Modeling and simulation of a double auction artificial financial market

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Abstract

We present a double-auction artificial financial market populated by heterogeneous agents who trade one risky asset in exchange for cash. Agents issue random orders subject to budget constraints. The limit prices of orders may depend on past market volatility. Limit orders are stored in the book whereas market orders give immediate birth to transactions. We show that fat tails and volatility clustering are recovered by means of very simple assumptions. We also investigate two important stylized facts of the limit order book, i.e., the distribution of waiting times between two consecutive transactions and the instantaneous price impact function. We show both theoretically and through simulations that if the order waiting times are exponentially distributed, even trading waiting times are also exponentially distributed.
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0. Introduction

In previous studies [1–4], we presented an artificial financial market where the price formation process was performed by means of a clearing house [5]. In the clearing house mechanism, buy and sell limit orders are accumulated over time and the market is cleared periodically at the intersection of demand and supply curves.

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The rationale of the clearing house is a key tenet of economic theory: prices are at an equilibrium when supply and demand match [6]. However, the clearing house is an unrealistic description of the way stock exchanges operate around the world. Progress in information technology and the increasing deregulation of exchanges have led to a wide adoption of the limit order book for price formation. The limit order book is based on a double auction mechanism, where traders freely announce bids and asks and may accept other traders' bids and asks as long as the market remains open. The double auction requires no auctioneer and both theoretical and empirical studies have found that it has remarkable power to promote price formation [7].

In this paper, we present a model of trading in an artificial stock market by means of a limit order book. Main statistical properties of simulated prices are also presented and discussed with regard to agent behavioral assumptions.

The limit order book is a snapshot at a given instant of the queues of all buy and sell limit orders, with their respective prices and volumes. Limit orders are organized in ascending order according to their limit prices. All buy limit orders are below the best buy limit order, i.e., the buy limit order with the highest limit price (the bid price). The best buy limit order is situated below the best sell limit order, i.e., the sell limit order with the lowest limit price (the ask price). All other sell limit orders are above the best sell limit order. Orders are stored in the book. A transaction occurs when a trader hits the quote (the bid or the ask price) on the opposite side of the market. If a trader issues a limit order, say a sell limit order, the order either adds to the book if its limit price is above the bid price, or generates a trade at the bid if it is below or equal to the bid price. In the latter case, the limit order becomes a marketable limit order or more simply, a market order. Conversely, if the order is a buy limit order it becomes a market order and is executed if its limit price is above the ask price, else it is stored in the book. Limit orders with the same limit price are prioritized by time of submission, with the oldest order given the highest priority. Order's execution often involves partial fills before it is completed, but partial fills do not change the time priority.

The paper is organized as follows: Section 1 presents the model and computational experiments are presented in Section 2. Section 3 provides a discussion of results and the conclusions of the study.

1. The model

A model of artificial trading by means of a limit order book is presented in this section. Agents trade one single stock in exchange for cash. They are modeled as liquidity traders. As a consequence, the decision making process is nearly random and depends on the finite amount of financial resources they own, i.e., cash + stocks. At the beginning of the simulation, cash and stocks are uniformly distributed among agents. Trading decisions can also depend on the volatility of the market.

Trading is organized in M daily sections. Each trading day is subdivided in T elementary time steps, say seconds. During the trading day, at given time steps t_h,

a trader i is randomly chosen for issuing an order. Order waiting times $\tau_h^o = t_h - t_{h-1}$ follow an exponential distribution with mean λ^o. Then, the order generation process is a Poisson process. Orders are cancelled after time Θ and at the beginning of each day the book is emptied.

A trader issues a buy or a sell order with probability 50%. Let us denote with $a(t_{h-1})$ and with $d(t_{h-1})$ the values of the ask and of the bid prices stored in the book at time step t_{h-1}. Let us suppose that the order issued at time step t_h be a sell order. Then we assume that the limit price $s_i(t_h)$ associated to the sell order is given by

$$s_i(t_h) = n_i(t_h)a(t_{h-1}) \,, \tag{1}$$

where $n_i(t_h)$ is a random draw by trader i at time step t_h from a Gaussian distribution with constant mean $\mu = 1$ and standard deviation σ. If $s_i(t_h) > d_i(t_{h-1})$, the limit order is stored in the book and no trades are recorded; else, the order becomes a market order and a transaction occurs at the price $p(t_h) = d(t_{h-1})$. In the latter case, the sell order is partially or totally fulfilled and the bid price is updated. The quantity of stocks offered for sale is a random fraction of the quantity of stocks owned by the trader.

If the order is a buy order, we assume that the associated limit price $b_i(t_h)$ is given by

$$b_i(t_h) = n_i(t_h)d(t_{h-1}) \,, \tag{2}$$

where $n_i(t_h)$ is determined as for sell orders. If $b_i(t_h) < a(t_{h-1})$, the limit order is stored in the book and no trades are recorded; otherwise the order becomes a market order and a transaction occurs at the price $p(t_h) = a(t_{h-1})$. The quantity of stocks ordered to be bought depends on cash endowment of the trader and on the value of $b_i(t_h)$.

It is worth noting that, in our framework, agents compete for the provision of liquidity. If an agent issues a buy order, its benchmark is the best limit buy order given by the bid price. Being $\mu = 1$, half times, he offers a more competitive buy order (if $b_i(t_h) > d(t_{h-1})$), which may result in a trade if $b_i(t_h) \geqslant a(t_{h-1})$). The same applies for sell limit orders.

2. Computational experiments

The timing parameters of every simulation have been set to $M = 1000$ daily sections, each characterized by a length of $T = 25,200\,\text{s}$ (corresponding to 7 h of trading activity). The average order waiting time λ^o have been set to 20 s and the order lifespan to $\Theta = 600\,\text{s} \gg \lambda^o$. The number of agents is set to 10,000. At the beginning of the simulation, the stock price is set at 100,00 units of cash, say dollars and each trader is endowed with an equal amount of cash and of shares of the risky stocks. These amounts are 100,000 dollars and 1000 shares, respectively.

Sell and buy limit prices are computed following Eqs. (1) and (2), respectively. The random number $n_i(t_h)$ is a random draw by trader i from a Gaussian distribution with constant mean $\mu = 1$ and standard deviation σ. We considered two cases for the parameter σ. In the first case, σ has been set to the constant value 0.005. In the

second, σ depends linearly on the intra-day volatility of the stock. This dependence, which can be viewed as a volatility feedback, was a key feature in the experiments reported in our previous studies [1–4] and was responsible for the fat tails in the distribution of returns and the clustering of volatility. Here, we find that the intra-day distribution of returns is leptokurtic even in the case of absence of volatility feedback and constant values of σ.

2.1. Experiment with absence of volatility feedback

We performed a computational experiment where σ has been set to the constant value of 0.005, see Figs. from 1 to 3. Fig. 1 presents the intraday price path (top) and the shape of the corresponding log-returns (bottom). It is worth noting that, with the limit order book as a mechanism for price formation, time intervals between consecutive trades are not constant. In order to deal with returns related to homogeneous time windows, log-returns have been computed according to the previous-tick interpolation [8, p. 37] with a time window equal to 60 s.

The intraday stock price is characterized by a sort of intermittent behavior with sudden jumps. This finding is confirmed by the probability density function of returns shown in Fig. 2. The empirical PDF shows a clear departure from the
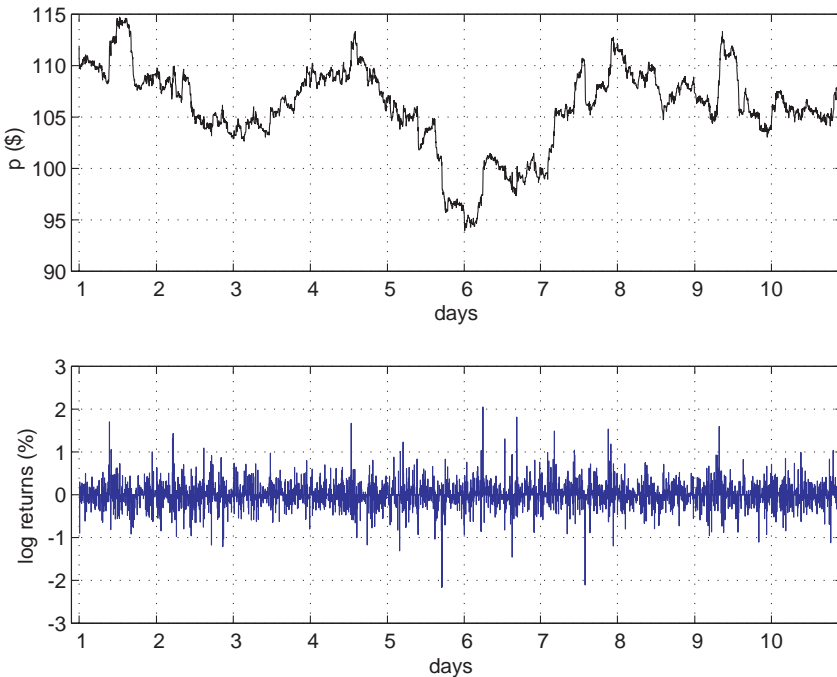

Fig. 1. A sample of the intraday price path (top) and corresponding log-returns (bottom) in the experiment with absence of volatility feedback.
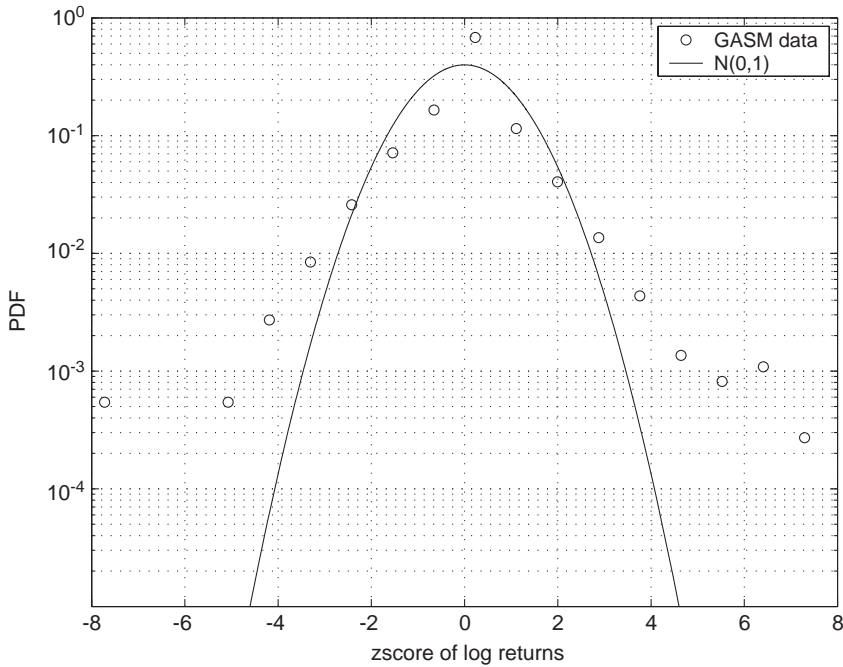
Fig. 2. Dots represent an estimate of the probability density function (PDF) for intraday log-returns in the experiment with absence of volatility feedback; the continuous line is the normal distribution. The picture shows a clear deviation from the hypothesis of normal distribution. The finding is confirmed by the Jarque–Bera test for normality.

assumption of normal distribution. Indeed, according to the Jarque–Bera test, the null hypothesis of normality for the distribution of returns is rejected at the significance level of 5%. Recent empirical studies on the form of the estimated PDF of real price returns have shown important deviations from the paradigm of the Gaussian distribution [9–11]. An important stylized fact of real stock prices is thus recovered by our model. It is worth noting that there are no particular ad hoc behavioral assumptions about the agents' decision making process, such as the dependence of limit prices on market volatility (volatility feedback) and/or the initial endowment of wealth among agents. In our previous studies [1–4], we showed that these ad hoc assumptions played a key role in the statistical properties of price series. In particular, they gave origins to leptokurtic returns distributions and to the clustering of volatility. In those experiments, the price formation was performed by means of the mechanism of the clearing house. Current experiments are characterized by a new structural feature: the limit order book. Within the double auction setting, fat tails in the PDF of intraday returns are recovered without any particular hypothesis on the behavior of agents. A possible conclusion could be that important statistical features of stock price series, such as the fat tails phenomenon,
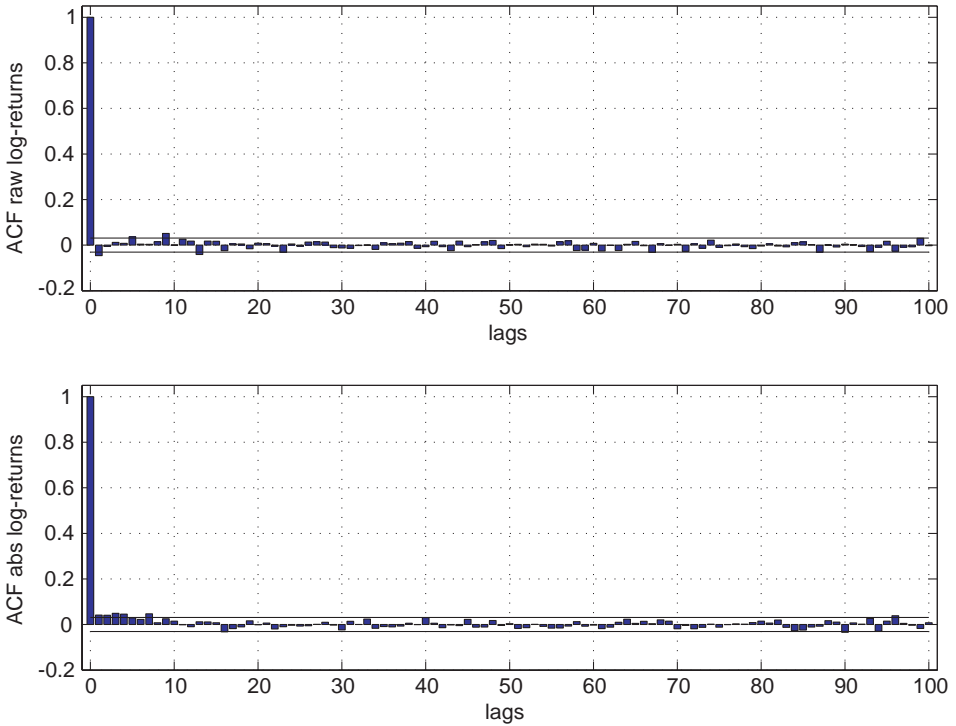
Fig. 3. Autocorrelation function (ACF) of raw intraday returns (top) and of absolute values of intraday returns (bottom) in the experiment with absence of volatility feedback. In the two cases the ACF decays to zero very rapidly and no long-range serial correlation is present.

may depend mainly on the market microstructure, as a recent paper by Li Calzi and Pellizzari [12] suggests.

Furthermore, daily close returns are normally distributed. A possible explanation is that the limit order book is emptied at the beginning of every trading day, so the book influences the statistics of prices only on an intraday basis.

As regarding the clustering of volatility, no phenomenon of persistence of fluctuations is present on both intraday returns (see Fig. 3), and daily close returns. Fig. 3 shows that the autocorrelation function (ACF) of raw returns decay to zero very rapidly as the autocorrelation function of the absolute values of returns. The two horizontal lines represent the noise levels[1] computed as $\pm 2/\sqrt{L}$, where L is the length of the time series considered.

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[1] The sample ACF of a pure white noise process of length L lies within the band $(-2/\sqrt{L}, +2/\sqrt{L})$ at every lag with a confidence level of 95%, see Ref. [13, p. 86] for further details.

2.2. Experiment with intraday volatility feedback

In this computational experiment, agent behaviors depend on the volatility of the market, as already discussed in our previous papers [1–4]. Traders are influenced by the volatility of the market in making limit orders. When volatility is high, uncertainty about the "true" value of a stock grows and traders place orders whose limit prices are characterized by a broader distribution.

In the current framework, the volatility feedback is on an intraday basis, i.e., order limit prices depend upon the volatility of intraday prices. The dependence is modeled by means of the value of σ, which is not constant but is described by

$$\sigma = k\sigma_{T_i}\,, \tag{3}$$

where k is a constant and σ_{T_i} is the volatility of intraday stock prices computed as the standard deviations of log-returns in the time window T_i. In the experiment reported, $k = 1.5$ and T_i is uniformly distributed among traders in a range between 600 and 6000 s.

Figs. from 4 to 6 present the statistics of intraday prices regarding this experiment. Fig. 4 presents a sample of the intraday price path (top) and of corresponding
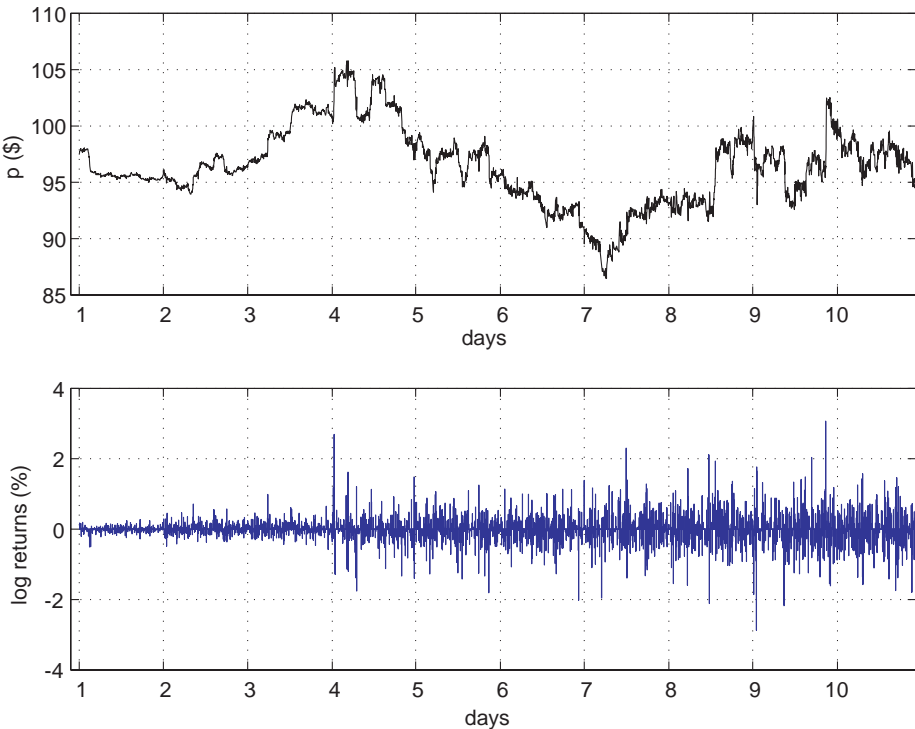

Fig. 4. A sample of the Intraday price path (top) and corresponding log-returns (bottom) in the experiment with intraday volatility feedback.
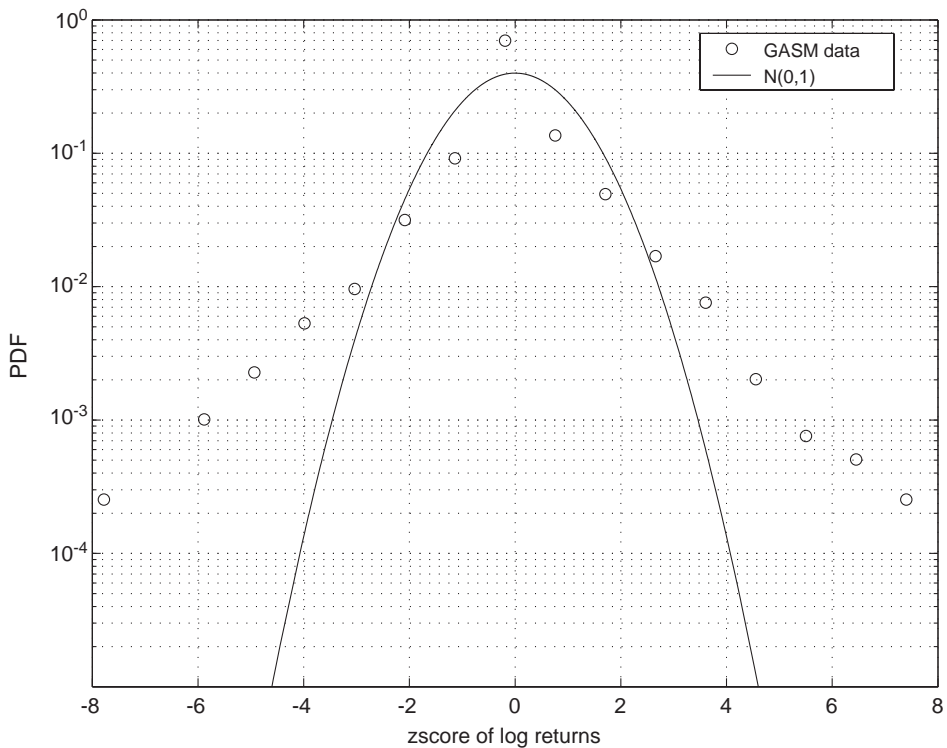
Fig. 5. Dots represent an estimate of the probability density function (PDF) for intraday log-returns in the intraday volatility feedback case, the continuous line is the normal distribution. The picture shows a clear deviation from the hypothesis of normal distribution. The finding is confirmed by the Jarque–Bera test for normality.

log-returns (bottom). The probability density function of returns is showed in Fig. 5. The departure of empirical PDF from a normal distribution is evident, besides, according to the Jarque–Bera test, the null hypothesis of normality is rejected at the significance level of 5%. The important stylized fact of fat tails is thus recovered even in this experimental setting. Indeed, the previous experiment suggests that fat tails may be caused simply by the double auction mechanism. On the other hand, our previous studies suggested that the volatility feedback mechanism is also a way to generate leptokurtic returns distributions.

As regarding the clustering of volatility, the bottom part of Fig. 6 shows a slow decay of the ACF of absolute values of intraday log-returns. It is worth noting that, in our previous studies, we showed that the volatility feedback mechanism is the cause of the volatility clustering. The same applies in this experiment where a volatility feedback on an intraday basis is responsible for a volatility clustering of intraday returns.

Statistics of daily close prices is similar to the statistics found in the previous experiment: normal distributions of returns and absence of volatility clustering.
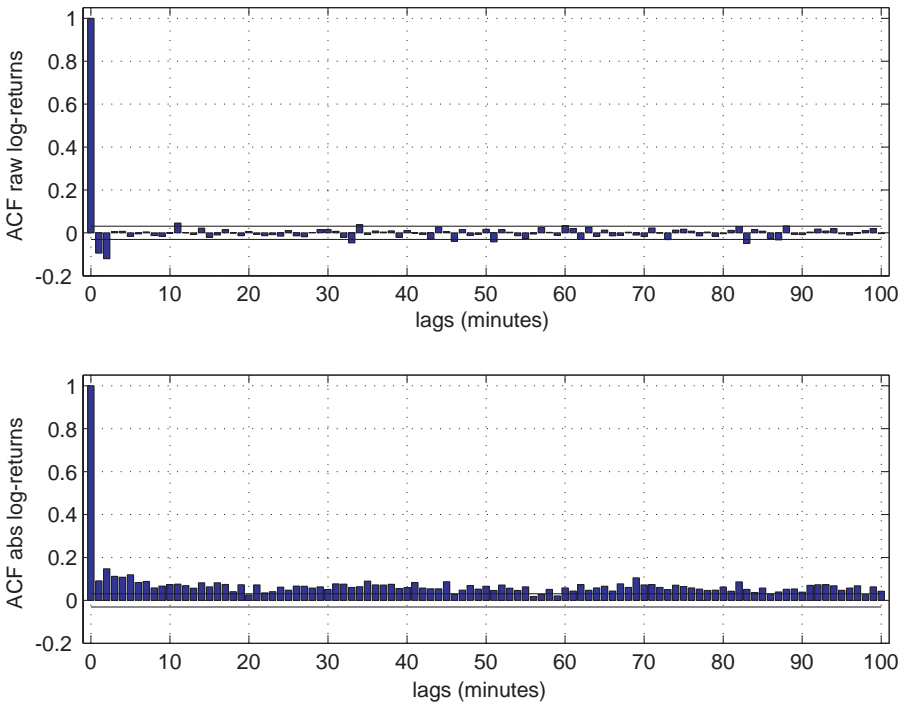
Fig. 6. Autocorrelation function (ACF) of raw intraday returns (top) and of absolute value of intraday returns (bottom) in the experiment with intraday volatility feedback. The ACF of absolute values of log-returns is characterized by a slow decay, thus showing the presence of long-range serial correlation in the magnitude of fluctuations.

2.3. Waiting times distribution of trading activity

We considered the waiting times distribution between two consecutive transactions. It is worth noting that an assumption of the model is that the order generation process is a Poisson process, so that the orders waiting times τ^o are exponential distributed with mean λ^o. In the two experiments reported in previous sections, also time intervals between consecutive trades τ_k are found to follow an exponential distribution. Fig. 7 shows the survival probability distribution of waiting times τ_k between two consecutive intraday transactions; dots represent an estimate of the distribution computed on the artificial time series, the continuous line is an exponential distribution with mean λ equal to the average of observed waiting times τ_k. The null hypothesis that the empirical distribution is well described by an exponential distribution is confirmed by the Anderson–Darling test at the significance level of 5%. The average λ of waiting times between consecutive transactions is of course greater than λ^o because many orders are not executed and are then canceled. In the experiments considered, we found λ equals nearly to 50 s with λ^o set to 20 s.
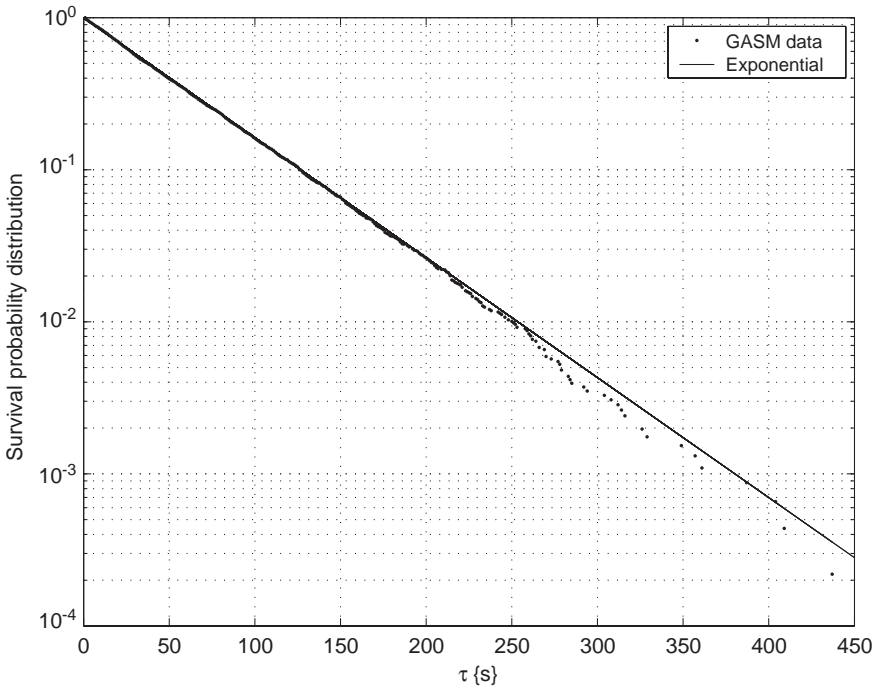
Fig. 7. Survival probability distribution of waiting times τ_k between two consecutive intraday transactions. Dots represent an estimate of the distribution computed on the artificial time series. The continuous line is an exponential distribution with mean λ equal to the average of observed waiting times τ_k.

The process of trading is then a Poisson process as the order arrival process. An identical conclusion follows from theoretical considerations. Consider now that every transaction occurs when a new order that arrives in the book finds a matching order in the queue of orders of the opposite type. Therefore, any new order will be satisfied or not in function of the state of the book at that moment. The state of the book varies for each moment and for each simulation path. However, given the absence of significant feedbacks in the market, it is reasonable to assume that the average state of the book is time invariant. Therefore, each incoming order will be satisfied on average with a constant probability and the trading process can be regarded as a random extraction from the Poisson process of order issuing. The procedure of random extraction from a Poisson process is called "thinning" and it is well known that a thinning from a Poisson process is a new Poisson process; see Ref. [14] for further details.

An empirical analysis made recently by one of the authors (MR) on intraday data of the General Electric stock, traded at the New York Stock Exchange, revealed that waiting times between consecutive trades are well fitted by a stretched exponential [15]. This apparent contradiction may be resolved considering that at the New York

Stock Exchange trading activity is only partially organized through limit orders, i.e., the great majority of trading is performed by means of market orders which are matched by specialist dealers. This different market microstructure can be responsible for this difference in the statistical features of waiting times between consecutive transactions.

3. Discussion and conclusion

This paper presented some computational experiments performed in an artificial financial market where a limit order book is employed as price formation mechanism. The computational experiments show that the fat tails of the returns distribution can be recovered simply as a consequence of the limit order book without any additional assumption on agents behavior. In addition, if a limited heterogeneity is added to agents so that each agent is characterized by his own feedback of market volatility on order decision, then volatility clustering is also recovered. As shown by our previous experiments, an artificial market with the clearing house mechanism does not show fat tails of returns unless specific assumptions are made regarding volatility feedback of wealth distribution among agents. The findings suggest that the microstructure of the market may play a key role in the formation of fat tails of returns.

In addition we have shown that the assumption of exponentially distributed waiting times between orders results in exponentially distributed waiting time, i.e., Poisson process, for trades.

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