Analytics Software Tools

Optimization Module

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Overview

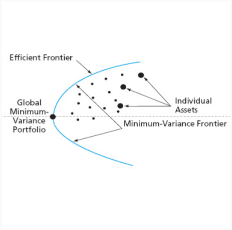
1. Introduction to Optimization
    - General Optimization
    - Linear Optimization
2. Overview of Optimization in Practice
    - Optimization Solvers
    - Algebraic Modeling Languages
3. Intro to Julia
4. Intro to JuMP
5. Guide to MIO modeling and solvers

Introduction to Optimization

What is Optimization?

Optimization is a powerful tool to solve many problems in the real world:

- Economics and finance (portfolio optimization)
- Artificial intelligence (self-driving cars)
- Civil engineering (infrastructure planning)
- Health care (operating room scheduling)



What is Optimization?

Mathematically, optimization can be defined as the following:

Maximize (or minimize) the **objective function** $f(\mathbf{x})$
by choosing the **decision variables** $\mathbf{x}$,
subject to some **constraints** $g(\mathbf{x}) \leq \mathbf{b}$.

What is Optimization?

A typical formulation looks like this:

$$\begin{aligned} \max \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & g(\mathbf{x}) \leq \mathbf{b} \\ & x_i \in \mathbb{R} \qquad \forall i = 1, 2, \ldots, n \end{aligned}$$

What is Optimization?

Or could look like this:

$$
\begin{aligned}
\min_{r,s,\theta,\mathbf{y}_1,\mathbf{y}_2,\mathbf{Z}} \quad & r+s \qquad\qquad \text{(D.13)}\\
s.t. \quad & \begin{pmatrix} r+\mathbf{y}_1^{+T}\hat{\mathbf{u}}^{(0)}-\mathbf{y}_1^{-T}\hat{\mathbf{u}}^{(N+1)} & \frac{1}{2}(\mathbf{q}-\mathbf{y}_1)^T, \\ \frac{1}{2}(\mathbf{q}-\mathbf{y}_1) & \mathbf{Z} \end{pmatrix} \succeq \mathbf{0},\\
& \begin{pmatrix} r+\mathbf{y}_2^{+T}\hat{\mathbf{u}}^{(0)}-\mathbf{y}_2^{-T}\hat{\mathbf{u}}^{(N+1)}+\theta t-1 & \frac{1}{2}(\mathbf{q}-\mathbf{y}_2-\theta\mathbf{v})^T, \\ \frac{1}{2}(\mathbf{q}-\mathbf{y}_2-\theta\mathbf{v}) & \mathbf{Z} \end{pmatrix} \succeq \mathbf{0},\\
& s \geq (\gamma_2^B\hat{\boldsymbol{\Sigma}}+\hat{\boldsymbol{\mu}}\hat{\boldsymbol{\mu}}^T)\circ\mathbf{Z}+\hat{\boldsymbol{\mu}}^T\mathbf{q}+\sqrt{\gamma_1^B}\|\mathbf{q}+2\mathbf{Z}\hat{\boldsymbol{\mu}}\|_{\hat{\boldsymbol{\Sigma}}^{-1}},\\
& \mathbf{y}_1=\mathbf{y}_1^+-\mathbf{y}_1^-,\quad \mathbf{y}_2=\mathbf{y}_2^+-\mathbf{y}_2^-,\quad \mathbf{y}_1^+,\mathbf{y}_1^-,\mathbf{y}_2^+,\mathbf{y}_2^-\theta\geq\mathbf{0}.
\end{aligned}
$$

Reference: Nathan Kallus (Operations Research Center Ph.D. '15) thesis

What is Optimization?

Or could look like this:

$$
\begin{array}{lll}
\min & \frac{1}{\hat{L}} \sum_{t \in \mathcal{T}_L} L_t + \alpha \cdot \sum_{t \in \mathcal{T}_B} \sum_{j=1}^{p} s_{jt} & \qquad (28) \\
\text{s.t.} & L_t \geq N_t - N_{kt} - n(1 - c_{kt}), & k = 1, \dots, K, \ \forall t \in \mathcal{T}_L, \\
& L_t \leq N_t - N_{kt} + n c_{kt}, & k = 1, \dots, K, \ \forall t \in \mathcal{T}_L, \\
& L_t \geq 0, & \forall t \in \mathcal{T}_L, \\
& N_{kt} = \frac{1}{2} \sum_{i=1}^{n} (1 + Y_{ik}) z_{it}, & k = 1, \dots, K, \ \forall t \in \mathcal{T}_L, \\
& N_t = \sum_{i=1}^{n} z_{it}, & \forall t \in \mathcal{T}_L, \\
& \sum_{k=1}^{K} c_{kt} = l_t, & \forall t \in \mathcal{T}_L, \\
& \mathbf{a}_m^T \mathbf{x}_i + \mu \leq b_m + (2 + \mu)(1 - z_{it}), & i = 1, \dots, n, \ \forall t \in \mathcal{T}_B, \ \forall m \in A_L(t), \\
& \mathbf{a}_m^T \mathbf{x}_i \geq b_m - 2(1 - z_{it}), & i = 1, \dots, n, \ \forall t \in \mathcal{T}_B, \ \forall m \in A_R(t), \\
& \sum_{t \in \mathcal{T}_L} z_{it} = 1, & i = 1, \dots, n, \\
& z_{it} \leq l_t, & \forall t \in \mathcal{T}_L, \\
& \sum_{i=1}^{n} z_{it} \geq N_{\min} l_t, & \forall t \in \mathcal{T}_L, \\
& \sum_{j=1}^{p} \hat{a}_{jt} \leq d_t, & \forall t \in \mathcal{T}_B, \\
& \hat{a}_{jt} \geq a_{jt}, & j = 1, \dots, p, \ \forall t \in \mathcal{T}_B, \\
& \hat{a}_{jt} \geq -a_{jt}, & j = 1, \dots, p, \ \forall t \in \mathcal{T}_B, \\
& -s_{jt} \leq a_{jt} \leq s_{jt}, & j = 1, \dots, p, \ \forall t \in \mathcal{T}_B \\
& s_{jt} \leq d_t, & j = 1, \dots, p, \ \forall t \in \mathcal{T}_B, \\
& \sum_{j=1}^{p} s_{jt} \geq d_t, & \forall t \in \mathcal{T}_B, \\
& -d_t \leq b_t \leq d_t, & \forall t \in \mathcal{T}_B, \\
& d_t \leq d_{p(t)}, & \forall t \in \mathcal{T}_B \setminus \{1\}, \\
& z_{it}, l_t \in \{0, 1\}, & i = 1, \dots, n, \ \forall t \in \mathcal{T}_L, \\
& d_t \in \{0, 1\}, & j = 1, \dots, p, \ \forall t \in \mathcal{T}_B.
\end{array}
$$

Reference: Dimitris Bertsimas and Jack Dunn. *Optimal classification trees.* Machine Learning (2017): 1-44.

Why Optimization?

It is widely used in these application areas because:

- a natural way to model the world
- extensive flexibility in modeling choices
- tractability and scalability of modern solvers
- Growing interests in the application of ML and AI
- Industry shift of focus from predictive to prescriptive

Linear Optimization

Linear optimization in particular is a simple yet still powerful type. The objective function and the constraints are *linear* in the decision variables.

Here is a general formulation:

$$\begin{aligned}
\min_{x} \quad & c_1x_1 + c_2x_2 + \ldots + c_nx_n \\
\text{s.t.} \quad & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \geq b_1 \\
& a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \geq b_2 \\
& \ldots \\
& a_{k1}x_1 + a_{k2}x_2 + \ldots + a_{kn}x_n \geq b_k
\end{aligned}$$

Or in matrix notation,

$$\begin{aligned}
\min \quad & \mathbf{c}'\mathbf{x} \\
\text{s.t.} \quad & \mathbf{A}\mathbf{x} \geq \mathbf{b}
\end{aligned}$$

Linear Optimization

Example: minimizing absolute sum of residuals (similar to least square regression). Given data **x** and **y**:

$$\min_{\boldsymbol{\beta}} \quad \sum_{i=1}^{n} |y_i - \boldsymbol{\beta}^T \mathbf{x}_i|$$

We can use a trick to linearize the absolute terms:

$$\begin{aligned} \min_{\boldsymbol{\beta}} \quad & \sum_{i=1}^{n} z_i \\ \text{s.t.} \quad & y_i - \boldsymbol{\beta}^T \mathbf{x}_i \le z_i \qquad \forall i = 1, \ldots, n \\ & -y_i + \boldsymbol{\beta}^T \mathbf{x}_i \le z_i \quad \forall i = 1, \ldots, n \end{aligned}$$

Now you just need to solve it!

Algorithm for Linear Optimization

Many algorithms, with different performance, exist to date:

1. the *simplex method* developed in 1947
2. the *ellipsoid method* by Shor (1970), Yudin and Nemirovskii (1977)
3. the *interior point method* by Karmarkar (1984)
4. ...

Optimization Workflow

1. Start with a question
2. Formulate the optimization problem
3. Collect data
4. **Solve it using an algorithm**
5. Evaluate the solutions and iterate

Topic of next section!

Overview of Optimization in Practice

Solving an optimization problem

We've analyzed a real-world problem, collected data, and formulated an optimization model that describes the scenario

How do we solve the problem? Should we just code up the simplex method from a textbook and pass in our problem? ✗

Real-world optimization is fraught with difficulties that don't exist in a textbook:

- Floating-point errors
- Ill-conditioned matrices and numerical instability
- Modifications to algorithms to speed things up or work around problems (e.g. degeneracy cycling)

Leave writing the optimization algorithms to the experts!

Use an off-the-shelf **optimization solver**

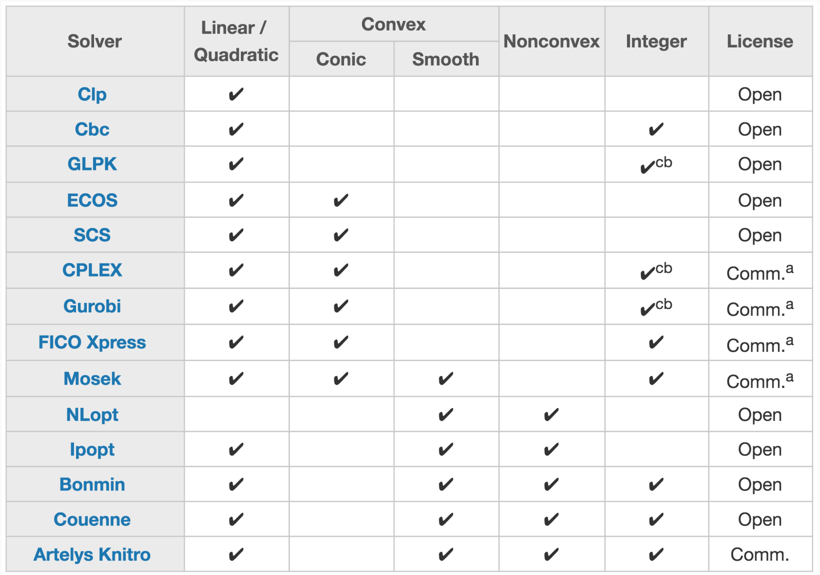
Types of optimization problem

- **Linear:** everything is linear/affine
- **Quadratic:** quadratic objective and constraints
- **Conic:** variables and constraints lie in cones
- **Convex:** constraints and objective are convex
- **Nonlinear:** everything else...

- **Integer:** some variables restricted to integer values

Optimization solvers

www.juliaopt.org



Solver	Linear / Quadratic	Convex		Nonconvex	Integer	License
		Conic	Smooth			
Clp	✔					Open
Cbc	✔				✔	Open
GLPK	✔				✔[cb]	Open
ECOS	✔	✔				Open
SCS	✔	✔				Open
CPLEX	✔	✔			✔[cb]	Comm.[a]
Gurobi	✔	✔			✔[cb]	Comm.[a]
FICO Xpress	✔	✔			✔	Comm.[a]
Mosek	✔	✔	✔		✔	Comm.[a]
NLopt			✔	✔		Open
Ipopt	✔		✔	✔		Open
Bonmin	✔		✔	✔	✔	Open
Couenne	✔		✔	✔	✔	Open
Artelys Knitro	✔		✔	✔	✔	Comm.

Limits of optimization

Some very rough limits of what problem sizes can be handled by state-of-the-art optimization solvers:

- **Linear:** 1,000,000s of variables
- **Quadratic:** 100,000s
- **Nonlinear:** 1000s*

- **Integer Linear:** 100,000s**

* (or 10s)

** (or 100s)

Using an optimization solver

How we see the problem:

$$\begin{aligned} \min \quad & \sum_{(i,j)\in E} c_{ij}x_{ij} \\ \text{s.t.} \quad & \sum_{(i,j)\in E} x_{ij} = \sum_{(j,k)\in E} x_{jk} \quad j = 2, \ldots, n-1 \\ & \sum_{(i,n)\in E} x_{in} = 1 \\ & 0 \leq x_{ij} \leq C_{ij} \end{aligned}$$

What the solver wants:

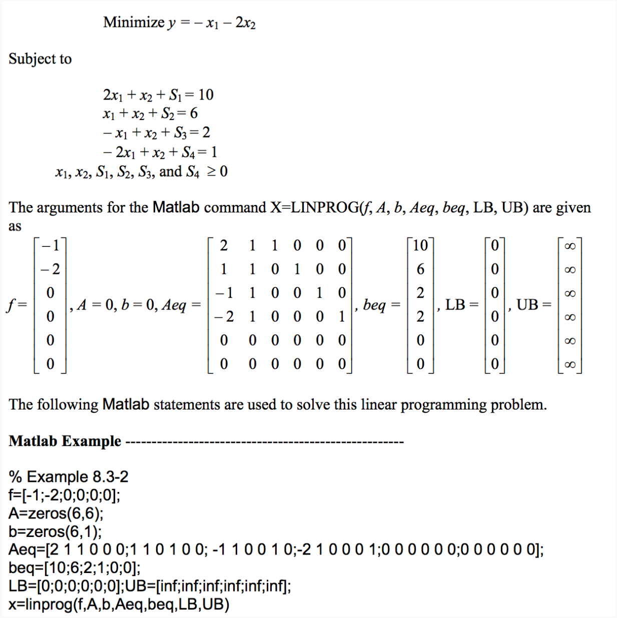
Inputs: (A, b, c)

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

To use the solver directly, we would need to create the A matrix and b/c vectors from our problem

- Slow (both to implement and when running)

Forming A/b/c ourselves?



$$\text{Minimize } y = -x_1 - 2x_2$$

Subject to

$$\begin{aligned} &2x_1 + x_2 + S_1 = 10 \\ &x_1 + x_2 + S_2 = 6 \\ &-x_1 + x_2 + S_3 = 2 \\ &-2x_1 + x_2 + S_4 = 1 \\ x_1, x_2, S_1, S_2, S_3, &\text{ and } S_4 \ge 0 \end{aligned}$$

The arguments for the Matlab command X=LINPROG(f, A, b, Aeq, beq, LB, UB) are given as

$$f = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, A = 0, b = 0, Aeq = \begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ -2 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, beq = \begin{bmatrix} 10 \\ 6 \\ 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \text{LB} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{UB} = \begin{bmatrix} \infty \\ \infty \\ \infty \\ \infty \\ \infty \\ \infty \end{bmatrix}$$

The following Matlab statements are used to solve this linear programming problem.

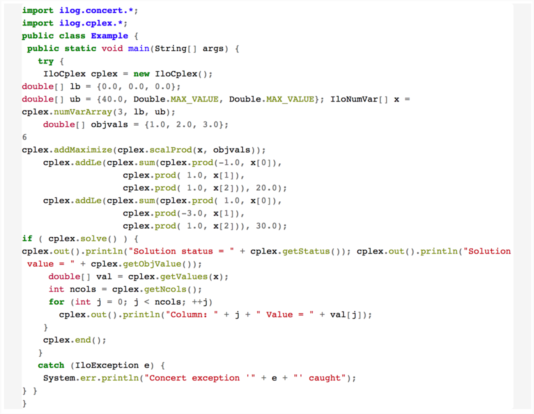
Matlab Example ---

```
% Example 8.3-2
f=[-1;-2;0;0;0;0];
A=zeros(6,6);
b=zeros(6,1);
Aeq=[2 1 1 0 0 0;1 1 0 1 0 0; -1 1 0 0 1 0;-2 1 0 0 0 1;0 0 0 0 0 0;0 0 0 0 0 0];
beq=[10;6;2;1;0;0];
LB=[0;0;0;0;0;0];UB=[inf;inf;inf;inf;inf;inf];
x=linprog(f,A,b,Aeq,beq,LB,UB)
```

Talk directly to a solver?

We can talk directly to solvers using a normal programming language (that the solver supports!)

- Tied to that solver — have to start over to try another solver
- Not easy to understand — made for coders not optimizers

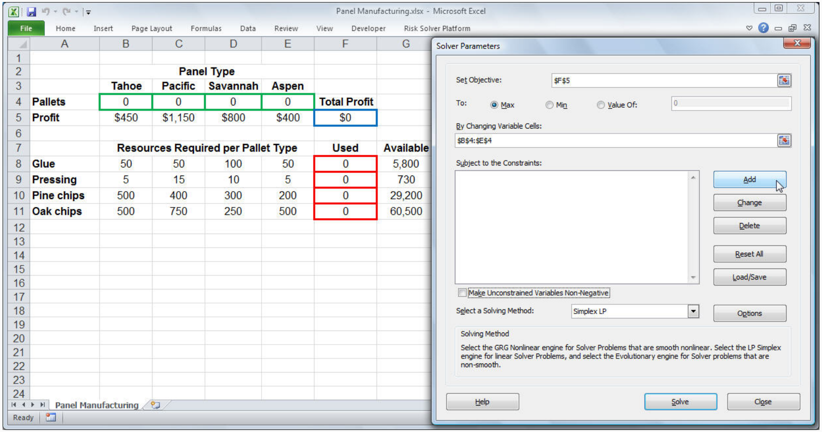


```
import ilog.concert.*;
import ilog.cplex.*;
public class Example {
 public static void main(String[] args) {
   try {
    IloCplex cplex = new IloCplex();
double[] lb = {0.0, 0.0, 0.0};
double[] ub = {40.0, Double.MAX_VALUE, Double.MAX_VALUE}; IloNumVar[] x =
cplex.numVarArray(3, lb, ub);
    double[] objvals = {1.0, 2.0, 3.0};
6
cplex.addMaximize(cplex.scalProd(x, objvals));
    cplex.addLe(cplex.sum(cplex.prod(-1.0, x[0]),
                   cplex.prod( 1.0, x[1]),
                   cplex.prod( 1.0, x[2])), 20.0);
    cplex.addLe(cplex.sum(cplex.prod( 1.0, x[0]),
                   cplex.prod(-3.0, x[1]),
                   cplex.prod( 1.0, x[2])), 30.0);
if ( cplex.solve() ) {
cplex.out().println("Solution status = " + cplex.getStatus()); cplex.out().println("Solution
 value = " + cplex.getObjValue());
     double[] val = cplex.getValues(x);
     int ncols = cplex.getNcols();
     for (int j = 0; j < ncols; ++j)
       cplex.out().println("Column: " + j + " Value = " + val[j]);
     }
    cplex.end();
   }
   catch (IloException e) {
    System.err.println("Concert exception '" + e + "' caught");
} }
}
```

Excel Solver?

Excel Solver gives a point-and-click interface for solving optimization problems in Excel

- Slow, hard to scale to large problems



Panel Manufacturing.xlsx - Microsoft Excel

	A	B	C	D	E	F	G
1							
2			Panel Type				
3		Tahoe	Pacific	Savannah	Aspen		
4	Pallets	0	0	0	0	Total Profit	
5	Profit	$450	$1,150	$800	$400	$0	
6							
7		Resources Required per Pallet Type				Used	Available
8	Glue	50	50	100	50	0	5,800
9	Pressing	5	15	10	5	0	730
10	Pine chips	500	400	300	200	0	29,200
11	Oak chips	500	750	250	500	0	60,500

Solver Parameters

Set Objective: $F$5

To: [✓ Max] Min Value Of: 0

By Changing Variable Cells: $B$4:$E$4

Subject to the Constraints:

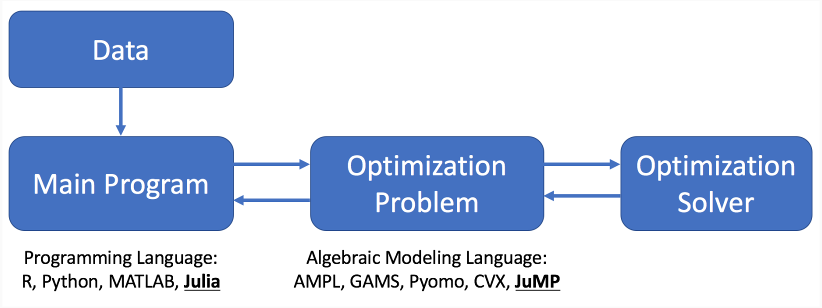
Make Unconstrained Variables Non-Negative

Select a Solving Method: Simplex LP

Solving Method

Select the GRG Nonlinear engine for Solver Problems that are smooth nonlinear. Select the LP Simplex engine for linear Solver Problems, and select the Evolutionary engine for Solver problems that are non-smooth.

Solving optimization problems in practice


Data
Main Program
Optimization Problem
Optimization Solver
Programming Language:
R, Python, MATLAB, **Julia**
Algebraic Modeling Language:
AMPL, GAMS, Pyomo, CVX, **JuMP**

In reality, we use an Algebraic Modeling Language to communicate with the solver

- We write down our model in code as we would on paper
- AML handles converting the model into solver-format and getting results from solver

Our problem in the JuMP AML

```
m = Model()
@variable(m, 0 <= x[e in E] <= e.capacity)
@constraint(m, flowcon[j=2:n-1],
  sum{x[e], e in E, e.to == j}
  ==
  sum{x[e], e in E, e.from == j}
)
@constraint(m, sum{x[e], e in edges, e.to == n} == 1)
@objective(m, Min, sum{e.cost * x[e], e in edges})
```

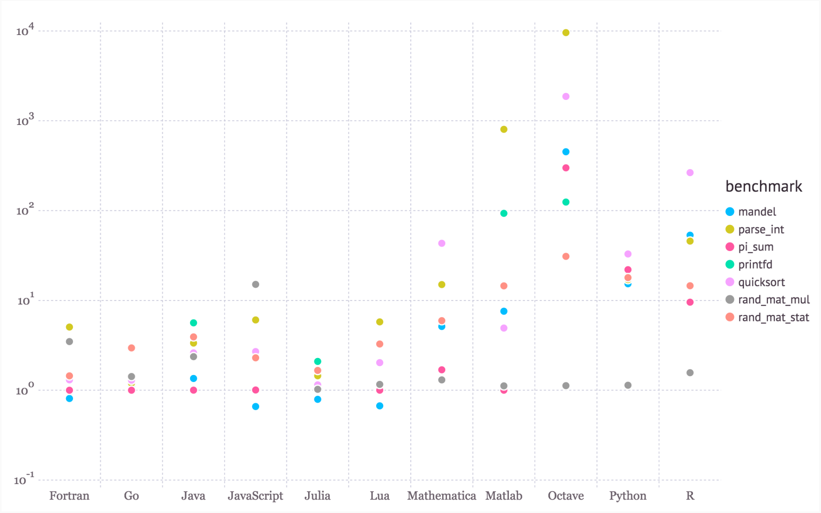
The code for our model reads almost the same as the mathematical version!

The Julia Programming Language

`www.julialang.org`

- High-level, high-performance, open-source dynamic language for technical computing
- Allows use to keep the productivity of dynamic languages without giving up any speed
- Developed here at MIT, relatively new (released in 2012)

How fast is Julia?



JuMP: Julia for Mathematical Programming

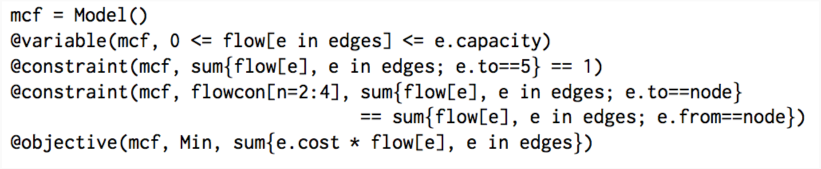
www.juliaopt.org

- Open-source, free, fast algebraic modeling language for Julia
- Created by MIT ORC students (Iain, Miles and Joey plus others)
- Easy to learn, use and read
- As fast as comparable commercial software

JuMP vs. others—Syntax

JuMP

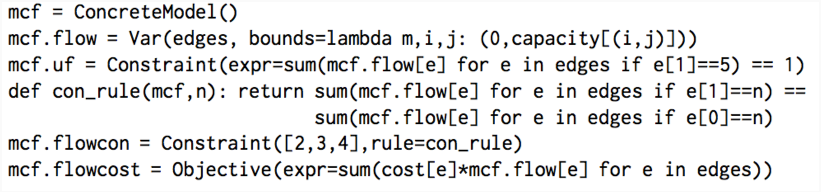
(open-source, Julia, 2013)



```
mcf = Model()
@variable(mcf, 0 <= flow[e in edges] <= e.capacity)
@constraint(mcf, sum{flow[e], e in edges; e.to==5} == 1)
@constraint(mcf, flowcon[n=2:4], sum{flow[e], e in edges; e.to==node}
                                == sum{flow[e], e in edges; e.from==node})
@objective(mcf, Min, sum{e.cost * flow[e], e in edges})
```

Pyomo

(open-source, Python, 2008)

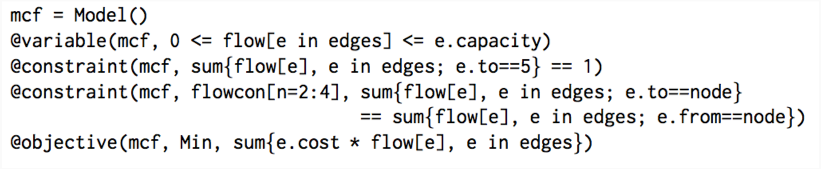


```
mcf = ConcreteModel()
mcf.flow = Var(edges, bounds=lambda m,i,j: (0,capacity[(i,j)]))
mcf.uf = Constraint(expr=sum(mcf.flow[e] for e in edges if e[1]==5) == 1)
def con_rule(mcf,n): return sum(mcf.flow[e] for e in edges if e[1]==n) ==
                            sum(mcf.flow[e] for e in edges if e[0]==n)
mcf.flowcon = Constraint([2,3,4],rule=con_rule)
mcf.flowcost = Objective(expr=sum(cost[e]*mcf.flow[e] for e in edges))
```

JuMP vs. others—Syntax

JuMP

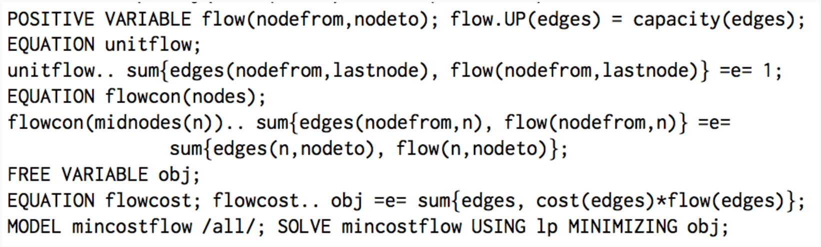
(open-source, Julia, 2013)



```
mcf = Model()
@variable(mcf, 0 <= flow[e in edges] <= e.capacity)
@constraint(mcf, sum{flow[e], e in edges; e.to==5} == 1)
@constraint(mcf, flowcon[n=2:4], sum{flow[e], e in edges; e.to==node}
                                 == sum{flow[e], e in edges; e.from==node})
@objective(mcf, Min, sum{e.cost * flow[e], e in edges})
```

GAMS

(commercial, custom language, 1976)



```
POSITIVE VARIABLE flow(nodefrom,nodeto); flow.UP(edges) = capacity(edges);
EQUATION unitflow;
unitflow.. sum{edges(nodefrom,lastnode), flow(nodefrom,lastnode)} =e= 1;
EQUATION flowcon(nodes);
flowcon(midnodes(n)).. sum{edges(nodefrom,n), flow(nodefrom,n)} =e=
               sum{edges(n,nodeto), flow(n,nodeto)};
FREE VARIABLE obj;
EQUATION flowcost; flowcost.. obj =e= sum{edges, cost(edges)*flow(edges)};
MODEL mincostflow /all/; SOLVE mincostflow USING lp MINIMIZING obj;
```

JuMP vs. others—Syntax

JuMP

(open-source, Julia, 2013)

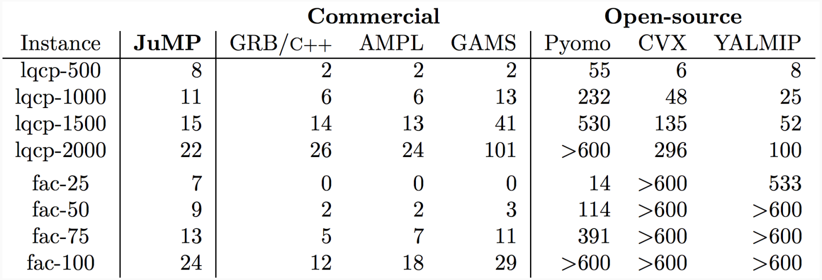
```
mcf = Model()
@variable(mcf, 0 <= flow[e in edges] <= e.capacity)
@constraint(mcf, sum{flow[e], e in edges; e.to==5} == 1)
@constraint(mcf, flowcon[n=2:4], sum{flow[e], e in edges; e.to==node}
                                 == sum{flow[e], e in edges; e.from==node})
@objective(mcf, Min, sum{e.cost * flow[e], e in edges})
```

AMPL

(commercial, custom language, 1985)

```
var flow{(i,j) in edges} >= 0.0, <= capacity[i,j];
subject to unitflow:  sum{(i,5) in edges} flow[i,5] == 1;
subject to flowconserve {n in 2..4}:
  sum{(i,n) in edges} flow[i,n] == sum{(n,j) in edges} flow[n,j];
minimize flowcost:    sum{(i,j) in edges} cost[i,j] * flow[i,j];
```

JuMP vs. others—Speed



		Commercial			Open-source		
Instance	**JuMP**	GRB/C++	AMPL	GAMS	Pyomo	CVX	YALMIP
lqcp-500	8	2	2	2	55	6	8
lqcp-1000	11	6	6	13	232	48	25
lqcp-1500	15	14	13	41	530	135	52
lqcp-2000	22	26	24	101	>600	296	100
fac-25	7	0	0	0	14	>600	533
fac-50	9	2	2	3	114	>600	>600
fac-75	13	5	7	11	391	>600	>600
fac-100	24	12	18	29	>600	>600	>600

Seconds taken to generate model and pass to solver

Traditionally, open-source AMLs have been significantly slower than their commercial counterparts—tradeoff between flexibility and speed

Why do we use JuMP?

JuMP has become the de-facto AML of choice here at the ORC

- We use it in our research
- We teach it in our classes

Why?

- It's free and open-source—easy to add features, no contention for licenses
- It's embedded in an existing programming language—no need to switch to custom language just for optimization
- It's about as fast as the best commercial AMLs out there!

Intro to Julia

Getting started with Julia

Open up a terminal window, and type in

```
julia
```

You should see the Julia welcome message

If this doesn't work, let me know so we can fix it

Basic arithmetic

Simple arithmetic:

```
3 + 7
4.8 / 9.3
11 ^ 5
```

Comparisons:

```
12 <= 5
!(5 == 8)
5 != 8
(2014 % 4) == 0
```

Variables

Variables:

```
x = 5
y = 12
z = x - y
z
w
```

Everything has a type:

```
typeof(z)         # Int64
typeof(z + 1.5)   # Float64
typeof(z >= 0)    # Bool
```

Arrays, Vectors and Matrices

1D arrays (vectors) are for sequences of values, indexed from 1 to n:

```
v = [1, 2, 3]
v[1]    # access elements using [ ]; starts from 1
v[0]    # gives an error!
v[end]  # last element
```

We can also make 2D arrays (matrices):

```
A = [ 1  1  1;
     -1 -1 -2;
      0  0  0]
B = [0.1 0.5 0.4;
     0.2 0.3 0.5;
     0.8 0.1 0.1]
```

```
B[2, 3]
A + B
B ^ 100
B[1, :]
```

More arrays

Modifying arrays:

```
a = [1, 2, 3]
a[end + 1] = 4  # won’t work!
```

Push/pop commands:

```
push!(a, 4)    # append 4 to end of a
pop!(a)        # remove last element of a; 4
a              # a is back to being [1, 2, 3]
z = []
push!(z, 5)    # what happens?
z = Int64[]
push!(z, 5)    # how about now?
push!(z, 1.5)  # and this?
```

Exercise 1

Create the arrays A, x and b as follows:

```
A = rand(10, 3); x = rand(3); b = rand(10)
```

Write an expression that evaluates to

- `true` if the inner product of the fourth row of `A` and `x` is greater than the fourth element of `b`; and
- `false` otherwise.

What do you find?

Hint: you might want to use the `dot` function to compute the inner product. Type `?dot` to access the Julia help for this function

Lessons from Exercise 1

```
dot(A[4,:], x)                         # 1.67087
dot(A[4,:], x) > b[4]                  # true
```

How about if we didn't use `dot`?

```
A[4,:] * x                               # error, wrong dim
A[4,:]' * x                      # works now
A[4,:]' * x > b[4]                   # error, num vs matrix
(A[4,:]' * x)[1] > b[4]          # works now
```

Creating arrays

```
ones(3)
zeros(4,5)
```

List comprehensions:

```
oddnumbers  = [2 * i - 1 for i in 1:10]
evennumbers = [j + 1 for j in oddNumbers]

A = [i + j for i in 1:5, j in 1:5]
```

Printing things

```
a = 123.0

# printing without line break
# use comma to separate terms
print("The value of a = ", a); print(".")

# printing with line break
# use dollar sign within string for Julia expression
println("a is $a, a-10 is $(a-10)."); println("That's it.")
```

Conditionals

If-then-else:

```
x = 2
if x > 6
  x = x + 20
elseif (x < 5)
  x = factorial(x)
else
  println("Not changing x")
end
```

Loops

for loops:

```
for k = 1:5
  println(k ^ 2)
end
```

while loops:

```
x = 1
while x < 2000
  print("*")
  x = 2 * x
end
```

Functions

Functions are defined using the keyword function:

```
function convexcomb(x, y, theta)
  (1 - theta) * x + theta * y
end
```

Functions by default return the last statement, but you can also use return:

```
function convexcomb2(x, y, theta)
  if (theta < 0 || theta > 1)
    println("Bad theta!!!")
  else
    return (1 - theta) * x + theta * y
  end
end
```

Exercise 2

Write a function that takes a matrix A, vectors x and b, and returns a vector containing the row indices for which Ax >= b does not hold.

There are many ways to do this!

Hints:

- `.<` or `.<=` performs element-wise comparisons
- `find` function returns indices of non-zero elements for Int64 or Float64 arrays or true elements for BitArrays
- Use `?` to access the help

Example Solutions to Exercise 2

```
function findinds(x, A, b)
  find(A * x .< b)
end

function findinds(x, A, b)
  inds = collect(1:length(b))
  falseinds = A * x .< b
  inds[falseinds]
end
```

Example Solutions to Exercise 2

```
function findinds(x, A, b)
  inds = []
  for i in 1:length(b)
    diff = (dot(A[i,:], x) - b[i])[1,1]
    if (diff < 0)
      push!(inds, i)
    end
  end
  return inds
end
```

Scripts in Julia

Julia scripts are plain text files with the extension ".jl"

test.jl

```
function findinds(x, A, b)
  find(A * x .< b)
end
println("findinds loaded")
```

In a terminal window, you can run one (say test.jl) by passing it as a command line argument to Julia:

```
julia test.jl
```

Loading a script inside Julia

Within a session, you can use the keyword `include` to load everything in a script.

In Julia session:

```
include("test.jl")
A = rand(10,3) / 3; x = rand(3); b = rand(10)
findinds(x, A, b)
```

IJulia and Jupyter

Another common way of interacting with Julia is using IJulia notebooks (similar to IPython)

To open IJulia:

```
using IJulia
notebook()
```

Navigate to the folder with the iris.csv file. We are going to do some data work in an IJulia notebook.

DataFrames in Julia

Like data frames in R, Julia also has a similar structure. You will need to load the package DataFrames first:

```
using DataFrames
iris = read_table("iris.csv", header=false)
iris[1:5,:]
```

Joining DataFrames

Use join function on data frames, with shared id Species:

```
species_price = DataFrame(Species = ["setosa",
                                     "versicolor",
                                     "virginica"],
                         Price = [2.5, 3.1, 3.2])
join(iris, species_price, on = :Species, kind = :left)
```

DataFramesMeta: the dplyr of Julia

DataFramesMeta package provides the functionality similar to dplyr in R:

Julia	dplyr
@where	filter
@transform	mutate
@by	
@groupby	group_by
@based_on	summarise/do
@orderby	arrange
@select	select

DataFramesMeta: the dplyr of Julia

Examples:

```
using DataFramesMeta
@where(iris, :SepalLength .> 5)
@select(iris, :SepalLength, :SepalWidth, :Species)
@transform(iris, logSepalLength = log(:SepalLength))
```

DataFramesMeta: chaining

You can even chain operations similar to the %>% in dplyr:

```
iris_summary = @linq iris |>
    transform(logSL = log(:SepalLength),
      logSW=log(:SepalWidth))|>
    where(:logSL .>= 1)|>
    by(:Species, meanLogSL = mean(:logSL),
      meanLogSW = mean(:logSW))|>
    orderby(:meanLogSL)|>
    select(var = :Species, :meanLogSL)
```

Exercise 3

Read the icecream data from RDataset using the following:

```
icecream = dataset("Ecdat", "Icecream")
```

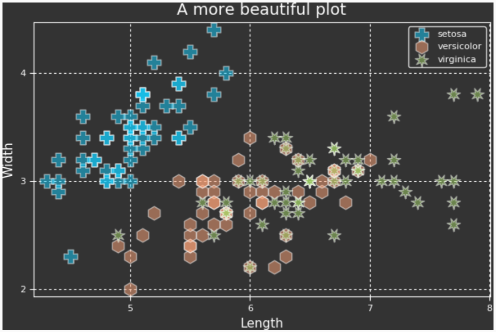
Inspect the dataset.

Check to see if on average the consumption is higher when the temperature is at least 50 degrees compared to lower temperature?

Plots and Statplots

To plot in Julia is easy:

```
using Plots
using StatPlots
scatter(iris, :SepalLength, :SepalWidth, group=:Species,
          ...)
```



Exercise 4

Continue with the icecream dataset and explore the following using visualization and/or data manipulations:

1. How is income related to Consumption?
2. Create the ‘Revenue‘ variable as the product between ‘Price‘ and ‘Cons‘. Do you see a positive relationship between the temperature and revenue?
3. Create a new variable ‘IncomeGroup‘ that groups income based on a few buckets (your choice). Plot the distribution of the consumption over the different groups. What do you find?

Intro to JuMP

Notebook 1

Go to IJulia notebook: “1 — Intro to JuMP”

Notebook 2

Go to IJulia notebook: “2 — Transportation Problem”

Notebook 3

Go to IJulia notebook: “3 — Airline Revenue Management”

Notebook 4

Go to IJulia notebook: “4 — Whiskas Cat Food”

Guide to MIO modeling and solvers

What is Integer Optimization?

Integer linear optimization is a surprisingly expressive class of problems of the form

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \in \mathbb{R}^m \times \mathbb{Z}^n \end{aligned}$$

Essentially, this is linear optimization with a new primitive: integrality conditions on some of the variables.

Often we further restrict integer variables to be binary variables, $x \in \{0,1\}$. This is a powerful way of modeling logical outcomes.

Notebook 5

Go to IJulia notebook: “5 — Sudoku”

Practical Integer Optimization

Most real-world problems inevitably end up having to incorporate integer variables

- Most common: binary variables used to model yes/no decisions

Unfortunately, optimization with integer variables is much more difficult, and getting it right is more of an art than a science

To have the most success with MIO, it's important to know both how to formulate problems in a good way, and how to work effectively with mixed-integer optimization solvers

Progress of MIO in the past 25 years

- Speed up between CPLEX 1.2 (1991) and CPLEX 11 (2007): 29,000 times
- Gurobi 1.0 (2009) comparable to CPLEX 11
- Speed up between Gurobi 1.0 and Gurobi 6.5 (2015): 73 times
- Total software speedup: 2,100,000 times
- A MIO problem that would have taken a year to solve 25 years ago can now be solved on the **same 25-year-old computer** in around 15 seconds.
- Hardware speedup: $10^{6.2} =$ 1,560,000 times
- Total Speedup: **3.3 trillion times!**

Guide to linear MIO solvers

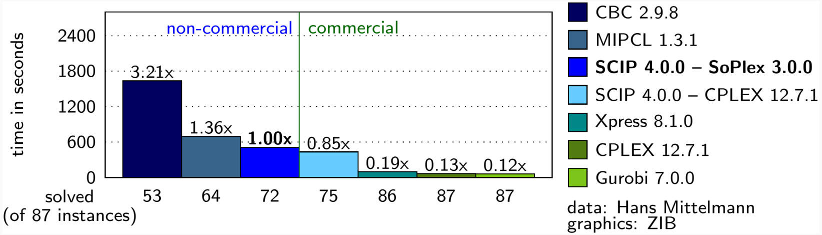
The commercial solvers are significantly faster than open-source alternatives:

- Gurobi is typically seen as the fastest, followed closely by
- CPLEX (owned by IBM), and then
- Xpress (owned by FICO)

Open-source MIO solvers:

- CBC is the best free and open-source solver
- GLPK is a much slower alternative
- SCIP is open-source, but only free for academic use

MIO solver speeds



Commercial MIO solvers are about 25x faster than CBC

Guide to Gurobi

We're going to focus on Gurobi:

- Fastest MIO solver
- Easy to get academic license
- Most research-friendly of the commercial solvers

Most of what we cover applies to the other solvers, with minor differences

Notebook 6

Go to IJulia notebook: “6 — Working with MIO Solvers”

MIO modeling techniques

Now that we're familiar with Gurobi, we'll take a look at just a few of the things we can model using MIO

- Binary choices and boolean logic
- Logical implications using big-M constraints

This is by no means going to be exhaustive! A great reference for MIO modeling techniques is MIP Formulations and Linearizations by FICO:

`http://www.fico.com/en/node/8140?file=5125`

Logical Constraints — Boolean Logic

There is a natural mapping from the boolean `TRUE` and `FALSE` to the binary variables `1` and `0`

It is also quite common to want to express logical constraints and implications in our optimization models, e.g.

- if this factory is built, we can build no other factories in this year
- either factory A or factory B can be built, but not both

The "trick" is expressing these logical constraints and implications as linear inequalities, and sometimes we need to introduce auxiliary variables

We'll start with modeling simple logical statements: `AND` & `OR`.

Boolean Logic — AND

z is `TRUE` if and only if x and y are both `TRUE` $\quad (z = x \texttt{ AND } y)$

- $0 \texttt{ AND } 0 = 0$
- $1 \texttt{ AND } 0 = 0$
- $0 \texttt{ AND } 1 = 0$
- $1 \texttt{ AND } 1 = 1$

We can write this as

$$z \geq x + y - 1, \quad z \leq x, \quad z \leq y, \quad 0 \leq z \leq 1$$

which we can check by putting the values in for each case

Boolean Logic — OR

z is `TRUE` if and only if either x or y or both are `TRUE` $\quad (z = x \texttt{ OR } y)$

- $0 \texttt{ OR } 0 = 0$
- $1 \texttt{ OR } 0 = 1$
- $0 \texttt{ OR } 1 = 1$
- $1 \texttt{ OR } 1 = 1$

This can also be expressed by linear inequalities, with no auxiliaries required:

$$z \geq x, \quad z \geq y, \quad z \leq x + y, \quad 0 \leq z \leq 1$$

Boolean Logic — XOR

z is `TRUE` if and only if either x or y is `TRUE` but not both ($z = x \texttt{ XOR } y$)

- $0 \texttt{ XOR } 0 = 0$
- $1 \texttt{ XOR } 0 = 1$
- $0 \texttt{ XOR } 1 = 1$
- $1 \texttt{ XOR } 1 = 0$

This one is a bit trickier. One way to do it is $z = x \texttt{ XOR } y = (x \texttt{ OR } y) - (x \texttt{ AND } y)$, but then we would need two auxiliary variables, e.g. $z_1 = x \texttt{ OR } y$, $z_2 = x \texttt{ AND } y$, and then finally $z = z_1 - z_2$

Exercise: Express $z = x \texttt{ XOR } y$ with using ONE auxiliary variable. Can you do it with NO auxiliary variables?

Logical Implications

We often need to model constraints that should only hold if a logical condition is satisfied. For example:

- We have a network with a ‘source’ and a ‘sink’, and want to maximize flow between them across the network. We have a set of arcs we can build, at a cost, so our objective is to maximize profit from the flow less the cost of building these arcs.

If we say that $x_{i,j}$ is the flow from i to j, and $z_{i,j}$ is a binary decision for whether we build the arc or not, we will have the constraint

$$x_{i,j} > 0 \implies z_{i,j} = 1$$

That is, if the flow on the arc is non-zero, we must build the arc

Big-M constraints for logical implications

Our logical constraint is:

$$x_{i,j} > 0 \implies z_{i,j} = 1$$

One way to express this as a linear equality is to write

$$0 \leq x_{i,j} \leq M z_{i,j}$$

where M is a sufficiently large constant. That means that M is greater than the largest value $x_{i,j}$ would take in an optimal solution.

- if $z_{i,j} = 0$, we have $0 \leq x_{i,j} \leq 0$, so $x_{i,j} = 0$
- if $z_{i,j} = 1$, we have $0 \leq x_{i,j} \leq M$, so $x_{i,j}$ is unrestricted

Solution to XOR question

$$\begin{aligned} z &\geq x - y \\ z &\geq y - x \\ z &\leq x + y \\ z &\leq 2 - x - y \end{aligned}$$

Check

- $x = 0, y = 0$: third constraint implies $z \leq 0$
- $x = 1, y = 0$: first constraint implies $z \geq 1$
- $x = 0, y = 1$: second constraint implies $z \geq 1$
- $x = 1, y = 1$: fourth constraint implies $z \leq 2 - 1 - 1 = 0$

Notebook 7

Go to IJulia notebook: “7 — Facility Location Problem”

Take-aways from Today

- Julia is a fast technical computing language with tools for data wrangling similar to R
- Algebraic modeling languages like JuMP make it easy to solve optimization problems
    - Why JuMP? It's fast, free, and embedded in a real programming language
- LO and MIO are powerful and flexible modeling frameworks, but...
- **Not all formulations are created equally!**
    - Different ways of expressing the same model can give very different behaviors
    - Understanding what the solver is doing is crucial for improving performance