

A Unifying Theory for Scaling Laws of Human Populations

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The spatial distribution of people exhibits clustering across a wide range of scales, from household ($\sim 10^{-2}$ km) to continental ($\sim 10^4$ km) scales. Empirical data indicates simple power-law scalings for the size distribution of cities (known as Zipf's law¹), the geographic distribution of friends², and the population density fluctuations as a function of scale. We derive a simple statistical model that explains all of these scaling laws based on a single unifying principle involving the random spatial growth of clusters of people on all scales. The model makes important new predictions for the spread of diseases and other social phenomena.

Human populations exhibit remarkably simple properties given the complexity of socioeconomic interactions between humans and their environments. One such example is the well known *Zipf's law*¹ for cities: the rank of a city is inversely proportional to the number of people who live in the city. If the most populous city in the United States has a population of $N_{\text{max,US}} \sim 8 \times 10^6$, the second most populous city will have a population of $\frac{1}{2}N_{\text{max,US}} \sim 4 \times 10^6$, the third $\frac{1}{3}N_{\text{max,US}} \sim 2.7 \times 10^6$, and so forth. This simple relation fits empirical data extremely well^{3,4}. A mathematically equivalent formulation of Zipf's law is that the underlying distribution of cities follows a power law⁵; namely, the probability that a city has a population N scales as $1/N^2$.

Another remarkable scaling law^{6,7} states that the probability a given person A befriends another person B is inversely proportional to the number of people $N_{B,A}$ geographically closer to A than B . $N_{B,A}$ is commonly referred to as the rank of B , so we will refer to this relationship as the *inverse-rank friendship law*. In the limiting case where people are distributed uniformly over a 2-dimensional surface, the probability will scale as $\sim 1/r^2$ where r is the separation between people. The general relation, which leads to a complicated spatial dependence when clustering is present, appears to be valid at least on the scale of towns and cities⁶.

On the theoretical side, recent work² used the inverse-rank friendship law to explain a variety of urban characteristics such as the length of phone calls made, the spread of sexually transmitted diseases, and economic activity, but did not comment on the theoretical origin of the law. On the other hand, the remarkable simplicity and empirical success of Zipf's law have attracted significant theoretical attention and debate^{3,8,9}, though there is no consensus on the origin of Zipf's law. Existing work treats cities as the fundamental entities of the theory, with population as a property of each city. Our approach is conceptually different: we treat the population density as the fundamental quantity, thinking of cities as objects that form when the population density exceeds a critical threshold. The situation is therefore conceptually and mathematically analogous to the formation of galaxies in the universe, where non-linear gravitational collapse occurs when the matter density exceed some critical value. Our conceptual advance here is also a practical one, since we can apply the mathematical tools developed in cosmology¹⁰ to the problem at hand.

To start, consider the human population density ρ as a function on \mathbb{R}^2 , the 2D Euclidean

plane. Since we will be interested in regions much smaller in size than the radius of the Earth, we will ignore the effects of curvature. The fluctuations relative to the average population density $\delta(\mathbf{x}) \equiv [(\rho(\mathbf{x})/\bar{\rho}) - 1]$ can be expanded in Fourier modes $\delta(\mathbf{x}) = \int d^2k \delta_{\mathbf{k}} e^{-i\mathbf{k}\mathbf{x}}/(2\pi)^2$. The amplitude of the population density fluctuations with wavenumber k (or an associated length scale $\sim 1/k$) is given by the spherically averaged power spectrum $P(k)$, defined by $\langle \delta_{\mathbf{k}} \delta_{\mathbf{k}'}^* \rangle = (2\pi)^2 \delta_D^2(\mathbf{k} - \mathbf{k}') P(k)$, where δ_D is the Dirac delta function (not to be confused with the fractional over-density δ).

It is conventional to define a dimensionless power spectrum in the number density $\Delta^2(k) \equiv k^2 P(k)/(2\pi)$, which represents the typical (squared) fractional over-density of people $(\delta\rho/\rho)^2$ on the spatial scale $\sim 1/k$. To make further progress, we must fix the functional form of $\Delta(k)$ by some theoretical principle. To this end, consider an over-density of size $\sim 1/k$. At a discrete time step, this over-density might grow or shrink in spatial coverage. As a concrete example, consider a collection of farms (with a characteristic population density of a few people per typical farm area) in otherwise relatively uninhabited countryside. At each time step, a farm could be added or destroyed. Therefore, the spatial size of the over-density might grow or shrink, while $\delta\rho/\rho$ (a number associated with farms) will be held constant. More precisely, we define a monotonically decreasing function $X(k)$ such that $\lim_{k \rightarrow \infty} X = 0$, which quantifies the spatial extent of an over-density. This function might represent the area of the over-density $X(k) \propto 1/k^2$ or its perimeter $X(k) \propto 1/k$, but our derivation will not depend on the detailed form of X . We can then perform a change of variables and view $\Delta(k)$ as a function of X : $\Delta(X(k)) = \Delta(k)$. The unifying principle is that all over-densities can grow or shrink spatially, executing a random walk in X . This process can

continue until the overdensity disappears ($X = 0$), or the over-density takes up some maximum X_{\max} , where $X_{\max} \equiv X(k_{\min})$ is set by the continental length scale $\sim 1/k_{\min}$. For a large ensemble of over-densities, this is a diffusion-like process with reflecting boundary conditions obeying

$$\frac{\partial \Delta}{\partial t} = D \frac{\partial^2 \Delta}{\partial X^2} \quad (1)$$

with some diffusion constant D . Any initial conditions will relax to the steady-state solution $\Delta(X) \rightarrow \text{constant}$ for $0 \leq X \leq X_{\max}$ on a timescale $T_{\text{relax}} \sim X_{\max}^2/D$. We intuitively expect T_{relax} to be reasonably short, since the geographic mobility timescale of ~ 5 yrs (in the United States, $\sim 35\%$ of people change residences within 5 years¹¹) is considerably shorter than, say, the population growth timescale ~ 30 yrs set by the typical age of parenthood. Any initial conditions set by antiquity or perturbations to the system (e.g. catastrophic events that displace many people) should be quickly erased. We therefore predict that on sufficiently long timescales,

$$P(k) \propto k^{-2}. \quad (2)$$

We test this prediction in Figure 1 against publicly available data^a from the Center for International Earth Science Information Network (CIESIN) and Centro Internacional de Agricultura Tropical (CIAT)¹². The results are in good agreement with the theoretical prediction across a broad range of spatial scales, from a few km to $\sim 10^3$ km.

Given a power spectrum $P(k)$, it is possible to calculate the number of cities as a function of their population N . To perform this calculation, we employ the Press-Schechter (PS) formalism¹⁰, traditionally used in the context of cosmology to predict the abundance of gravitationally-bound

^aURL: <http://sedac.ciesin.columbia.edu/gpw>

objects given a power spectrum of the fluctuations in the cosmic matter density. The excursion set formalism¹³ provides a more rigorous derivation, but the PS formalism has the benefit of simplicity. The end result is identical in either case.

We picture cities of area A as discrete objects which form when the population density as a function of spatial coordinates $\rho(\mathbf{x})$, or equivalently $\delta(\mathbf{x})$, averaged over an area A surpasses a critical threshold,^b δ_C . To this end, we define a smoothed density field $\delta_A(\mathbf{x}) = \int d^2k W_A(k) \delta_{\mathbf{k}} e^{-i\mathbf{k}\mathbf{x}} / (2\pi)^2$, where the low-pass window function $W_A(k) = 1$ if $k \leq 1/\sqrt{A}$ and $W_A(k) = 0$ otherwise. For a fixed \mathbf{x} , we model $\delta_A(\mathbf{x})$ a random variable with probability distribution $p_A(\delta_A)$. The key insight of the PS formalism is to identify the fraction f_A of people living in cities of area A or larger with the cumulative probability $f = 2 \int_{\delta_C}^{\infty} p_A d\delta$. This is illustrated in Figure 2.

To make further progress, we must assume something about the functional form of p_A . The conventional PS formalism assumes that p_A is a Gaussian with mean 0 and variance $\sigma^2(A) \equiv \int_{k_{\min}}^{1/\sqrt{A}} dk k P(k) / (2\pi) \propto \ln k/k_{\min}$; our results here will be more general. In particular, we only assume that p_A has a universal shape for all A . Since the mean of δ_A is zero for all A by definition, and since for any random variable θ the associated standard deviation obeys $\sigma_{a\theta} = a\sigma_\theta$, this allows us to write $p_A(\delta) = g(\delta/\sigma(A))/\sigma(A)$ for some general probability density function g . Differentiating f_A yields $n(N)$, the number of cities on Earth's surface with population $N = \bar{\rho}A$

^bThis assumption can be relaxed, allowing for the average population density of a city to vary systematically with size. In our model, this corresponds to a critical threshold that varies with A . In this case, the excursion set formalism can be used with a moving barrier¹⁴. We will ignore this subtlety, since Zipf's law is still obtained in the limit that $\delta_C \ll \sigma$.

per unit area per unit population:

$$n(N) = -\nu g(\nu) \frac{\rho}{N} \frac{d \ln \sigma}{dN} \propto \frac{1}{N^2} \frac{g(\nu)}{\ln(N_{\max}/N)}, \quad (3)$$

where we have defined $\nu \equiv \delta_C/\sigma(N)$, the number of standard deviations associated with city formation. Note that for $\nu \ll 1$, $g(\nu)$ is a slowly varying function of N for two reasons: the first derivative of g around $\nu = 0$ is small for small deviations from the mode, and ν is only a weak function of N . Thus, equation (3) implies that the logarithmic slope $d \log n / d \log N$ tends to -2 in the limit of $N \ll N_{\max}$. This limit is empirically justified, since even the largest cities in the world contain only $\sim 10^{-3}$ of the world's population. Thus the number of cities above a certain population threshold has a logarithmic slope of -1 . This statement is equivalent to Zipf's law: the rank of a city is inversely proportional to its size.

As a second application of our formalism, let us calculate the average number of friends a person has in a given region. We again adopt a very simplistic model, where “communities” forms if the population density exceeds some critical value $\delta \geq \delta_f$, and two people are friends if they both are members of the same community. We know from the previous discussions that in such a model, the probability distribution of community sizes p_c scales $\propto 1/N^2$. Let us now consider an arbitrary person in a region with a total of N_R people. A randomly chosen person will have on average N_f friends given by

$$N_f = \frac{1}{1 - \int_{N_f}^{\infty} p_c(N) dN} \int_1^{N_R} N p_c(N) dN \propto \log N_R + \mathcal{O}\left(\frac{\log N_R}{N_R}\right) \quad (4)$$

For large N_R , the dependence is logarithmic in N_R . Interpreting the derivative as the probability that a person will acquire a friend, we have thus derived the inverse-rank friendship law

$dN_f/dN_R \propto 1/N_R$, previously proposed⁶ to fit empirical data. We stress that our derivation is based entirely on theoretical considerations with minimal input from empirical data.

As a final application of our formalism, we consider the spread of disease. In particular, we will be interested in epidemics: if the average person infects more than one person before recovering, the disease will spread uncontrollably. This has long been understood using the language of random graph theory. The size of the largest connected component of a random graph in the Erdos-Renyi model diverges when the average number of edges that each node has reaches unity¹⁵.

We extend this idea to a realistic population distribution, which is not easily modeled by a random graph. To be concrete, we will focus on the venerable susceptible-infectious-recovery (SIR) model¹⁶ of disease (see Ref. [17] for a review). We split the population into three categories: susceptible S , infectious I , and recovered R . In our formalism, there is an associated density with these populations such that $S(\mathbf{x}) + I(\mathbf{x}) + R(\mathbf{x}) = \rho(\mathbf{x})$. The SIR model is given by $\dot{S} = -\beta SI$, $\dot{I} = (\beta S - \gamma)I$, and $\dot{R} = \gamma I$ where β and γ determine the rates of infection and recovery, respectively. For an epidemic to occur, one needs $\beta S - \gamma > 0$. If the entire population starts out as susceptible, this defines a critical density at which epidemics occur: $\rho_e = \gamma/\beta$ or equivalently $\delta_e \equiv \rho_e/\bar{\rho} - 1$. Therefore, we can predict the distribution of epidemics using the PS formalism. The answer is simply equation (3) with δ_C replaced by δ_e . However, the interpretation of equation (3) in this context is very different.

Equation (3) can be used as a starting point for addressing many non-trivial questions about epidemics. For example, if a person is chosen at random and infected, what is the probability

$p_I(N)$ that the disease will eventually infect N people? The probability must be proportional to the number of people that are in a potential epidemic cluster of size N , so $p_I(N) \propto N \times n_e(N) \propto 1/N$ for $N \ll N_{\max}$ where $n_e(N)$ is the number density of epidemic clusters. We note that any social behavior that requires a community of people (for example, organized crime or the spread of ideas) may be modeled within our framework.

In summary, we have presented a derivation of two known laws, Zipf's law and the inverse-rank friendship law, and successfully predicted the power spectrum $P(k)$ of population density fluctuations in the continental US. These derivations stemmed from the simple, unifying principle of random spatial growth of clusters of people. Remarkably, there is a wide range of possible models for $X(k)$ and an even wider range of initial conditions to which our results are insensitive. This is an appealing feature, enabling us to forgo any fine-tuning arguments in explaining the empirical data. Just as the development of models for non-linear structure formation in the universe led to a wealth of theoretical and observational work in cosmology^{18,19}, future work here could include the calculation of new observables such as the bias factor²⁰ for the spread of epidemics and refinements to the detailed theory of clustering based on additional data.

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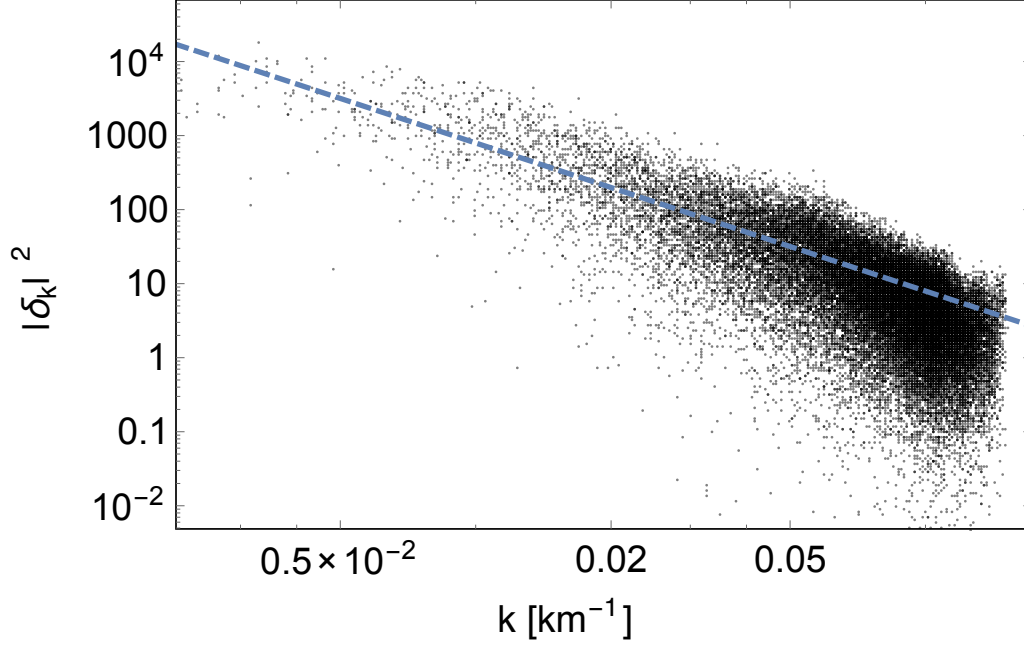


Figure 1: Empirically measured power spectrum $P(k) \propto |\delta_k|^2$ of population density fluctuations as a function of the spatial wavenumber k . The blue dashed line depicts the predicted slope $P(k) \propto k^{-2}$ with an arbitrary normalization. The data was obtained by taking the discrete Fourier transform of a 1000×1000 arcmin² map of the population density of a section of the continental United States. The area was selected to minimize artifacts due to boundary conditions defined by lakes and oceans.

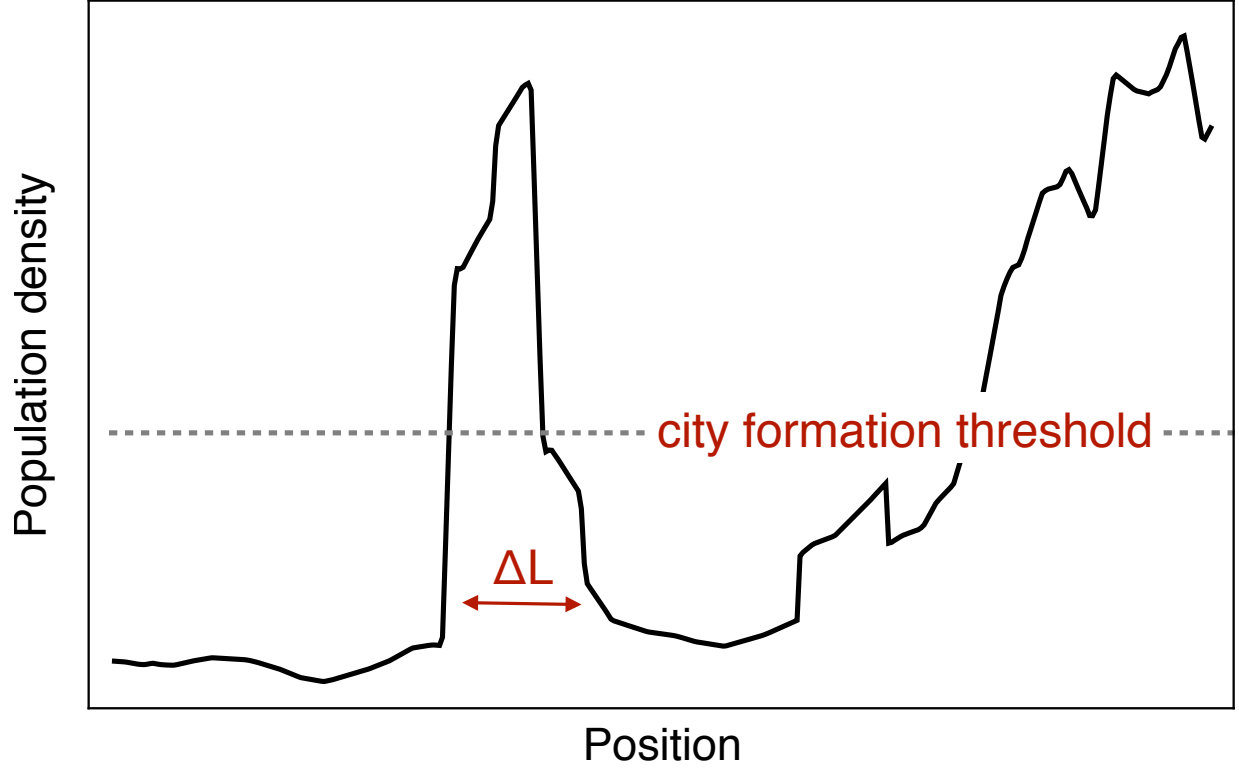


Figure 2: Illustration of our approach. Plotted vertically is the smoothed spatial population density over a strip of the continental United States. A region of length ΔL where the average population density is greater than some threshold value is declared a city of size ΔL . Given the power spectrum of population density fluctuations $P(k)$, we analytically compute the number distribution of cities. We use a similar approach for calculating the sizes of “friendship” communities and the size distribution of epidemics.