A Study of Gödel's Ontological Argument

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I. INTRODUCTION

Kurt Gödel is considered to be one of the greatest logicians of all time. He is best known for two important developments in the history of logic: his completeness theorem [] (proven in 1929 as part of his doctoral thesis []) and his incompleteness theorems (proven in 1931) - due to which which John von Neumann claimed "the subject of logic has certainly completely changed its nature and possibilities" [1]. His work in mainstream logic was "monumental" [1] but perhaps his most ambitious work was his ontological argument; a claimed proof of the existence of God (defined in a slightly unorthodox way[1]).

Herein I present and examine Gödel's Ontological Argument in an atypical order[2], to allow for closer examination of the axioms and hopefully a more pedagogical presentation of the definitions and argument. Furthermore, I provide natural deductions of all theorems.

I assume only a knowledge of first-order logic [] and its associated Gentzen-style natural deduction system [].

Notation and conventions???

II. DEFINITIONS

A. Positive Properties

B. Definition 1: Defining God and God-like Objects

1. Informal Expression of Definition One

The purpose of definition one is to define God-like objects. An object x is denoted as being God-like by the expression $G(x)$ and this is defined, in Eqn. 1, as object x having all positive properties.

2. Logical Expression of Definition One

Definition one is denoted as **Df.1** and can be expressed logically as:

$$G(x) \leftrightarrow \forall\phi(P(\phi) \rightarrow \phi(x)) \tag{1}$$

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[1]

[2] There is a standard presentation that mixed the axioms, definitions, and theorems but here I separate them

3. Discussion of Definition One

How well definition one serves as a definition of God is a very deep question. One that I'm not nearly even close to having the slightest chance in hell of answering.

C. Definition 2: Defining Essence

1. Informal Expression of Definition Two

Definition two defines the essences of an object. A property, ϕ, is said to be an essence of an object, x, (which is denoted as ϕ *ess* x) if x has the property ϕ and x having any property, ψ, implies that necessarily every object with property ϕ has the property ψ i.e. an essence of an object is a property possessed by it and implying any and all of its properties.

2. Logical Expression of Definition Two

Definition two is denoted as **Df.2** and can be expressed logically as:

$$\phi \textit{ ess } x \leftrightarrow \phi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y))) \tag{2}$$

D. Definition 3: Defining Necessary Existence

1. Informal Expression of Definition Three

Definition three defines necessary existence, denoted as E. The necessary existence of an object, x, is defined as for every property of x which is that for any essence of x there necessarily exists an object with that property.

2. Logical Expression of Definition Three

Definition three is denoted as **Df.3** and can be expressed logically as:

$$E(x) \leftrightarrow \forall\phi(\phi \textit{ ess } x \rightarrow \Box\exists y\phi(y)) \tag{3}$$

III. AXIOMS

A. Axiom One

1. Logical Expression of Axiom One

The logical expression of axiom one is denoted as **Ax.1** and is:

$$(P(\phi) \land \Box\forall x(\phi(x) \to \psi(x))) \to P(\psi) \tag{4}$$

2. Informal Expression of Axiom One

3. Discussion of Axiom One

B. Axiom Two

1. Logical Expression of Axiom Two

The logical expression of axiom two is denoted as **Ax.2** and is:

$$P(\neg\phi) \leftrightarrow \neg P(\phi) \tag{5}$$

2. Informal Expression of Axiom Two

Axiom Two can be expressed as any property is either positive, or it's negation is, but not both.

3. Discussion of Axiom Two

C. Axiom Three

1. Logical Expression of Axiom Three

The logical expression of axiom three is denoted as **Ax.3** and is:

$$P(G) \tag{6}$$

2. Informal Expression of Axiom Three

Axiom three is fairly simple, it simply states that being God-like is a positive property.

3. Discussion of Axiom Three

Axiom three is - I believe - actually fairly easy to defend: being God-like is defined as having all positive qualities and so it stands to reason it is itself positive. It would be strange if having positive qualities was not positive.

D. Axiom Four

1. Logical Expression of Axiom Four

The logical expression of axiom four is denoted as **Ax.4** and is:

$$P(\phi) \to \Box P(\phi) \tag{7}$$

2. Informal Expression of Axiom Four

Any property that is positive is necessarily positive. I.e. a positive property is positive in all possible worlds.

3. Discussion of Axiom Four

I think Axiom four is another relatively easy one to defend: it essentially claims that positive properties - e.g. being moral - are not positive by chance; there was not a contingent event that made them positive.

E. Axiom Five

1. Logical Expression of Axiom Five

The logical expression of axiom five is denoted as **Ax.5** and is:

$$P(E) \tag{8}$$

2. Informal Expression of Axiom Five

3. Discussion of Axiom Five

IV. CONSTRUCTION OF REQUIRED RULES

While we start from the standard natural deduction for first order logic (which is known to be both sound and complete), but we are required - to produce natural deductions for Gödel's ontological argument - to reason with both quantification over ??? and with modal operators ($\Box$ and $\Diamond$).
We handle the modal operators first. It may be wise to simply look up a system for handling the required logics but a - very - brief search gave me nothing great and a bit of DIY seems like it could be fun! Also, constructing the required rules myself keeps this manuscript self-contained and pedagogical.

A. Natural deduction and Modal Operators

1. Initial Definitions

Our first step is to define a perspective to take in deriving these rules. We take the set of all possible worlds, $\mathbb{W}$, and define them as the universe of a new logic. This new logic, which we denote as $\mathcal{L}_1'$, corresponds to the first order logic being used in each individual universe (which must be consistent across all possible worlds) which we label $\mathcal{L}_1$ by the rule: For every formula in $\mathcal{L}_1$, ϕ, there exists a corresponding formula for each possible world in $\mathbb{W}$, denoted as $\phi(x)$ where $x \in \mathbb{W}$, in $\mathcal{L}_1'$. $\phi(x)$ is read as ϕ is true in the possible world x. We can then define the two modal operators using quantification over the possible worlds.

Definition 1. For any $\phi \in \mathcal{L}_1$,

$$\Box\phi \equiv \forall x \phi(x) \tag{9}$$

$$\Diamond\phi \equiv \exists x \phi(x) \tag{10}$$

2. *Deriving Functional Rules*

Lemma 1. $\forall A$

$$\neg\Box\neg A \vdash \Diamond A \tag{11}$$

Proof.

$$\begin{prooftree}
\AxiomC{$[A(a)]^1$}
\RightLabel{$\exists$ Intro}
\UnaryInfC{$\exists x A(x)$}
\AxiomC{$[\neg\exists x A(x)]^2$}
\RightLabel{$\neg$ Elim}
\BinaryInfC{$\bot$}
\end{prooftree}$$

□

3. *Soundness*

Lemma 2. *The system of natural deduction for modal operators, defined above, is sound.*

B. Limited Extension to Second Order

V. NATURAL DEDUCTION OF GÖDEL'S ONTOLOGICAL ARGUMENT

A. Derivation of Theorem 1

Informal Statement of Theorem 1. For any positive property, there is at least one possible world where there exists an object with that property.

Theorem 1.

$$\{\textbf{\textit{Ax.1}}, \textbf{\textit{Ax.2}}\} \vdash P(\phi) \rightarrow \Diamond\exists x \phi(x) \tag{12}$$

Proof. Let δ be the derivation entailed to exist by Lemma **??** i.e. the derivation of $\neg\Box\neg A \vdash \Diamond A$, for any A.
Additionally, let δ' be the following derivation:

$$\begin{prooftree}
\AxiomC{$\forall x \neg\phi(x)$}
\RightLabel{$\forall$ Elim}
\UnaryInfC{$\neg\phi(a)$}
\AxiomC{$[\exists x\phi(x)]^1$}
\RightLabel{$\exists$ Elim}
\UnaryInfC{$\phi(a)$}
\RightLabel{$\neg$ Elim}
\BinaryInfC{$\bot$}
\LeftLabel{(1)}
\RightLabel{$\bot_c$}
\UnaryInfC{$\neg\exists x\phi(x)$}
\end{prooftree}$$

This derivation shows that $\forall x \neg\phi(x) \vdash \neg\exists x\phi(x)$ and is then used within a world in the below derivation showing the full theorem.

$$\begin{prooftree}
\AxiomC{$[\Box\forall x\neg\phi(x)]^2$}
\RightLabel{$\Box$ Elim}
\UnaryInfC{$\forall x\neg\phi(x)$}
\RightLabel{$\forall$ Elim}
\UnaryInfC{$\neg\phi(a)$}
\AxiomC{$[\phi(a)]^1$}
\RightLabel{$\neg$ Elim}
\BinaryInfC{$\bot$}
\RightLabel{$\bot_I$}
\UnaryInfC{$\neg\phi(a)$}
\LeftLabel{(1)}
\RightLabel{$\rightarrow$ Intro}
\UnaryInfC{$\phi(a)\rightarrow\neg\phi(a)$}
\RightLabel{$\forall$ Intro}
\UnaryInfC{$\forall x(\phi(x)\rightarrow\neg\phi(x))$}
\AxiomC{$[P(\phi)]^3$}
\RightLabel{$\wedge$ Intro}
\BinaryInfC{$P(\phi)\wedge\forall x(\phi(x)\rightarrow\neg\phi(x))$}
\AxiomC{$\textbf{Ax.1}$}
\RightLabel{$\rightarrow$ Elim}
\BinaryInfC{$P(\neg\phi)$}
\AxiomC{$\textbf{Ax.2}$}
\RightLabel{$\rightarrow$ Elim}
\BinaryInfC{$\neg P(\phi)$}
\AxiomC{$[P(\phi)]^3$}
\RightLabel{$\neg$ Elim}
\BinaryInfC{$\bot$}
\LeftLabel{(2)}
\RightLabel{$\bot_c$}
\UnaryInfC{$\neg\Box\forall x\neg\phi(x)$}
\RightLabel{rww, δ'}
\UnaryInfC{$\neg\Box\neg\exists x\phi(x)$}
\LeftLabel{δ}
\RightLabel{By Lemma 1}
\UnaryInfC{$\Diamond\exists x\phi(x)$}
\LeftLabel{(3)}
\RightLabel{$\rightarrow$ Intro}
\UnaryInfC{$P(\phi)\rightarrow\Diamond\exists x\phi(x)$}
\end{prooftree}$$

□

B. Derivation of Theorem 2

Informal Statement of Theorem 2. There exists a possible world where a God-like object exists.

Theorem 2.

$$\{\textbf{\textit{Th.1}},\textbf{\textit{Ax.3}}\} \vdash \Diamond\exists x G(x) \tag{13}$$

Proof. This proof is a fairly straightforward application of

$$\dfrac{\textbf{Th.1} \qquad \textbf{Ax.3}}{\Diamond\exists x G(x)} \rightarrow \text{Intro}$$

□

C. Derivation of Theorem 3

Informal Statement of Theorem 3. For any God-like object, being God-like is an essential property of that object.

Theorem 3.

$$\{\textbf{\textit{Ax.2}},\textbf{\textit{Ax.4}}\} \vdash G(x) \rightarrow G\ ess\ x \tag{14}$$

Proof. Let δ_2 denote the natural deduction:

$$\dfrac{{\scriptstyle(1)}\dfrac{\dfrac{\dfrac{\dfrac{\textbf{Df.1} \qquad [G(a)]^1}{\forall\phi(P(\phi)\rightarrow\phi(a))}\rightarrow\text{Elim}}{P(\Gamma)\rightarrow\Gamma(a)}\forall\text{Elim} \qquad P(\Gamma)}{\Gamma(a)}\rightarrow\text{Elim}}{G(a)\rightarrow\Gamma(a)}\rightarrow\text{Intro}}{\forall y(G(y)\rightarrow\Gamma(y))}\forall\text{Intro}$$

I'll skip the natural deductions (as they are standard textbook derivations) but let δ_2' be the deduction that for any property, Γ, $\neg\neg\Gamma = \Gamma$ and δ_2'' be that an implication implies its contrapositive.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\textbf{Df.1} \qquad [G(x)]^1}{\forall\phi(P(\psi)\rightarrow\psi(x))}}{P(\neg\Gamma)\rightarrow\neg\Gamma(x)}\forall\text{Elim}}{\Gamma(x)\rightarrow\neg P(\neg\Gamma)}\delta_2'' \qquad [\Gamma(x)]^2}{\neg P(\neg\Gamma)}\rightarrow\text{Elim} \qquad \textbf{Ax.2}}{\neg\neg P(\Gamma)}}{P(\Gamma)}\delta_2' \qquad \textbf{Ax.4}}{\Box P(\Gamma)}}{\Gamma(x)\rightarrow\Box P(\Gamma)}\rightarrow\text{Intro}}{\Box\forall y(G(y)\rightarrow\Gamma(y))}\text{rww}, \delta}{{\scriptstyle(2)}\ \Gamma(x)\rightarrow(\Box\forall y(G(y)\rightarrow\Gamma(y)))}\rightarrow\text{Intro}}{\forall\psi(\psi(x)\rightarrow(\Box\forall y(G(y)\rightarrow\psi(y))))}\forall\text{Intro} \qquad [G(x)]^1}{G(x)\land\forall\psi(\psi(x)\rightarrow(\Box\forall y(G(y)\rightarrow\psi(y))))}\land\text{Intro} \qquad \textbf{Df.2}}{{\scriptstyle(1)}\dfrac{G\ ess\ x}{G(x)\rightarrow G\ ess\ x}\rightarrow\text{Intro}}$$

□

D. Derivation of Theorem 4

Informal Statement of Theorem 4. There exists a God-like object in every possible world.

Theorem 4.

$$\{\textbf{\textit{Ax.5}},\textbf{\textit{Th.3}}\} \vdash \Box\exists y G(y) \tag{15}$$

Proof. Let δ_4 denote the below derivation of $\{\textbf{Ax.5}, \textbf{Th.3}, \exists y G(y)\} \vdash \Box\exists y G(y)$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{[G(a)] \qquad \textbf{Df.1}}{\forall\psi(P(\psi) \to \psi(a))}}{P(E) \to E(a)}\ \forall\ \text{Elim} \qquad \textbf{Ax.5}}{E(a)}\ \to \text{Elim} \qquad \textbf{Df.3}}{\dfrac{\dfrac{\forall\phi(\phi\ ess\ a \to \Box\exists y\phi(y))}{G\ ess\ a \to \Box\exists y G(y)}\ \forall\ \text{Elim} \qquad \textbf{Th.3}}{(1)\ \dfrac{\Box\exists y G(y) \qquad \exists y G(y)}{\Box\exists y G(y)}\ \exists\ \text{Elim}}\ \to \text{Elim}}
$$

Then, $\Box$

$$
\dfrac{\dfrac{\Diamond\exists y G(y)}{\Diamond\Box\exists y G(y)}\ \text{rww},\ \delta_4}{\Box\exists y G(y)}\ \Diamond\Box = \Box
$$

VI. DISCUSSION

VII. ACKNOWLEDGEMENTS

[1] P. R. Halmos, The legend of john von neumann, The American Mathematical Monthly **80**, 382 (1973).