Exponential function

In calculus one defines the exponential function e^t by

$$e^t := \lim_{n\to\infty}(1+\frac{t}{n})^n \,, t\in\mathbb{R}.$$

Suppose one took the point of view of differential equations and *defined* e^t to be the (unique) solution to the ODE

$$\frac{dE}{dt} = E \tag{2.54}$$

that satisfies the initial condition $E(0) = 1$.[15] Prove that the addition formula

$$e^{t+s} = e^t e^s$$

follows from the ODE definition. [Hint: Define

$$\phi(t) := E(t+s) - E(t)E(s)$$

where $E(t)$ is the above unique solution to the ODE satisfying $E(0) = 1$. Show that ϕ satisfies the ODE

$$\frac{d\phi}{dt} = \phi(t)$$

From this conclude that necessarily $\phi(t) = 0$ for all t.]

Using the above ODE definition of $E(t)$ show that

$$\int_0^t E(s)\,ds = E(t) - 1.$$

Let $E_0(t) = 1$ and define $E_n(t)$, $n \geq 1$ by

$$E_{n+1}(t) = 1 + \int_0^t E_n(s)\,ds, \;\; n = 0, 1, 2, \ldots. \tag{2.55}$$

Show that

$$E_n(t) = 1 + t + \frac{t^2}{2!} + \cdots + \frac{t^n}{n!}.$$

By the ratio test this sequence of partial sums converges as $n \to \infty$. Assuming one can take the limit $n \to \infty$ inside the integral (2.55),[16]conclude that

$$e^t = E(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}$$

[15]That is, we are taking the point of view that we define e^t to be the solution $E(t)$. Here is a proof that given a solution to (2.54) satisfying the initial condition $E(0) = 1$, that such a solution is unique. Suppose we have found two such solutions: $E_1(t)$ and $E_2(t)$. Let $y(t) = E_1(t)/E_2(t)$, then

$$\frac{dy}{dt} = \frac{1}{E_2}\frac{dE_1}{dt} - \frac{E_1}{E_2^2}\frac{dE_2}{dt} = \frac{E_1}{E_2} - \frac{E_1}{E_2^2}E_2 = 0$$

Thus $y(t) =$ constant. But we know that $y(0) = E_1(0)/E_2(0) = 1$. Thus $y(t) = 1$, or $E_1(t) = E_2(t)$.

[16]The series $\sum_{n\geq 0} s^n/n!$ converges *uniformly* on the closed interval $[0, t]$. From this fact it follows that one is allowed to interchange the sum and integration. These convergence topics are normally discussed in an advanced calculus course.

#11. Addition formula for the tangent function

Suppose we wish to find a real-valued, differentiable function $F(x)$ that satisfies the functional equation

$$F(x+y) = \frac{F(x)+F(y)}{1-F(x)F(y)} \tag{2.56}$$

1. Show that such an F necessarily satisfies $F(0) = 0$. Hint: Use (2.56) to get an expression for $F(0+0)$ and then use fact that we seek F to be real-valued.
2. Set $\alpha = F'(0)$. Show that F must satisfy the differential equation

   $$\frac{dF}{dx} = \alpha(1+F(x)^2) \tag{2.57}$$

   Hint: Differentiate (2.56) with respect to y and then set $y = 0$.
3. Use the method of separation of variables to solve (2.57) and show that

   $$F(x) = \tan(\alpha x).$$

Chapter 3

Second Order Linear Equations

Figure 3.1: $e^{ix} = \cos + i \sin x$, Leonhard Euler, *Introductio in Analysin Infinitorum*, 1748

3.1 Theory of second order equations

3.1.1 Vector space of solutions

First order linear differential equations are of the form

$$\frac{dy}{dx} + p(x)y = f(x). \tag{3.1}$$

Second order linear differential equations are linear differential equations whose highest derivative is second order:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = f(x). \tag{3.2}$$

If $f(x) = 0$,

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0, \tag{3.3}$$

the equation is called *homogeneous*. For the discussion here, we assume p and q are continuous functions on a closed interval $[a, b]$. There are many important examples where this condition fails and the points at which either p or q fail to be continuous are called *singular points*. An introduction to singular points in ordinary differential equations can be found in Boyce & DiPrima [4]. Here are some important examples where the continuity condition fails.

Legendre's equation

$$p(x) = -\frac{2x}{1-x^2}, \;\; q(x) = \frac{n(n+1)}{1-x^2}\,.$$

At the points $x = \pm 1$ both p and q fail to be continuous.

Bessel's equation

$$p(x) = \frac{1}{x}, \;\; q(x) = 1 - \frac{\nu^2}{x^2}\,.$$

At the point $x = 0$ both p and q fail to be continuous.

We saw that a solution to (3.1) was uniquely specified once we gave one initial condition,

$$y(x_0) = y_0.$$

In the case of second order equations we must give two initial conditions to specify uniquely a solution:

$$y(x_0) = y_0 \;\text{ and }\; y'(x_0) = y_1. \tag{3.4}$$

This is a basic theorem of the subject. It says that if p and q are continuous on some interval (a, b) and $a < x_0 < b$, then there exists an unique solution to (3.3) satisfying the initial conditions (3.4).[1] We will not prove this theorem in this class. As an example of the

[1] See Theorem 3.2.1 in the [4], pg. 131 or chapter 6 of [3]. These theorems dealing with the existence and uniqueness of the initial value problem are covered in an advanced course in differential equations.

appearance to two constants in the general solution, recall that the solution of the harmonic oscillator

$$\ddot{x} + \omega_0^2 x = 0$$

contained x_0 and v_0.

Let $\mathcal{V}$ denote the set of all solutions to (3.3). The most important feature of $\mathcal{V}$ is that it is a *two-dimensional vector space*. That it is a vector space follows from the linearity of (3.3). (If y_1 and y_2 are solutions to (3.3), then so is $c_1 y_1 + c_2 y_2$ for all constants c_1 and c_2.) To prove that the dimension of $\mathcal{V}$ is two, we first introduce two special solutions. Let Y_1 and Y_2 be the unique solutions to (3.3) that satisfy the initial conditions

$$Y_1(0) = 1,\ \ Y_1'(0) = 0, \ \ \text{and}\ \ Y_2(0) = 0,\ \ Y_2'(0) = 1,$$

respectively.

We claim that $\{Y_1, Y_2\}$ forms a basis for $\mathcal{V}$. To see this let $y(x)$ be any solution to (3.3).[2] Let $c_1 := y(0)$, $c_2 := y'(0)$ and

$$\Delta(x) := y(x) - c_1\, Y_1(x) - c_2\, Y_2(x).$$

Since y, Y_1 and Y_2 are solutions to (3.3), so too is Δ. ($\mathcal{V}$ is a vector space.) Observe

$$\Delta(0) = 0 \ \ \text{and}\ \ \Delta'(0) = 0. \tag{3.5}$$

Now the function $y_0(x) :\equiv 0$ satisfies (3.3) and the initial conditions (3.5). Since solutions are unique, it follows that $\Delta(x) \equiv y_0 \equiv 0$. That is,

$$y = c_1\, Y_1 + c_2\, Y_2.$$

To summarize, we've shown every solution to (3.3) is a linear combination of Y_1 and Y_2. That Y_1 and Y_2 are linearly independent follows from their initial values: Suppose

$$c_1 Y_1(x) + c_2 Y_2(x) = 0.$$

Evaluate this at $x = 0$, use the initial conditions to see that $c_1 = 0$. Take the derivative of this equation, evaluate the resulting equation at $x = 0$ to see that $c_2 = 0$. Thus, Y_1 and Y_2 are linearly independent. We conclude, therefore, that $\{Y_1, Y_2\}$ is a basis and $\dim \mathcal{V} = 2$.

3.1.2 Wronskians

Given two solutions y_1 and y_2 of (3.3) it is useful to find a simple condition that tests whether they form a basis of $\mathcal{V}$. Let φ be the solution of (3.3) satisfying $\varphi(x_0) = \varphi_0$ and $\varphi'(x_0) = \varphi_1$. We ask are there constants c_1 and c_2 such that

$$\varphi(x) = c_1 y_1(x) + c_2 y_2(x)$$

for all x? A necessary and sufficient condition that such constants exist at $x = x_0$ is that the equations

$$\begin{aligned} \varphi_0 &= c_1\, y_1(x_0) + c_2\, y_2(x_0), \\ \varphi_1 &= c_1\, y'(x_0) + c_2\, y_2'(x_0), \end{aligned}$$

[2] We assume for convenience that $x = 0$ lies in the interval (a, b).

have a unique solution $\{c_1, c_2\}$. From linear algebra we know this holds if and only if the determinant

$$\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} \neq 0.$$

We define the Wronskian of two solutions y_1 and y_2 of (3.3) to be

$$W(y_1, y_2; x) := \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x). \end{vmatrix} = y_1(x)y_2'(x) - y_1'(x)y_2(x). \tag{3.6}$$

From what we have said so far one would have to check that $W(y_1, y_2; x) \neq 0$ for all x to conclude $\{y_1, y_2\}$ forms a basis.

We now derive a formula for the Wronskian that will make the check necessary at only one point. Since y_1 and y_2 are solutions of (3.3), we have

$$y_1'' + p(x)y_1' + q(x)y_1 = 0, \tag{3.7}$$
$$y_2'' + p(x)y_2' + q(x)y_2 = 0. \tag{3.8}$$

Now multiply (3.7) by y_2 and multiply (3.8) by y_1. Subtract the resulting two equations to obtain

$$y_1y_2'' - y_1''y_2 + p(x)\left(y_1y_2' - y_1'y_2\right) = 0. \tag{3.9}$$

Recall the definition (3.6) and observe that

$$\frac{dW}{dx} = y_1y_2'' - y_1''y_2.$$

Hence (3.9) is the equation

$$\frac{dW}{dx} + p(x)W(x) = 0, \tag{3.10}$$

whose solution is

$$W(y_1, y_2; x) = c\,\exp\left(-\int^x p(s)\,dx\right). \tag{3.11}$$

Since the exponential is never zero we see from (3.11) that either $W(y_1, y_2; x) \equiv 0$ or $W(y_1, y_2; x)$ is never zero.

To summarize, to determine if $\{y_1, y_2\}$ forms a basis for $\mathcal{V}$, one needs to check at *only one point* whether the Wronskian is zero or not.

Applications of Wronskians

1. Claim: Suppose $\{y_1, y_2\}$ form a basis of $\mathcal{V}$, then they cannot have a common point of inflection in $a < x < b$ unless $p(x)$ and $q(x)$ simultaneously vanish there. To prove this, suppose x_0 is a common point of inflection of y_1 and y_2. That is,

$$y_1''(x_0) = 0 \quad\text{and}\quad y_2''(x_0) = 0.$$

Evaluating the differential equation (3.3) satisfied by both y_1 and y_2 at $x = x_0$ gives

$$\begin{aligned} p(x_0)y_1'(x_0) + q(x_0)y_1(x_0) &= 0, \\ p(x_0)y_2'(x_0) + q(x_0)y_2(x_0) &= 0. \end{aligned}$$

Assuming that $p(x_0)$ and $q(x_0)$ are not both zero at x_0, the above equations are a set of homogeneous equations for $p(x_0)$ and $q(x_0)$. The only way these equations can have a nontrivial solution is for the determinant

$$\begin{vmatrix} y_1'(x_0) & y_1(x_0) \\ y_2'(x_0) & y_2(x_0) \end{vmatrix} = 0.$$

That is, $W(y_1, y_2; x_0) = 0$. But this contradicts that $\{y_1, y_2\}$ forms a basis. Thus there can exist no such common inflection point.

2. Claim: Suppose $\{y_1, y_2\}$ form a basis of $\mathcal{V}$ and that y_1 has *consecutive* zeros at $x = x_1$ and $x = x_2$. Then y_2 has one and only one zero between x_1 and x_2. To prove this we first evaluate the Wronskian at $x = x_1$,

$$W(y_1, y_2; x_1) = y_1(x_1)y_2'(x_1) - y_1'(x_1)y_2(x_1) = -y_1'(x_1)y_2(x_1)$$

since $y_1(x_1) = 0$. Evaluating the Wronskian at $x = x_2$ gives

$$W(y_1, y_2; x_2) = -y_1'(x_2)y_2(x_2).$$

Now $W(y_1, y_2; x_1)$ is either positive or negative. (It can't be zero.) Let's assume it is positive. (The case when the Wronskian is negative is handled similarly. We leave this case to the reader.) Since the Wronskian is always of the same sign, $W(y_1, y_2; x_2)$ is also positive. Since x_1 and x_2 are consecutive zeros, the signs of $y_1'(x_1)$ and $y_1'(x_2)$ are opposite of each other. But this implies (from knowing that the two Wronskian expressions are both positive), that $y_2(x_1)$ and $y_2(x_2)$ have opposite signs. Thus there exists at least one zero of y_2 at $x = x_3$, $x_1 < x_3 < x_2$. If there exist two or more such zeros, then between any two of these zeros apply the above argument (with the roles of y_1 and y_2 reversed) to conclude that y_1 has a zero between x_1 and x_2. But x_1 and x_2 were assumed to be consecutive zeros. Thus y_2 has one and only one zero between x_1 and x_2.

In the case of the harmonic oscillator, $y_1(x) = \cos\omega_0 x$ and $y_2(x) = \sin\omega_0 x$, and the fact that the zeros of the sine function interlace those of the cosine function is well known.

Here is a second example: Consider the Airy differential equation

$$\frac{d^2y}{dx^2} - xy = 0 \tag{3.12}$$

Two linearly independent solutions to the Airy DE are plotted in Figure 3.2. We denote these particular linearly independent solutions by $y_1(x) := \mathrm{Ai}(x)$ and $y_2(x) := \mathrm{Bi}(x)$. The function $\mathrm{Ai}(x)$ is the solution approaching zero as $x \to +\infty$ in Figure 3.2. Note the interlacing of the zeros.

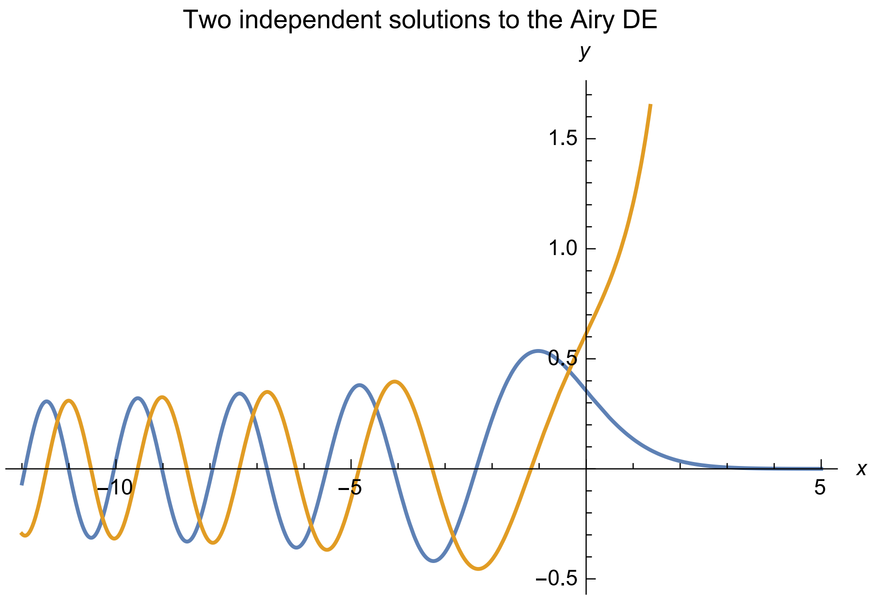


Figure 3.2: The Airy functions $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ are plotted. Note that the between any two zeros of one solutions lies a zero of the other solution.

3.2 Reduction of order

Suppose y_1 is a solution of (3.3). Let

$$y(x) = v(x) y_1(x).$$

Then

$$y' = v' y_1 + v y_1' \text{ and } y'' = v'' y_1 + 2v' y_1' + v y_1''.$$

Substitute these expressions for y and its first and second derivatives into (3.3) and make use of the fact that y_1 is a solution of (3.3). One obtains the following differential equation for v:

$$v'' + \left(p + 2\frac{y_1'}{y_1}\right) v' = 0,$$

or upon setting $u = v'$,

$$u' + \left(p + 2\frac{y_1'}{y_1}\right) u = 0.$$

This last equation is a first order linear equation. Its solution is

$$u(x) = c \exp\left(-\int \left(p + 2\frac{y_1'}{y_1}\right) dx\right) = \frac{c}{y_1^2(x)} \exp\left(-\int p(x)\, dx\right).$$

This implies

$$v(x) = \int u(x)\, dx,$$

so that

$$y(x) = c y_1(x) \int u(x)\, dx.$$

The point is, we have shown that if one solution to (3.3) is known, then a second solution can be found—expressed as an integral.

3.3 Constant coefficients

We assume that $p(x)$ and $q(x)$ are constants independent of x. We write (3.3) in this case as[3]

$$a y'' + b y' + c y = 0. \tag{3.13}$$

We "guess" a solution of the form

$$y(x) = e^{\lambda x}.$$

Substituting this into (3.13) gives

$$a\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + c e^{\lambda x} = 0.$$

Since $e^{\lambda x}$ is never zero, the only way the above equation can be satisfied is if

$$a\lambda^2 + b\lambda + c = 0. \tag{3.14}$$

Let $\lambda_\pm$ denote the roots of this quadratic equation, i.e.

$$\lambda_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

We consider three cases.

1. Assume $b^2 - 4ac > 0$ so that the roots $\lambda_\pm$ are both real numbers. Then $\exp(\lambda_+ x)$ and $\exp(\lambda_- x)$ are two linearly independent solutions to (3.14). That they are solutions follows from their construction. They are linearly independent since

   $$W(e^{\lambda_+ x}, e^{\lambda_- x}; x) = (\lambda_- - \lambda_+) e^{\lambda_+ x} e^{\lambda_- x} \neq 0$$

   Thus in this case, every solution of (3.13) is of the form

   $$c_1 \exp(\lambda_+ x) + c_2 \exp(\lambda_- x)$$

   for some constants c_1 and c_2.

2. Assume $b^2 - 4ac = 0$. In this case $\lambda_+ = \lambda_-$. Let λ denote their common value. Thus we have one solution $y_1(x) = e^{\lambda x}$. We could use the method of reduction of order to show that a second linearly independent solution is $y_2(x) = x e^{\lambda x}$. However, we choose to present a more intuitive way of seeing this is a second linearly independent solution. (One can always make it rigorous at the end by verifying that that it is

[3] This corresponds to $p(x) = b/a$ and $q(x) = c/a$. For applications it is convenient to introduce the constant a.

indeed a solution.) Suppose we are in the distinct root case but that the two roots are very close in value: $\lambda_+ = \lambda + \varepsilon$ and $\lambda_- = \lambda$. Choosing $c_1 = -c_2 = 1/\varepsilon$, we know that

$$\begin{aligned} c_1 y_1 + c_2 y_2 &= \frac{1}{\varepsilon} e^{(\lambda+\varepsilon)x} - \frac{1}{\varepsilon} e^{\lambda x} \\ &= e^{\lambda x} \, \frac{e^{\varepsilon x} - 1}{\varepsilon} \end{aligned}$$

is also a solution. Letting $\varepsilon \to 0$ one easily checks that

$$\frac{e^{\varepsilon x} - 1}{\varepsilon} \to x,$$

so that the above solution tends to

$$x e^{\lambda x},$$

our second solution. That $\{e^{\lambda x}, x e^{\lambda x}\}$ is a basis is a simple Wronskian calculation.

3. We assume $b^2 - 4ac < 0$. In this case the roots $\lambda_\pm$ are complex. At this point we review the the exponential of a complex number.

Complex exponentials

Let $z = x + iy$ (x, y real numbers, $i^2 = -1$) be a complex number. Recall that x is called the real part of z, $\Re z$, and y is called the imaginary part of z, $\Im z$. Just as we picture real numbers as points lying in a line, called the real line $\mathbb{R}$; we picture complex numbers as points lying in the plane, called the complex plane $\mathbb{C}$. The coordinates of z in the complex plane are (x, y). The absolute value of z, denoted $|z|$, is equal to $\sqrt{x^2 + y^2}$. The complex conjugate of z, denoted $\overline{z}$, is equal to $x - iy$. Note the useful relation

$$z\,\overline{z} = |z|^2 .$$

In calculus, or certainly an advanced calculus class, one considers (simple) functions of a complex variable. For example the function

$$f(z) = z^2$$

takes a complex number, z, and returns it square, again a complex number. (Can you show that $\Re f = x^2 - y^2$ and $\Im f = 2xy$?). Using complex addition and multiplication, one can define *polynomials* of a complex variable

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 .$$

The next (big) step is to study power series

$$\sum_{n=0}^{\infty} a_n z^n .$$

With power series come issues of convergence. We defer these to your advanced calculus class.

With this as a background we are (almost) ready to define the exponential of a complex number z. First, we recall that the exponential of a real number x has the power series expansion

$$e^x = \exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (0! := 1).$$

In calculus classes, one normally defines the exponential in a different way[4] and then proves e^x has this Taylor expansion. However, one could *define* the exponential function by the above formula and then *prove* the various properties of e^x follow from this definition. This is the approach we take for defining the exponential of a complex number except now we use a power series in a complex variable:[5]

$$\boxed{e^z = \exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad z \in \mathbb{C}} \tag{3.15}$$

We now derive some properties of $\exp(z)$ based upon this definition.

- Let $\theta \in \mathbb{R}$, then

$$\begin{aligned}
\exp(i\theta) &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(i\theta)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(i\theta)^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n+1}}{(2n+1)!} \\
&= \cos\theta + i \sin\theta.
\end{aligned}$$

  This last formula is called *Euler's Formula*. See Figure 3.3. Two immediate consequences of Euler's formula (and the facts $\cos(-\theta) = \cos\theta$ and $\sin(\theta) = -\sin\theta$) are

$$\begin{aligned}
\exp(-i\theta) &= \cos\theta - i \sin\theta \\
\overline{\exp(i\theta)} &= \exp(-i\theta)
\end{aligned}$$

  Hence

$$|\exp(i\theta)|^2 = \exp(i\theta)\, \exp(-i\theta) = \cos^2\theta + \sin^2\theta = 1$$

  That is, the values of $\exp(i\theta)$ lie on the unit circle. The coordinates of the point $e^{i\theta}$ are $(\cos\theta, \sin\theta)$.

- We claim the addition formula for the exponential function, well-known for real values, also holds for complex values

[4] A common definition is $e^x = \lim_{n\to\infty}(1 + x/n)^n$.

[5] It can be proved that this infinite series converges for all complex values z.

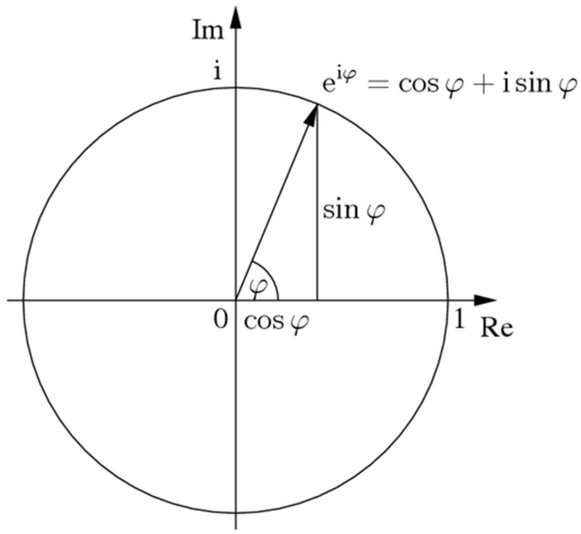


Figure 3.3: Euler's formula.

$$\boxed{\exp(z+w) = \exp(z)\,\exp(w), \quad z, w \in \mathbb{C}.} \tag{3.16}$$

We are to show

$$\begin{aligned}
\exp(z+w) &= \sum_{n=0}^{\infty} \frac{1}{n!}(z+w)^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k} \quad \text{(binomial theorem)}
\end{aligned}$$

is equal to

$$\begin{aligned}
\exp(z)\,\exp(w) &= \sum_{k=0}^{\infty} \frac{1}{k!} z^k \sum_{m=0}^{\infty} \frac{1}{m!} w^m \\
&= \sum_{k,m=0}^{\infty} \frac{1}{k!m!} z^k w^m \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{k!(n-k)!} z^k w^{n-k} \quad n := k+m \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} z^k w^{n-k}\,.
\end{aligned}$$

Since

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}\,,$$

we see the two expressions are equal as claimed.

- We can now use these two properties to understand better $\exp(z)$. Let $z = x+iy$, then
$$\exp(z) = \exp(x+iy) = \exp(x)\exp(iy) = e^x\left(\cos y + i\sin y\right).$$
Observe the right hand side consists of functions from calculus. Thus with a calculator you could find the exponential of any complex number using this formula.[6]
A form of the complex exponential we frequently use is if $\lambda = \sigma + i\mu$ and $x \in \mathbb{R}$, then
$$\exp(\lambda x) = \exp\left((\sigma + i\mu)x)\right) = e^{\sigma x}\left(\cos(\mu x) + i\sin(\mu x)\right).$$

Returning to (3.13) in case $b^2 - 4ac < 0$ and assuming a, b and c are all real, we see that the roots $\lambda_\pm$ are of the form[7]
$$\lambda_+ = \sigma + i\mu \;\text{ and }\; \lambda_- = \sigma - i\mu.$$
Thus $e^{\lambda_+ x}$ and $e^{\lambda_- x}$ are linear combinations of
$$e^{\sigma x}\cos(\mu x) \;\text{ and }\; e^{\sigma x}\sin(\mu x).$$
That they are linear independent follows from a Wronskian calculuation. To summarize, we have shown that every solution of (3.13) in the case $b^2 - 4ac < 0$ is of the form
$$c_1 e^{\sigma x}\cos(\mu x) + c_2 e^{\sigma x}\sin(\mu x)$$
for some constants c_1 and c_2.

Remarks: The MatLab function `exp` handles complex numbers. For example,

```
>> exp(i*pi)
ans =
-1.0000 + 0.0000i
```

The imaginary unit i is `i` in MatLab . You can also use `sqrt(-1)` in place of `i`. This is sometimes useful when `i` is being used for other purposes. There are also the functions

`abs, angle, conj, imag real`

For example,

```
>> w=1+2*i

w =

   1.0000 + 2.0000i
```

[6]Of course, this assumes your calculator doesn't overflow or underflow in computing e^x.
[7]$\sigma = -b/2a$ and $\mu = \sqrt{4ac - b^2}/2a$.

```
>> abs(w)

ans =

    2.2361

>> conj(w)

ans =

   1.0000 - 2.0000i

>> real(w)

ans =

     1

>> imag(w)

ans =

     2

>> angle(w)

ans =

    1.1071
```

3.4 Forced oscillations of the mass-spring system

The forced mass-spring system is described by the differential equation

$$m\,\frac{d^2x}{dt^2} + \gamma\,\frac{dx}{dt} + k\,x = F(t) \tag{3.17}$$

where $x = x(t)$ is the displacement from equilibrium at time t, m is the mass, k is the constant in Hooke's Law, $\gamma > 0$ is the coefficient of friction, and $F(t)$ is the forcing term. In these notes we examine the solution when the forcing term is periodic with period $2\pi/\omega$. (ω is the frequency of the forcing term.) The simplest choice for a periodic function is either sine or cosine. Here we examine the choice

$$F(t) = F_0\,\cos\omega t$$

where F_0 is the amplitude of the forcing term. All solutions to (3.17) are of the form

$$x(t) = x_p(t) + c_1 x_1(t) + c_2 x_2(t) \tag{3.18}$$

where x_p is a particular solution of (3.17) and $\{x_1, x_2\}$ is a basis for the solution space of the *homogeneous* equation.

The homogeneous solutions have been discussed earlier. We know that both x_1 and x_2 will contain a factor

$$e^{-(\gamma/2m)t}$$

times factors involving sine and cosine. Since for all $a > 0$, $e^{-at} \to 0$ as $t \to \infty$, the homogeneous part of (3.18) will tend to zero. That is, for all initial conditions we have for *large* t to good approximation

$$x(t) \approx x_p(t).$$

Thus we concentrate on finding a particular solution x_p.

With the right-hand side of (3.17) having a cosine term, it is natural to guess that the particular solution will also involve $\cos\omega t$. If one guesses

$$A\cos\omega t$$

one quickly sees that due to the presence of the frictional term, this cannot be a correct since sine terms also appear. Thus we guess

$$x_p(t) = A\cos\omega t + B\sin\omega t \tag{3.19}$$

We calculate the first and second dervatives of (3.19) and substitute the results together with (3.19) into (3.17). One obtains the equation

$$\left[-A\omega^2 m + B\omega\gamma + kA\right] \cos\omega t + \left[-B\omega^2 m - A\omega\gamma + kB\right] \sin\omega t = F_0\cos\omega t$$

This equation must hold for all t and this can happen only if

$$\left[-A\omega^2 m + B\omega\gamma + kA\right] = F_0 \quad \text{and} \quad \left[-B\omega^2 m - A\omega\gamma + kB\right] = 0$$

These last two equations are a pair of linear equations for the unknown coefficients A and B. We now solve these linear equations. First we rewrite these equations to make subsequent steps clearer:

$$\begin{aligned} \left(k - \omega^2 m\right) A + \omega\gamma\, B &= F_0, \\ -\omega\gamma\, A + \left(k - \omega^2 m\right) B &= 0. \end{aligned}$$

Using Cramer's Rule we find (check this!)

$$\begin{aligned} A &= \frac{k - m\omega^2}{(k - m\omega^2)^2 + \gamma^2\omega^2}\, F_0 \\ B &= \frac{\gamma\omega}{(k - m\omega^2)^2 + \gamma^2\omega^2}\, F_0 \end{aligned}$$

We can make these results notationally simpler if we recall that the natural frequency of a (frictionless) oscillator is

$$\omega_0^2 = \frac{k}{m}$$

and define

$$\Delta(\omega) = \sqrt{m^2(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2} \tag{3.20}$$

so that

$$A = \frac{m(\omega_0^2 - \omega^2)}{\Delta(\omega)^2}\, F_0 \quad \text{and} \quad B = \frac{\gamma\omega}{\Delta(\omega)^2}\, F_0$$

Using these expressions for A and B we can substitute into (3.19) to find our particular solution x_p. The form (3.19) is not the best form in which to understand the properties of the solution. (It is convenient for performing the above calculations.) For example, it is not obvious from (3.19) what is the *amplitude* of oscillation. To answer this and other questions we introduce polar coordinates for A and B:

$$A = R\cos\delta \quad \text{and} \quad B = R\sin\delta.$$

Then

$$\begin{aligned} x_p(t) &= A\cos\omega t + B\sin\omega t \\ &= R\cos\delta\,\cos\omega t \,+\, R\sin\delta\,\sin\omega t \\ &= R\cos(\omega t - \delta) \end{aligned}$$

where in the last step we used the cosine addition formula. Observe that R is the amplitude of oscillation. The quantity δ is called the *phase angle*. It measures how much the oscillation lags (if $\delta > 0$) the forcing term. (For example, at $t = 0$ the amplitude of the forcing term is a maximum, but the maximum oscillation is delayed until time $t = \delta/\omega$.)

Clearly,

$$A^2 + B^2 = R^2\cos^2\delta + R^2\sin^2\delta = R^2$$

and

$$\tan\delta = \frac{B}{A}$$

Substituting the expressions for A and B into the above equations give

$$\begin{aligned} R^2 &= \frac{m^2(\omega_0^2 - \omega^2)}{\Delta^4}\, F_0^2 + \frac{\gamma^2\omega^2}{\Delta^4}\, F_0^2 \\ &= \frac{\Delta^2}{\Delta^4}\, F_0^2 \\ &= \frac{F_0^2}{\Delta^2} \end{aligned}$$

Thus

$$R = \frac{F_0}{\Delta} \tag{3.21}$$

where we recall Δ is defined in (3.20). Taking the ratio of A and B we see that

$$\tan\delta = \frac{\gamma\omega}{m(\omega_0^2 - \omega^2)}$$

3.4.1 Resonance

We now examine the behavior of the amplitude of oscillation, $R = R(\omega)$, as a function of the frequency ω of the driving term.

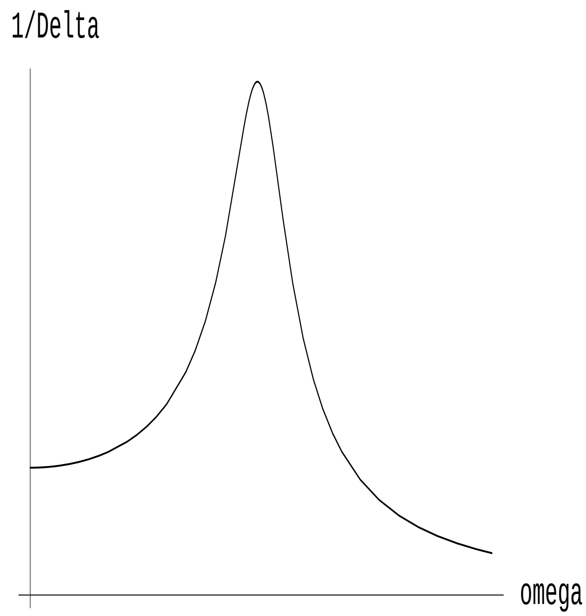


Figure 3.4: $1/\Delta(\omega)$ as a function of ω.

Low frequencies: When $\omega \to 0$, $\Delta(\omega) \to m\omega_0^2 = k$. Thus for low frequencies the amplitude of oscillation approaches F_0/k. This result could have been anticipated since when $\omega \to 0$, the forcing term tends to F_0, a constant. A particular solution in this case is itself a constant and a quick calculation shows this constant is eqaul to F_0/k.

High frequencies: When $\omega \to \infty$, $\Delta(\omega) \sim m\omega^2$ and hence the amplitude of oscillation $R \to 0$. Intuitively, if you shake the mass-spring system too quickly, it does not have time to respond before being subjected to a force in the opposite direction; thus, the overall effect is no motion. Observe that greater the mass (inertia) the smaller R is for large frequencies.

Maximum Oscillation: The amplitude R is a maximum (as a function of ω) when $\Delta(\omega)$ is a minimum. Δ is a minimum when Δ^2 is a minimum. Thus to find the frequency corresponding to maximum amplitude of oscillation we must minimize

$$m^2 \left(\omega^2 - \omega_0^2\right)^2 + \gamma^2\omega^2.$$

To find the minimum we take the derivative of this expression with respect to ω and set it equal to zero:

$$2m^2(\omega^2 - \omega_0^2)(2\omega) + 2\gamma^2\omega = 0.$$

Factoring the left hand side gives

$$\omega \left[\gamma^2 + 2m^2(\omega^2 - \omega_0^2)\right] = 0.$$

Since we are assuming $\omega \neq 0$, the only way this equation can equal zero is for the expression in the square brackets to equal zero. Setting this to zero and solving for ω^2

gives the frequency at which the amplitude is a maximum. We call this ω_{max}:

$$\omega_{\mathrm{max}}^2 = \omega_0^2 - \frac{\gamma^2}{2m^2} \; = \; \omega_0^2 \left(1 - \frac{\gamma^2}{2km}\right).$$

Taking the square root gives

$$\omega_{\mathrm{max}} = \omega_0 \sqrt{1 - \frac{\gamma^2}{2km}}\,.$$

Assuming $\gamma \ll 1$ (the case of very small friction), we can expand the square root to get the approximate result

$$\omega_{\mathrm{max}} = \omega_0 \left(1 - \frac{\gamma^2}{4km} + \mathrm{O}(\gamma^4)\right).$$

That is, when ω is very close to the natural frequency ω_0 we will have maximum oscillation. This phenomenon is called *resonance*. A graph of $1/\Delta$ as a function of ω is shown in Fig. 3.4.

3.5 Exercises

#1. Euler's formula

Using Euler's formula prove the trig identity

$$\cos(4\theta) = \cos^4\theta - 6\cos^2\theta\,\sin^2\theta + \sin^4\theta.$$

Again using Euler's formula find a formula for $\cos(2n\theta)$ where $n = 1, 2, \ldots$. In this way one can also get identities for $\cos(2n+1)\theta$ as well as $\sin n\theta$.

#2. Roots of unity

Show that the n (distinct) solutions to the polynomial equation

$$x^n - 1 = 0$$

are $e^{2\pi i k/n}$ for $k = 1, 2, \ldots, n$. For $n = 6$ draw a picture illustrating where these roots lie in the complex plane.

#3. Constant coefficient ODEs

In each case find the unique solution $y = y(x)$ that satisfies the ODE with stated initial conditions:

1. $y'' - 3y' + 2y = 0$, $y(0) = 1$, $y'(0) = 0$.
2. $y'' + 9y = 0$, $y(0) = 1$, $y'(0) = -1$.
3. $y'' - 4y' + 4y = 0$, $y(0) = 2$, $y'(0) = 0$.

#4. Higher order equations

The third order homogeneous differential equation with constant coefficients is

$$a_3 y''' + a_2 y'' + a_1 y' + a_0 y = 0 \tag{3.22}$$

where a_i are constants. Assume a solution of the form

$$y(x) = e^{\lambda x}$$

and derive an equation that λ must satisfy in order that y is a solution. (You should get a cubic polynomial.) What is the form of the general solution to (3.22)?

#5. Euler's equation

A differential equation of the form

$$t^2 y'' + aty' + by = 0,\ t > 0 \tag{3.23}$$

where a, b are real constants, is called Euler's equation.[8] This equation can be transformed into an equation with constant coefficients by letting $x = \ln t$. Solve

$$t^2 y'' + 4ty' + 2y = 0 \tag{3.24}$$

#6 Forced undamped system

Consider a forced undamped system described by

$$\ddot{y} + y = 3\cos(\omega t)$$

with initial conditions $y(0) = 1$ and $\dot{y}(0) = 1$. Find the solution for $\omega \neq 1$.

#7. Driven damped oscillator

Let

$$\ddot{y} + 3\dot{y} + 2y = 0$$

be the equation of a damped oscillator. If a forcing term is $F(t) = 10\cos t$ and the oscillator is initially at rest at the origin, what is the solution of the equation for this driven damped oscillator? What is the phase angle?

#8. Damped oscillator

A particle is moving according to

$$\ddot{y} + 10\dot{y} + 16y = 0$$

with the initial condition $y(0) = 1$ and $\dot{y}(0) = 4$. Is this oscillatory ? What is the maximum value of y?

[8] There is perhaps no other mathematician whose name is associated to so many functions, identities, equations, numbers, ... as Euler.

#9. Wronskian

Consider (3.3) with $p(x)$ and $q(x)$ continuous on the interval $[a, b]$. Prove that if two solutions y_1 and y_2 have a maximum or minimum at the same point in $[a, b]$, they cannot form a basis of $\mathcal{V}$.

#10. Euler's equation (revisited) from physics

In Exercise 2.3.9 we obtained a set of three first-order differential equations for Ω_1, Ω_2 and Ω_3, which are called the Euler equations when there is no torque. Let us assume that $I_1 = I_2 \neq I_3$. (The body with these moments of inertia is called *a free symmetric top.*) In this case we have

$$I_1\dot{\Omega}_1 = (I_2 - I_3)\Omega_2\Omega_3 \tag{3.25}$$

$$I_2\dot{\Omega}_2 = (I_3 - I_1)\Omega_3\Omega_1 \tag{3.26}$$

$$I_3\dot{\Omega}_3 = 0 \tag{3.27}$$

Notice that Ω_3 is a constant from (3.27). Show that Ω_1 and Ω_2 have the form of

$$\begin{aligned} \Omega_1(t) &= A\sin(\omega t + \theta_0); \\ \Omega_2(t) &= A\cos(\omega t + \theta_0) \end{aligned}$$

where A and θ_0 are some constants. Here Ω_1, Ω_2 and Ω_3 are three components of the angular velocity vector $\vec{\Omega}$. Show that it follows that the *magnitude* (length) of $\vec{\Omega}$ is a constant. Find an explicit expression for ω in terms of I_i and the constant Ω_3.

Chapter 4

Difference Equations

Figure 4.1: Leonardo Fibonacci, c. 1170–c. 1250.

Science is what we understand well enough to explain to a computer. Art is everything else we do.

D.E. Knuth in the preface of $A{=}B$ by H. Wilf & D. Zeilberger

4.1 Introduction

We have learned that the general inhomogeneous second order linear differential equation is of the form

$$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = f(x).$$

The independent variable, x, takes values in $\mathbf{R}$. (We say x is a continuous variable.) Many applications lead to problems where the independent variable is *discrete*; that is, it takes values in the integers. Instead of $y(x)$ we now have y_n, n an integer. The discrete version of the above equation, called an *inhomogeneous second order linear difference equation*, is

$$a_n\, y_{n+2} + b_n\, y_{n+1} + c_n\, y_n = f_n \tag{4.1}$$

where we assume the sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{f_n\}$ are known. For example,

$$(n^2+5)y_{n+2} + 2y_{n+1} + \frac{3}{n+1}y_n = e^n, \;\; n = 0, 1, 2, 3, \ldots$$

is such a difference equation. Usually we are given y_0 and y_1 (the initial values), and the problem is to solve the difference equation for y_n.

In this chapter we consider the special case of *constant coefficient* difference equations:

$$a\, y_{n+2} + b\, y_{n+1} + c\, y_n = f_n$$

where a, b, and c are constants independent of n. If $f_n = 0$ we say the difference equation is *homogeneous*. An example of a homogeneous second order constant coefficient difference equation is

$$6y_{n+2} + \frac{1}{3}y_{n+1} + 2y_n = 0.$$

4.2 Constant coefficient difference equations

4.2.1 Solution of constant coefficient difference equations

In this section we give an algorithm to solve all second order homogeneous constant coefficient difference equations

$$a\, y_{n+2} + b\, y_{n+1} + c\, y_n = 0. \tag{4.2}$$

The method is the discrete version of the method we used to solve contant coefficient differential equations. We first guess a solution of the form

$$y_n = \lambda^n, \;\; \lambda \neq 0.$$

(For differential equations we guessed $y(x) = e^{\lambda x}$.) We now substitute this into (4.2) and require the result equal zero,

$$\begin{aligned} 0 &= a\lambda^{n+2} + b\lambda^{n+1} + c\lambda^n \\ &= \lambda^n \left(a\lambda^2 + b\lambda + c\right). \end{aligned}$$

This last equation is satisfied if and only if

$$a\lambda^2 + b\lambda + c = 0. \tag{4.3}$$

Let λ_1 and λ_2 denote the roots of this quadratic equation. (For the moment we consider only the case when the roots are distinct.) Then

$$\lambda_1^n \text{ and } \lambda_2^n$$

are both solutions to (4.2). Just as in our study of second order ODEs, the linear combination

$$c_1\lambda_1^n + c_2\lambda_2^n$$

is also a solution and every solution of (4.2) is of this form. The constants c_1 and c_2 are determined once we are given the initial values y_0 and y_1:

$$\begin{aligned} y_0 &= c_1 + c_2, \\ y_1 &= c_1\lambda_1 + c_2\lambda_2, \end{aligned}$$

are two equation that can be solved for c_1 and c_2.

4.2.2 Fibonacci numbers

Consider the sequence of numbers

$$0 \quad 1 \quad 1 \quad 2 \quad 3 \quad 5 \quad 8 \quad 13 \quad 21 \quad 34 \quad 55 \quad 89 \quad 144 \quad 233 \cdots$$

that is, each number is the sum of the preceding two numbers starting with

$$0 \quad 1$$

as initial values. These integers are called *Fibonacci numbers* and the nth Fibonacci number is denoted by F_n. The numbers grow very fast, for example,

$$F_{100} = 354\,224\,848\,179\,261\,915\,075.$$

From their definition, F_n satisfies the difference equation

$$F_{n+1} = F_n + F_{n-1} \text{ for } n \geq 1$$

with

$$F_0 = 0, F_1 = 1.$$

The quadratic equation we must solve is

$$\lambda^2 = \lambda + 1,$$

whose roots are

$$\lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2}.$$

Setting

$$F_n = c_1\lambda_1^n + c_2\lambda_2^n,$$

we see that at $n = 0$ and 1 we require

$$\begin{aligned} 0 &= c_1 + c_2, \\ 1 &= c_1\lambda_1 + c_2\lambda_2. \end{aligned}$$

Solving these we find

$$c_1 = \frac{1}{\sqrt{5}}, \;\; c_2 = -\frac{1}{\sqrt{5}}\,,$$

and hence

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n .$$

Since $\lambda_1 > 1$ and $|\lambda_2| < 1$, λ_1^n grows with increasing n whereas $\lambda_2^n \to 0$ as $n \to \infty$. Thus for large n

$$F_n \sim \frac{1}{\sqrt{5}}\,\lambda_1^n\,,$$

and

$$\lim_{n\to\infty} \frac{F_{n-1}}{F_n} = \frac{1}{\lambda_1} := \omega.$$

The number

$$\omega = \frac{\sqrt{5}-1}{2} = 0.61803398\ldots\ldots$$

is called the golden mean.[1]

4.3 Inhomogeneous difference equations

In a completely analogous way to the ODE case, one proves that every solution to the inhomogeneous linear difference equation (4.1) is of the form

$$(y_n)_{\text{homo}} + (y_n)_{\text{part}}$$

where $(y_n)_{\text{homo}}$ is a solution to the homogeneous equation (4.1) with $f_n = 0$ and $(y_n)_{\text{part}}$ is a particular solution to the inhomogeneous equation (4.1).

[1]More often the number

$$\phi = 1/\omega = \frac{1+\sqrt{5}}{2} = 1.6180339887\ldots$$

is called the golden mean or golden ratio. Two quantities a and b are said to be in the *golden ratio* ϕ if

$$\frac{a+b}{a} = \frac{a}{b} = \phi.$$

In words, $a + b$ is to a as a is to b. A *golden rectangle* is a rectangle whose ratio of the longer side to the shorter side is the golden mean ϕ. Since the ratio of Fibonacci numbers F_n/F_{n-1} converges to ϕ as $n \to \infty$, rectangles whose long side is of length F_n and whose short side is of length F_{n-1} are approximate golden rectangles. For example, here are some approximate golden rectangles: 1×1, 1×2, 2×3, 3×5, 5×8, 8×13, and so on. The higher we go in this sequence the closer we come to the golden rectangle.

4.4 Exercises

#1. Degenerate roots

Consider the constant coefficient difference equation (4.2) but now assume the two roots $\lambda_{1,2}$ are equal. Show that

$$n\lambda_1^n$$

is a second linearly independent solution to (4.2).

#2. Rational approximations to $\sqrt{2}$

Solve the difference equation

$$x_{n+1} = 2x_n + x_{n-1}, \;\; n \geq 1$$

with initial conditions $x_0 = 0$ and $x_1 = 1$ that corresponds to the sequence 0, 1, 2, 5, 12, 29,.... Show that

$$\lim_{n\to\infty} \frac{x_{n+1} - x_n}{x_n} = \sqrt{2}.$$

The rational numbers

$$\frac{x_{n+1} - x_n}{x_n}$$

provide us with very good rational approximations to the square root of two.[2]

#3. Catalan numbers

Many times nonlinear recurrence relations arise. For example, *Catalan numbers* T_n satisfy the nonlinear recurrence relation

$$T_n = \sum_{k=0}^{n-1} T_k T_{n-1-k}, \;\; n = 1, 2, \ldots$$

[2]The square root of two is irrational. Here is a proof that $\sqrt{2}$ is irrational. Suppose not; that is, we suppose that $\sqrt{2}$ is a rational number. All rational numbers can be written as a ratio of two integers. Thus we assume there are integers p and q such that $\sqrt{2} = p/q$. By canceling out common factors in p and q we can assume that the fraction is reduced. By definition of the square root,

$$2 = \frac{p^2}{q^2}$$

which implies $p^2 = 2q^2$. Thus p^2 is an even integer since it is two times q^2. If p^2 is even then it follows that p is even. (The square of an odd integer is an odd integer.) Since p is even it can be written as $p = 2n$ for some integer n. Substituting this into $p^2 = 2q^2$ and canceling the common factor of two, we obtain

$$q^2 = 2p^2$$

But this last equation means q^2; and hence q, is an even integer. Thus both p and q have a common factor of two. But we assumed that all common factors were cancelled. Thus we arrive at a contradiction to the assertion that $\sqrt{2}$ is rational.

From Wikipedia: Pythagoreans discovered that the diagonal of a square is incommensurable with its side, or in modern language, that the square root of two is irrational. Little is known with certainty about the time or circumstances of this discovery, but the name of Hippasus of Metapontum is often mentioned. For a while, the Pythagoreans treated as an official secret the discovery that the square root of two is irrational, and, according to legend, Hippasus was murdered for divulging it. The square root of two is occasionally called "Pythagoras' number" or "Pythagoras' Constant."

where $T_0 := 1$. Define

$$T(z) = \sum_{n=0}^{\infty} T_n z^n.$$

Show that

$$T(z) = \frac{1 - \sqrt{1-4z}}{2z}.$$

From this prove that

$$T_n = \frac{1}{n+1}\binom{2n}{n}$$

where $\binom{n}{k}$ is the binomial coefficient. Catalan numbers arise in a variety of combinatorial problems. Here is one example:

> Suppose $2n$ points are placed in fixed positions, evenly distributed on the circumference of a circle. Then there are T_n ways to join n pairs of the points so that the resulting chords do not intersect.

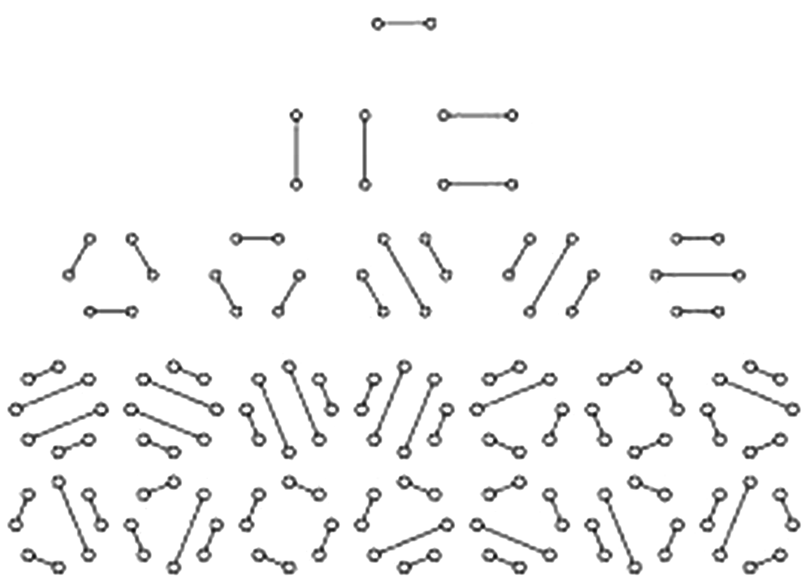


Figure 4.2: The Catalan numbers $T_1 = 1$, $T_2 = 2$, $T_3 = 5$ and $T_4 = 14$ are illustrated in the above counting problem of ways to join n pairs so that the chords do not intersect. Here $n = 1, 2, 3, 4$. A more pictorial presentation is suppose there are 8 people at a table, then ask how many ways can they shake hands with one other person and so that no arms are crossed? The answer is $T_4 = 14$.

One can easily make a table of values of T_n using, say, the MATHEMATICA command (this gives T_1 through T_{10}).

```
Table[{n, Binomial[2*n, n]/(n + 1)}, {n, 1, 10}]
```

Chapter 5

Matrix Differential Equations

Figure 5.1: Vladimir Arnold, 1937–2010.

Linear systems are almost the only large class of differential equations for which there exists a definitive theory. This theory is essentially a branch of linear algebra, and allows us to solve all autonomous linear equations.

V. A. Arnold, *Ordinary Differential Equations* [2]

5.1 The matrix exponential

Let A be a $n \times n$ matrix with constant entries. In this chapter we study the matrix differential equation

$$\boxed{\frac{dx}{dt} = Ax \quad \text{where} \quad x \in \mathbf{R}^n.} \tag{5.1}$$

We will present an algorithm that reduces solving (5.1) to problems in linear algebra.

The exponential of the matrix tA, $t \in \mathbf{R}$, is defined by the infinite series[1]

$$\boxed{e^{tA} = \exp(tA) := I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots .} \tag{5.2}$$

Remark: In an advanced course you will prove that this infinite series of matrices converges to a $n \times n$ matrix.

It is important to note that for matrices A and B, in general,

$$\exp(tA)\exp(tB) \neq \exp(tA + tB).$$

If A and B commute ($AB = BA$) then it is the case that

$$\exp(tA)\exp(tB) = \exp(tA + tB).$$

This last fact can be proved by examining the series expansion of both sides—on the left hand side one has to multiply two infinite series. You will find that by making use of $AB = BA$ the result follows precisely as in the case of complex exponentials.

Here are some examples:

1.
$$A = D = \text{diagonal matrix} = \operatorname{diag}\left(\lambda_1, \lambda_2, \ldots, \lambda_n\right).$$

   Observe that
$$D^k = \operatorname{diag}\left(\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k\right).$$

   Thus
$$\sum_{k=0}^{\infty} \frac{t^k}{k!} D^k = \operatorname{diag}\left(e^{t\lambda_1}, e^{t\lambda_2}, \ldots, e^{t\lambda_n}\right).$$

[1] We put the scalar factor t directly into the definition of the matrix exponential since it is in this form we will use the matrix exponential.

2. Suppose that A is a diagonalizable matrix; that is, there exist matrices U and D with U invertible and D diagonal such that

$$A = UDU^{-1}.$$

Observe

$$A^2 = (UDU^{-1})(UDU^{-1}) = UD^2U^{-1},$$

and more generally,

$$A^k = UD^kU^{-1}.$$

Thus

$$\begin{aligned} \exp(tA) &= \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!} UD^kU^{-1} \\ &= U \sum_{k=0}^{\infty} \frac{t^k}{k!} D^k U^{-1} \\ &= U \exp(tD) U^{-1}. \end{aligned} \tag{5.3}$$

In the next to the last equality, we used the fact that U and U^{-1} do not depend upon the summation index k and can therefore be brought outside of the sum. The last equality makes use of the previous example where we computed the exponential of a diagonal matrix. This example shows that if one can find such U and D, then the computation of the $\exp(tA)$ is reduced to matrix multiplications. This last result, (5.3), is quite suitable for using MatLab or Mathematica.

3. Let

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Matrix multiplication shows

$$A^2 = -I,$$

and thus

$$A^{2k} = \left(A^2\right)^k = (-I)^k = (-1)^k I,$$

$$A^{2k+1} = A^{2k} A = (-1)^k A.$$

Hence

$$
\begin{aligned}
\exp(tA) &= \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k && (5.4)\\
&= \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} A^{2k} + \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} A^{2k+1}\\
&= \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} (-1)^k I + \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} (-1)^k A\\
&= \cos t\, I + \sin t\, A\\
&= \begin{pmatrix} \cos t & 0 \\ 0 & \cos t \end{pmatrix} + \begin{pmatrix} 0 & -\sin t \\ \sin t & 0 \end{pmatrix}\\
&= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}. && (5.5)
\end{aligned}
$$

Remark: You can also compute

$$\exp\left(t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)$$

by the method of Example #2. Try it!

5.2 Application of e^{tA} to differential equations

5.2.1 Derivative of e^{tA} with respect to t

The following is the basic property of the exponential that we apply to differential equations. As before, A denotes a $n \times n$ matrix with constant coefficients.

$$\boxed{\frac{d}{dt} \exp(tA) = A \exp(tA) = \exp(tA) A.} \qquad (5.6)$$

Here is the proof: Differentiate

$$e^{tA} = I + tA + \frac{t^2}{2!} A^2 + \frac{t^3}{3!} A^3 + \cdots$$

term-by-term[2] with the result

$$
\begin{aligned}
\frac{d}{dt} e^{tA} &= A + tA^2 + \frac{t^2}{2!} A^3 + \cdots\\
&= A \left(I + tA + \frac{t^2}{2!} A^2 + \cdots \right)\\
&= A\, e^{tA}\\
&= e^{tA} A.
\end{aligned}
$$

[2]In a complex analysis course you will prove that convergent complex power series can be differentiated term-by-term and the resulting series has the same radius of convergence. Note there really is something to prove here since there is an interchange of two limits.

The last equality follows by factoring A out on the right instead of the left.

5.2.2 Solution to matrix ODE with constant coefficients

We now use (5.6) to prove

Theorem: Let

$$\frac{dx}{dt} = Ax \tag{5.7}$$

where $x \in \mathbf{R}^n$ and A is a $n \times n$ matrix with constant coefficients. Then every solution of (5.7) is of the form

$$x(t) = \exp(tA)x_0 \tag{5.8}$$

for some constant vector $x_0 \in \mathbf{R}^n$.

Proof: (i) First we show that $x(t) = e^{tA}x_0$ is a solution:

$$\begin{aligned}\frac{dx}{dt} &= \frac{d}{dt}\left(e^{tA}x_0\right) = \left(\frac{d}{dt}e^{tA}\right)x_0 \\ &= Ae^{tA}x_0 \\ &= Ax(t).\end{aligned}$$

(ii) We now show that *every* solution of (5.7) is of the form (5.8). Let $y(t)$ be any solution to (5.7). Let

$$\Delta(t) := e^{-tA}y(t).$$

If we can show that $\Delta(t)$ is independent of t—that it is a constant vector which we call x_0, then we are done since multiplying both sides by e^{tA} shows

$$e^{tA}x_0 = e^{tA}\Delta(t) = e^{tA}e^{-tA}y(t) = y(t).$$

(We used the fact that tA and $-tA$ commute so that the addition formula for the matrix exponential is valid.) To show that $\Delta(t)$ is independent of t we show its derivative with respect to t is zero:

$$\begin{aligned}\frac{d\Delta}{dt} &= \frac{d}{dt}\left\{e^{-tA}y(t)\right\} \\ &= \left(\frac{d}{dt}e^{-tA}\right)y(t) + e^{-tA}\,\frac{dy}{dt} \quad \text{(product rule)} \\ &= \left(-e^{-tA}A\right)y(t) + e^{-tA}\,\left(Ay(t)\right) \quad (y(t) \text{ satisfies ODE}) \\ &= 0.\end{aligned}$$

The next theorem relates the solution $x(t)$ of (5.7) to the eigenvalues and eigenvectors of the matrix A (in the case A is diagonalizable).

Theorem: Let A be a diagonalizable matrix. Any solution to (5.7) can be written as

$$x(t) = c_1e^{t\lambda_1}\psi_1 + c_2e^{t\lambda_2}\psi_2 + \cdots + c_ne^{t\lambda_n}\psi_n \tag{5.9}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A with associated eigenvectors $\psi_1, \ldots, \psi_n$, and $c_1, \ldots, c_n$ are constants.

Proof: All solutions of (5.7) are of the form (5.8). Since A is diagonalizable, the eigenvectors of A can be used to form a basis: $\{\psi_1, \ldots, \psi_n\}$. Since this is a basis there exist constants $c_1, \ldots, c_n$ such that

$$x_0 = c_1\psi_1 + c_2\psi_2 + \cdots + c_n\psi_n.$$

(x_0 is the constant vector appearing in (5.8).)

For any eigenvector ψ of A with eigenvalue λ we have

$$e^{tA}\psi = e^{t\lambda}\psi.$$

(This can be proved by applying the infinite series (5.2) to the eigenvector ψ and noting $A^k\psi = \lambda^k\psi$ for all positive integers k.) Thus

$$\begin{aligned} e^{tA}x_0 &= c_1 e^{tA}\psi_1 + \cdots c_n e^{tA}\psi_n \\ &= c_1 e^{t\lambda_1}\psi_1 + \cdots + c_n e^{t\lambda_n}\psi_n. \end{aligned}$$

Here are two immediate corollaries of this theorem:

1. If A is diagonalizable and has only *real* eigenvalues, then any solution $x(t)$ of (5.1) will have no oscillations.
2. If A is diagonalizable and the real part of every eigenvalue is negative, then

   $$x(t) \to 0 \ \text{ (zero vector), as } \ t \to +\infty$$

   To see this recall that if $\lambda = \sigma + i\mu$ (σ and μ both real), then

   $$e^{\lambda t} = e^{\sigma t} e^{i\mu t}.$$

   If $\sigma < 0$, $e^{\sigma t} \to 0$ as $t \to +\infty$. Now apply preceding theorem.

Example: Here is an example using MatLab or Mathematica to solve a system of equations. Consider

$$\frac{dx}{dt} = Ax \ \text{ where } \ A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 2 & 0 \\ 3 & 1 & 2 \end{pmatrix}.$$

The solution $x(t)$ to this DE that satisfies the initial condition $x(0) = x_0$ is

$$x(t) = e^{tA}x_0$$

so we must compute the 3×3 matrix e^{tA}. Here is a step-by-step approach:

Step 1: Find the eigenvalues of A

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 & 2 \\ 2 & 2-\lambda & 0 \\ 3 & 1 & 2-\lambda \end{vmatrix} = -\lambda^3 + 5\lambda^2 - 4\lambda = -\lambda(\lambda-1)(\lambda-4)$$

Thus the eigenvalues of A are

$$\lambda_1 = 4; \lambda_2 = 1, \lambda_3 = 0.$$

Since the matrix is 3×3 and there are three distinct eigenvalues, we know that A is diagonalizable.

Step 2: Find the eigenvectors of A. That is we must solve the linear equations

$$A\psi_1 = \lambda_1 \psi_1, \;\; A\psi_2 = \lambda_2 \psi_2, \;\; A\psi_3 = \lambda_3 \psi_3$$

where eigenvalues are given in Step 1. Note that the solutions will be determined only up to an overall constant, e.g. if $A\psi = \lambda\psi$, then $c\psi$ is also an eigenvector where c is a constant. Solving the above equations one finds

$$\psi_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \;\; \psi_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \;\; \psi_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

Step 3: Form matrices U and D and compute U^{-1}. D is the diagonal matrix

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and U is the matrix formed from the eigenvectors:

$$U = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ \psi_1 & \psi_2 & \psi_3 \\ \downarrow & \downarrow & \downarrow \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}.$$

We now compute the inverse of U:

$$U^{-1} = \begin{pmatrix} 1/3 & 0 & 1/3 \\ 1/3 & 1 & -2/3 \\ -1 & -1 & 1 \end{pmatrix}$$

Thus we have

$$A = UDU^{-1}$$

Step 4: Compute e^{tA} from formula

$$\begin{aligned} e^{tA} &= Ue^{tD}U^{-1} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} e^{4t} & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1/3 & 0 & 1/3 \\ 1/3 & 1 & -2/3 \\ -1 & -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 - \frac{e^t}{3} + \frac{e^{4t}}{3} & 1 - e^t & -1 + \frac{2e^t}{3} + \frac{e^{4t}}{3} \\ -1 + \frac{2e^t}{3} + \frac{e^{4t}}{3} & -1 + 2e^t & 1 - \frac{4e^t}{3} + \frac{e^{4t}}{3} \\ -1 + \frac{e^t}{3} + \frac{2e^{4t}}{3} & -1 + e^t & 1 - \frac{2e^t}{3} + \frac{2e^{4t}}{3} \end{pmatrix} \end{aligned}$$

Step 5: If the initial condition is vector

$$x_0 = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

then the solution to the differential equation $dx/dt = Ax$ with initial condition $x(0) = x_0$ is

$$\begin{aligned} x(t) &= e^{tA}x_0 = \begin{pmatrix} 1 - \frac{e^t}{3} + \frac{e^{4t}}{3} & 1 - e^t & -1 + \frac{2e^t}{3} + \frac{e^{4t}}{3} \\ -1 + \frac{2e^t}{3} + \frac{e^{4t}}{3} & -1 + 2e^t & 1 - \frac{4e^t}{3} + \frac{e^{4t}}{3} \\ -1 + \frac{e^t}{3} + \frac{2e^{4t}}{3} & -1 + e^t & 1 - \frac{2e^t}{3} + \frac{2e^{4t}}{3} \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \\ &= \begin{pmatrix} \left(1 - \frac{e^t}{3} + \frac{e^{4t}}{3}\right) c_1 + \left(1 - e^t\right) c_2 + \left(-1 + \frac{2e^t}{3} + \frac{e^{4t}}{3}\right) c_3 \\ \left(-1 + \frac{2e^t}{3} + \frac{e^{4t}}{3}\right) c_1 + \left(-1 + 2e^t\right) c_2 + \left(1 - \frac{4e^t}{3} + \frac{e^{4t}}{3}\right) c_3 \\ \left(-1 + \frac{e^t}{3} + \frac{2e^{4t}}{3}\right) c_1 + \left(-1 + e^t\right) c_2 + \left(1 - \frac{2e^t}{3} + \frac{2e^{4t}}{3}\right) c_3 \end{pmatrix} \end{aligned}$$

Step 6: An alternative to Step 5 is to simply say the general solution is of the form

$$x(t) = c_1 e^{4t}\psi_1 + c_2 e^t \psi_2 + c_3 \psi_3$$

where c_1, c_2 and c_3 are arbitrary constants (not the constants c_j appearing in Step 5!) and ψ_j are the eigenvectors found above.

5.3 Relation to earlier methods of solving constant coefficient DEs

Earlier we showed how to solve

$$a\ddot{y} + b\dot{y} + cy = 0$$

where a, b and c are constants. Indeed, we proved that the general solution is of the form

$$y(t) = c_1 e^{t\lambda_1} + c_2 e^{t\lambda_2}$$

where λ_1 and λ_2 are the roots to

$$a\lambda^2 + b\lambda + c = 0.$$

(We consider here only the case of distinct roots.)

Let's analyze this familiar result using matrix methods. The $x \in \mathbf{R}^2$ is

$$x(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y \\ dy/dt \end{pmatrix}$$

Therefore,

$$\begin{aligned} \frac{dx}{dt} &= \begin{pmatrix} dy/dt \\ d^2y/dt^2 \end{pmatrix} \\ &= \begin{pmatrix} x_2 \\ -\frac{b}{a}x_2 - \frac{c}{a}x_1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \end{aligned}$$

This last equality defines the 2×2 matrix A. The characteristic polynomial of A is

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -\frac{c}{a} & -\frac{b}{a} - \lambda \end{vmatrix} = \lambda^2 + \frac{b}{a}\lambda + \frac{c}{a}.$$

Thus the eigenvalues of A are the same quantities λ_1 and λ_2 appearing above. Since

$$x(t) = e^{tA} x_0 = S \begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{pmatrix} S^{-1} x_0,$$

$x_1(t)$ is a linear combination of $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$.

5.4 Problem from Markov processes

In the theory of continuous time Markov processes (see, e.g., Chapter 2 in [7]), one has a set of *states* and one asks for the *transition probability* from state $\mathbf{i}$ to state $\mathbf{j}$ in time t. Here is an example[3]

Suppose we have three states that we label **1**, **2** and **3**. We are given that the *rates* of transition from $\mathbf{i} \longrightarrow \mathbf{j}$ are known numbers q_{ij}. In this example suppose

$$q_{12} = 1, q_{13} = 1, q_{21} = 1, q_{23} = 0, q_{31} = 2, q_{32} = 1.$$

The theory of Markov processes tells us that we form a Q-matrix whose off-diagonal elements are q_{ij}, $i \neq j$, and whose row sums equal zero. In this example

$$Q = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & -3 \end{pmatrix} \tag{5.10}$$

Denote by $p_{ij}(t)$ the transition probability from state $\mathbf{i}$ to state $\mathbf{j}$ in time t. Form the matrix

$$P(t) = (p_{ij}(t)),$$

then the theory of Markov processes tells us that $P(t)$ satisfies the matrix DE

$$\frac{dP}{dt} = QP \tag{5.11}$$

where $P(0) = I$. We know that the solution to this DE is

$$P(t) = \exp(tQ). \tag{5.12}$$

So we must compute $\exp(tQ)$ where Q is given by (5.10).

[3] This example is taken from [7], Chapter 2.

5.4.1 Computing $\exp(tQ)$

Step 1. We first compute the eigenvalues of Q. To do this we compute the characteristic polynomial

$$p(\lambda) = \det(Q-\lambda I) = \begin{vmatrix} -2-\lambda & 1 & 1 \\ 1 & -1-\lambda & 0 \\ 2 & 1 & -3-\lambda \end{vmatrix} = -8\lambda - 6\lambda^2 - \lambda^3 = -\lambda(\lambda+2)(\lambda+4)$$

Thus the eigenvalues are 0, -2 and -4. Since there are three distinct eigenvalues and the matrix is 3×3, we know that Q is a diagonalizable matrix, see (5.3).[4]

Step 2. We next compute the eigenvectors corresponding to the eigenvalues. That is, we must solve the linear equations

$$Q\Psi_k = \lambda_k \Psi_k, \;\; k = 1, 2, 3$$

where $\lambda_1 = 0$, $\lambda_2 = -2$, $\lambda_3 = -4$. Linear algebra computations show[5]

$$\Psi_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \;\; \Psi_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \;\; \Psi_3 = \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix}. \tag{5.13}$$

Step 3. Form matrices U and U^{-1}. We know that U is formed from the eigenvectors Ψ_k:

$$U = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ \Psi_1 & \Psi_2 & \Psi_3 \\ \downarrow & \downarrow & \downarrow \end{pmatrix} = \begin{pmatrix} 1 & 1 & -3 \\ 1 & -1 & 1 \\ 1 & 1 & 5 \end{pmatrix}$$

We now compute U^{-1}. A linear algebra computation gives[6]

$$U^{-1} = \begin{pmatrix} \frac{3}{8} & \frac{1}{2} & \frac{1}{8} \\ \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \\ -\frac{1}{8} & 0 & \frac{1}{8} \end{pmatrix}$$

Step 4. We now have enough information to compute $\exp(tQ)$. We let D denote the diagonal matrix

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$

[4] If one first defines the matrix Q, then the command `Det[Q-λ*IdentityMatrix[3]]` in Mathematica finds the characteristic polynomial.

[5] Using a computer software package makes this step easier. For example, the command `Eigenvectors[Q]` in Mathematica gives the eigenvectors of the matrix Q.

[6] After defining the matrix U, the command `Inverse[U]` in Mathematica computes the inverse matrix U^{-1}.

From (5.3) we know that

$$\begin{aligned}
P(t) = \exp(tQ) &= U \exp(tD) U^{-1} \\
&= \begin{pmatrix} 1 & 1 & -3 \\ 1 & -1 & 1 \\ 1 & 1 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{e}^{-2t} & 0 \\ 0 & 0 & \mathrm{e}^{-4t} \end{pmatrix} \begin{pmatrix} \frac{3}{8} & \frac{1}{2} & \frac{1}{8} \\ \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \\ -\frac{1}{8} & 0 & \frac{1}{8} \end{pmatrix} \\
&= \begin{pmatrix} \frac{3}{8} + \frac{3e^{-4t}}{8} + \frac{e^{-2t}}{4} & \frac{1}{2} - \frac{e^{-2t}}{2} & \frac{1}{8} - \frac{3e^{-4t}}{8} + \frac{e^{-2t}}{4} \\ \frac{3}{8} - \frac{e^{-4t}}{8} - \frac{e^{-2t}}{4} & \frac{1}{2} + \frac{e^{-2t}}{2} & \frac{1}{8} + \frac{e^{-4t}}{8} - \frac{e^{-2t}}{4} \\ \frac{3}{8} - \frac{5e^{-4t}}{8} + \frac{e^{-2t}}{4} & \frac{1}{2} - \frac{e^{-2t}}{2} & \frac{1}{8} + \frac{5e^{-4t}}{8} + \frac{e^{-2t}}{4} \end{pmatrix} \qquad (5.14)
\end{aligned}$$

Thus, for example, the transition probability from state **1** to **1** in time t is given by the $(1,1)$ matrix element of $P(t)$:

$$p_{11}(t) = \frac{3}{8} + \frac{1}{4}\mathrm{e}^{-2t} + \frac{3}{8}\mathrm{e}^{-4t}.$$

and the transition probability from state **3** to **2** in time t is given by the $(3,2)$ matrix element of $P(t)$:

$$p_{32}(t) = \frac{1}{2} - \frac{1}{2}\mathrm{e}^{-2t}.$$

5.4.2 Some observations

1. Observe that the row sums of $P(t)$ are all equal to one. For example in the first row

   $$p_{11}(t)+p_{12}(t)+p_{13}(t) = \left[\frac{3}{8} + \frac{3}{8}\mathrm{e}^{-4t} + \frac{1}{4}\mathrm{e}^{-2t}\right] + \left[\frac{1}{2} - \frac{1}{2}\mathrm{e}^{-2t}\right] + \left[\frac{1}{8} - \frac{3}{8}\mathrm{e}^{-4t} + \frac{1}{4}\mathrm{e}^{-2t}\right] = 1$$

   This is simply the statement that the probability of starting in an initial state **i** and being in *any* state **j** is one.

2. Observe that

   $$\lim_{t\to\infty} P(t) = \begin{pmatrix} \frac{3}{8} & \frac{1}{2} & \frac{1}{8} \\ \frac{3}{8} & \frac{1}{2} & \frac{1}{8} \\ \frac{3}{8} & \frac{1}{2} & \frac{1}{8} \end{pmatrix}$$

   This means that after a long time no matter what is the initial state **i**, the transition probability to any state **j** is independent of the initial state.

5.5 Application of matrix DE to radioactive decays

Consider a radioactive decay of type

$$\mathbf{A} \longrightarrow \mathbf{B} \longrightarrow \mathbf{C}$$

where **C** is assumed to be stable. We denote the decay rate of **A** by λ_A and the decay rate of **B** by λ_B. (Since **C** is assumed to be stable, the decay rate of **C** equals zero.) An example of such a radioactive decay (by β-decay) is Iodine–133 decays to Xenon–133 which then decays to Cesium–133.

Introduce $p_\alpha(t)$ equal to the amount of material $\alpha = \mathbf{A}, \mathbf{B}, \mathbf{C}$ present at time t. We start with an initial amount of material $\mathbf{A}$ which for simplicity we take to be 1. Thus the initial conditions are

$$p_A(0) = 1,\;\; p_B(0) = 0,\;\; p_C(0) = 0.$$

The differential equations satisfied by the p_α, $\alpha = A, B, C$, are

$$\begin{aligned} \frac{dp_A}{dt} &= -\lambda_A\, p_A, \\ \frac{dp_B}{dt} &= \lambda_A\, p_A - \lambda_B\, p_B, \\ \frac{dp_C}{dt} &= \lambda_B p_B. \end{aligned}$$

We can write this in matrix form

$$\frac{dP}{dt} = AP$$

where

$$P = \begin{pmatrix} p_A \\ p_B \\ p_C \end{pmatrix}$$

and

$$A = \begin{pmatrix} -\lambda_A & 0 & 0 \\ \lambda_A & -\lambda_B & 0 \\ 0 & \lambda_B & 0 \end{pmatrix}.$$

We know that

$$P(t) = \exp(tA)P(0)$$

so we need to compute $\exp(tA)$.

Step 1. Since A is a lower triangular matrix, the eigenvalues of A are $-\lambda_A$, $-\lambda_B$ and 0.

Step 2. The corresponding eigenvectors of A are

$$\Psi_A = \begin{pmatrix} \frac{\lambda_A - \lambda_B}{\lambda_B} \\ -\frac{\lambda_A}{\lambda_B} \\ 1 \end{pmatrix}, \;\; \Psi_B = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \;\; \Psi_C = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(Mathematica does these calculations easily.)

Step 3. The columns of U are the vectors Ψ_A, Ψ_B, Ψ_C.

Step 4.

$$\begin{aligned} \exp(tA) &= U \exp(tD) U^{-1} \\ &= \begin{pmatrix} (\lambda_A - \lambda_B)/\lambda_B & 0 & 0 \\ -\lambda_A/\lambda_B & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-\lambda_A t} & 0 & 0 \\ 0 & \mathrm{e}^{-\lambda_B t} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} (\lambda_A - \lambda_B)/\lambda_B & 0 & 0 \\ -\lambda_A/\lambda_B & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \mathrm{e}^{-\lambda_A t} & 0 & 0 \\ \lambda_A(\mathrm{e}^{-\lambda_B t} - \mathrm{e}^{-\lambda_A t}) & \mathrm{e}^{-\lambda_B t} & 0 \\ (\lambda_A - \lambda_B)^{-1}\left[\lambda_A(1 - \mathrm{e}^{-\lambda_B t}) - \lambda_B\left(1 - \mathrm{e}^{-\lambda_A t}\right)\right] & 1 - \mathrm{e}^{-\lambda_B t} & 1 \end{pmatrix} \end{aligned}$$

Step 5. To find the solution $P(t)$ we must apply the matrix $\exp(tA)$ to the initial vector

$$P(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

This results in

$$\begin{aligned} p_A(t) &= \mathrm{e}^{-\lambda_A t}, \\ p_B(t) &= \frac{\lambda_A}{\lambda_B - \lambda_A}\left(\mathrm{e}^{-\lambda_A t} - \mathrm{e}^{-\lambda_B t}\right), \\ p_C(t) &= 1 - \frac{1}{\lambda_A - \lambda_B}\left(\lambda_A \mathrm{e}^{-\lambda_B t} - \lambda_B \mathrm{e}^{-\lambda_A t}\right). \end{aligned}$$

It is an exercise in calculus to show that the maximum of $p_B(t)$ occurs at

$$t_{\max} = \frac{\log(\lambda_A/\lambda_B)}{\lambda_A - \lambda_B}.$$

We now apply this to the example of Iodine–133 decay. The half-life of A:=Iodine-133 is 20.81 hours and the half-life of B:=Xenon–133 is 5.243 days. From this we can calculate the rates after we decide whether our unit of time is hours or days—let's take days. Then[7]

$$\lambda_A = 0.7994, \quad \lambda_B = 0.1322.$$

Thus, for example, the maximum amount of Xenon–133 occurs in approximately 2.70 days. In the figure we show the populations of the three nuclides as a function of time.

5.6 Inhomogenous matrix equations

Consider the inhomogenous equation

$$\frac{dx}{dt} = Ax + f(t),\; x(0) = x_0 \tag{5.15}$$

where x is a vector of dimension n, A a $n \times n$ constant coefficient matrix, and $f(t)$ is a given vector which in general depends, say continuously, upon the independent variable t. We use the method of variation of parameters to find a particular solution to (5.15). Let[8]

$$x(t) = e^{tA} y(t)$$

Then

$$\frac{dx}{dt} = Ae^{tA} y(t) + e^{tA}\frac{dy}{dt}$$

To satisfy the differential equation this must equal

$$Ae^{tA} y(t) + f(t)$$

[7]Recall the formula $T_{1/2} = \frac{\log 2}{\lambda}$.

[8]If $f(t) = 0$ then we know that $y(t)$ would be a constant vector. For nonzero $f(t)$ we are allowing for the possibility that y can depend upon t; hence the name variation of parameters.

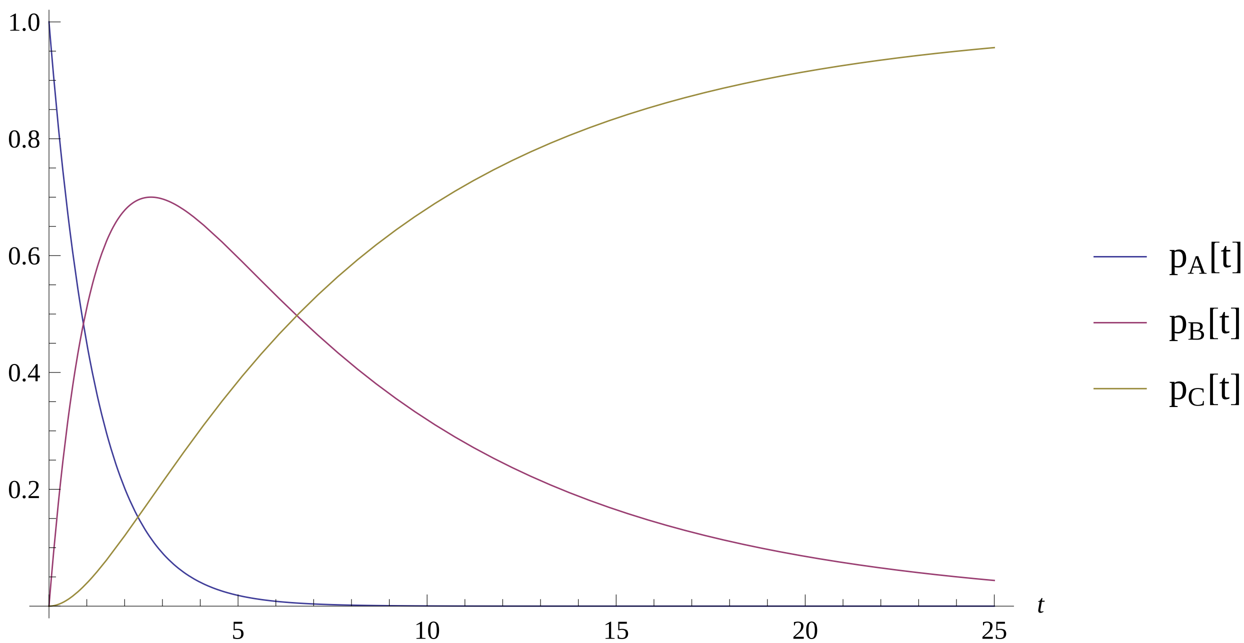


Figure 5.2: The population of A=Iodine–133, B=Xenon–133 and C=Cesium–133 starting with an initial amount of Iodine–133. The unit of time is in days.

and hence we must have

$$e^{tA}\frac{dy}{dt} = f(t)$$

Solving this for dy/dt:

$$\frac{dy}{dt} = e^{-tA}f(t)$$

The right hand side of the above equation is expressed in terms of known quantities. Integrating gives

$$y(t) = \int_0^t e^{-sA}f(s)\,ds$$

and hence the particular solution

$$x_{\mathrm{part}}(t) = e^{tA}\int_0^t e^{-sA}f(s)\,ds$$

Thus the solution satisfying the initial condition is

$$\boxed{x(t) = e^{tA}\int_0^t e^{-sA}f(s)\,ds + e^{tA}x_0} \qquad (5.16)$$

Observe that the solution of (5.15) has been reduced in (5.16) to matrix calculations and integration.

5.6.1 Nonautonomous linear equations

Consider the inhomogeneous equation

$$\frac{dx}{dt} = A(t)x + f(t) \qquad (5.17)$$

where now we assume the $n \times n$ matrix $A(t)$ depends upon the independent variable t and $f(t)$, as before, is a given column vector of size n which in general depends upon t. We assume that the coefficients of $A(t)$ are continuous on some closed interval $[a, b]$ and for simplicity we take $a < 0 < b$. Everything we say below is for t in the interval $a \le t \le b$.

The homogeneous version of (5.17) is

$$\frac{dx}{dt} = A(t)x. \qquad (5.18)$$

Since A is now assumed to depend upon t, the solution to (5.18) satisfying the initial condition $x(0) = x_0$ is *not* $\exp(tA)x_0$. So what plays the role of $\exp(tA)$ when A is a function of t?

Let $x_j(t)$, a column vector of size n, denote the solution to (5.18) satisfying the initial condition $x_j(0) = \mathbf{e}_j$, $j = 1, 2, \ldots, n$, where $\{\mathbf{e}_k\}_{k=1,2,\ldots n}$ denotes the standard basis of

$\mathbb{R}^n$. It can be shown that the functions $\{x_j(t)\}_{j=1,\ldots,n}$ form a basis for the vector space of solutions to (5.18). Form the $n \times n$ matrix $\mathcal{X}(t)$ whose columns are the $x_j(t)$:

$$\mathcal{X}(t) = \begin{pmatrix} \uparrow & \uparrow & \cdots & \uparrow \\ x_1(t) & x_2(t) & \cdots & x_n(t) \\ \downarrow & \downarrow & \cdots & \downarrow \end{pmatrix}.$$

$\mathcal{X}(t)$ satisfies

$$\mathcal{X}(0) = \mathrm{I}_n,$$

where I_n is the $n \times n$ identity matrix. (This follows immediately from the fact that $x_j(0) = \mathbf{e}_j$.) Furthermore, it can be shown that $\mathcal{X}(t)$ is invertible for all t. Note that $\mathcal{X}(t)$ satisfies the matrix differential equation

$$\frac{d\mathcal{X}(t)}{dt} = A(t)\mathcal{X}(t),$$

a fact that follows immediately from the fact that for each j, $dx_j/dt = A(t)x_j$. The $n \times n$ matrix $\mathcal{X}(t)$ is called the *fundamental matrix*. In the case when A has constant coefficients, the fundamental matrix is $\exp(tA)$.

We now use the matrix $\mathcal{X}(t)$ to construct a particular solution, $x_{\mathrm{part}}(t)$, to the inhomogeneous equation (5.17) by following the ideas for the constant coefficient case. We seek solutions to (5.17) of the form

$$x_{\mathrm{part}}(t) = \mathcal{X}(t)y(t).$$

We differentiate $x_{\mathrm{part}}(t)$ with respect to t:

$$\begin{aligned} \frac{dx_{\mathrm{part}}}{dt} &= \frac{d}{dt}\left(\mathcal{X}(t)y(t)\right) \\ &= \frac{d\mathcal{X}}{dt}\, y(t) + \mathcal{X}(t)\,\frac{dy}{dt} \\ &= A(t)\mathcal{X}(t)y(t) + \mathcal{X}(t)\,\frac{dy}{dt}. \end{aligned}$$

We want this to equal $A(t)x_{\mathrm{part}}(t) + f(t)$ so we get the equation

$$A(t)\mathcal{X}(t)\, y(t) + \mathcal{X}(t)\,\frac{dy}{dt} = A(t)x_{\mathrm{part}}(t) + f(t) = A(t)\mathcal{X}(t)y + f(t)$$

or

$$\mathcal{X}(t)\,\frac{dy}{dt} = f(t).$$

Multiplying both sides of the above equation by $[\mathcal{X}(t)]^{-1}$ gives

$$\frac{dy}{dt} = [\mathcal{X}(t)]^{-1} f(t).$$

Now integrate the above from 0 to t

$$y(t) = \int_0^t [\mathcal{X}(s)]^{-1} f(s)\, ds,$$

and recalling $x_{\mathrm{part}}(t) = \mathcal{X}(t)y$ gives

$$x_{\mathrm{part}}(t) = \mathcal{X}(t)y(t) = \mathcal{X}(t) \int_0^t [\mathcal{X}(s)]^{-1} f(s)\, ds.$$

Since the general solution to (5.17) is a particular solution plus a solution to the homogeneous equation, we have

$$\boxed{x(t) = \mathcal{X}(t) \int_0^t [\mathcal{X}(s)]^{-1} f(s)\, ds + \mathcal{X}(t) x_0} \tag{5.19}$$

solves (5.17) with initial condition $x(0) = x_0$. This formula should be compared with (5.16).

To summarize,

> To solve the *inhomogeneous equation* (5.17), one first finds the basis of solutions $\{x_j(t)\}_{j=1,\dots,n}$, $x_j(0) = \mathbf{e}_j$, to the *homogeneous* equation (5.18) and constructs the $n \times n$ fundamental matrix $\mathcal{X}(t)$. Then the solution to the inhomogeneous equation satisfying the initial condition $x(0) = x_0$ is given by (5.19). Thus the real difficulty is in solving the homogeneous problem, i.e. finding the fundamental matrix $\mathcal{X}(t)$.

5.7 Exercises

#1. Harmonic oscillator via matrix exponentials

Write the oscillator equation

$$\ddot{x} + \omega_0^2 x = 0$$

as a first order system (5.1). (Explicitly find the matrix A.) Compute $\exp(tA)$ and show that $x(t) = \exp(tA)x_0$ gives the now familiar solution. Note that we computed $\exp(tA)$ in (5.5) for the case $\omega_0 = 1$.

#2. Exponential of nilpotent matrices

1. Using the series expansion for the matrix exponential, compute $\exp(tN)$ where
$$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$
Answer the same question for
$$N = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$
How do these answers differ from $\exp(tx)$ where x is any real number?
2. A $n \times n$ matrix N is called nilpotent[9] if there exists a positive integer k such that
$$N^k = 0$$
where the 0 is the $n \times n$ zero matrix. If N is nilpotent let k be the smallest integer such that $N^k = 0$. Explain why $\exp(tN)$ is a matrix whose entries are polynomials in t of degree at most $k - 1$.

[9] In an advanced course in linear algebra, it will be proved that every matrix A can be written *uniquely*

#3. Computing e^{tA}

Let

$$A = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} \tag{5.20}$$

1. Find the eigenvalues and eigenvectors of A. (You can use any software package you like and merely quote the results.)
2. Use these to compute e^{tA}.

#4.

Consider the system of linear differential equations

$$\frac{dx}{dt} = Ax$$

where A is the 4×4 matrix

$$A = \begin{pmatrix} -5/2 & 1 & 1/2 & -1/2 \\ 3/4 & -5/2 & 0 & 3/4 \\ 1 & 2 & -3 & 1 \\ 0 & 2 & -1/2 & -2 \end{pmatrix} \tag{5.21}$$

Prove that all solutions $x(t)$ to this DE tend to zero as $t \to \infty$. Hint: You need not compute e^{tA}. You can prove this statement simply by computing the eigenvalues of A. (Why?)

#5.

Consider the system of linear differential equations

$$\frac{dx}{dt} = Ax$$

where A is the 4×4 matrix

$$A = \begin{pmatrix} 0 & 0 & -3/2 & 2 \\ -3/4 & 1/2 & 0 & -3/4 \\ -1 & -2 & 1 & -1 \\ 1/2 & -3 & 3/2 & -3/2 \end{pmatrix} \tag{5.22}$$

Find a subspace V of $\mathbb{R}^4$ such that if $x(0) \in V$, then $x(t) \to 0$ as $t \to \infty$. Hint: The subspace V is described in terms of (some of) the eigenvectors of A.

---

as $D + N$ where D is a diagonalizable matrix, N is a nilpotent matrix, *and* $DN = ND$. Furthermore, an algorithm will be given to find the matrices D and N from the matrix A. Once this is done then one can compute $\exp(tA)$ as follows

$$\exp(tA) = \exp(tD + tN) = \exp(tD)\exp(tN).$$

We showed above how to reduce the computation of $\exp(tD)$, D a diagonalizable matrix, to linear algebra. This problem shows that $\exp(tN)$ reduces to finitely many matrix multiplications. Thus the computation of both $\exp(tD)$ and $\exp(tN)$ are reduced to linear algebra and hence so is $\exp(tA)$. Observe that it is crucial that we know $DN = ND$.

#6.

Consider the system of linear differential equations

$$\frac{dx}{dt} = Ax$$

where A is the 2×2 matrix

$$A = \begin{pmatrix} 1 & \alpha \\ -\alpha & 3 \end{pmatrix} \tag{5.23}$$

For what values of α will the solutions exhibit oscillatory behavior?

#7. Radioactive decay & first introduction to Laplace transforms

Birth processes have been used since the time of Rutherford to model radioactive decay. (Radioactive decay occurs when an unstable isotope transforms to a more stable isotope, generally by emitting a subatomic particle.) In many cases a radioactive nuclide A decays into a nuclide B which is also radioactive; and hence, B decays into a nuclide C, etc. The nuclides B, C, etc. are called the progeny (formerly called daughters). This continues until the decay chain reaches a stable nuclide. For example, uranium-238 decays through α-emission to thorium-234 which in turn decays to protactinium-234 through β-emission. This chain continues until the stable nuclide lead-206 is reached.

1. Let the decay states be $E_1 \to E_2 \to \cdots \to E_N$ where E_N is the final stable state. We can relabel these states to be simply $1, 2, \ldots, N$. (That is, we write E_j as simply j.) Let $\mathcal{N}(t)$ denote the state of the nuclide at time t. $\mathcal{N}(t)$ is a random process (called a Markov process) due to the fact that radioactive decay is inherently random. Thus we introduce

$$\begin{aligned} p_j(t) &= \mathbb{P}(\mathcal{N}(t) = j | N(0) = 1) \\ &= \text{probability that nuclide is in state } j \text{ at time } t \\ & \text{ given it starts in state } 1 \text{ at time } t = 0. \end{aligned}$$

   These probabilities $p_j(t)$ satisfy differential equations called the *Kolmogorov forward equations*:

$$\frac{dp_j}{dt} = \lambda_{j-1} p_{j-1}(t) - \lambda_j p_j(t),\ j = 1, 2, \ldots, N. \tag{5.24}$$

   The constants λ_j are called the *decay rates*. A decay rate λ is related to the half-life, $T_{1/2}$, of the nuclide by the well-known formula

$$T_{1/2} = \frac{\log 2}{\lambda},\ \log 2 = 0.693147\ldots \tag{5.25}$$

   We assume $\lambda_i \neq \lambda_j$ for $i, j = 1, \ldots, N-1$. We set $\lambda_0 = 0$ and $\lambda_N = 0$. (λ_N is set equal to zero since the final state N is a stable nuclide and does not decay.)

   In applications to radioactive decay, if N_1 is the number of initial nuclides (the number of nuclides in state E_1), then $N_1 p_j(t)$ is the number of nuclides in state E_j at time t.

2. Introduce the Laplace transform[10]

$$\hat{p}_j(s) = \int_0^\infty e^{-ts} p_j(t)\, dt$$

and show that the Laplace transform of (5.24) is

$$s\hat{p}_j(s) - \delta_{j,1} = \lambda_{j-1}\hat{p}_{j-1}(s) - \lambda_j \hat{p}_j(s),\ j = 1, \ldots, N. \tag{5.26}$$

Solve these equations for $\hat{p}_j(s)$ and show that

$$\hat{p}_j(s) = \frac{\lambda_1}{s+\lambda_1}\frac{\lambda_2}{s+\lambda_2}\cdots\frac{\lambda_{j-1}}{s+\lambda_{j-1}}\frac{1}{s+\lambda_j}$$

3. Using the above expression for $\hat{p}_j(s)$ partial fraction the result:

$$\hat{p}_j(s) = \sum_{k=1}^{j} \frac{c_{j,k}}{s+\lambda_k}$$

See if you can find expressions for $c_{j,k}$. You might want to take some special cases to see if you can make a guess for the $c_{j,k}$. (The Mathematica command `Apart` will prove useful.)

4. From the partial fraction decomposition of $\hat{p}_j(s)$ explain why you can almost immediately conclude

$$p_j(t) = \sum_{k=1}^{j} c_{j,k}\, e^{-\lambda_k t},\ j = 1, 2, \ldots, N. \tag{5.27}$$

5. For the special case of $N = 4$: $E_1 \to E_2 \to E_3 \to E_4$ find explicitly the probabilities $p_j(t)$. (You can use Mathematica if you wish. Note there is a command `InverseLaplaceTransform`.)

6. Show that $p_2(t)$ has a maximum at $t = t_m$

$$t_m = \frac{\log(\lambda_1/\lambda_2)}{\lambda_1 - \lambda_2} > 0.$$

In terms of the radioactive decay interpretation, this is the time when the first progeny has a maximum population.

7. Using Mathematica (recall the command `Series`) show that as $t \to 0$

$$\begin{aligned} p_1(t) &= 1 - \lambda_1 t + \mathrm{O}(t^2) \\ p_2(t) &= \lambda_1 t + \mathrm{O}(t^2) \\ p_3(t) &= \frac{1}{2}\lambda_1\lambda_2 t^2 + \mathrm{O}(t^3) \\ p_4(t) &= \frac{1}{3!}\lambda_1\lambda_2\lambda_3 t^3 + \mathrm{O}(t^4) \end{aligned}$$

[10]See Chapter 8 of these Notes and Boyce & Diprima, Chapter 6 [4].

8. Radon 222 gas is a chemically inert radioactive gas that is part of the Uranium 238 decay chain. Radon and its radioactive progeny are known carcinogens. Here is part of the decay chain[11]

$$\cdots \longrightarrow \text{Rn } 222 \longrightarrow \text{Po } 218 \longrightarrow \text{Pb } 214 \longrightarrow \text{Bi } 214 \longrightarrow \cdots$$

The half-life of each nuclide is known (recall (5.25)):

$$\begin{aligned} \text{Rn } 222{:} &\quad T_{1/2} = 3.8235\,\text{days} \\ \text{Po } 218{:} &\quad T_{1/2} = 3.10\,\text{minutes} \\ \text{Pb } 214{:} &\quad T_{1/2} = 26.8\,\text{minutes} \\ \text{Bi } 214{:} &\quad T_{1/2} = 19.9\,\text{minutes} \end{aligned}$$

Let N_{Rn} denote the initial amount of Rn 220 and assume the other nuclides are not present at time $t = 0$. Solve the Kolmogorov forward equations for this particular birth process. (Note that here the probabilities do not sum to one since the Bi 214 also decays.) This is not so messy if you use MATHEMATICA. Find the times when each of the progeny have maximum population. (Highest probability) You might want to use MATHEMATICA'S `FindRoot`.

[11] Po=polonium, Pb=lead, Bi=bismuth.

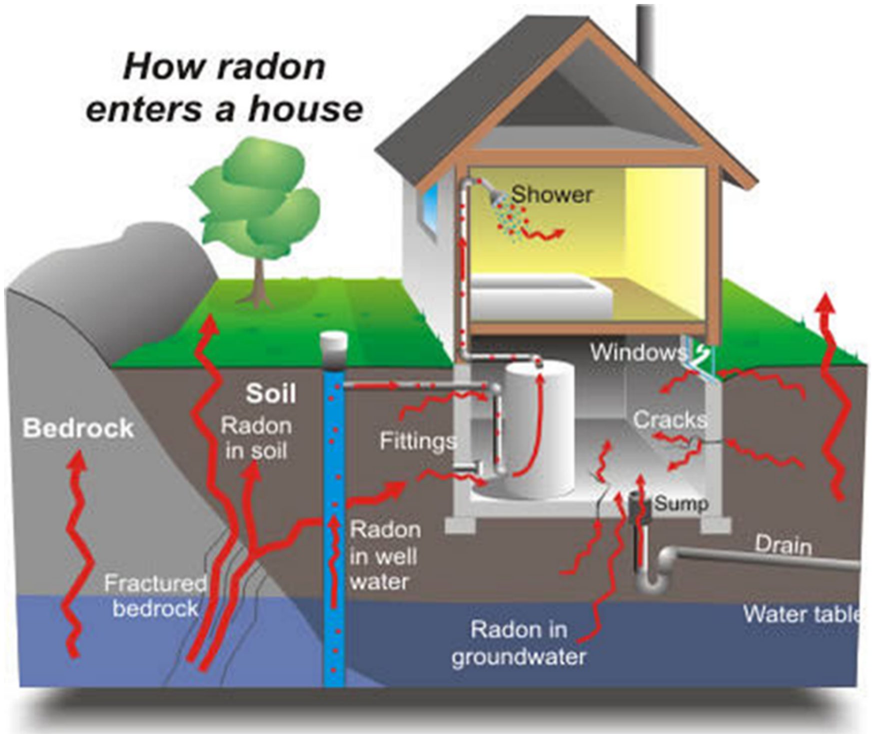


Figure 5.3: From the EPA website: Radon is a radioactive gas. It comes from the natural decay of uranium that is found in nearly all soils. It typically moves up through the ground to the air above and into your home through cracks and other holes in the foundation. Your home traps radon inside, where it can build up. Any home may have a radon problem. This means new and old homes, well-sealed and drafty homes, and homes with or without basements. Radon from soil gas is the main cause of radon problems. Sometimes radon enters the home through well water. In a small number of homes, the building materials can give off radon, too. However, building materials rarely cause radon problems by themselves.

Chapter 6

Weighted String

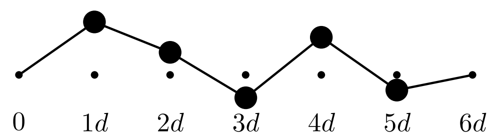
Figure 6.1: Hermann von Helmholtz, 1821–1894.

Because linear equations are easy to solve and study, the theory of linear oscillations is the most highly developed area of mechanics. In many nonlinear problems, linearization produces a satisfactory approximate solution. Even when this is not the case, the study of the linear part of a problem is often a first step, to be followed by the study of the relation between motions in a nonlinear system and its linear model.

V. I. Arnold, *Mathematical Methods of Classical Mechanics* [1]

6.1 Derivation of differential equations

The *weighted string* is a system in which the mass is concentrated in a set of equally spaced mass points, N in number with spacing d, imagined to be held together by massless springs of equal tension T. We further assume that the construction is such that the mass points move only in the vertical direction (y direction) and there is a constraining force to keep the mass points from moving in the horizontal direction (x direction). We call it a "string" since these mass points give a discrete string—the tension in the string is represented by the springs. The figure below illustrates the weighted string for $N = 5$.



The string is "tied down" at the endpoints 0 and $(N + 1)d$. The horizontal coordinates of the mass points will be at $x = d, 2d, \ldots, Nd$. We let u_j denote the vertical displacement of the j^{th} mass point and F_j the transverse force on the j^{th} particle. To summarize the variables introduced so far:

$$\begin{aligned}
m &= \text{mass of particle,}\\
N &= \text{total number of particles,}\\
T &= \text{tension of spring,}\\
d &= \text{horizontal distance between two particles,}\\
u_j &= \text{vertical displacement of } j^{th} \text{ particle,} \quad j = 1, 2, \ldots N,\\
F_j &= \text{transverse force on } j^{th} \text{ particle,} \quad j = 1, 2, \ldots N.
\end{aligned}$$

To impose the boundary conditions that the ends of the string are rigidly fixed at $x = 0$ and $x = (N + 1)d$, we take

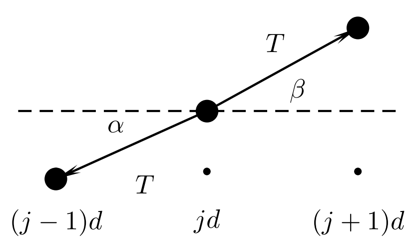
$$u_0 = 0 \quad \text{and} \quad u_{N+1} = 0.$$

Newton's equations for these mass points are

$$F_j = m\frac{d^2u_j}{dt^2}, \quad j = 1, 2, \ldots, N.$$

This is a system of N second order differential equations. We now find an expression for the transverse force F_j in terms of the vertical displacements.

In the diagram below, the forces acting on the j^{th} particle are shown.



From the diagram,

$$F_j = T\sin\beta - T\sin\alpha.$$

We make the assumption that the angles α and β are small. (The string is not stretched too much!) In this small angle approximation we have

$$\sin\alpha \approx \tan\alpha \quad \text{and} \quad \sin\beta \approx \tan\beta.$$

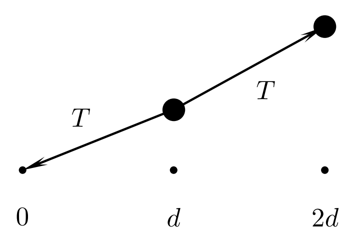
Therefore, in this small angle approximation

$$\begin{aligned} F_j &\approx T\tan\beta - T\tan\alpha \\ &= T\left(\frac{u_{j+1}-u_j}{d}\right) - T\left(\frac{u_j - u_{j-1}}{d}\right). \end{aligned}$$

Thus,

$$m\frac{d^2u_j}{dt^2} = \frac{T}{d}\left(u_{j+1} - 2u_j + u_{j-1}\right), \quad j = 1, 2, \ldots, N. \tag{6.1}$$

Note that these equations are valid for $j = 1$ and $j = N$ when we interpret $u_0 = 0$ and $u_{N+1} = 0$. For example, for $j = 1$ the force F_1 is determined from the diagram:



$$\begin{aligned} F_1 &= T\frac{(u_2 - u_1)}{d} - T\frac{u_1}{d} \\ &= \frac{T}{d}\left(u_2 - 2u_1 + u_0\right), \quad u_0 = 0. \end{aligned}$$

Equation (6.1) is a system of N second order linear differential equations. Thus the dimension of the vector space of solutions is $2N$; that is, it takes $2N$ real numbers to specify the initial conditions (N initial positions and N initial velocities). Define the $N \times N$ matrix

$$V_N = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix} \tag{6.2}$$

and the column vector $\mathbf{u}$

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ u_N \end{pmatrix} . \tag{6.3}$$

Then (6.1) can be written in the compact matrix form

$$\boxed{\frac{d^2\mathbf{u}}{dt^2} + \frac{T}{md} V_N \mathbf{u} = 0.} \tag{6.4}$$

Note: We could also have written (6.1) as a *first* order matrix equation of the form

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} \tag{6.5}$$

where A would be a $2N \times 2N$ matrix. However, for this application it is simpler to develop a special theory for (6.4) rather than to apply the general theory of (6.5) since the matrix manipulations with V_N will be a bit clearer than they would be with A.

6.2 Reduction to an eigenvalue problem

Equation (6.4) is the matrix version of the harmonic oscillator equation

$$\frac{d^2x}{dt^2} + \frac{k}{m} x = 0, \quad \omega_0^2 = \frac{k}{m} . \tag{6.6}$$

Indeed, we will show that (6.4) is precisely N harmonic oscillators (6.6)—once one chooses the correct coordinates. We know that solutions to (6.6) are linear combinations of

$$\cos \omega_0 t \quad \text{and} \quad \sin \omega_0 t.$$

Thus we "guess" that solutions to (6.4) are linear combinations of the form

$$\cos \omega t \, \mathbf{f} \quad \text{and} \quad \sin \omega t \, \mathbf{f}$$

where ω is to be determined and $\mathbf{f}$ is a column vector of length N. (Such a "guess" can be theoretically deduced from the theory of the matrix exponential when (6.4) is rewritten in the form (6.5).)

Thus setting

$$\mathbf{u} = e^{i\omega t} \mathbf{f},$$

we see that (6.4) becomes the matrix equation

$$V_N \mathbf{f} = \frac{md}{T} \omega^2 \mathbf{f}.$$

That is, we must find the eigenvalues and eigenvectors of the matrix V_N. Since V_N is a real symmetric matrix, it is diagonalizable with real eigenvalues. To each eigenvalue λ_n, i.e.

$$V_N \mathbf{f_n} = \lambda_n \mathbf{f_n}, \quad n = 1, 2, \ldots, N,$$

there will correspond a positive frequency

$$\omega_n^2 = \frac{T}{md} \lambda_n, \quad n = 1, 2, \ldots, N,$$

and a solution of (6.4) of the form

$$\mathbf{u_n} = (a_n \cos(\omega_n t) + b_n \sin(\omega_n t))\, \mathbf{f_n}$$

where a_n and b_n are constants. This can now be easily verified by substituting this above expression into the differential equation. To see we have enough constants of integration we observe that we have two constants, a_n and b_n, for each (vector) solution $\mathbf{u_n}$. And we have N vector solutions $\mathbf{u_n}$—thus $2N$ constants in all. We now turn to an explicit evaluation of the frequencies ω_n^2—such frequencies are called *normal modes*.

6.3 Computation of the eigenvalues of V_N

We introduce the characteristic polynomial of the matrix V_N:

$$D_N(\lambda) = D_N = \det\left(V_N - \lambda I\right).$$

Expanding the determinant D_N in the last column, we see that it is a sum of two terms—each a determinant of matrices of size $(N-1) \times (N-1)$. One of these determinants equals $(2-\lambda)D_{N-1}$ and the other equals D_{N-2} as is seen after expanding again, this time by the last row. In this way one deduces

$$D_N = (2-\lambda)D_{N-1} - D_{N-2}, \quad N = 2, 3, 4, \ldots$$

with

$$D_0 = 1 \quad \text{and} \quad D_1 = 2 - \lambda.$$

We now proceed to solve this constant coefficient difference equation (in N). From earlier work we know that the general solution is of the form

$$c_1 \mu_1^N + c_2 \mu_2^N$$

where μ_1 and μ_2 are the roots of

$$\mu^2 - (2-\lambda)\mu + 1 = 0.$$

Solving this quadratic equation gives

$$\mu_{1,2} = 1 - \frac{\lambda}{2} \pm \frac{1}{2}\sqrt{(2-\lambda)^2 - 4}\,.$$

It will prove convenient to introduce an auxiliary variable θ through

$$2 - \lambda = 2\cos\theta,$$

A simple computation now shows

$$\mu_{1,2} = e^{\pm i\theta}.$$

Thus

$$D_N = c_1 e^{iN\theta} + c_2 e^{-iN\theta}.$$

To determine c_1 and c_2 we require that

$$D_0 = 1 \quad \text{and} \quad D_1 = 2 - \lambda.$$

That is,

$$\begin{aligned} c_1 + c_2 &= 1, \\ c_1 e^{i\theta} + c_2 e^{-i\theta} &= 2 - \lambda = 2\cos\theta. \end{aligned}$$

Solving for c_1 and c_2,

$$\begin{aligned} c_1 &= \frac{e^{i\theta}}{e^{i\theta} - e^{-i\theta}}, \\ c_2 &= -\frac{e^{-i\theta}}{e^{i\theta} - e^{-i\theta}}. \end{aligned}$$

Therefore,

$$\begin{aligned} D_N &= \frac{1}{e^{i\theta} - e^{-i\theta}} \left(e^{i(N+1)\theta} - e^{-i(N+1)\theta} \right) \\ &= \frac{\sin\left((N+1)\theta\right)}{\sin\theta}. \end{aligned}$$

The eigenvalues of V_N are solutions to

$$D_N(\lambda) = \det(V_N - \lambda I) = 0.$$

Thus we require

$$\sin\left((N+1)\theta\right) = 0,$$

which happens when

$$\theta = \theta_n := \frac{n\pi}{N+1}, \quad n = 1, 2, \ldots, N.$$

Thus the eigenvalues of V_N are

$$\lambda_n = 2 - 2\cos\theta_n = 4\sin^2(\theta_n/2), \;\; n = 1, 2, \ldots, N. \tag{6.7}$$

The eigenfrequencies are

$$\begin{aligned} \omega_n^2 &= \frac{T}{md}\lambda_n = \frac{2T}{md}(1 - \cos\theta_n) \\ &= \frac{2T}{md}\left(1 - \cos\frac{n\pi}{N+1}\right) = \frac{4T}{md}\sin^2\left(\frac{n\pi}{2(N+1)}\right). \end{aligned} \tag{6.8}$$

Remark: We know there are at most N distinct eigenvalues of V_N. The index n does not start at zero because this would imply $\theta = 0$, but $\theta = 0$—due to the presence of $\sin\theta$ in the denominator of D_N—is *not* a zero of the determinant and hence does not correspond to an eigenvalue of V_N. We conclude there are N distinct eigenvalues of V_N. These *eigenfrequencies* are also called *normal modes* or *characteristic oscillations*.

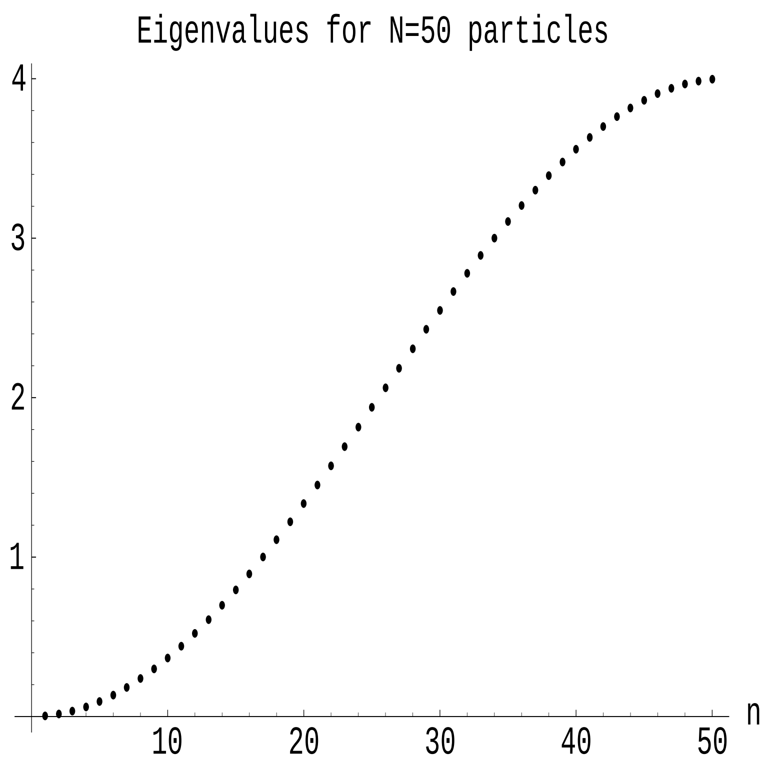


Figure 6.2: Eigenvalues λ_n, (6.7), for $N = 50$ particles.

6.4 The eigenvectors

6.4.1 Constructing the eigenvectors $\mathbf{f_n}$

We now find the eigenvector $\mathbf{f}_n$ corresponding to eigenvalue λ_n. That is, we want a column vector $\mathbf{f_n}$ that satisfies

$$V_N \mathbf{f_n} = 2(1-\cos\theta_n)\mathbf{f_n}, \quad n = 1, 2, \ldots, N.$$

Setting,

$$\mathbf{f_n} = \begin{pmatrix} f_{n1} \\ f_{n2} \\ \cdot \\ \cdot \\ f_{nN} \end{pmatrix},$$

the above equation in component form is

$$-f_{n,j-1} + 2f_{n,j} - f_{n,j+1} = 2(1-\cos\theta_n)f_{n,j}$$

with

$$f_{n,0} = f_{n,N+1} = 0.$$

This is a constant coefficient difference equation *in the j index*. Assume, therefore, a solution of the form

$$f_{n,j} = e^{ij\varphi}.$$

The recursion relation becomes with this guess

$$-2\cos\varphi + 2 = 2(1-\cos\theta_n),$$

i.e.

$$\varphi = \pm\theta_n.$$

The $f_{n,j}$ will be linear combinations of $e^{\pm ij\theta_n}$,

$$f_{n,j} = c_1 \sin(j\theta_n) + c_2\cos(j\theta_n).$$

We require $f_{n,0} = f_{n,N+1} = 0$ which implies $c_2 = 0$.

To summarize,

$$\boxed{\begin{aligned} V_N\mathbf{f_n} &= \frac{md}{T}\omega_n^2\mathbf{f_n}, \quad n = 1, 2, \ldots, N, \\ \omega_n^2 &= \frac{2T}{md}(1-\cos\theta_n), \quad \theta_n = \frac{n\pi}{N+1}, \\ \mathbf{f_n} &= \begin{pmatrix} \sin(\theta_n) \\ \sin(2\theta_n) \\ \cdot \\ \cdot \\ \sin(N\theta_n) \end{pmatrix} \quad n = 1, 2, \ldots, N. \end{aligned}} \tag{6.9}$$

The general solution to (6.4) is

$$\mathbf{u}(t) = \sum_{n=1}^{N} \left(a_n \cos(\omega_n t) + b_n \sin(\omega_n t)\right) \mathbf{f_n},$$

or in component form,

$$u_j(t) = \sum_{n=1}^{N} \left(a_n \cos(\omega_n t) + b_n \sin(\omega_n t)\right) \sin(j\theta_n). \tag{6.10}$$

Thus every oscillation of the weighted string is a sum of characteristic oscillations.

6.4.2 Orthogonality of eigenvectors

The set of eigenvectors $\{\mathbf{f_n}\}_{n=1}^{N}$ forms a basis for $\mathbf{R}^N$ since the matrix V_N is symmetric. (Another reason they form a basis is that the eigenvalues of V_N are distinct.) We claim the eigenvectors have the additional (nice) property that they are orthogonal, i.e.

$$\mathbf{f_n} \cdot \mathbf{f_m} = 0, \quad n \neq m,$$

where $\cdot$ denotes the vector dot product. The orthogonality is a direct result of the fact that V_N is a symmetric matrix. Another way to prove this is to use (6.9) to compute

$$\mathbf{f_n} \cdot \mathbf{f_m} = \sum_{j=1}^{N} \sin(j\theta_n) \sin(j\theta_m). \tag{6.11}$$

To see that this is zero for $n \neq m$, we leave as an exercise to prove the trigonometric identity

$$\sum_{j=1}^{N} \sin\left(\frac{nj\pi}{N+1}\right) \sin\left(\frac{mj\pi}{N+1}\right) = \frac{1}{2}(N+1)\delta_{n,m}$$

where $\delta_{n,m}$ is the Kronecker delta function. (One way to prove this identity is first to use the formula $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i$ to rewrite the above sum as a sum of exponentials. The resulting sums will be finite geometric series.) From this identity we also get that the length of each vector, $\|\mathbf{f}_n\|$, is

$$\|\mathbf{f}_n\| = \sqrt{\frac{N+1}{2}}\,.$$

6.5 Determination of constants a_n and b_n

Given the initial vectors $\mathbf{u}(0)$ and $\dot{\mathbf{u}}(0)$, we now show how to determine the constants a_n and b_n. At $t = 0$,

$$\mathbf{u}(0) = \sum_{n=1}^{N} a_n \mathbf{f_n}.$$

Dotting the vector $\mathbf{f_p}$ into both sides of this equation and using the orthogonality of the eigenvectors, we see that

$$a_p = \frac{2}{N+1} \sum_{j=1}^{N} \sin\left(\frac{pj\pi}{N+1}\right) u_j(0), \quad p = 1, 2, \ldots, N. \tag{6.12}$$

Differentiating $\mathbf{u}(t)$ with respect to t and then setting $t = 0$, we have

$$\dot{\mathbf{u}}(0) = \sum_{n=1} \omega_n b_n \mathbf{f_n}.$$

Likewise dotting $\mathbf{f_p}$ into both sides of this equation results in

$$b_p = \frac{2}{N+1} \frac{1}{\omega_p} \sum_{j=1}^{N} \sin\left(\frac{pj\pi}{N+1}\right) \dot{u}_j(0), \quad p = 1, 2, \ldots, N. \tag{6.13}$$

If we assume the weighted string starts in an initial state where all the initial velocities are zero,

$$\dot{u}_j(0) = 0,$$

then the solution $\mathbf{u}(t)$ has components

$$u_j(t) = \sum_{n=1}^{N} a_n \cos(\omega_n t) \sin(j\theta_n) \tag{6.14}$$

where the constants a_n are given by (6.12) in terms of the initial displacements $u_j(0)$. The special solutions obtained by setting all the a_n except for one to zero, are called the *normal modes of oscillation* for the weighted string. They are most interesting to graph as a function both in space (the j index) and in time (the t variable). In figures we show a "snapshot" of various normal mode solutions at various times t.

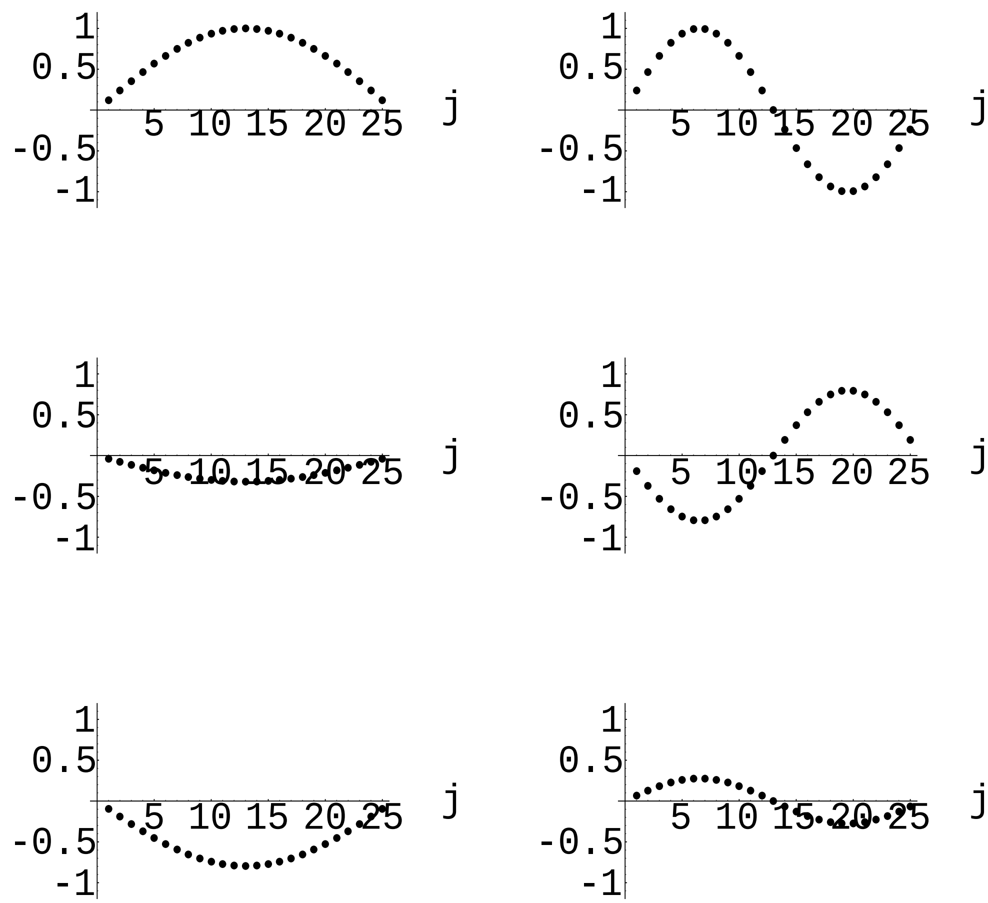


Figure 6.3: Vertical displacements u_j for the two lowest ($n = 1$ and $n = 2$) normal modes are plotted as function of the horizontal position index j. Each column gives the same normal mode but at different times t. System is for $N = 25$ particles.

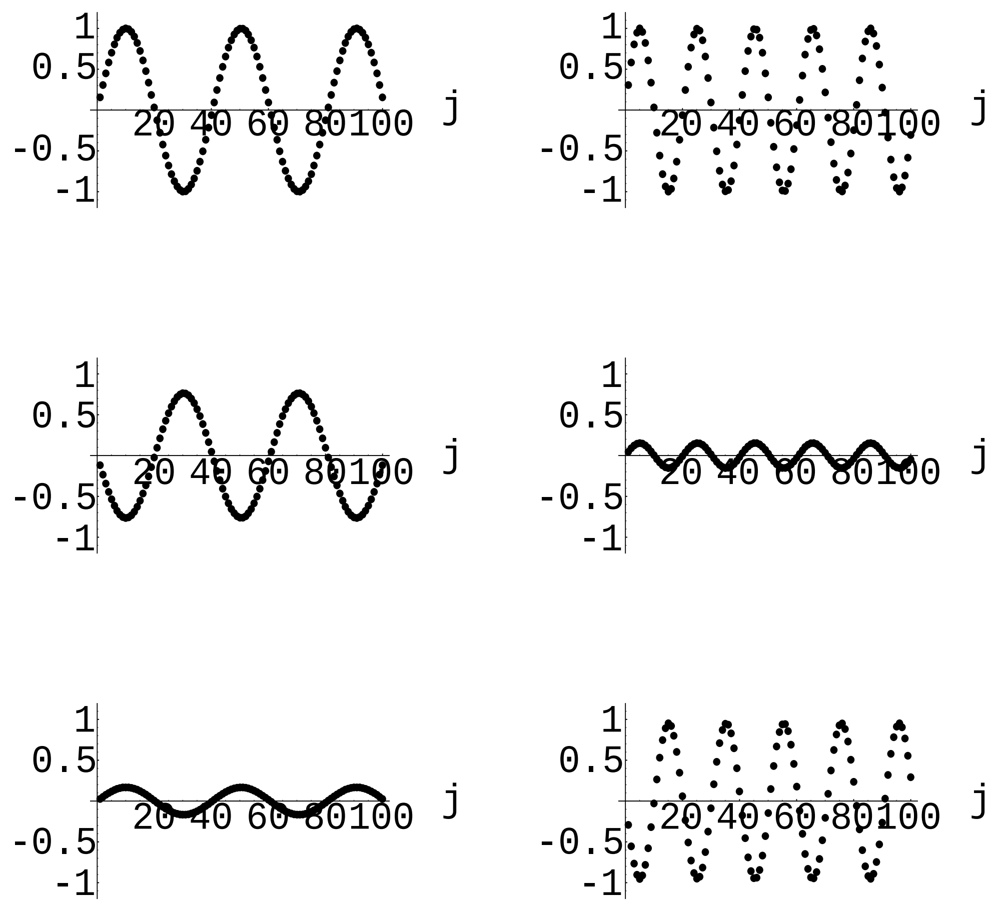


Figure 6.4: Vertical displacements u_j for the two normal modes $n = 5$ and $n = 10$ are plotted as function of the horizontal position index j. Each column gives the same normal mode but at different times t. System is for $N = 100$ particles.

6.6 Continuum limit: The wave equation

As the pictures illustrate, when the number of particles N becomes large and the distance d between the particles becomes small, there appear limiting curves that describe the oscillations of the entire system. These limiting curves describe the oscillations of the string. Let us pursue this in more detail. We assume

$$N \to \infty \text{ and } d \to 0 \text{ such that } Nd \to L$$

where L is the length of the string (under no tension). We assume that the mass of the string is given by μL where μ is the mass per unit length. Thus we assume

$$mN \to \mu L$$

The positions of the particles, jd, $j = 1, 2, \ldots, N$, are then assumed to approach a continuous position variable x:

$$jd \to x$$

We now examine the continuum limit of the system of ordinary differential equations

$$\frac{d^2 u_j}{dt^2} = \frac{T}{md}\left(u_{j-1} - 2u_j + u_{j+1}\right)$$

To do this we assume there exists a function $u(x,t)$ such that

$$u_j(t) = u(jd, t)$$

Then, since d is small,

$$u_{j-1} = u(jd - d, t) = u(x,t) - d\,\frac{\partial u}{\partial x}(x,t) + \frac{1}{2} d^2\,\frac{\partial^2 u}{\partial x^2}(x,t) + O(d^3)$$

and similarly

$$u_{j+1} = u(jd + d, t) = u(x,t) + d\,\frac{\partial u}{\partial x}(x,t) + \frac{1}{2} d^2\,\frac{\partial^2 u}{\partial x^2}(x,t) + O(d^3)$$

and hence

$$u_{j-1} - 2u_j + u_{j+1} = d^2\,\frac{\partial^2 u}{\partial x^2}(x,t) + O(d^3)$$

Substituting this into our differential equations we obtain

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\mu}\,\frac{\partial^2 u}{\partial x^2}$$

Note that since $m = \mu L / N$,

$$\frac{Td^2}{md} = \frac{Td}{m} = \frac{TdN}{\mu L} = \frac{T}{\mu}$$

Also observe that T/μ has the dimensions of (velocity)^2. Thus let's call

$$v^2 = \frac{T}{\mu}$$

so that we have

$$\boxed{\frac{\partial^2 u}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0.} \tag{6.15}$$

This is the one-dimensional *wave equation*. It is an example of a *partial differential equation*. Given our analysis of the weighted string, we can anticipate that if we studied solutions of the *single* partial differential equation (6.15), then $u = u(x,t)$ would describe the oscillations of a string. Note that we would have the two boundary conditions

$$u(0,t) = u(L,t) = 0$$

which corresponds to the statement that the string is tied down at $x = 0$ and at $x = L$ for all times t. In addition, we specify at $t = 0$ the initial displacement of the string: $u(x,0) = f(x)$ where f is a given function as well as the initial velocity $\frac{\partial u}{\partial t}(x,0)$. The problem then is to find the solution to (6.15) satisfying these conditions. In the next section we show how the methods we've developed so far permit us to find such a solution.

6.6.1 Solution to the wave equation

We first look for solutions of the form (called separation of variables)

$$u(x,t) = X(x)\,T(t)$$

where X is only a function of x and T is only a function of t. Since

$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2X}{dx^2}\,T(t) \ \text{ and } \ \frac{\partial^2 u}{\partial t^2} = X(x)\,\frac{d^2T}{dt^2},$$

we have, upon substituting these expressions into (6.15) and dividing by $X\,T$ the condition

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{v^2}\frac{1}{T}\frac{d^2T}{dt^2}.$$

The left-hand side of the above equation is a function only of x and the right-hand side of the same equation is a function only of t. The only way this can be true is for both sides to equal the same constant. (We will see below that this constant has to be negative to satisfy the boundary conditions. Anticipating this fact we write the constant as $-k^2$.) That is to say, we have

$$\frac{1}{X}\frac{d^2X}{dx^2} = -k^2 = \frac{1}{v^2}\frac{1}{T}\frac{d^2T}{dt^2}$$

This gives us *two* ordinary differential equations:

$$\frac{d^2X}{dx^2} + k^2X = 0, \quad \frac{d^2T}{dt^2} + k^2v^2T = 0.$$

The solution to the first equation is

$$X(x) = c_1\cos(kx) + c_2\sin(kx).$$

We want $u(0,t)=0$ which implies $c_1=0$. We also require $u(L,t)=0$. If we set $c_2=0$ then X is identically zero and we have the trivial solution. Thus we must require

$$\sin(kL)=0.$$

This is satisfied if

$$kL=n\pi, n=1,2,3,\ldots.$$

(Note that $n=-1,-2,\ldots$ give the same solution up to a sign and $n=0$ corresponds to X identically zero.) The solution to the T equation is also a linear combination of sines and cosines. Thus for each value of n we have found a solution satisfying the conditions $u(0,t)=u(L,t)=0$ of the form

$$u_n(x,t)=\sin(\frac{n\pi}{L}x)\left[a_n\cos(\frac{n\pi v}{L}t)+b_n\sin(\frac{n\pi v}{L}t)\right]$$

where a_n and b_n are constants. Since the wave equation is *linear*, linear supposition of solutions results in a solution. Thus

$$u(x,t)=\sum_{n=1}^{\infty}\sin(\frac{n\pi}{L}x)\left[a_n\cos(\frac{n\pi v}{L}t)+b_n\sin(\frac{n\pi v}{L}t)\right]$$

is a solution satisfying $u(0,t)=u(L,t)=0$. We now require that $u(x,0)=f(x)$. That is we want

$$u(x,0)=\sum_{n=1}^{\infty}a_n\sin(\frac{n\pi}{L}x)=f(x)$$

We now use the fact that the for $m,n=1,2,\ldots$

$$\int_0^L\sin(\frac{m\pi}{L}x)\sin(\frac{n\pi}{L}x)\,dx=\frac{L}{2}\,\delta_{m,n}$$

to find

$$a_n=\frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi}{L}x)\,dx. \tag{6.16}$$

This determines the constants a_n. If we further assume (for simplicity) that

$$\frac{\partial u}{\partial t}(x,0)=0$$

(initial velocity is zero), then a very similar calculation gives $b_n=0$. Thus we have shown

$$u(x,t)=\sum_{n=1}^{\infty}a_n\ \sin(\frac{n\pi}{L}x)\ \cos(\frac{n\pi v}{L}t) \tag{6.17}$$

where a_n are given by (6.16).

It is instructive to compare this solution of the wave equation to the solution (6.14) of the weighted string. We take the $N\to\infty$ limit directly in (6.14) and use the same scaling as we have in the above derivation of the wave equation. In this limit we can replace

$$d\longrightarrow\frac{L}{N},\ m\longrightarrow\frac{\mu L}{N},\ j\longrightarrow\frac{xN}{L}$$

Thus

$$\omega_n^2 = \frac{4T}{md}\sin^2\left(\frac{n\pi}{2(N+1)}\right) \sim \frac{4T}{md}\,\frac{n^2\pi^2}{4(N+1)^2} \sim \frac{T}{\mu}\,\frac{n^2\pi^2}{L^2}$$

so that

$$\omega_n \longrightarrow v\,\frac{n\,\pi}{L}.$$

(Recall the definition $v = \sqrt{T/\mu}$.) Similarly,

$$j\theta_n = \frac{nj\pi}{N+1} = \frac{N}{N+1}\,\frac{n\pi}{L}\,x \longrightarrow \frac{n\pi}{L}\,x.$$

Putting these limiting expressions into (6.14) and taking the $N \to \infty$ limit we see that (6.14) becomes (6.17). The only point that needs further checking is to show the a_n as given by (6.12) approaches the a_n as given by (6.16). This requires the natural assumption that the initial conditions $u_j(0)$ can be written in the form $u_j(0) = f(jd)$ for some smooth function f. This is the f of $u(x,0) = f(x)$. A calculation then shows that (6.12) is the Riemann sum approximation to (6.16) and approaches (6.16) as $N \to \infty$.

The take home message is that the oscillations described by the solution to the wave equation can be equivalently viewed as an infinite system of harmonic oscillators.

6.7 Inhomogeneous problem

The inhomogeneous version of (6.4) is

$$\frac{d^2\mathbf{u}}{dt^2} + \frac{T}{md}V_N\mathbf{u} = \mathbf{F}(t) \tag{6.18}$$

where $\mathbf{F}(t)$ is a given driving term. The j^{th} component of $\mathbf{F}(t)$ is the external force acting on the particle at site j. An interesting case of (6.18) is

$$\mathbf{F}(t) = \cos\omega t\,\mathbf{f}$$

where $\mathbf{f}$ is a constant vector. The general solution to (6.18) is the sum of a particular solution and a solution to the homogeneous equation. For the particular solution we assume a solution of the form

$$\mathbf{u}_p(t) = \cos\omega t\,\mathbf{g}.$$

Substituting this into the differential equation we find that $\mathbf{g}$ satisfies

$$\left(V_N - \frac{md}{T}\omega^2 I\right)\mathbf{g} = \frac{md}{T}\mathbf{f}.$$

For $\omega^2 \neq \omega_n^2$, $n = 1, 2, \ldots, N$, the matrix

$$\left(V_N - \frac{md}{T}\omega^2 I\right)$$

is invertible and hence

$$\mathbf{g} = \frac{md}{T}\left(V_N - \frac{md}{T}\omega^2 I\right)^{-1}\mathbf{f}\,.$$

Writing (possible since the eigenvectors form a basis)

$$\mathbf{f} = \sum_{n=1}^{N} \alpha_n \mathbf{f_n},$$

we conclude that

$$\mathbf{g} = \sum_{n=1}^{N} \frac{\alpha_n}{\omega_n^2 - \omega^2} \mathbf{f_n}$$

for $\omega^2 \neq \omega_n^2$, $n = 1, 2, \ldots, N$. The solution with initial values

$$\mathbf{u}(0) = 0, \quad \dot{\mathbf{u}}(0) = 0 \tag{6.19}$$

is therefore of the form

$$\mathbf{u}(t) = \cos\omega t \sum_{n=1}^{N} \frac{\alpha_n}{\omega_n^2 - \omega^2} \mathbf{f_n} + \sum_{n=1} \left(a_n \cos(\omega_n t) + b_n \sin(\omega_n t)\right) \mathbf{f_n}\,.$$

Imposing the initial conditions (6.19) we obtain the two equations

$$\sum_{n=1} \left(\frac{\alpha_n}{\omega_n^2 - \omega^2} + a_n \right) \mathbf{f_n} \quad = \quad 0, \tag{6.20}$$

$$\sum_{n=1} \omega_n b_n \mathbf{f_n} \quad = \quad 0. \tag{6.21}$$

From the fact that $\{\mathbf{f_n}\}_{n=1}^{N}$ is a basis we conclude

$$a_n = -\frac{\alpha_n}{\omega_n^2 - \omega^2}, \quad b_n = 0 \quad \text{for} \quad n = 1, 2, \ldots, N.$$

Thus the solution is

$$\mathbf{u}(t) \quad = \quad \sum_{n=1}^{N} \frac{\alpha_n}{\omega_n^2 - \omega^2} \left(\cos(\omega t) - \cos(\omega_n t)\right) \mathbf{f_n} \tag{6.22}$$

$$= \quad \sum_{n=1}^{N} \frac{2\alpha_n}{\omega_n^2 - \omega^2} \sin\left(\frac{1}{2}(\omega_n + \omega)t\right) \sin\left(\frac{1}{2}(\omega_n - \omega)t\right) \mathbf{f_n}\,. \tag{6.23}$$

We observe that there is a *beat* whenever the driving frequency ω is close to a normal mode of oscillation ω_n. Compare this discussion with that of Boyce & DiPrima [4].

6.8 Vibrating membrane

6.8.1 Helmholtz equation

In the previous section we discussed the vibrating string. Recall that we have a string of unstretched length L that is tied down at ends 0 and L. If $u = u(x;t)$ denotes the vertical displacement of the string at position x, $0 \le x \le L$, at time t, then we showed that for small displacements u satisfies the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0$$

where $v^2 = T/\mu$, T equals the tension in string and μ is the density of the string. We solved this equation subject to the boundary conditions $u(0,t) = u(L,t) = 0$ for all t and with initial conditions $u(x,0) = f(x)$ and $\frac{\partial u}{\partial t}(x,0) = g(x)$ where f and g are given.

Now we imagine a uniform, flexible membrane, of mass ρ per unit area, stretched under a uniform tension T per unit length over a region Ω in the plane whose boundary $\partial\Omega$ is a smooth curve (with a possible exception of a finite number of corners).

We now let $U = U(x,y;t)$ denote the vertical displacement of the membrane at position $(x,y) \in \Omega$ at time t from its equilibrium position. We again assume that the membrane is tied down at the boundary; that is[1]

$$U(x,y;t) = 0 \text{ for } (x,y) \in \partial\Omega.$$

The motion of $U = U(x,y;t)$ is governed by the two-dimensional wave equation:

$$\boxed{\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} - \frac{1}{v^2}\frac{\partial^2 U}{\partial t^2} = 0 \text{ for } (x,y) \in \Omega} \tag{6.24}$$

where $v^2 = T/\rho$. One recognizes $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}$ as the two-dimensional Laplacian. So if we introduce

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

the wave equation takes the form

$$\Delta U - \frac{1}{v^2}\frac{\partial^2 U}{\partial t^2} = 0.$$

We proceed as before and look for solutions of (6.24) in which the variables separate

$$U(x,y;t) = u(x,y)T(t).$$

Substituting this into (6.24), and then dividing by uT gives

$$\frac{1}{u}\Delta u = \frac{1}{v^2}\frac{1}{T}\frac{d^2 T}{dt^2}.$$

The right-hand side depends only upon t where as the left-hand side depends only upon x, y. Thus for the two sides to be equal they must equal the same constant. Call this constant $-k^2$. Thus we have the two equations

[1] In one dimension $\Omega = (0, L)$ and the boundary of Ω consists of the two points 0 and L.

$$\begin{aligned}\frac{d^2T}{dt^2} + \omega^2 T &= 0 \text{ where } \omega = kv, \\ \Delta u + k^2 u &= 0. \end{aligned} \tag{6.25}$$

The differential equation for T has our well-known solutions

$$e^{i\omega t} \text{ and } e^{-i\omega t}.$$

The second equation (6.25), called the *Helmholtz equation*, is a partial differential equation for $u = u(x,y)$. We wish to solve this subject to the boundary condition

$$u(x,y) = 0 \text{ for } (x,y) \in \partial\Omega.$$

6.8.2 Rectangular membrane

Consider the rectangular domain

$$\Omega = \{(x,y) : 0 < x < a, 0 < y < b\} \tag{6.26}$$

For this rectangular domain the Helmholtz equation can be solved by the method of separation of variables. If one assumes a solution of the form (variables x and y separate)

$$u(x,y) = X(x)Y(y)$$

then the problem is reduced to two one-dimensional problems. It is an exercise to show that the allowed frequencies are

$$\omega_{m,n} = \pi v \left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 \right]^{1/2}, \quad m,n = 1,2,3,\ldots \tag{6.27}$$

6.8.3 Circular membrane: The drum

We now consider the circular domain

$$\Omega = \left\{(x,y) : x^2 + y^2 < a^2\right\}$$

so the boundary of Ω is the circle $\partial\Omega : x^2 + y^2 = a^2$. Even though the variables separate in the Cartesian coordinates x and y, this is of no use since the boundary is circular and we would not be able to apply the BC $u = 0$ on the circular boundary $\partial\Omega$. Since the domain is circular it is natural to introduce polar coordinates

$$x = r\cos\theta, \quad y = r\sin\theta.$$

It is an exercise in the chain rule to show that in polar coordinates the 2D Laplacian is

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2};$$

and hence, the Helmholtz equation in polar coodinates is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + k^2 u = 0. \tag{6.28}$$

We write $u = u(r,\theta)$.[2]

[2]Possible point of confusion: We wrote $u = u(x,y)$ so really our new function of r and θ is $u(r\cos\theta, r\sin\theta)$. Technically we should give this function of r and θ a new name but that would be rather pedantic.

Separation of variables

We now show that the variables separate. So we look for a solution of the form

$$u(r,\theta) = R(r)\Theta(\theta).$$

Substituting this into (6.28), multiplying by $r^2/R\Theta$ we have

$$\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{r}{R}\frac{dR}{dr} + k^2r^2 = -\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}$$

By the now familiar argument we see that each of the above sides must equal a constant, call it m^2, to obtain the two differential equations

$$\frac{d^2\Theta}{d\theta^2} + m^2\Theta = 0 \tag{6.29}$$

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + (k^2 - \frac{m^2}{r^2})R = 0 \tag{6.30}$$

Two linearly independent solutions to (6.29) are

$$e^{im\theta} \quad \text{and} \quad e^{-im\theta}$$

The point with polar coordinates (r,θ) is the same point as the one with polar coordinates $(r,\theta+2\pi)$. Thus our solution $u(r,\theta)$ and $u(r,\theta+2\pi)$ must be the same solution. This requires

$$e^{im\theta+im2\pi} = e^{im\theta}$$

or $e^{2\pi im} = 1$. That is, m must be an integer. If $m = 0$ the general solution to (6.29) is $c_1 + c_2\theta$. But the $\theta \to \theta + 2\pi$ argument requires we take $c_2 = 0$. Thus the general solution to (6.29) is

$$a_m\cos(m\theta) + b_m\sin(m\theta), \quad m = 0, 1, 2, \ldots$$

We now return to (6.30), called the *Bessel equation*, which is a second order linear differential equation. General theory tells us there are two linearly independent solutions. Tradition has it we single out two solutions. One solution, called $J_m(kr)$, is finite as $r \to 0$ and the other solution, called $Y_m(kr)$ goes to infinity as $r \to 0$. Both of these functions are called *Bessel functions*. It can be shown that the Bessel function $J_m(z)$ is given by the series expansion

$$J_m(z) = \left(\frac{z}{2}\right)^m \sum_{j=0}^{\infty}(-1)^j \frac{1}{j!(m+j)!}\left(\frac{z}{2}\right)^{2j} \tag{6.31}$$

A plot of the Bessel function $J_0(x)$ for $0 \le x \le 40$ is given in Figure 6.5. In Mathematica, Bessel functions $J_m(z)$ are called by the command `BesselJ[m,z]`. Since $u(r,\theta)$ is well-defined at $r = 0$ (center of the drum), this requires we only use the J_m solutions. Thus we have shown that

$$J_m(kr)\left(a_m\cos(m\theta) + b_m\sin(m\theta)\right), \quad m = 0, 1, 2, \ldots$$

are solutions to (6.28). We now require that these solutions vanish on $\partial\Omega$. That is, when $r = a$ and for all θ we require the above solution to vanish. This will happen if

$$J_m(ka) = 0.$$

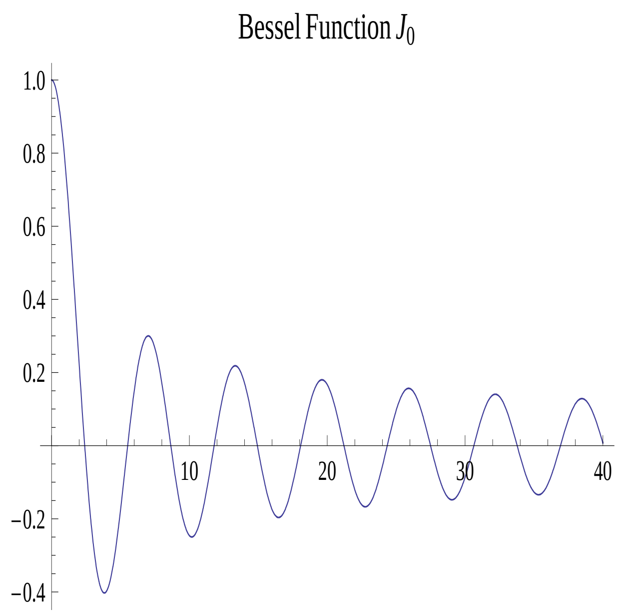


Figure 6.5: The Bessel function $J_0(x)$. First zero occurs at approximately 2.4048, the second zero at 5.5201, the third zero at 8.6537, ...

That is we have to be at a *zero* of the Bessel function J_m. It is known that J_m has an infinite number of real zeros, call them $j_{m,n}$, $n = 1, 2, \ldots$. Thus the frequencies that the drum can oscillate at are

$$\omega_{m,n} = \frac{v}{a}\, j_{m,n}, \;\; m = 0, 1, 2, \ldots; n = 1, 2, \ldots$$

where $j_{m,n}$ is the nth zero of the Bessel function $J_m(z)$. These zeros can be found in Mathematica using the command `BesselJZero[m,n]`.

6.8.4 Comments on separation of variables

For general domains Ω one cannot solve the Helmholtz equation (6.25) by the method of separation of variables. In general if one makes the transformations $x = f(\xi, \eta)$ and $y = g(\xi, \eta)$ then one would want the curves of constant ξ (or constant η) to describe the boundary $\partial\Omega$ *and* for Helmholtz's equation to separate variables in the new variables ξ and η. In general there are no such coordinates. For an *elliptical membrane* the Helmholtz equation does separate in what are called *elliptic coordinates*

$$x = \frac{c}{2} \cosh\mu \, \cos\theta, \quad y = \frac{c}{2} \sinh\mu \, \sin\theta$$

where $c \in \mathbb{R}^+$, $0 < \mu < \infty$ and $0 \le \theta \le 2\pi$. The curves $\mu =$ constant and $\theta =$ constant are confocal ellipses and hyperbolas, respectively. Qualitative new phenomena arise for elliptical (and more generally convex) membranes: the existence of *whispering gallery modes* and *bouncing ball modes*. In the whispering gallery mode the eigenfunction is essentially nonzero only in a thin strip adjacent to the boundary of Ω. Thus a person who speaks near the wall of a convex room can be heard across the room near the wall, but not in the interior of the room. For further information see [5] and references therein.

6.9 Exercises

#1. Weighted string on a circle

We consider the same weighted string problem but now assume the masses lie on a circle; this means that the first mass is coupled to the last mass by a string. The effect of this is that (6.1) remains the same if we *now* interpret $u_0 = u_N$ and $u_{N+1} = u_1$. Explain why this is the case. What is the matrix V_N in this case? Show that the differential equations can still be written in the matrix form (6.4) where now the V_N is your new V_N. Does the reduction to an eigenvalue problem, as in §6.2, remain the same? Explain.

#2. Diagonalization of V_N from problem #1

Let V_N be the $N \times N$ matrix found in the previous problem. Show that the eigenvalue problem

$$V_N \mathbf{f} = \lambda \mathbf{f}$$

becomes in component form

$$-f_{j-1} + 2f_j - f_{j+1} = \lambda f_j,\ j = 1, 2, \ldots, N \tag{6.32}$$

where $f_0 = f_N$ and $f_{N+1} = f_1$. Let ω denote an N^{th} root of unity; that is, any of the values $e^{2\pi i n/N}$, $n = 0, 1, \ldots, N-1$. For each such choice of ω, define

$$\hat{f}_\omega = \sum_{j=1}^{N} f_j\, \omega^j \tag{6.33}$$

Multiply (6.32) by ω^j and sum the resulting equation over $j = 1, 2, \ldots, N$. Show that the result is

$$2(1 - \cos\phi)\hat{f}_\omega = \lambda \hat{f}_\omega$$

where $\omega = e^{i\phi}$. From this we conclude that the eigenvalues are

$$\lambda_n = 2\left(1 - \cos\left(\frac{2\pi n}{N}\right)\right), n = 0, 1, \ldots, N-1$$

Explain why this is so. This should be compared with (6.7). Find an eigenvector $\mathbf{f}_n$ corresponding to eigenvalue λ_n. (Hint: Follow the method in §6.4.1.)

#3. Coupled pendulums

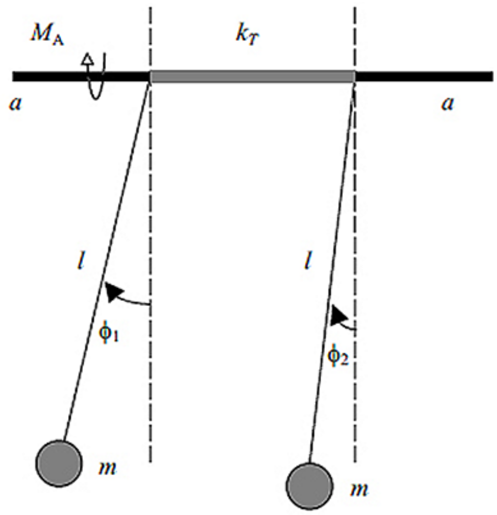


Figure 1 - Two identical pendulums of mass m and length l, coupled by a massless rod with torsional spring constant k_T and freely rotating about the axis $a-a$. A torque M_A is applied on the system. The angular deviations of the pendulums from their vertical equilibrium positions are indicated as ϕ_1 and ϕ_2, respectively. The mechanical system is immersed in a fluid with damping constant b.

Figure 6.6: Coupled pendulums. Here we assume the damping force is zero.

Consider the system of two mathematical pendulums of lengths ℓ_1 and ℓ_2 and masses m_1 and m_2, respectively, in a gravitional field mg which move in two parallel vertical planes perpendicular to a common flexible support such as a string from which they are suspended. Denote by θ_1 (θ_2) the angle of deflection of pendulum #1 (#2). The kinetic energy of this system is

$$\mathrm{KE} = \frac{1}{2} m_1 \ell_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \ell_2^2 \dot{\theta}_2^2 \,,$$

and the potential energy is

$$\mathrm{PE} = m_1 g\, \ell_1 (1-\cos\theta_1) + m_2 g\, \ell_2 (1-\cos\theta_2) + V_{int}$$

where V_{int} is the *interaction* potential energy.[3] If there is no twist of the support, then there is no interaction of the two pendulums. We also expect the amount of twist to depend upon the difference of the angles θ_1 and θ_2. It is reasonable to assume V_{int} to be an even function of $\theta_1 - \theta_2$. Thus

$$V_{int}(0) = 0, \quad V_{int}'(0) = 0.$$

[3]These expressions should be compared with (2.26).

For small deflection angles (the only case we consider) the simplest assumption is then to take

$$V_{int}(\theta_1-\theta_2)=\frac{1}{2}\kappa(\theta_1-\theta_2)^2$$

where κ is a positive constant. Since we are assuming the angles are small, the potential energy is then given, to a good approximation, by

$$\mathrm{PE}=\frac{1}{2}m_1 g\ell_1\,\theta_1^2+\frac{1}{2}m_2 g\ell_2\,\theta_2^2+\frac{1}{2}\kappa(\theta_1-\theta_2)^2.$$

Under these assumptions it can be shown that Newton's equations are

$$\begin{aligned} m_1\ell_1^2\ddot{\theta}_1 &= -(m_1 g\ell_1+\kappa)\theta_1+\kappa\theta_2\,, \\ m_2\ell_2^2\ddot{\theta}_2 &= \kappa\theta_1-(m_2 g\ell_2+\kappa)\theta_2\,. \end{aligned}$$

Observe that for $\kappa=0$ the ODEs reduce to two uncoupled equations for the *linearized* mathematical pendulum. To simplify matters somewhat, we introduce

$$\omega_1^2=\frac{g}{\ell_1},\ \ \omega_2=\frac{g}{\ell_2}\ \ k_1=\frac{\kappa}{m_1\ell_1^2},\ \ k_2=\frac{\kappa}{m_2\ell_2^2}\,.$$

Then it is not difficult to show (you need not do this) that the above differential equations become

$$\boxed{\begin{aligned} \ddot{\theta}_1 &= -(\omega_1^2+k_1)\theta_1+k_1\theta_2 \\ \ddot{\theta}_2 &= k_2\theta_1-(\omega_2^2+k_2)\theta_2\,. \end{aligned}} \tag{6.34}$$

We could change this into a system of first order DEs (the matrix A would be 4×4). However, since equations of this form come up frequently in the theory of small oscillations, we proceed to develop a "mini theory" for these equations. Define

$$\Theta=\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}.$$

Show that the equations (6.34) can be written as

$$\ddot{\Theta}=A\Theta \tag{6.35}$$

where A is a 2×2 matrix. Find the matrix A. Assume a solution of (6.35) to be of the form

$$\Theta(t)=e^{i\omega t}\begin{pmatrix}a_1\\ a_2\end{pmatrix}. \tag{6.36}$$

Using (6.36) in (6.35) show that (6.35) reduces to

$$A\Theta=-\omega^2\Theta. \tag{6.37}$$

This is an eigenvalue problem. Show that ω^2 must equal

$$\begin{aligned} \omega_\pm^2 &= \frac{1}{2}(\omega_1^2+\omega_2^2+k_1+k_2) \\ &\quad \pm\frac{1}{2}\sqrt{(\omega_1^2-\omega_2^2)^2+2(\omega_1^2-\omega_2^2)(k_1-k_2)+(k_1+k_2)^2}\,. \end{aligned} \tag{6.38}$$

Show that an eigenvector for ω_+^2 is

$$f_1 = \begin{pmatrix} 1 \\ -k_2(\omega_+^2 - \omega_2^2 - k_2)^{-1} \end{pmatrix}, \tag{6.39}$$

and an eigenvector corresponding to ω_-^2 is

$$f_2 = \begin{pmatrix} -k_1(\omega_-^2 - \omega_1^2 - k_1)^{-1} \\ 1 \end{pmatrix}. \tag{6.40}$$

Now show that the general solution to (6.34) is

$$\begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix} = (c_1 \cos(\omega_+ t) + c_2 \sin(\omega_+ t))\, f_1 + (c_3 \cos(\omega_- t) + c_4 \sin(\omega_- t))\, f_2 \tag{6.41}$$

where c_i are real constants. One can determine these constants in terms of the initial data

$$\theta_1(0),\ \dot\theta_1(0),\ \theta_2(0),\ \dot\theta_2(0).$$

To get some feeling for these rather complicated expressions, we consider the special case

$$\theta_1(0) = \theta_0,\ \dot\theta_1(0) = 0,\ \theta_2(0) = 0,\ \dot\theta_2(0) = 0 \tag{6.42}$$

with

$$m_1 = m_2 = m,\ \ell_1 = \ell_2 = \ell. \tag{6.43}$$

These last conditions imply

$$\omega_1 = \omega_2 := \omega_0.$$

Explain in words what these initial conditions, (6.42), correspond to in the physical set up.

If we define

$$k = \frac{\kappa}{m\,\ell^2},$$

show that in the special case (6.42) and (6.43) that

$$\omega_+ = \sqrt{\omega_0^2 + 2k} \ \text{ and } \ \omega_- = \omega_0. \tag{6.44}$$

In this same case solve for the coefficients c_1, c_2, c_3 and c_4 and show that

$$c_1 = \frac{1}{2}\theta_0,\ c_2 = 0,\ c_3 = \frac{1}{2}\theta_0,\ c_4 = 0,$$

and hence (6.41) becomes

$$\begin{aligned} \theta_1(t) &= \theta_0 \cos\left(\frac{1}{2}(\omega_+ + \omega_-)t\right) \cos\left(\frac{1}{2}(\omega_+ - \omega_-)t\right), \\ \theta_2(t) &= \theta_0 \sin\left(\frac{1}{2}(\omega_+ + \omega_-)t\right) \sin\left(\frac{1}{2}(\omega_+ - \omega_-)t\right). \end{aligned}$$

Suppose further that

$$\frac{k}{\omega_0^2} \ll 1. \tag{6.45}$$

What does this correspond to physically? Under assumption (6.45), show that *approximately*

$$\begin{aligned} \theta_1(t) &\approx \theta_0 \cos(\omega_0 t) \cos\left(\frac{k}{2\omega_0} t\right), \\ \theta_2(t) &\approx \theta_0 \sin(\omega_0 t) \sin\left(\frac{k}{2\omega_0} t\right). \end{aligned} \tag{6.46}$$

Discuss the implications of (6.46) in terms of the periods

$$T_0 = \frac{2\pi}{\omega_0} \quad \text{and} \quad T_1 = \frac{2\pi}{k/2\omega_0}.$$

Show that in this approximation

$$T_1 \gg T_0.$$

Draw plots of $\theta_1(t)$ and $\theta_2(t)$ using the approximate expressions (6.46).

#4. The Toda chain and Lax pairs

Consider N particles on a circle (periodic boundary conditions) whose positions $x_n(t)$ at time t satisfy the *Toda equations*

$$\frac{d^2 x_n}{dt^2} = \exp\left(-(x_n - x_{n-1})\right) - \exp\left(-(x_{n+1} - x_n)\right), \quad n = 1, 2, \ldots, N, \tag{6.47}$$

where $x_{N+1} = x_1$ and $x_0 = x_N$. These equations are nonlinear and admit certain solutions, called *solitons*, which are stable pulses. This system of equations has been extensively studied. Here we give only a brief introduction to some of these results.[4]

To make the problem easier we now set $N = 5$ but everything that follows can be generalized to any positive integer N.

Define

$$a_n = \frac{1}{2} \exp\left(-(x_{n+1} - x_n)/2\right) \quad \text{and} \quad b_n = \frac{1}{2} \frac{dx_n}{dt}, n = 1, \ldots, 5. \tag{6.48}$$

Show that if x_n satisfies the Toda equations (6.47), then a_n and b_n satisfy the differential equations

$$\frac{da_n}{dt} = a_n\left(b_n - b_{n+1}\right) \quad \text{and} \quad \frac{db_n}{dt} = 2\left(a_{n-1}^2 - a_n^2\right). \tag{6.49}$$

Define two 5×5 matrices L and B, they are called a *Lax pair*, by

$$L = \begin{pmatrix} b_1 & a_1 & 0 & 0 & a_5 \\ a_1 & b_2 & a_2 & 0 & 0 \\ 0 & a_2 & b_3 & a_3 & 0 \\ 0 & 0 & a_3 & b_4 & a_4 \\ a_5 & 0 & 0 & a_4 & b_5 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & -a_1 & 0 & 0 & a_5 \\ a_1 & 0 & -a_2 & 0 & 0 \\ 0 & a_2 & 0 & -a_3 & 0 \\ 0 & 0 & a_3 & 0 & -a_4 \\ -a_5 & 0 & 0 & a_4 & 0 \end{pmatrix}. \tag{6.50}$$

[4] See, for example, *Theory of Nonlinear Lattices* by Morikazu Toda, Springer-Verlag, 1981.

Show (6.49) can be written as the matrix equation

$$\frac{dL}{dt} = BL - LB \tag{6.51}$$

Define the matrix $U = U(t)$ to be the solution to the differential equation

$$\frac{dU}{dt} = BU$$

that satisfies the initial condition $U(0) = I$.

Show that $U(t)$ is a unitary matrix; that is, $U^*(t)U(t) = I$ for all t where U^* is the adjoint matrix.[5] Hint: Observe that $B^* = -B$. Use this to first show that

$$\frac{dU^*}{dt} = -U^*B$$

and then show $\frac{d}{dt}U^*(t)U(t) = 0$.

Now prove that

$$\frac{d}{dt}\left(U^*L(t)U(t)\right) = 0$$

and hence that

$$U^*(t)L(t)U(t) = L(0)$$

That is, $L(0)$ and $L(t)$ are unitarily equivalent. From this conclude

The eigenvalues of $L(t)$ are independent of t

Thus the eigenvalues of the Lax matrix L are first integrals of motion of the Toda chain. For general N this means that we have found N integrals of the motion. This is a remarkable result since normally one can only find a limited number of integrals of the motion (energy, angular momentum, etc.).

#5. Wave equation

In the section "Solution to the Wave Equation" it was claimed that a similar argument shows that the coefficients b_n are equal to zero. (See discussion between (6.16) and (6.17).) Prove that $b_n = 0$.

#6. Weighted string with friction

We now assume that the particles in the *weighted string problem* are subject to a force due to the presence of friction. (Imagine the particles are moving in a medium which offers resistance to the motion of the particles.) Assuming the frictional force is proportional to the velocity, the system of differential equations describing the motion is

$$m\,\frac{d^2u_j}{dt^2} = \frac{T}{d}\,\left(u_{j+1} - 2u_j + u_{j-1}\right) - \gamma\,\frac{du_j}{dt}, \quad j = 1, 2, \ldots, N \tag{6.52}$$

where γ is positive and, as before, $u_0 = u_{N+1} = 0$.

[5] Recall that if X is any matrix then X^* is the matrix obtained by taking the complex conjugate of each element in X and then taking the transpose.

1. Rewrite the system (6.52) in matrix form such that when $\gamma = 0$ the equation becomes identical to the matrix equation (6.4).

2. Assume a solution of the form
$$\mathbf{u}(t) = e^{i\omega t}\mathbf{f} \tag{6.53}$$
where $\mathbf{f}$ is a column vector independent of t and ω is to be determined. For what values of ω is (6.53) a solution to the matrix equation derived in part (1)?

   NOTE: This will not require a *complete* reworking of the eigenvalues since you may use the information we already have proved about V_N to find the eigenvalues in this new problem. You should not have to solve anything more complicated than a quadratic equation.

3. Explain the significance of the fact that the ω's you obtain are *complex* numbers.

4. For a large system $N \gg 1$ explain why you expect some of the allowed ω's to be *purely imaginary*. Explain the significance of this result, i.e. what is the implication for the motion?

#7. Rectangular membrane

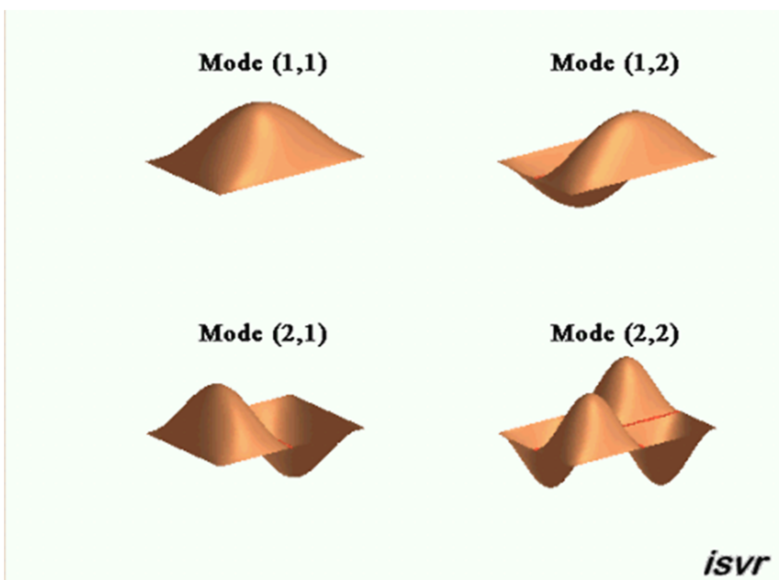


Figure 6.7: Four lowest normal modes of a rectangular drum.

In this section we obtain the solution of (6.25) in the case of a rectangular domain (6.26).

1. By assuming that the solution can be written as $u(x, y) = X(x)Y(y)$, obtain a 2nd order DE for X with independent variable x and similarly a DE for Y with independent variable y.

2. We assume the membrane is tied down at the boundary of the domain Ω. (This implies boundary conditions on the solutions we seek.)

3. Show that the eigenvalues and the corresponding eigenfunctions of the differential equations with boundary conditions in parts (1) and (2) are

$$\mu_m = \frac{m^2\pi^2}{a^2}; \quad X_m(x) = A_m \sin\left(\frac{m\pi x}{a}\right), \quad m = 1, 2, \cdots \tag{6.54}$$

$$\nu_n = \frac{n^2\pi^2}{b^2}; \quad Y_n(y) = B_n \sin\left(\frac{n\pi y}{b}\right), \quad n = 1, 2, \cdots \tag{6.55}$$

4. Show that the eigenfrequencies (normal modes) of the rectangular membrane are given by (6.27). (By dimensional analysis conclude where the factor v, which was set equal to one here, must appear.)

5. Find the general solution to (6.25) for this rectangular domain.

#8. Alternating mass-spring: Acoustic and optical phonons

Consider $2N$ particles on a circle interacting via a spring connnecting adjacent particles. We assume the particles on the odd sites have mass m_1 and the particles on the even sites have mass m_2. If u_j denotes the displacement from equilibrium of particle j, the differential equations describing the motion are

$$m_j \frac{d^2 u_j}{dt^2} + k\left(-u_{j-1} + 2u_j - u_{j+1}\right) = 0 \text{ for } j = 1, 2, 3, \ldots, 2N, \tag{6.56}$$

where because the particles are on a circle

$$u_{2N+1} = u_1 \text{ and } u_0 = u_{2N}.$$

Here k is the spring constant for the spring connecting any two particles. We are interested in finding the frequencies at which the system can oscillate.

1. Assume a solution of the form

$$u_j(t) = e^{i\omega t} v_j, \quad v_j \text{ independent of } t,$$

and show that (6.56) becomes

$$-m_j \omega^2 v_j + k\left(-v_{j-1} + 2v_j - v_{j+1}\right) = 0 \text{ for } j = 1, 2, 3, \ldots, 2N, \tag{6.57}$$

2. For $j = 1, 2, \ldots, N$ define the vectors

$$V_j = \begin{pmatrix} v_{2j-1} \\ v_{2j} \end{pmatrix}.$$

Show that (6.57) can be written equivalently as

$$-\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \omega^2 V_j + k\left\{-\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} V_{j-1} + \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} V_j - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} V_{j+1}\right\} = 0 \tag{6.58}$$

for $j = 1, 2, \ldots, N$.

3. Let η denote any Nth root of unity, i.e. $\eta^N = 1$ so η is of the form $\eta = e^{i\phi} = e^{2\pi i j/N}$ for some integer $j = 0, 1, \ldots, N-1$. Define

$$\hat{V}_\eta = \sum_{j=1}^{N} V_j \eta^j$$

Show that $\hat{V}_\eta$ satisfies the equation

$$\left\{ -\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \omega^2 + k \left[-\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \eta + \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \eta^{-1} \right] \right\} \hat{V}_\eta = 0. \tag{6.59}$$

4. What is the condition for nontrivial solutions $\hat{V}_\eta$ to exist for (6.59)? Hint: Equation (6.59) is of the form $A\hat{V}_\eta = 0$ where the matrix A is the 2×2 matrix inside the curly brackets of (6.59). Using the condition you just found, show that the normal modes of vibration are given by

$$\omega^2_{\pm,j} = k\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \pm k\left[\frac{1}{m_1^2} + \frac{1}{m_2^2} + \frac{2}{m_1 m_2}\cos\left(\frac{2\pi j}{N}\right)\right]^{1/2} \tag{6.60}$$

where $j = 0, 1, 2, \ldots N-1$.

5. Show that the frequencies derived in (6.60) lie on two curves, called *dispersion curves.* These two curves should be compared with the one dispersion curve for the equal mass problem. Plot the two dispersion curves.[6] The curve that is zero at $j = 0$ is called the *acoustic mode* and the other is called the *optical mode*.[7] This is a model of a one-dimensional lattice vibrations of a diatomic system.

#9. Energy of the vibrating string

The vibrating string has the total energy $E(t)$ at time t

$$E(t) = \int_0^L \left(\frac{1}{2}\mu\, u_t^2(x,t) + \frac{1}{2}T\, u_x^2(x,t)\right) dx$$

Explain why the first term is the kinetic energy and the second term is the potential energy of the vibrating string. Recall the solution $u(x,t)$ of the vibrating string problem, i.e. (6.17). Above we use the notation

$$u_t(x,t) := \frac{\partial u(x,t)}{\partial t} \text{ and } u_x(x,t) := \frac{\partial u(x,t)}{\partial x}.$$

You may assume as given the following integrals:

$$\int_0^L \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx = \frac{1}{2}L\,\delta_{m,n} \tag{6.61}$$

[6] In plotting you might want to fix some values of m_1, m_2 and k.

[7] The acoustic modes correspond to sound waves in the lattice. The optical modes, which are nonzero at $j = 0$, are called "optical" because in ionic crystals they are excited by light. The quantized version of these excitations are called acoustic phonons and optical phonons.

and

$$\int_0^L \cos\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)dx = \frac{1}{2}L\,\delta_{m,n} \tag{6.62}$$

Use (6.61) and (6.62) to show

$$E(t) = \frac{\pi^2 T}{4L}\sum_{n=1}^{\infty} n^2 a_n^2. \tag{6.63}$$

Note that the result is independent of t, i.e. the energy of the vibrating string is conserved. Give a physical interpretation of this expression for E in terms of harmonic oscillators.

Chapter 7

Quantum Harmonic Oscillator

Figure 7.1: Erwin Schrödinger, 1887–1961 and Paul Dirac, 1902–1984.

A simple and interesting example of a dynamical system in quantum mechanics is the harmonic oscillator. This example is of importance for general theory, because it forms a corner-stone in the theory of radiation.

P. A. M. Dirac, *The Principles of Quantum Mechanics*

7.1 Schrödinger equation

In *classical mechanics* the state of a system consisting of N particles is specified by the position $\vec{x}$ and momentum $\vec{p} = m\vec{v}$ of each particle. The time evolution of this state is determined by solving Newton's Second Law (or equivalently, say Hamilton's equations). Thus, for example, a one particle system moving in three-dimensions (three degrees of freedom) determines a curve in 6-dimensional space: namely, $(\vec{x}(t), \vec{p}(t))$. For the familiar harmonic oscillator (mass-spring system) there is only one-degree of freedom (the movement of the mass is in one dimension only) and the position and momentum are given by the now familiar formulas[1]

$$x(t) = x_0 \cos(\omega_0 t) + \frac{p_0}{m\omega_0} \sin(\omega_0 t), \tag{7.1}$$

$$p(t) = p_0 \cos(\omega_0 t) - m\omega_0 x_0 \sin(\omega_0 t). \tag{7.2}$$

In *quantum mechanics* the notion of the state of the system is more abstract. The state is specified by a vector Ψ in some abstract vector space $\mathcal{H}$. This vector space has an inner product $(\cdot, \cdot)$.[2] Thus every state $\Psi \in \mathcal{H}$ satisfies

$$\| \Psi \| := (\Psi, \Psi)^{1/2} < \infty. \tag{7.3}$$

The importance of (7.3) is that in the Born interpretation $|(\Psi, \Phi)|^2$ is interpreted as a probability; and hence, must be finite (and less than or equal to one).[3] In what is called the Schrödinger representation, one can describe the state Ψ as a function $\Psi(x)$ where x is the position (of say the particle). Then $|\Psi(x)|^2$ is the probability density of finding the particle in some small neighborhood of the point x. Integrating this over all possible positions must then give one.

The evolution of the state Ψ with time is determined by solving the Schrödinger equation:

$$\boxed{i\hbar \frac{\partial \Psi}{\partial t} = H\Psi.} \tag{7.4}$$

Here $\hbar$ is the Planck's constant[4] (divided by 2π) and H is the quantum mechanical Hamiltonian, a linear self-adjoint operator on the space $\mathcal{H}$.[5]

[1] Actually, the second may look a little different from the earlier formulas. This is due to the fact that we are using momentum p instead of velocity v to describe the second coordinate of (x, p). Here p_0 is the initial momentum and is related to the initial velocity by $p_0 = mv_0$.

[2] Such vector spaces are called *Hilbert spaces*.

[3] This assumes that states Ψ are normalized so that their "length" is one, i.e. $\| \Psi \| = 1$.

[4] In the cgs system, $\hbar = 1.05457 \times 10^{-27}$ erg-sec. A quantity that has the units of energy$\times$time is called an *action*. In modern particle physics a unit system is adopted such that in these units $\hbar = 1$. Max Planck received the Nobel prize in 1919 for "his discovery of energy quanta".

[5] An operator H is self-adjoint if $(H\psi, \psi) = (\psi, H\psi)$ for all $\psi \in \mathcal{H}$. It is the generalization to Hilbert spaces of the notion of a Hermitian matrix. There are some additional subtle questions regarding the domain of the operator H. In these notes we ignore such questions and assume H is well-defined on all states $\Psi \in \mathcal{H}$.

7.2 Harmonic oscillator

7.2.1 Harmonic oscillator equation

We illustrate the notions of quantum mechanics and its relationship to differential equations in the context of the harmonic oscillator. The harmonic oscillator is one of the most important simple examples in quantum mechanics. In this case the vector space $\mathcal{H}$ is the space of square-integrable functions. This space consists of all (complex valued) functions $\psi(x)$ such that

$$\int_{-\infty}^{\infty} |\psi(x)|^2 \, dx < \infty.$$

This space is denoted by $L^2(\mathbb{R})$ and it comes equipped with an inner product

$$(\psi, \varphi) = \int_{-\infty}^{\infty} \psi(x)\bar{\varphi}(x)\, dx$$

where $\bar{\varphi}$ is the complex conjugate of φ. (Note that in most physics books the complex conjugation is on the first slot.) The first observation, and an important one at that, is that the state space is infinite dimensional. For example, it can be proved that the infinite sequence of functions

$$x^j e^{-x^2}, \;\; j = 0, 1, 2 \ldots$$

are linearly independent elements of $L^2(\mathbb{R})$. Thus in quantum mechanics one quickly goes beyond linear algebra which is traditionally restricted to finite-dimensional vector spaces.

The operator H which describes the harmonic oscillator can be defined once we give the *quantization procedure*—a heuristic that allows us to go from a classical Hamiltonian to a quantum Hamiltonian. As mentioned above, classically the state is given by the vector $(x, p) \in \mathbb{R}^2$. In quantum mechanics the position and momentum are replaced by operators $\hat{x}$ and $\hat{p}$. For the vector space of states $\mathcal{H} = L^2(\mathbb{R})$, the position operator acts on $L^2(\mathbb{R})$ by multiplication,

$$(\hat{x}\psi)(x) = x\psi(x), \;\; \psi \in L^2(\mathbb{R})$$

and the momentum operator $\hat{p}$ acts by differentiation followed by multiplication by the constant $-i\hbar$,

$$(\hat{p}\psi)(x) = -i\hbar \frac{\partial \psi}{\partial x}(x), \;\; \psi \in L^2(\mathbb{R}).$$

Since $\hat{x}$ is multiplication by x we usually don't distinguish between x and $\hat{x}$. From this we observe that in quantum mechanics the position operator and the momentum operator do not commute. To see this, let $\psi \in L^2(\mathbb{R})$, then

$$\begin{aligned}
(\hat{x}\hat{p} - \hat{p}\hat{x})\,\psi(x) &= -i\hbar x \frac{\partial \psi}{\partial x} + i\hbar \frac{\partial}{\partial x}\,(x\psi(x)) \\
&= -i\hbar x \frac{\partial \psi}{\partial x} + i\hbar x \frac{\partial \psi}{\partial x} + i\hbar\,\psi(x) \\
&= i\hbar\,\psi(x).
\end{aligned}$$

Introducing the commutator; namely, for any two operators A and B we define $[A, B] = AB - BA$, the above can be written more compactly as[6]

$$[\hat{x}, \hat{p}] = i\hbar\,\mathrm{id} \tag{7.5}$$

[6] Just as in linear algebra, if A and B are two linear operators and it holds for all vectors ψ that $A\psi = B\psi$, then we can conclude that as operators $A = B$.

where by id we mean the identity operator. Equation (7.5) is at the heart of the famous *Heisenberg Uncertainty Relation*.

With these rules we can now define the quantum harmonic oscillator Hamiltonian given the classical Hamiltonian (energy). Classically,[7]

$$\begin{aligned} E &= \text{KE} + \text{PE} \\ &= \frac{1}{2m}\,p^2 + \frac{1}{2}\,m\omega_0^2\,x^2. \end{aligned}$$

Replacing $p \to \hat{p}$ and x by multiplication by x we have

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega_0^2 x^2$$

so that Schrödinger's equation becomes

$$i\hbar\,\frac{\partial\Psi}{\partial t} = H\Psi = -\frac{\hbar^2}{2m}\,\frac{d^2\Psi}{dx^2} + \frac{1}{2}\,m\omega_0^2 x^2\Psi. \tag{7.6}$$

We first look for solutions in which the variables x and t separate

$$\Psi(x,t) = A(t)\psi(x).$$

Substituting this into (7.6) and dividing the result by $A(t)\psi(x)$ we find

$$i\hbar\,\frac{1}{A}\,\frac{dA}{dt} = \frac{1}{\psi}\,H\psi.$$

Since the left hand side is only a function of t and the right hand side is only a function of x both sides must equal a common constant. Calling this constant E (observe this constant has the units of energy), we find

$$\begin{aligned} \frac{dA}{dt} &= -\frac{iE}{\hbar}\,A, \\ H\psi &= E\psi. \end{aligned}$$

The first equation has solution

$$A(t) = e^{-iEt/\hbar}$$

so that

$$\Psi(x,t) = e^{-iEt/\hbar}\,\psi(x). \tag{7.7}$$

We now examine

$$H\psi = E\psi \tag{7.8}$$

in detail. The first observation is that (7.8) is an eigenvalue problem in $L^2(\mathbb{R})$. Thus the eigenvalues of the operator H are interpreted as energies. It is convenient to introduce dimensionless variables to simplify notationally the differential equation. Let

$$\xi = x\,\sqrt{\frac{m\omega_0}{\hbar}}, \quad \varepsilon = \frac{2E}{\hbar\omega_0}.$$

[7] Recall the potential energy for the harmonic oscillator is $V(x) = \frac{1}{2}kx^2 = \frac{1}{2}\,m\omega_0^2 x^2$.

Performing this change of variables, the Schrödinger equation $H\psi = E\psi$ becomes

$$-\frac{d^2\psi}{d\xi^2} + (\xi^2 - \varepsilon)\psi = 0. \tag{7.9}$$

We want solutions to (7.9) that are square integrable. It is convenient to also perform a change in the dependent variable[8]

$$\psi(\xi) = e^{-\xi^2/2}\, v(\xi).$$

Then a straightforward calculation shows that v must satisfy the equation

$$\frac{d^2 v}{d\xi^2} - 2\xi\,\frac{dv}{d\xi} + (\varepsilon - 1)v = 0. \tag{7.10}$$

Observe that (7.10) is not a constant coefficient differential equation, so that the methods we have developed do not apply to this equation.

7.2.2 Hermite polynomials

To find solutions of (7.10) we look for solutions that are of the form[9]

$$v(\xi) = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 + \cdots = \sum_{k=0}^{\infty} a_k \xi^k. \tag{7.11}$$

The idea is to substitute this into (7.10) and to find conditions that the coefficients a_k must satisfy. Since

$$\frac{dv}{d\xi} = a_1 + 2a_2\xi + 3a_3\xi^2 + \cdots = \sum_{k=1}^{\infty} k a_k\,\xi^{k-1}$$

and

$$\frac{d^2v}{d\xi^2} = 2a_2 + 6a_3\xi + \cdots = \sum_{k=2}^{\infty} k(k-1)a_k\,\xi^{k-2} = \sum_{k=0}^{\infty}(k+1)(k+2)a_{k+2}\,\xi^k,$$

we have

$$\frac{d^2v}{d\xi^2} - 2\xi\,\frac{dv}{d\xi} + (\varepsilon-1)v = 2a_2 + (\varepsilon-1)a_0 + \sum_{k=1}^{\infty}\left\{(k+2)(k+1)a_{k+2} + (\varepsilon - 1 - 2k)a_k\right\}\xi^k.$$

For a power series to be identically zero, each of the coefficients must be zero. Hence we obtain[10]

$$(k+2)(k+1)a_{k+2} + (\varepsilon - 1 - 2k)\,a_k = 0,\ \ k = 0, 1, 2, \ldots \tag{7.12}$$

Thus once a_0 is specified, the coefficients $a_2, a_4, a_6, \ldots$ are determined from the above recurrence relation. Similarly, once a_1 is specified the coefficients $a_3, a_5, a_7, \ldots$ are determined. The recurrence relation (7.12) can be rewritten as

$$\frac{a_{k+2}}{a_k} = \frac{2k - \varepsilon + 1}{(k+2)(k+1)},\ \ k = 0, 1, 2, \ldots \tag{7.13}$$

[8]This change of variables makes the recursion relation derived below simpler.

[9]This is called the power series method.

[10]Note that the $k=0$ condition is $2a_2 + (\varepsilon - 1)a_0 = 0$.

Our first observation from (7.13) is that

$$\lim_{k\to\infty}\left|\frac{a_{k+2}}{a_k}\right| = 0$$

and so by the ratio test for power series, the radius of convergence of (7.11) is infinite. (This is good since we want our functions ψ to be defined for all ξ.)

Now comes a crucial point. We have shown for any choices of a_0 and a_1 and for any choice of the parameter (dimensionless energy) ε, that the function $\psi(\xi) = e^{-\xi^2/2}v(\xi)$ solves the differential equation (7.9) where v is given by (7.11) and the coefficients a_k satisfy (7.13). However, a basic requirement of the quantum mechanical formalism is that $\psi(\xi)$ is an element of the state space $L^2(\mathbb{R})$; namely, it is square integrable. Thus the question is whether $e^{-\xi^2/2}v(\xi)$ is square integrable. We will show that we have square integrable functions for only certain values of the energy ε; namely, we will find the *quantization of energy.*

The ratio of the series coefficients, a_{k+1}/a_k, in the function

$$e^{\alpha z} = \sum_{k=0}^{\infty} a_k z^k = \sum_{k=0}^{\infty}\frac{\alpha^k}{k!}\, z^k$$

is $\alpha/(k+1) \sim \alpha/k$ as $k\to\infty$. For the series (recall given a_0 we can determine $a_2, a_4, \ldots$)

$$v(\xi) = \sum_{k=0}^{\infty} a_{2k}\xi^{2k} = \sum_{k=0}^{\infty} b_k z^k,\ \ b_k = a_{2k}, z = \xi^2,$$

the ratio of coefficients, b_{k+1}/b_k, is (we use (7.13) to get the second equality)

$$\frac{b_{k+1}}{b_k} = \frac{a_{2k+2}}{a_{2k}} = \frac{4k-\varepsilon+1}{(2k+2)(2k+1)} \sim \frac{1}{k},\ \ k\to\infty.$$

This suggests in comparing the series for v with the series for $e^{\alpha z}$, and it can be proved,[11] that

$$v(\xi) \sim e^{\xi^2},\ \ \xi\to\infty.$$

Similar remarks hold for the series $\sum_{k=0}^{\infty} a_{2k+1}\xi^{2k+1}$. This means our solution $\psi(\xi) = v(\xi)e^{-\xi^2/2}$ is not square integrable since it grows as $e^{\xi^2/2}$. Hence ψ is not a valid state in quantum mechanics. *There is a way out of this*: If the coefficients a_k would vanish identically from some point on, then the solution $v(\xi)$ will be a polynomial and thus ψ will be square integrable. From the recurrence relation (7.13) we see that this will happen if the numerator vanishes for some value of k. That is, if

$$\varepsilon = 2n+1$$

for some nonnegative integer n, then $a_{n+2} = a_{n+4} = \cdots = 0$. It is traditional to choose a normalization (which amounts to choices of a_0 and a_1) so that the coefficient of the highest

[11] This asymptotic analysis can be made rigorous using the theory of irregular singular points.

power is 2^n. With this normalization the polynomials are called *Hermite polynomials* and are denoted by $H_n(\xi)$. The first few polynomials are[12]

$$\begin{aligned}
H_0(\xi) &= 1, \\
H_1(\xi) &= 2\xi, \\
H_2(\xi) &= 4\xi^2 - 2, \\
H_3(\xi) &= 8\xi^3 - 12\xi, \\
H_4(\xi) &= 16\xi^4 - 48\xi^2 + 12, \\
H_5(\xi) &= 32\xi^5 - 160\xi^3 + 120\xi, \\
H_6(\xi) &= 64\xi^6 - 480\xi^4 + 720\xi^2 - 120.
\end{aligned}$$

Thus we have found solutions[13]

$$\psi_n(\xi) = N_n H_n(\xi) e^{-\xi^2/2} \tag{7.14}$$

to (7.9); namely,

$$H\psi_n = \frac{\hbar\omega_0}{2}(2n+1)\psi_n, \ \ n = 0, 1, 2, \ldots$$

We have solved an eigenvalue problem in the infinite dimensional space $L^2(\mathbb{R})$. It is convenient to choose the overall normalization constant N_n such that

$$\parallel \psi_n \parallel = 1, \ \ n = 0, 1, 2, \ldots$$

That is, N_n is chosen so that

$$N_n^2 \int_{-\infty}^{\infty} H_n(\xi)^2 e^{-\xi^2} \, d\xi = 1. \tag{7.15}$$

It can be shown that

$$N_n = \left[\sqrt{\pi}\, n!\, 2^n\right]^{-1/2}.$$

7.2.3 Quantization of energy

The quantized energy levels are

$$\boxed{E_n = \frac{1}{2}\hbar\omega_0\, \varepsilon_n = \hbar\omega_0\, (n + 1/2), \ \ n = 0, 1, 2, \ldots.}$$

That is to say, the energy of the quantum oscillator cannot have arbitrary real values (as in the case of the classical oscillator), but must be one of the discrete set of numbers

$$\frac{1}{2}\hbar\omega_0, \ \frac{3}{2}\hbar\omega_0, \ \frac{5}{2}\hbar\omega_0, \ldots$$

[12] One can compute a Hermite polynomial in MATHEMATICA by the command `HermiteH[n,x]` where n is a nonnegative integer.

[13] Here N_n is an overall normalization constant which we choose below.

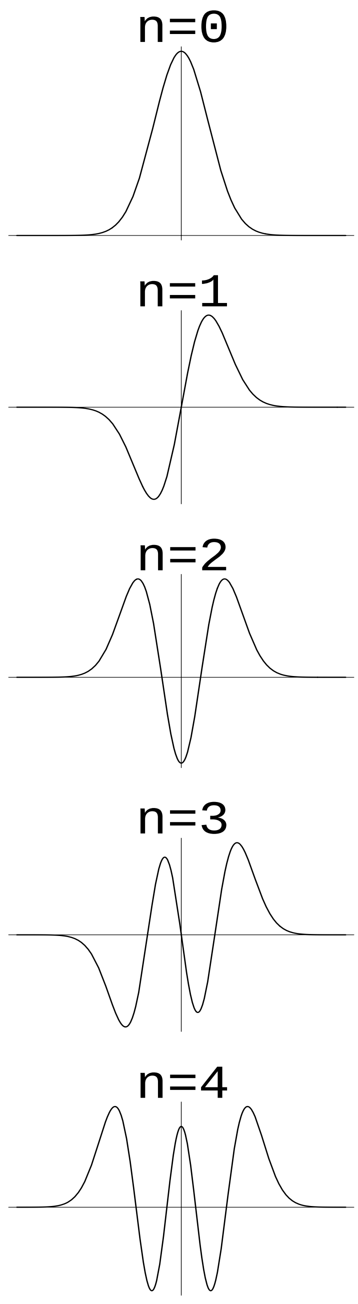


Figure 7.2: Harmonic Oscillator Wave Functions $\psi_n(x)$ for $n = 0, 1, 2, 3, 4$.

The lowest energy, $\frac{1}{2}\hbar\omega_0$, is called the *ground state energy* and has associated wave function

$$\psi_0(\xi) = \frac{1}{\pi^{1/4}}\, e^{-\xi^2/2}.$$

Thus the ground state energy of the quantum harmonic oscillator is nonzero. In the classical harmonic oscillator, we can have $p = x = 0$ which corresponds to $E = 0$.

7.2.4 Some properties of Hermite polynomials

Solution of recurrence relation

To obtain a more explicit formula for the Hermite polynomials we must solve the recurrence relation (7.13). The polynomials $H_n(x)$ are normalized so that the coefficient of the highest power is 2^n. This will determine a_0 (when n is even) and a_1 (when n is odd). We treat here the case of n even and leave the case of n odd to the reader. First

$$\frac{a_n}{a_0} = \frac{a_2}{a_0}\,\frac{a_4}{a_2}\cdots\frac{a_n}{a_{n-2}}$$

The right hand side of this expression is determined from (7.13) and equals

$$\frac{2(n)}{1\cdot 2}\,\frac{2(n-2)}{3\cdot 4}\,\frac{2(n-4)}{5\cdot 6}\cdots\frac{2(2)}{(n-1)n}$$

This can be rewritten as

$$\frac{2^{n/2}n(n-2)(n-4)\cdots 4\cdot 2}{n!}$$

This is the ratio a_n/a_0. Requiring that $a_n = 2^n$ gives

$$a_0 = 2^{n/2}\,(n-1)(n-3)(n-5)\cdots 5\cdot 3\cdot 1$$

We now determine a_m—the coefficient of x^m— (when n is even we can take m even too). Proceeding in a similar manner we write

$$\frac{a_m}{a_0} = \frac{a_2}{a_0}\,\frac{a_4}{a_2}\cdots\frac{a_m}{a_{m-2}}$$

and again note the right hand side is determined from the recurrence relation (7.13); namely,

$$(-1)^{m/2}\,\frac{2(n)}{1\cdot 2}\,\frac{2(n-2)}{3\cdot 4}\,\frac{2(n-4)}{5\cdot 6}\cdots\frac{2(n-m+2)}{(m-1)\cdot m}$$

Multiplying this by the value of a_0 we get that a_m equals

$$(-1)^{m/2}\frac{2^{(n+m)/2}}{m!}\,\left[n(n-2)(n-4)\cdots(n-m+2)\right]\left[(n-1)(n-3)\cdots 5\cdot 3\cdot 1\right]$$

The product of the two quantities in square brackets can be rewritten as

$$\frac{n!}{(n-m)!}\,(n-m-1)(n-m-3)(n-m-5)\cdots 5\cdot 3\cdot 1$$

Now let $m \to n-m$ (so a_m is the coefficient of x^{n-m}) to find that a_m equals

$$(-1)^{m/2} 2^{n-m/2} \binom{n}{m} 1 \cdot 3 \cdot 5 \cdots (m-1)$$

where $\binom{n}{m}$ is the binomial coefficient. Since m is even and runs over $0, 2, 4, \ldots n$, we can let $m \to 2m$ to get the final formula[14]

$$H_n(x) = n! \sum_{m=0}^{[n/2]} \frac{(-1)^m}{m!(n-2m)!} (2x)^{n-2m}. \tag{7.16}$$

This same formula holds for n odd if we interpret $[n/2] = (n-1)/2$ when n is odd. From (7.16) we can immediately derive the differentiation formula

$$\frac{dH_n}{dx} = 2n\, H_{n-1}(x). \tag{7.17}$$

Orthogonality properties

The harmonic oscillator Hamiltonian H is self-adjoint with distinct eigenvalues. Just as we proved for matrices, it follows that that eigenfunctions ψ_n are orthogonal. The normalization constant N_n is chosen so that they are orthonormal. That is if ψ_n are defined by (7.14), then

$$(\psi_n, \psi_m) = N_n^2 \int_{-\infty}^{\infty} H_n(x)\, H_m(x)\, e^{-x^2}\, dx = \delta_{m,n} \tag{7.18}$$

where N_n are defined in (7.15) and $\delta_{m,n}$ is the Kronecker delta function.[15] The functions ψ_n are called the *harmonic oscillator wave functions*.

From the orthogonality relations we can derive what is called the *three-term recursion relation*; namely, we claim that

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0. \tag{7.19}$$

Since the highest power of H_n has coefficient 2^n, we see that

$$H_{n+1}(x) - 2xH_n(x)$$

must be a polynomial of degree less than or equal to n. Using (7.16) we can see that the highest power is the same as the highest power of $2n\, H_{n-1}(x)$. Thus the left hand side of (7.19) is a polynomial of degree less than or equal to $n-2$. It can be written as the linear combination

$$c_0\, H_0(x) + c_1\, H_1(x) + \cdots + c_{n-2}\, H_{n-2}(x).$$

We now multiply both sides of this resulting equation by $H_k(x)\, e^{-x^2}$, $0 \le k \le n-2$, and integrate over all of $\mathbb{R}$. Using the orthogonality relation one concludes that $c_k = 0$.[16]

[14] We used the fact that

$$(2m-1)!!/(2m)! = 1/(2^m\, m!)$$

where $(2m-1)!! = (2m-1)(2m-3) \cdots 5 \cdot 3 \cdot 1$.

[15] $\delta_{m,n}$ equals 1 if $m = n$ and 0 otherwise.

[16] Perhaps the only point that needs clarification is why

$$\int_{\mathbb{R}} 2xH_n(x)H_k(x)e^{-x^2}\, dx$$

For applications to the harmonic oscillator, it is convenient to find what (7.16) and (7.19) imply for the oscillator wave functions ψ_n. It is an exercise to show that[17]

$$x\,\psi_n(x) = \sqrt{\frac{n}{2}}\,\psi_{n-1}(x) + \sqrt{\frac{n+1}{2}}\,\psi_{n+1}(x)\,, \tag{7.20}$$

$$\frac{d\psi_n(x)}{dx} = \sqrt{\frac{n}{2}}\,\psi_{n-1}(x) - \sqrt{\frac{n+1}{2}}\,\psi_{n+1}(x)\,. \tag{7.21}$$

7.2.5 Completeness of the harmonic oscillator wave functions $\{\psi_n\}_{n\geq 0}$

In finite-dimensional vector spaces, we understand the notion of a basis. In particular, we've seen the importance of an orthonormal basis. In Hilbert spaces these concepts are more subtle and a full treatment will not be given here. Here is what can be proved. For any vector $\Psi \in L^2(\mathbb{R})$ we can find coefficients a_n such that

$$\Psi = \sum_{n=0}^{\infty} a_n \psi_n. \tag{7.22}$$

Since this is an infinite sum we must say in what sense this sum converges. If we define the partial sums

$$\Psi_n = \sum_{k=0}^{n} a_k \psi_k,$$

then we say $\Psi_n \to \Psi$ as $n \to \infty$ if

$$\lim_{n\to\infty} \parallel \Psi - \Psi_n \parallel = 0.$$

(Observe that Ψ_n, being a sum of a finite number of terms is well-defined.) Recall that the norm $\parallel \cdot \parallel$ in $L^2(\mathbb{R})$ is

$$\parallel \Psi - \Psi_n \parallel^2 = \int_{\mathbb{R}} \left|\Psi(x) - \Psi_n(x)\right|^2\, dx.$$

It is in this sense the series converges. Since ψ_n form an orthonormal sequence, the coefficients a_n are given simply by

$$a_n = (\psi_n, \Psi)\,.$$

Observe that since ψ_n form an orthonormal basis, the vector Ψ in (7.22) satisfies

$$\parallel \Psi \parallel^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

---

is zero for $0 \leq k \leq n-2$. Since $2xH_k(x)$ is a polynomial of degree $k+1 \leq n-1$, it too can be expanded in terms of Hermite polynomials of degree less than or equal to $n-1$; but these are all orthogonal to H_n. Hence the expansion coefficients must be zero.

[17] These formulas are valid for $n = 0$ if we define $\psi_{-1}(x) = 0$.

7.3 Some properties of the harmonic oscillator

In quantum mechanics if $\mathcal{O}$ is an observable (mathematically, a self-adjoint operator on the Hilbert space $\mathcal{H}$), then the average (or expected) value of $\mathcal{O}$ in the state Ψ is

$$\langle \mathcal{O} \rangle = (\mathcal{O}\Psi, \Psi)\,.$$

For the quantum oscillator, the average position in the eigenstate ψ_n is

$$\langle x \rangle = (x\psi_n, \psi_n) = \int_{\mathbb{R}} x\psi_n(x)^2\,dx = 0.$$

(The integral is zero since ψ_n^2 is an even function of x so that $x\psi_n(x)^2$ is an odd function.) The average of the square of the position in the eigenstate ψ_n is

$$\left\langle x^2 \right\rangle = \left(x^2\psi_n, \psi_n\right).$$

This inner product (integral) can be evaluated by first using (7.20) twice to write $x^2\psi_n$ as a linear combination of the ψ_k's:

$$\begin{aligned}
x^2\psi_n &= x\left\{\sqrt{\frac{n}{2}}\,\psi_{n-1} + \sqrt{\frac{n+1}{2}}\,\psi_{n+1}\right\} \\
&= \sqrt{\frac{n}{2}}\left\{\sqrt{\frac{n-1}{2}}\,\psi_{n-2} + \sqrt{\frac{n}{2}}\,\psi_n\right\} + \sqrt{\frac{n+1}{2}}\left\{\sqrt{\frac{n+1}{2}}\,\psi_n + \sqrt{\frac{n+1}{2}}\,\psi_{n+2}\right\} \\
&= \frac{1}{2}\sqrt{n(n-1)}\,\psi_{n-2} + (n+\frac{1}{2})\,\psi_n + \frac{1}{2}\sqrt{n+1)(n+2)}\,\psi_{n+2}.
\end{aligned}$$

The inner product can now be calculated using the orthonormality of the wave functions to find[18]

$$\left\langle x^2 \right\rangle = n + \frac{1}{2}\,.$$

A very similar calculation with $\hat{p} = -i\frac{d}{dx}$ (but this time using (7.21)) gives[19]

$$\begin{aligned}
\langle \hat{p} \rangle &= 0, \\
\left\langle \hat{p}^2 \right\rangle &= (n + \frac{1}{2})\,.
\end{aligned}$$

If we define

$$\begin{aligned}
\Delta x &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2}\,, \\
\Delta p &= \sqrt{\langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2}\,,
\end{aligned}$$

[18] This is the dimensionless result. Putting back in the dimensions, the average is

$$(n+1/2)\,\frac{\hbar}{m\omega_0}$$

.

[19] Again these are the dimensionless results. Putting back the units the second average is

$$\left\langle \hat{p}^2 \right\rangle = (n + \frac{1}{2})\,m\omega_0\hbar.$$

then (in physical units) we have in state ψ_n

$$\Delta x\, \Delta p = \left(n + \frac{1}{2}\right) \hbar \geq \frac{\hbar}{2}.$$

This is the *Heisenberg Uncertainty Principle* for the harmonic oscillator. The inequality part of the statement can be shown to be valid under very general conditions.

7.3.1 Averages $\langle \hat{x}(t) \rangle$ and $\langle \hat{p}(t) \rangle$

Let Ψ be any state of the system

$$\Psi(x) = \sum_{n=0}^{\infty} a_n \psi_n(x)$$

such that

$$A := \sum_{n\geq 0} \sqrt{\frac{n}{2}}\, a_n \bar{a}_{n-1}, \;\; B := \sum_{n\geq 0} \sqrt{\frac{n+1}{2}}\, a_n \bar{a}_{n+1}$$

are convergent sums. (Here, as throughout, the ψ_n are the harmonic oscillator wave functions.) The time evolution of Ψ is then given by

$$\Psi(x,t) = \sum_{n\geq 0} a_n e^{-iE_n t/\hbar}\, \psi_n(x) \tag{7.23}$$

which follows from the above discussion of separation of variables.

In the state $\Psi(x,t)$ we are interested in computing the average values of $\hat{x}$ and $\hat{p}$. For notationally convenience let's define

$$x_{\text{avg}}(t) = \langle \hat{x} \rangle = (\hat{x}\Psi(x,t), \Psi(x,t))$$

and

$$p_{\text{avg}}(t) = \langle \hat{p} \rangle = (\hat{p}\Psi(x,t), \Psi(x,t))\,.$$

Let

$$x_0 := x_{\text{avg}}(0) = (\hat{x}\Psi(x,0), \Psi(x,0))$$

and

$$p_0 := p_{\text{avg}}(0) = (\hat{p}\Psi(x,0), \Psi(x,0))\,.$$

We first calculate x_0 and p_0.

$$\begin{aligned}
x_0 &= \sum_{m,n\geq 0} a_n \bar{a}_m\, (x\psi_n, \psi_m) \\
&= \sum_{m,n\geq 0} a_n \bar{a}_m \left[\sqrt{\frac{n}{2}}\, (\psi_{n-1}, \psi_m) + \sqrt{\frac{n+1}{2}}\, (\psi_{n+1}, \psi_m) \right] \\
&= \sum_{m,n\geq 0} a_n \bar{a}_m \left[\sqrt{\frac{n}{2}}\, \delta_{n-1,m} + \sqrt{\frac{n+1}{2}}\, \delta_{n+1,m} \right] \\
&= A + B
\end{aligned}$$

where we use the orthonormality of the functions ψ_n. Similarly,

$$p_0 = -iA + iB.$$

We now calculate $x_{\text{avg}}(t)$ Now the state is (7.23). Proceeding as in the $t = 0$ case we see

$$x_{\text{avg}}(t) = \sum_{m,n\geq 0} a_n \bar{a}_m e^{-i(E_n - E_m)t/\hbar} \, (x\psi_n, \psi_m) \, .$$

The calculation of the inner products $(x\psi_n, \psi_m)$ was done in the $t = 0$ case. Noting that

$$E_n - E_{n-1} = \hbar\omega_0 \text{ and } E_n - E_{n+1} = -\hbar\omega_0,$$

we see that

$$x_{\text{avg}}(t) = e^{-i\omega_0 t}\, A + e^{i\omega_0 t}\, B. \tag{7.24}$$

Similarly, we find

$$p_{\text{avg}}(t) = -ie^{-i\omega_0 t}\, A + ie^{i\omega_0 t}\, B. \tag{7.25}$$

Writing these averages in terms of sines and cosines and using the above expressions for x_0 and p_0, we see that the average position and momentum in the state $\Psi(x, t)$ evolve according to[20]

$$x_{\text{avg}}(t) = x_0 \cos(\omega_0 t) + \frac{p_0}{m\omega_0} \sin(\omega_0 t) \tag{7.26}$$

$$p_{\text{avg}}(t) = p_0 \cos(\omega_0 t) - m\omega_0 x_0 \sin(\omega_0 t) \tag{7.27}$$

One should now compare the time evolution of the quantum averages (7.26) and (7.27) with the time evolution of the classical position and momentum (7.1) and (7.2). They are identical. It is a special property of the quantum harmonic oscillator that the quantum averages *exactly* follow the classical trajectories. More generally, one expects this to occur only for states whose wave function remains localized in a region of space.

[20] We restore the physical units in these last equations

7.4 The Heisenberg Uncertainty Principle

> The more precisely the position is determined, the less precisely the momentum is known in this instant, and vice versa. Werner Heisenberg, 1927.

In §7.3 we proved the Heisenberg Uncertainty Principle for the special case of the harmonic oscillator. Here we show this is a general feature of quantum mechanics.[21] First we recall some basic facts about complex vector spaces.

1. If Ψ and Φ are any two states in our Hilbert space of states $\mathcal{H}$, we have an inner product defined (Ψ, Φ) that satisfies the properties
   (a) $\overline{(\Psi, \Phi)} = (\Phi, \Psi)$ where $\overline{z}$ denotes the complex conjugate of z.
   (b) $(c_1\Psi_1 + c_2\Psi_2, \Phi) = c_1 (\Psi_1, \Phi) + c_2 (\Psi_2, \Phi)$ for all states Ψ_1, Ψ_2 and all complex numbers c_1, c_2.
   (c) The length or *norm* of the state Ψ is defined to be $\|\Psi\|^2 = (\Psi, \Psi) \geq 0$ with $\|\Psi\| = 0$ if and only if $\Psi = 0$, the zero vector in $\mathcal{H}$.
2. An operator A is called *Hermitian* (or self-adjoint) if
   $$(A\Psi, \Phi) = (\Psi, A\Phi)$$
   for all states Ψ, Φ. In quantum mechanics observables are assumed to be Hermitian. Note this makes the expected value of the observable A in state Ψ a real number
   $$\begin{aligned} \langle A \rangle &:= (A\Psi, \Psi) \\ &= (\Psi, A\Psi) \\ &= \overline{(A\Psi, \Psi)} \\ &= \overline{\langle A \rangle}. \end{aligned}$$
   Sometimes one writes $\langle A \rangle_\Psi$ to denote the state in which the expected value is computed.
3. Just as in linear algebra, we have the Cauchy-Schwarz inequality
   $$|(\Psi, \Phi)|^2 \leq \|\Psi\|^2 \, \|\Phi\|^2 \tag{7.28}$$
   for all states $\Psi, \Phi \in \mathcal{H}$.

We now assume we have observables A and B that satisfy the commutation relation

$$AB - BA = i \text{ id} \tag{7.29}$$

where id is the identity operator and i is the imaginary number, $i^2 = -1$. We showed earlier that in units where $\hbar = 1$ the position and momentum operators satisfy such a commutation relation. For a given state Ψ and observable A we define[22]

$$\Delta A = \sqrt{\left\langle (A - \langle A \rangle)^2 \right\rangle} = \sqrt{\langle A^2 \rangle - \langle A \rangle^2} \geq 0.$$

[21] This is an optional section and is not covered in the class lectures.

[22] In classical probability ΔA is called the standard deviation of A. The quantity ΔA is a measure of the deviation of A from its expected value $\langle A \rangle$.

We now prove that if observables A and B satisfy (7.29) then

$$\Delta A \cdot \Delta B \geq \frac{1}{2}. \tag{7.30}$$

Proof: Let Ψ denote any normalized state vector, i.e. $\|\Psi\| = 1$. Apply (7.29) to Ψ to obtain

$$AB\Psi - BA\Psi = i\Psi$$

Now take the inner product of each side with the state Ψ to obtain

$$(AB\Psi, \Psi) - (BA\Psi, \Psi) = i\,(\Psi, \Psi)$$

which simplifies to[23]

$$(B\Psi, A\Psi) - (A\Psi, B\Psi) = i \tag{7.31}$$

Let t denote any real number, then by the Cauchy-Schwarz inequality (7.28)

$$|(\Psi, A\Psi + itB\Psi)|^2 \leq \|A\Psi + itB\Psi\|^2 \tag{7.32}$$

since $\|\Psi\| = 1$. Let's simplify the left-hand side of (7.32)

$$(\Psi, A\Psi + itB\Psi) = (\Psi, A\Psi) - it(\Psi, B\Psi) = \langle A\rangle - it\langle B\rangle.$$

The absolute value squared of this is

$$\langle A\rangle^2 + t^2\langle B\rangle^2.$$

We now examine the right-hand side of (7.32)

$$\begin{aligned} \|A\Psi + itB\Psi\|^2 &= (A\Psi + itB\Psi, A\Psi + itB\Psi) \\ &= \|A\Psi\|^2 + it\,\{(B\Psi, A\Psi) - (A\Psi, B\Psi)\} + t^2\|B\Psi\|^2 \\ &= \|A\Psi\|^2 - t + t^2\|B\Psi\|^2 \text{ by use of (7.31).} \end{aligned} \tag{7.33}$$

Thus the inequality (7.32) becomes

$$\langle A\rangle^2 + t^2\langle B\rangle^2 \leq \|A\Psi\|^2 - t + t^2\|B\Psi\|^2.$$

Using the fact that $\|A\Psi\|^2 = (A\Psi, A\Psi) = (A^2\Psi, \Psi) = \langle A^2\rangle$ (and similarly for B) and the definition of ΔA (and similarly for ΔB), the above inequality can be rewritten as

$$t^2\,(\Delta B)^2 - t + (\Delta A)^2 \geq 0.$$

This holds for all real t. The above is a quadratic polynomial in t that is always nonnegative. This means that the discriminant of the quadratic polynomial must be nonpositive, i.e. $b^2 - 4ac \leq 0$. That is,

$$1 - 4\,(\Delta A)^2\,(\Delta B)^2 \leq 0$$

which implies that

$$\Delta A \cdot \Delta B \geq \frac{1}{2}$$

which is what we want to prove.

When A is the position operator and B is the momentum operator we get the Heisenberg Uncertainty Principle which states

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2}$$

where we have returned to physical units. The appearance of Planck's constant $\hbar$ in the right hand side shows that $\hbar$ sets the scale for quantum mechanical phenomena.

[23] Note $(AB\Psi, \Psi) = (B\Psi, A\Psi)$ and $(BA\Psi, \Psi) = (A\Psi, B\Psi)$ since A and B are Hermitian. Also note on the right hand side we used the fact that $(\Psi, \Psi) = \|\Psi\|^2 = 1$.

7.5 Comparison of three problems

	Weighted String	**Vibrating String**	**Quantum Harmonic Oscillator**
Vector space $\mathcal{V}$	$\mathbb{R}^N$	Functions on $[0,L]$ that vanish at the endpoints 0 and L	$\mathcal{H}=$ Square integrable functions on $\mathbb{R}$
Inner product	$(\vec{f},\vec{g})=\vec{f}\cdot\vec{g}=\sum_{i=1}^N f_i\overline{g_i}$	$(f,g)=\int_0^L f(x)\overline{g(x)}\,dx$	$(f,g)=\int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx$
Norm	$\Vert\vec{f}\Vert=\left[(\vec{f},\vec{f})\right]^{1/2}$	$\Vert f\Vert=\left[\int_0^L \vert f(x)\vert^2\,dx\right]^{1/2}$	$\Vert f\Vert=\left[\int_{-\infty}^{\infty} \vert f(x)\vert^2\,dx\right]^{1/2}$
Operator	$V_N,\ N\times N$ tridiagonal matrix	$L=\frac{d^2}{dx^2}$	$H=-\frac{d^2}{dx^2}+x^2$
Symmetry	$\left(V_N\vec{f},\vec{g}\right)=\left(\vec{f},V_N\vec{g}\right)$	$(Lf,g)=(f,Lg)$	$(Hf,g)=(f,Hg)$
Eigenvalues & Eigenvectors	$V_N\vec{f}_n=\lambda_n\vec{f}_n, n=1,2,\dots,N$ $\lambda_n=2\left(1-\cos(\frac{n\pi}{N+1})\right)$ $\vec{f}_n=(f_{n,j})_{1\leq j\leq N}$ $f_{n,j}=\sqrt{\frac{2}{N+1}}\sin(\frac{n\pi}{N+1}j)$	$Lu_n=-k_n^2u_n, n=1,2,3,\dots$ $k_n=\frac{n\pi}{L}$ $u_n(x)=\sqrt{\frac{2}{L}}\sin(\frac{n\pi}{L}x)$	$H\psi_n=\epsilon_n\psi_n, n=0,1,2,\dots$ $\epsilon_n=2n+1$ $\psi_n(x)=N_nH_n(x)\mathrm{e}^{-x^2/2}$ $H_n(x)=$ Hermite polynomial $N_n=[\sqrt{\pi}\,n!\,2^n]^{-1/2}$
Orthonormal	$(\vec{f}_m,\vec{f}_n)=\delta_{m,n}$	$(u_m,u_n)=\delta_{m,n}$	$(\psi_m,\psi_n)=\delta_{m,n}$
Completeness	$\vec{u}\in\mathbb{R}^N,\ \vec{u}=\sum_{j=1}^N a_j\vec{f}_j$	$u\in\mathcal{V},\ u(x)=\sum_{j=1}^\infty a_ju_j(x)$	$\Psi\in\mathcal{H},\ \Psi(x)=\sum_{j=0}^\infty a_j\psi_j(x)$
Coefficients	$a_j=\left(\vec{f}_j,\vec{u}\right)$	$a_j=(u,u_j)$	$a_j=(\Psi,\psi_j)$

7.6 Exercises

#1.

Using (7.17) and (7.19), prove (7.20) and (7.21).

#2. Averages $\langle\hat{x}^4\rangle$ and $\langle\hat{p}^4\rangle$:

For the state ψ_n, compute the averages $\langle\hat{x}^4\rangle$ and $\langle\hat{p}^4\rangle$.

#3.

Prove (7.25). (The proof is similar to the proof of (7.24).)

#4.

Define the operators

$$\begin{aligned} a &= \frac{1}{\sqrt{2}}\left(x + \frac{d}{dx}\right) \\ a^* &= \frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right) \end{aligned}$$

That is, if $\psi = \psi(x)$, then

$$(a\psi)(x) = \frac{1}{\sqrt{2}}\left(x\psi(x) + \frac{d\psi}{dx}\right)$$

and similarly for a^*. Using (7.20) and (7.21) show that for the harmonic oscillator wave functions ψ_n

$$\begin{aligned} a\psi_n &= \sqrt{n}\,\psi_{n-1},\ \ n \geq 1,\ \ a\psi_0 = 0, \\ a^*\psi_n &= \sqrt{n+1}\,\psi_{n+1},\ \ n = 0, 1, \ldots, \\ a^*a\psi_n &= n\,\psi_n,\ \ n = 0, 1, \ldots, \\ (aa^* - a^*a)\,\psi_n &= \psi_n,\ \ n = 0, 1, \ldots. \end{aligned}$$

Explain why this last equation implies the operator equation

$$[a, a^*] = \mathrm{id}.$$

In quantum mechanics the operator a is called an *annihilation operator* and the operator a^* is called a *creation operator*. On the basis of this exercise, why do you think they have these names?

#5. Hermite polynomials

We obtained the Hermite polynomials from the recurrence relation (7.13). Alternatively, we have a *generating formula* for the Hermite polynomials. Starting with this (which many books take as the definition of the Hermite polynomials), we may obtain the Schrödinger equation.

1. Verify that the first three Hermite polynomials $H_0(\xi), H_1(\xi)$ and $H_2(\xi)$ are given using the generating formula
$$H_n(\xi) = (-1)^n e^{\xi^2}\left(\frac{d^n}{d\xi^n} e^{-\xi^2}\right) \tag{7.34}$$
2. The generating function (7.34) can be used to give an alternative generating function for Hermite polynomials. Show that
$$e^{-z^2+2z\xi} = \sum_{n=0}^{\infty} \frac{z^n}{n!} H_n(\xi)\,. \tag{7.35}$$
   Hint: Let $F(z) = e^{-z^2}$ and consider the Taylor expansion of $e^{\xi^2} F(z - \xi)$ about the point $z = 0$.

3. Derive (7.17) from (7.35). Now derive (7.19) using (7.34) and the newly derived (7.17).

4. Use (7.17) and (7.19) to show that the Hermite polynomials are solutions of the *Hermite equation*
$$\frac{d^2}{d\xi^2}H_n(\xi) - 2\xi\frac{d}{d\xi}H_n(\xi) + 2nH_n(\xi) = 0. \tag{7.36}$$

5. We know that the Hermite polynomials satisfy
$$N_n^2\int_{-\infty}^{\infty} H_n(\xi)H_m(\xi)e^{-\xi^2}d\xi = \delta_{nm}. \tag{7.37}$$
Here by setting $\psi_n(\xi) = N_nH_n(\xi)e^{-\xi^2/2}$ we see that $\psi_n(\xi)$ are orthonormal in $L^2(\mathbb{R})$. Use (7.36) to obtain the differential equation that $\psi_n(\xi)$ satisfy. You should obtain (7.9) with $\varepsilon = 2n+1$. This implies that $\psi_n(\xi) = N_nH_n(\xi)e^{-\xi^2/2}$ is the eigenfunction corresponding to the eigenvalue of the Hamiltonian operator.

Chapter 8

Heat Equation

Figure 8.1: Joseph Fourier (1768–1830) a French mathematician and physicist best known for initiating the study of Fourier series with applications to the theory of oscillatory systems and heat transfer.

8.1 Introduction

The *heat equation* is the partial differential equation

$$\boxed{\frac{\partial u}{\partial t} - \Delta u = 0} \tag{8.1}$$

where Δ is the Laplacian. In three-dimensions if (x, y, z) are Cartesian coordinates, the heat equation reads

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} = 0,$$

whereas in one-dimension the heat equation reads

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0. \tag{8.2}$$

The heat equation has the following physical interpretation: the temperature $u(\vec{x}, t)$ at position $\vec{x}$ and time t satisfies (8.1). This assumes that the medium is homogeneous and the *thermal diffusivity* has been set equal to one. The heat equation is fundamental in probability theory due to its connection with Brownian motion. In this chapter we show how to solve the (8.2) subject to the initial condition $u(x, 0) = f(x)$. The function $f(x)$ can be interpreted as the initial distribution of temperature.

8.2 Fourier transform

Suppose $f = f(x)$, $x \in \mathbb{R}$, is smooth (say continuous) and decays sufficiently fast at $\pm\infty$ so that the integral

$$\boxed{\widehat{f}(\xi) := \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-2\pi \mathrm{i} x \xi}\, dx, \ \ \xi \in \mathbb{R},} \tag{8.3}$$

exists. The function $\widehat{f}(\xi)$ is called the *Fourier transform of f*. For example, if $f(x) = \mathrm{e}^{-a|x|}$, $a > 0$, then

$$\widehat{f}(\xi) = \frac{2a}{a^2 + 4\pi^2 \xi^2}.$$

The most important property of the Fourier transform is the *Fourier inversion formula* which says given $\widehat{f}(\xi)$ we can find $f(x)$ from the formula

$$\boxed{f(x) = \int_{-\infty}^{\infty} \widehat{f}(\xi) \mathrm{e}^{2\pi \mathrm{i} x \xi}\, d\xi, \ \ x \in \mathbb{R}.} \tag{8.4}$$

Thus, for example, we have

$$\int_{-\infty}^{\infty} \frac{2a}{a^2 + 4\pi^2 \xi^2} \mathrm{e}^{2\pi \mathrm{i} x \xi}\, d\xi = \mathrm{e}^{-a|x|}.$$

8.3 Solving the heat equation by use of the Fourier transform

We consider the 1D heat equation (8.2) for $x \in \mathbb{R}$ subject to the initial condition $u(x,0) = f(x)$. Write $u(x,t)$ in terms of its Fourier transform $\widehat{u}(\xi,t)$:

$$u(x,t) = \int_{-\infty}^{\infty} \widehat{u}(\xi,t)\mathrm{e}^{2\pi \mathrm{i} x\xi}\, d\xi.$$

Then

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \int_{-\infty}^{\infty} \mathrm{e}^{2\pi \mathrm{i} x\xi} \left[\frac{\partial \widehat{u}(\xi,t)}{\partial t} + 4\pi^2\xi^2\widehat{u}(\xi,t)\right] d\xi$$

We want this to equal zero so this will certainly be the case if

$$\frac{\partial \widehat{u}(\xi,t)}{\partial t} + 4\pi^2\xi^2\widehat{u}(\xi,t) = 0.$$

We solve this last equation:

$$\widehat{u}(\xi,t) = \mathrm{e}^{-4\pi^2\xi^2 t}\widehat{u}(\xi,0).$$

Thus we have

$$u(x,t) = \int_{-\infty}^{\infty} \mathrm{e}^{2\pi \mathrm{i} x\xi - 4\pi^2\xi^2 t}\widehat{u}(\xi,0)\, d\xi. \tag{8.5}$$

Specializing to $t = 0$ and using the initial condition gives

$$f(x) = u(x,0) = \int_{-\infty}^{\infty} \mathrm{e}^{2\pi \mathrm{i} x\xi}\widehat{u}(\xi,0)\, d\xi.$$

By the Fourier inversion formula

$$\widehat{u}(\xi,0) = \int_{-\infty}^{\infty} \mathrm{e}^{-2\pi \mathrm{i} x\xi} f(x)\, dx = \widehat{f}(\xi).$$

We now substitute this expression for $\widehat{u}(\xi,0)$ into (8.5) to obtain

$$\begin{aligned}
u(x,t) &= \int_{-\infty}^{\infty} \mathrm{e}^{2\pi \mathrm{i} x\xi - 4\pi^2\xi^2 t}\widehat{f}(\xi)\, d\xi \\
&= \int_{-\infty}^{\infty} \mathrm{e}^{2\pi \mathrm{i} x\xi - 4\pi^2\xi^2 t}\left[\int_{-\infty}^{\infty} f(x')\mathrm{e}^{-2\pi \mathrm{i} x'\xi}\, dx'\right] d\xi \\
&= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \mathrm{e}^{2\pi \mathrm{i}(x-x')\xi - 4\pi^2\xi^2 t}\, d\xi\right] f(x')\, dx'
\end{aligned} \tag{8.6}$$

We now evaluate the integral

$$\mathcal{I} := \int_{-\infty}^{\infty} \mathrm{e}^{2\pi \mathrm{i}(x-x')\xi - 4\pi^2\xi^2 t}\, d\xi.$$

To do this we first evaluate the integral

$$\mathcal{I}_1 = \int_{-\infty}^{\infty} \mathrm{e}^{-\alpha\xi^2}\, d\xi,\ \ \alpha > 0.$$

We look at $\mathcal{I}_1^2$ and transform the resulting double integral to polar coordinates:

$$\begin{aligned} \mathcal{I}_1^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-\alpha\xi_1^2}\mathrm{e}^{-\alpha\xi_2^2}\,d\xi_1 d\xi_2 \\ &= \int_0^{\infty}\int_0^{2\pi} \mathrm{e}^{-\alpha r^2} r\,dr d\theta \\ &= 2\pi\int_0^{\infty} r\mathrm{e}^{-\alpha r^2}\,dr \\ &= \frac{\pi}{\alpha} \end{aligned}$$

Thus

$$\mathcal{I}_1 = \sqrt{\frac{\pi}{\alpha}}.$$

We are now ready to evaluate $\mathcal{I}$. Let $y = x - x'$ so the integral we wish to evaluate is

$$\int_{-\infty}^{\infty} \mathrm{e}^{2\pi \mathrm{i} y\xi - 4\pi^2\xi^2 t}\,d\xi$$

We make the change of variables $\xi = \xi' + a$ and choose a so that the linear term in ξ' vanishes:

$$2\pi\mathrm{i}y\xi - 4\pi^2\xi^2 t = 2\pi\mathrm{i}y(\xi'+a) - 4\pi^2 t(\xi'^2 + 2\xi' a + a^2) = -4\pi^2 t\xi'^2 + \left[2\pi\mathrm{i}y - 8\pi^2 ta\right]\xi' + 2\pi\mathrm{i}ya - 4\pi^2 ta^2$$

Thus we choose

$$a = \frac{2\pi\mathrm{i}y}{8\pi^2 t} = \frac{\mathrm{i}y}{4\pi t}.$$

With this value of a we then have

$$2\pi\mathrm{i}y\xi - 4\pi^2\xi^2 t = -4\pi^2 t\xi'^2 - \frac{y^2}{4t}$$

Thus

$$\begin{aligned} \int_{-\infty}^{\infty} \mathrm{e}^{2\pi\mathrm{i}y\xi - 4\pi^2\xi^2 t}\,d\xi &= \int_{-\infty}^{\infty} \mathrm{e}^{-4\pi^2 t\xi'^2 - y^2/(4t)}\,d\xi' \\ &= \mathrm{e}^{-y^2/(4t)}\int_{-\infty}^{\infty} \mathrm{e}^{-4\pi^2 t\xi'^2}\,d\xi' = \frac{1}{\sqrt{4\pi t}}\,\mathrm{e}^{-y^2/(4t)} \end{aligned}$$

We summarize what we have found in the following theorem:

Theorem. Let $f(x)$ be continuous and decaying sufficiently fast at infinity so that its Fourier transform exists. Then the solution to the one-dimensional heat equation (8.2) satisfying the initial condition $u(x,0) = f(x)$ is

$$\boxed{u(x,t) = \int_{-\infty}^{\infty} K(x,y;t) f(y)\,dy} \tag{8.7}$$

where

$$\boxed{K(x,y;t) = \frac{1}{\sqrt{4\pi t}}\,\mathrm{e}^{-(x-y)^2/(4t)}.} \tag{8.8}$$

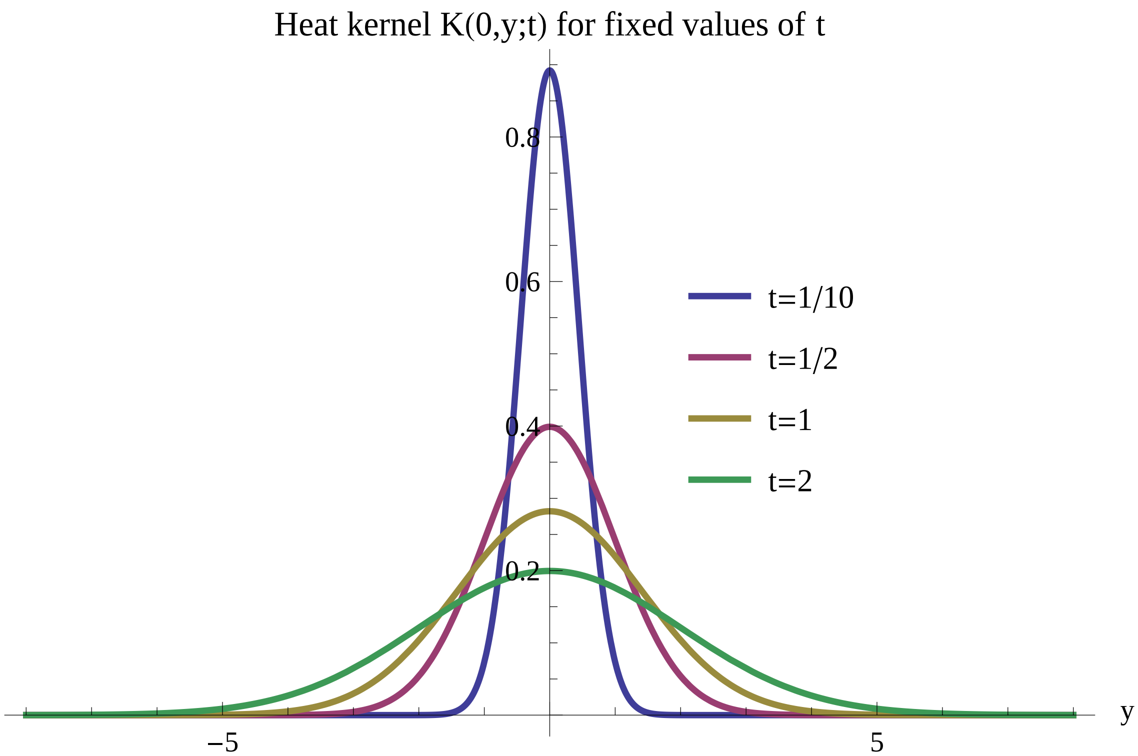


Figure 8.2: The heat kernel $K(x, y; t)$, (8.8), for $x = 0$ as a function of y for various fixed values of t. Note that for small t the functions is concentrated near $x = 0$ but for large t, K is spread out over a wide range of values.

8.3.1 Comments on (8.7) and (8.8)

1. For all $x \in \mathbb{R}$ and all $t > 0$ we have
$$\int_{-\infty}^{\infty} K(x, y; t)\, dy = 1.$$

2. Let's take the initial condition to be (see Figure 8.3)
$$f(x) = \begin{cases} 0 & \text{if} \quad x < -1, \\ 1 + x & \text{if} \quad -1 \leq x < 0, \\ 1 - x & \text{if} \quad 0 \leq x < 1, \\ 0 & \text{if} \quad x \geq 1. \end{cases} \tag{8.9}$$

   In Figure 8.4 we plot the solution $u(x, t)$ as a function of x for various values of t.

3. It is convenient to record the Gaussian integral we evaluated:
$$\int_{-\infty}^{\infty} \mathrm{e}^{-ax^2+2bx}\, dx = \sqrt{\frac{\pi}{a}}\, \mathrm{e}^{b^2/a}, \;\; a > 0. \tag{8.10}$$

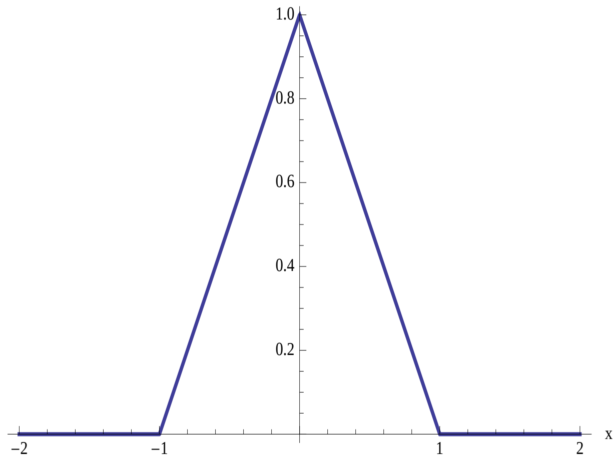


Figure 8.3: The initial temperature distribution $f(x)$ as defined in (8.9). Note that the distribution is not smooth at zero.

8.3.2 Semigroup property

We rewrite (8.7) in terms of an operator P_t; namely,

$$(P_t f)(x) := \int_{-\infty}^{\infty} K(x,y;t) f(y)\, dy, \;\; t > 0. \tag{8.11}$$

We now examine the effect of applying this operator twice; first at time t and secondly at time s:

$$\begin{aligned}
(P_s P_t f)(x) &= \int_{-\infty}^{\infty} K(x,y;s)\,(P_t f)(y)\, dy \\
&= \int_{-\infty}^{\infty} K(x,y;s) \left[\int_{-\infty}^{\infty} K(y,z;t) f(z)\, dz\right] dy \\
&= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} K(x,y;s) K(y,z;t)\, dy\right] f(z)\, dz \\
&= \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{4\pi s}\sqrt{4\pi t}} \mathrm{e}^{-x^2/(4s)-z^2/(4t)} \int_{-\infty}^{\infty} \mathrm{e}^{-(1/(4s)+1/(4t))y^2+2(x/(4s)+z/(4t))y}\, dy\right] f(z)\, dz \\
&= \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{4\pi s}\sqrt{4\pi t}} \mathrm{e}^{-x^2/(4s)-z^2/(4t)} \sqrt{\frac{\pi}{1/(4s)+1/(4t)}}\, \mathrm{e}^{(x/(4s)+z/(4t))^2/(1/(4s)+1/(4t))}\right] f(z)\, dz \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi(s+t)}} \mathrm{e}^{-(x-z)^2/(4(s+t))} f(z)\, dz = \int_{-\infty}^{\infty} K(x,z;s+t) f(z)\, dz \qquad (8.12) \\
&= (P_{s+t} f)(x) \qquad (8.13)
\end{aligned}$$

We used (8.10) in going from the fourth line to the fifth line above; and then, simplified the result to obtain (8.12). We summarize this result by the equation

$$P_s P_t = P_{s+t} = P_t P_s \tag{8.14}$$

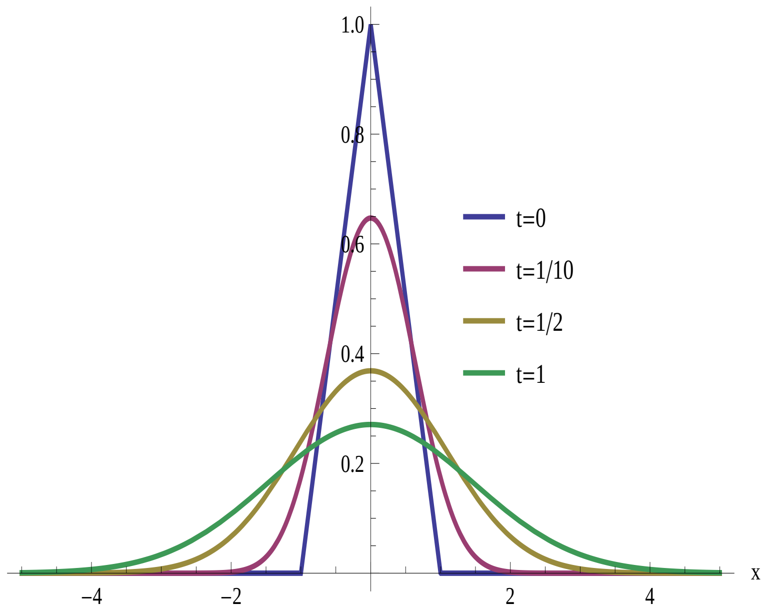


Figure 8.4: The solution $u(x,t)$ with initial condition (8.9). Note that the solution is "smoothed" by the action of the heat operator.

which is called the *semigroup property*. In words it says the effect of applying P_t and then applying P_s is the same as applying P_{s+t}.

We have seen this property before in Chapter 5:

$$\mathrm{e}^{sA}\mathrm{e}^{tA} = \mathrm{e}^{(s+t)A} = \mathrm{e}^{tA}\mathrm{e}^{sA}$$

where A is any $n \times n$ matrix. This analogy can be developed further: The solution to the equation $dx/dt = Ax$ that satisfies the initial condition $x(0) = x_0$ is obtained by applying the semigroup e^{tA} to the initial condition, i.e. $x(t) = \mathrm{e}^{tA}x_0$. For the heat equation we want the solution $u(x,t)$ that satisfies the initial condition $u(x,0) = f(x)$. This solution is obtained by applying the semigroup P_t to f; namely (8.13). This raises the question what is the analog of the matrix A for the heat equation? From the power series expansion of e^{tA} we know that A can be recovered from the formula

$$\lim_{t\to 0^+} \frac{1}{t}\left[\mathrm{e}^{tA}f - f\right] = Af.$$

This suggests we compute

$$\lim_{t\to 0+} \frac{1}{t}\left[(P_t f)(x) - f(x)\right].$$

First recall that

$$\int_{-\infty}^{\infty} K(x,y;t)\,dy = 1.$$

Therefore we can write

$$\begin{aligned} P_t f(x) - f(x) &= \int_{-\infty}^{\infty} K(x,y;t) f(y)\, dy - \left[\int_{-\infty}^{\infty} K(x,y;t) f(y)\, dy\right] f(x) \\ &= \int_{-\infty}^{\infty} K(x,y;t) \left(f(y) - f(x)\right)\, dy. \end{aligned} \tag{8.15}$$

Now for small t the Gaussian $\frac{1}{4\pi t}\mathrm{e}^{(x-y)^2/(4t)}$ is sharply peaked around $y = x$. This suggests that the main contribution in the above integral comes in the vicinity of $y = x$. Therefore we expand $f(y)$ about the point x:

$$f(y) = f(x) + (y-x)f'(x) + \frac{1}{2}f''(x)(y-x)^2 + \frac{1}{3!}f'''(x)(y-x)^3 + \frac{1}{4!}(y-x)^4 f^{(iv)}(x) + \cdots.$$

We now substitute this into (8.15). The integral

$$\int_{-\infty}^{\infty} (y-x) K(x,y;t)\, dy$$

as is easily seen to equal zero. (Make the change of variables $z = x - y$ and then note one is integrating an odd function over the real line; hence equal to zero.) The next integral is (make the same change of variables $z = x - y$)

$$\int_{-\infty}^{\infty} (y-x)^2 K(x,y;t)\, dy = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} z^2 \mathrm{e}^{-z^2/(4t)}\, dz = 2t.$$

All the remaining terms with odd powers of $(y - x)$ are zero. The higher even powers, e.g. $(y - x)^4$, integrate to give higher powers of t. Thus

$$\frac{1}{t}\left[P_t f(x) - f(x0\right] = \frac{1}{t}\left[\frac{1}{2}f''(x)2t + \frac{1}{4!}f^{(iv)}(x)12t^2 + \cdots\right]$$

so that

$$\lim_{t\to 0^+} \frac{1}{t}\left[P_t f(x) - f(x)\right] = \frac{d^2 f}{dx^2} \tag{8.16}$$

Thus the second-order differential operator $\frac{d^2}{dx^2}$ is the analogue of the matrix A.

A point that might need clarification is how one computes integrals

$$\int_{-\infty}^{\infty} x^{2n} \mathrm{e}^{-ax^2}\, dx, \;\; a > 0.$$

Start with the known result (8.10)

$$\int_{-\infty}^{\infty} \mathrm{e}^{-ax^2}\, dx = \sqrt{\frac{\pi}{a}}.$$

Now differentiate both sides with respect to a:

$$-\int_{-\infty}^{\infty} x^2 \mathrm{e}^{-ax^2}\, dx = -\frac{1}{2}\frac{\sqrt{\pi}}{a^{3/2}}.$$

Differentiating again with respect to a gives

$$\int_{-\infty}^{\infty} x^4 \mathrm{e}^{-ax^2}\, dx = \frac{3}{4}\frac{\sqrt{\pi}}{a^{5/2}}.$$

8.4 Heat equation on the half-line

We now consider the heat equation but with the spatial variable x restricted to positive axis—the semi-infinite rod or half-line. Thus we wish to solve (8.2) with an initial heat distribution $f(x)$, $x > 0$; and we further assume, that the end of the rod, $x = 0$, is fixed to zero temperature for all $t > 0$.

For the line we found a solution of the form

$$u(x,t) = \int_{-\infty}^{\infty} K(x,y;t) f(y)\, dy.$$

For the half-line we look for a solution of the form

$$u(x,t) = \int_{0}^{\infty} K_{\mathrm{semi}}(x,y;t) f(y)\, dy. \tag{8.17}$$

We want $K_{\mathrm{semi}}(x,y;t)$ to satisfy the heat equation as a function of (x,t) but we also want

$$K_{\mathrm{semi}}(0,y;t) = 0 \text{ for all } t.$$

This last condition will guarantee that $u(0,t) = 0$ for all $t > 0$. Observe that

$$\begin{aligned} K_{\mathrm{semi}}(x,y;t) &= K(x,y;t) - K(x,-y;t) \\ &= \frac{1}{\sqrt{4\pi t}} \mathrm{e}^{-(x-y)^2/4t} - \frac{1}{4\pi t} \mathrm{e}^{-(x+y)^2/4t} \end{aligned} \tag{8.18}$$

satisfies these conditions. What we have done is to place a second "heat pulse" at $-y$. This technique is called the *method of images*.

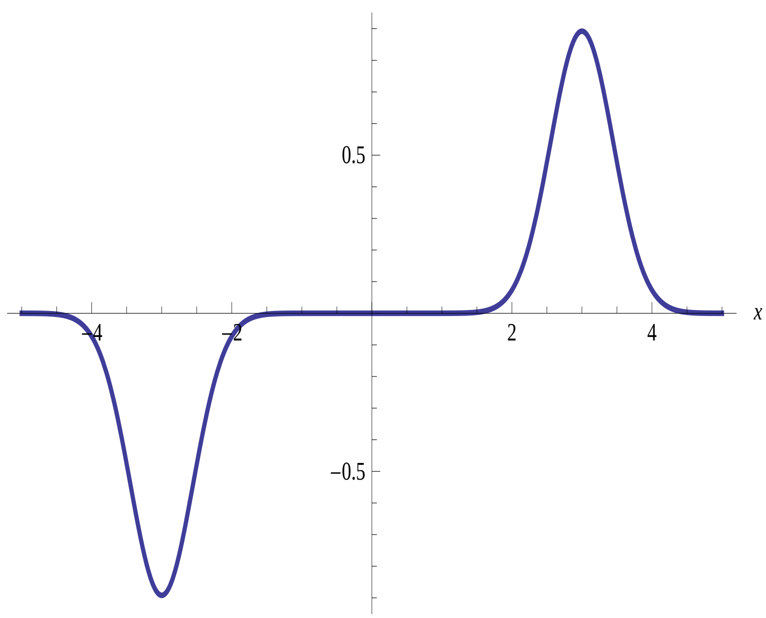


Figure 8.5: The kernel $K_{\mathrm{semi}}(x,x';t)$ as a function of x for the values $y = 3$ and $t = 1/10$.

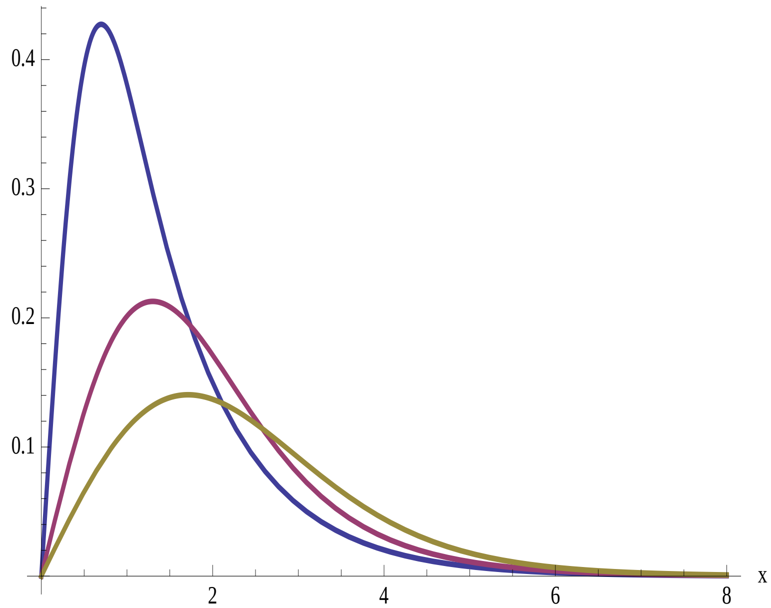


Figure 8.6: Plot of $u(x,t)$ as a function of x for three values of $t = 1/10, 1/2, 1$ with initial heat distribution $f(x) = \mathrm{e}^{-x}$. As t increases the heat distribution becomes flatter and less peaked.

8.5 Heat equation on the circle

Suppose now that the spatial variable x is restricted to a circle; that is, we want a solution to (8.2) that is periodic in x

$$u(x+1,t) = u(x,t)$$

(we've taken the period to equal 1 for simplicity). We are given an initial distribution $f(x)$, $0 \le x \le 1$. Can we use the method of images to find a

$$K_{\text{circle}}(x,y;t),$$

so that the solution we seek is of the form

$$u(x,t) = \int_0^1 K_{\text{circle}}(x,y;t) f(y)\, dy \tag{8.19}$$

and satisfies $u(x,0) = f(x)$ and $u(x+1,t) = u(x)$?

We use the kernel $K(x,y;t)$ and place image pulses at all the integers:

$$\begin{aligned} K_{\text{circle}}(x,y;t) &= \sum_{n=-\infty}^{\infty} K(x,y+n;t) \\ &= \frac{1}{\sqrt{4\pi t}} \sum_{n=-\infty}^{\infty} \mathrm{e}^{-(x-y-n)^2/(4t)} \end{aligned} \tag{8.20}$$

We could also proceed by the method of separation of variables: Look for solutions that are of the form

$$u(x,t) = X(x)T(t).$$

Substituting this into the one-dimensional heat equation gives

$$\frac{1}{T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2}$$

and the usual separation of variables argument concludes

$$\frac{1}{T}\frac{dT}{dt} = -k^2 \quad \text{and} \quad \frac{1}{X}\frac{d^2X}{dx^2} = -k^2$$

for some constant k. The general solution of the X is equation is a linear combination of $\mathrm{e}^{\mathrm{i}kx}$ and $\mathrm{e}^{-\mathrm{i}kx}$. We require that our solution be periodic; namely, $X(x+1) = X(x)$ (this way the function is defined on the circle). This requires

$$\mathrm{e}^{\mathrm{i}k} = 1$$

from which we conclude that $k = 2\pi n$ where $n = 0, \pm 1, \pm 2, \ldots$. The solution of the T equation is

$$T(t) = \mathrm{e}^{-k^2 t} = \mathrm{e}^{-4\pi^2 n^2 t},$$

so we've found solutions

$$u_n(x,t) = \mathrm{e}^{2\pi\mathrm{i}nx}\mathrm{e}^{-4\pi^2 n^2 t}.$$

Taking linear combinations we arrive at the general solutions

$$u(x,t) = \sum_{n=-\infty}^{\infty} a_n \mathrm{e}^{2\pi\mathrm{i}nx}\mathrm{e}^{-4\pi^2 n^2 t} \tag{8.21}$$

where a_n are constants to be determined from the initial condition. We require at $t = 0$ that $u(x,0) = f(x)$; namely,

$$f(x) = \sum_{n=-\infty}^{\infty} a_n \mathrm{e}^{2\pi\mathrm{i}nx}$$

Multiplying this last equation by $\mathrm{e}^{-2\pi\mathrm{i}mx}$ and then integrating the resulting expression from 0 to 1 gives

$$\int_0^1 \mathrm{e}^{-2\pi\mathrm{i}mx} f(x)\, dx = \sum_{n=-\infty}^{\infty} a_n \int_0^1 \mathrm{e}^{2\pi\mathrm{i}(n-m)x}\, dx = a_m \tag{8.22}$$

since $\int_0^1 \mathrm{e}^{2\pi\mathrm{i}(n-m)x}\, dx$ equals zero if $m \neq n$ and equals 1 if $m = n$.

We now use (8.22) in (8.21) to find

$$\begin{aligned} u(x,t) &= \sum_{n=-\infty}^{\infty} \left[\int_0^1 \mathrm{e}^{-2\pi\mathrm{i}ny} f(y)\, dy\right] \mathrm{e}^{2\pi\mathrm{i}nx}\mathrm{e}^{-4\pi^2 n^2 t} \\ &= \int_0^1 \left[\sum_{n=-\infty}^{\infty} \mathrm{e}^{2\pi\mathrm{i}n(x-y)}\mathrm{e}^{-4\pi^2 n^2 t}\right] f(y)\, dy \end{aligned} \tag{8.23}$$

Comparing (8.23) with (8.19) and (8.20), and noting this hold for all initial conditions $f(x)$, we conclude that

$$\begin{aligned} K_{\text{circle}}(x,y;t) &= \frac{1}{\sqrt{4\pi t}} \sum_{n=-\infty}^{\infty} \mathrm{e}^{-(x-y-n)^2/(4t)} \\ &= \sum_{n=-\infty}^{\infty} \mathrm{e}^{2\pi \mathrm{i} n(x-y)} \mathrm{e}^{-4\pi^2 n^2 t} \end{aligned}$$

Observe that for small t the first sum converges rapidly whereas the second sum converges slowly; however, for large t the first sum converges slowly and the second sum converges rapidly! For $x = y = 0$ we get the identity

$$\sum_{n=-\infty}^{\infty} \mathrm{e}^{-4\pi^2 n^2 t} = \frac{1}{\sqrt{4\pi t}} \sum_{n=-\infty}^{\infty} \mathrm{e}^{-n^2/(4t)}.$$

8.6 Exercises

#1.

Consider the partial differential equation for $u = u(x,t)$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + (1-a)\frac{\partial u}{\partial x}, \;\; t > 0, -\infty < x < \infty, \tag{8.24}$$

with initial condition $u(x,0) = f(x)$. When $a = 1$ this reduces to the heat equation. Following the method (Fourier transforms) used to solve the heat equation in §8.3, find the analogue of equations (8.7) and (8.8).

#2.

In analogy with the operator P_t defined in (8.13) for the heat equation on the line, we define for the half-line the operator Q_t by

$$\left(Q_t f\right)(x) := \int_0^\infty K_{\text{semi}}(x,y;t) f(y)\, dy, \;\; t > 0, \tag{8.25}$$

where K_{semi} is defined in (8.18). Show that Q_t satisfies the semigroup property

$$Q_{s+t} = Q_s Q_t.$$

Chapter 9

Laplace Transform

Figure 9.1: Pierre-Simon Laplace (1749–1827) was a French mathematician known for his contributions to celestial mechanics, probability theory and analysis. Both the Laplace equation and the Laplace transform are named after Pierre-Simon Laplace.

9.1 Matrix version of the method of Laplace transforms for solving constant coefficient DEs

The Laplace transform of a function $f(t)$ is

$$\boxed{F(s) = \mathcal{L}(f)(s) = \int_0^\infty e^{-ts} f(t)\, dt} \tag{9.1}$$

for $\Re(s)$ sufficiently large and positive. For the Laplace transform to make sense the function f cannot grow faster that an exponential near infinity. Thus, for example, the Laplace transform of e^{x^2} is not defined.

We extend (9.1) to vector-valued functions $f(t)$,

$$f(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix} \tag{9.2}$$

by

$$F(s) = \mathcal{L}(f)(s) = \begin{pmatrix} \int_0^\infty e^{-ts} f_1(t)\, dt \\ \int_0^\infty e^{-ts} f_2(t)\, dt \\ \vdots \\ \int_0^\infty e^{-ts} f_n(t)\, dt \end{pmatrix}. \tag{9.3}$$

Integration by parts shows that

$$\boxed{\mathcal{L}(\frac{df}{dt})(s) = s\mathcal{L}(f)(s) - f(0).} \tag{9.4}$$

We now explain how *matrix Laplace transforms* are used to solve the matrix ODE

$$\frac{dx}{dt} = Ax + f(t) \tag{9.5}$$

where A is a constant coefficient $n \times n$ matrix, $f(t)$ is a vector-valued function of the independent variable t ("forcing term") with initial condition

$$x(0) = x_0. \tag{9.6}$$

First, we take the Laplace transform of both sides of (9.5). From (9.4) we see that the Laplace transform of the LHS of (9.5) is

$$\mathcal{L}(\frac{dx}{dt}) = s\mathcal{L}(x) - x_0.$$

The Laplace transform of the RHS of (9.5) is

$$\begin{aligned}\mathcal{L}(Ax+f) &= \mathcal{L}(Ax)+\mathcal{L}(f)\\ &= A\mathcal{L}(x)+F(s)\end{aligned}$$

where we set $F(s)=\mathcal{L}(f)(s)$ and we used the fact that A is independent of t to conclude[1]

$$\mathcal{L}(Ax)=A\mathcal{L}(x). \tag{9.7}$$

Thus the Laplace transform of (9.5) is

$$s\mathcal{L}(x)-x_0=A\mathcal{L}(x)+F,$$

or

$$(sI_n-A)\mathcal{L}(x)=x_0+F(s) \tag{9.8}$$

where I_n is the $n\times n$ identity matrix. Equation (9.8) is a linear system of algebraic equations for $\mathcal{L}(x)$. We now proceed to solve (9.8). This can be done once we know that (sI_n-A) is invertible. Recall that a matrix is invertible if and only if the determinant of the matrix is nonzero. The determinant of the matrix in question is

$$p(s):=\det(sI_n-A), \tag{9.9}$$

which is the characteristic polynomial of the matrix A. We know that the zeros of $p(s)$ are the eigenvalues of A. If s is larger than the absolute value of the largest eigenvalue of A; in symbols,

$$s>\max|\lambda_i|, \tag{9.10}$$

then $p(s)$ cannot vanish and hence $(sI_n-A)^{-1}$ exists. We assume s satisfies this condition. Then multiplying both sides of (9.8) by $(sI_n-A)^{-1}$ results in

$$\boxed{\mathcal{L}(x)(s)=(sI_n-A)^{-1}x_0+(sI_n-A)^{-1}F(s).} \tag{9.11}$$

Equation (9.11) is the basic result in the application of Laplace transforms to the solution of constant coefficient differential equations with an inhomogeneous forcing term. Equation (9.11) will be a quick way to solve initial value problems once we learn efficient methods to (i) compute $(sI_n-A)^{-1}$, (ii) compute the Laplace transform of various forcing terms $F(s)=\mathcal{L}(f)(s)$, and (iii) find the inverse Laplace transform. Step (i) is easier if one uses software packages such as MatLab . Steps (ii) and(iii) are made easier by the use of extensive Laplace transform tables or symbolic integration packages such as Mathematica. It should be noted that many of the DE techniques one learns in engineering courses can be described as efficient methods to do these three steps for examples that are of interest to engineers.

We now give two examples that apply (9.11).

[1] You are asked to prove (9.7) in an exercise.

9.1.1 Example 1

Consider the scalar ODE

$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t) \tag{9.12}$$

where b and c are constants. We first rewrite this as a system

$$x(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$$

so that

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ -c & -b \end{pmatrix} x + \begin{pmatrix} 0 \\ f(t) \end{pmatrix}.$$

Then

$$sI_2 - A = \begin{pmatrix} s & -1 \\ c & s+b \end{pmatrix},$$

and

$$(sI_2 - A)^{-1} = \frac{1}{s^2+bs+c}\begin{pmatrix} s+b & 1 \\ -c & s \end{pmatrix}.$$

Observe that the characteristic polynomial

$$p(s) = \det(sI_2 - A) = s^2 + bs + c$$

appears in the denominator of the matrix elements of $(sI_2 - A)^{-1}$. (This factor in Laplace transforms should be familiar from the scalar treatment—here we see it is the characteristic polynomial of A.) By (9.11)

$$\mathcal{L}(x)(s) = \frac{1}{s^2+bs+c}\begin{pmatrix} (s+b)y(0) + y'(0) \\ -cy(0) + sy'(0) \end{pmatrix} + \frac{F(s)}{s^2+bs+c}\begin{pmatrix} 1 \\ s \end{pmatrix}$$

where $F(s) = \mathcal{L}(f)(s)$. This implies that the Laplace transform of $y(t)$ is given by

$$\mathcal{L}(y)(s) = \frac{(s+b)y(0) + y'(0)}{s^2+bs+c} + \frac{F(s)}{s^2+bs+c}. \tag{9.13}$$

This derivation of (9.13) may be compared with the derivation of equation (16) on page 302 of Boyce and DiPrima [4] (in our example $a = 1$).

9.1.2 Example 2

We consider the system (9.5) for the special case of $n = 3$ with $f(t) = 0$ and A given by

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 1 & -1 & -1 \end{pmatrix}. \tag{9.14}$$

The characteristic polynomial of (9.14) is

$$p(s) = s^3 - 2s^2 + s - 2 = (s^2+1)(s-2) \tag{9.15}$$

and so the matrix A has eigenvalues $\pm i$ and 2. A rather long linear algebra computation shows that

$$(sI_3 - A)^{-1} = \frac{1}{p(s)} \begin{pmatrix} s^2 - s - 1 & 1 & -s+2 \\ s+2 & s^2 & s-2 \\ s-3 & -s+1 & s^2-3s+2 \end{pmatrix}. \tag{9.16}$$

If one writes a partial fraction decomposition of each of the matrix elements appearing in (9.16) and collects together terms with like denominators, then (9.16) can be written as

$$\begin{aligned} (sI_3 - A)^{-1} &= \frac{1}{s-2} \begin{pmatrix} 1/5 & 1/5 & 0 \\ 4/5 & 4/5 & 0 \\ -1/5 & -1/5 & 0 \end{pmatrix} \\ &\quad + \frac{1}{s^2+1} \begin{pmatrix} (3+4s)/5 & -(2+s)/5 & -1 \\ -(3+4s)/5 & (2+s)/5 & 1 \\ (7+s)/5 & (-3+s)/5 & -1+s \end{pmatrix}. \end{aligned} \tag{9.17}$$

We now apply (9.17) to solve (9.5) with the above A and $f = 0$ for the case of initial conditions

$$x_0 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}. \tag{9.18}$$

We find

$$\mathcal{L}(x)(s) = (sI_3 - A)^{-1} x_0 = \frac{1}{s-2} \begin{pmatrix} -1/5 \\ -4/5 \\ 1/5 \end{pmatrix} + \frac{s}{s^2+1} \begin{pmatrix} 6/5 \\ -6/5 \\ 4/5 \end{pmatrix} + \frac{1}{s^2+1} \begin{pmatrix} 2/5 \\ -2/5 \\ 8/5 \end{pmatrix}. \tag{9.19}$$

To find $x(t)$ from (9.19) we use Table 6.2.1 on page 300 of Boyce and DiPrima [4]; in particular, entries 2, 5, and 6. Thus

$$x(t) = e^{2t} \begin{pmatrix} -1/5 \\ -4/5 \\ 1/5 \end{pmatrix} + \cos t \begin{pmatrix} 6/5 \\ -6/5 \\ 4/5 \end{pmatrix} + \sin t \begin{pmatrix} 2/5 \\ -2/5 \\ 8/5 \end{pmatrix}.$$

One can also use MATHEMATICA to compute the inverse Laplace transforms. To do so use the command `InverseLaplaceTransform`. For example if one inputs `InverseLaplaceTransform[1/(s-2),s,t]` then the output is e^{2t}.

We give now a second derivation of (9.19) using the eigenvectors of A. As noted above, the eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = i$, and $\lambda_3 = -i$. If we denote by ϕ_j an eigenvector associated to eigenvalue λ_j ($j = 1, 2, 3$), then a routine linear algebra computation gives the following possible choices for the ϕ_j:

$$\phi_1 = \begin{pmatrix} -1 \\ -4 \\ 1 \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} (1+i)/2 \\ -(1+i)/2 \\ 1 \end{pmatrix}, \quad \phi_3 = \begin{pmatrix} (1-i)/2 \\ (-1+i)/2 \\ 1 \end{pmatrix}.$$

Now for any eigenvector ϕ corresponding to eigenvalue λ of a matrix A we have

$$(sI_n - A)^{-1}\phi = (s - \lambda)^{-1}\phi.$$

To use this observation we first write

$$x_0 = c_1\phi_1 + c_2\phi_2 + c_3\phi_3.$$

A computation shows that

$$c_1 = 1/5, \quad c_2 = 2/5 - 4i/5, \quad \text{and} \quad c_3 = 2/5 + 4i/5.$$

Thus

$$(sI_3 - A)^{-1}x_0 = \frac{1}{5}(s-2)^{-1}\phi_1 + \frac{2-4i}{5}(s-i)^{-1}\phi_2 + \frac{2+4i}{5}(s+i)^{-1}\phi_3.$$

Combining the last two terms gives (9.19).

9.2 Structure of $(sI_n - A)^{-1}$ when A is diagonalizable

In this section we assume that the matrix A is diagonalizable; that is, we assume a set of linearly independent eigenvectors of A form a basis. Recall the following two theorems from linear algebra: (1) If the $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable; and (2) If the matrix A is symmetric (hermitian if the entries are complex), then A is diagonalizable.

Since A is assumed to be diagonalizable, there exists a nonsingular matrix P such that

$$A = PDP^{-1}$$

where D is

$$D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

and each eigenvalue λ_i of A appears as many times as the (algebraic) multiplicity of λ_i. Thus

$$\begin{aligned} sI_n - A &= sI_n - PDP^{-1} \\ &= P(sI_n - D)P^{-1}, \end{aligned}$$

so that

$$\begin{aligned} (sI_n - A)^{-1} &= \left(P(sI_n - D)P^{-1})\right)^{-1} \\ &= P(sI_n - D)^{-1}P^{-1}. \end{aligned}$$

Since P and P^{-1} are independent of s, the s dependence of $(sI_n - A)^{-1}$ resides in the diagonal matrix $(sI_n - D)^{-1}$. This tells us that the partial fraction decomposition of the matrix $(sI_n - A)^{-1}$ is of the form

$$(sI_n - A)^{-1} = \sum_{j=1}^{n} \frac{1}{s - \lambda_j} P_j$$

where

$$P_j = PE_jP^{-1}$$

and E_j is the diagonal matrix with all zeros on the main diagonal except for 1 at the (j, j)th entry. This follows from the fact that

$$(sI_n - D)^{-1} = \sum_{j=1}^{n} \frac{1}{s - \lambda_j} E_j$$

Note that P_j have the property that

$$P_j^2 = P_j.$$

Such matrices are called *projection operators*.

In general, it follows from Cramer's method of computing the inverse of a matrix, that the general structure of $(sI_n - A)^{-1}$ will be $1/p(s)$ times a matrix whose entries are polynomials of at most degree $n - 1$ in s. When an eigenvalue, say λ_1, is degenerate and of (algebraic) multiplicity m_1, then the characteristic polynomial will have a factor $(s - \lambda_1)^{m_1}$. We have seen that if the matrix is diagonalizable, upon a partial fraction decomposition only a single power of $(s - \lambda_1)$ will appear in the denominator of the partial fraction decompostion. Finally, we conclude by mentioning that when the matrix A is not diagonalizable, then this is reflected in the partial fraction decomposition of $(sI_n - A)^{-1}$ in that some powers of $(s - \lambda_j)$ occur to a higher degree than 1.

9.3 Exercises

#1.

Use the Laplace transform to find the solution of the initial value problem

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ 12 \\ 0 \end{pmatrix}, \quad \mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

#2.

Let A be a $n \times n$ matrix whose entries are real numbers and $x \in \mathbf{R}^n$. Prove that

$$\mathcal{L}(Ax) = A\mathcal{L}(x)$$

where $\mathcal{L}$ denotes the Laplace transform.

#3.

Let E_j denote the diagonal $n \times n$ matrix with all zeros on the main diagonal except for 1 at the (j, j) entry.

- Prove that $E_j^2 = E_j$.
- Show that if P is any invertible $n \times n$ matrix, then $P_j^2 = P_j$ where $P_j := PE_jP^{-1}$.

#4.

It is a fact that you will learn in an advanced linear algebra course, that *if* a 2×2 matrix A is *not* diagonalizable, then there exists a nonsingular matrix P such that

$$A = P\,B\,P^{-1}$$

where

$$B = \left(\begin{array}{cc} \lambda & 1 \\ 0 & \lambda \end{array} \right)$$

for some constant λ.

- Show that λ must be an eigenvalue of A with algebraic multiplicity 2.
- Find an eigenvector of A (in terms of the matrix P), and show that A has no other eigenvectors (except, of course, scalar multiples of the vector you have already found).
- Show that
$$(sI_2 - A)^{-1} = \frac{1}{s-\lambda} PE_1P^{-1} + \frac{1}{s-\lambda} PE_2P^{-1} + \frac{1}{(s-\lambda)^2} PNP^{-1}$$
where
$$N = \left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right).$$
- Relate what is said here to the remarks in the footnote in Exercise 5.5.2.

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