Alternative Taylor Series Derivation

1 Notation

The only notation used herein that may not have been seen at A-level is the use of $f^{(j)}(x)$ to denote the jth derivative of the function $f(x)$.

2 A Useful Integral

Define a function based on an integral:

$$J[x,n] = \int_0^x \Big(p^n f^{(n+1)}(x-p)\Big)dp$$

Then, integrating by parts:

$$\begin{aligned} J[x,n] &= \left[\frac{p^{n+1}}{n+1} f^{(n+1)}(x-p)\right]_0^x + \int_0^x \left(\frac{p^{n+1}}{n+1} f^{(n+2)}(x-p)\right)dp && (1) \\ &= \frac{x^{n+1}}{n+1} f^{(n+1)}(0) + \frac{1}{n+1}\int_0^x \Big(p^{n+1} f^{(n+2)}(x-p)\Big)dp && (2) \\ &= \frac{x^{n+1}}{n+1} f^{(n+1)}(0) + \frac{J[x,n+1]}{n+1} && (3) \end{aligned}$$

3 Deriving Maclaurin Series

The fundamental theorem of calculus states:

$$f(x) - f(0) = \int_0^x \Big(f'(t)\Big)dt \tag{4}$$

Then, noticing that the above integral is equal to $J[x,0]$, as defined above (you can check this using the substitution $t \to x - p$), and rearranging Equation 4:

$$f(x) = f(0) + J[x,0] \tag{5}$$

Then using Equation 3 repeatedly:

$$f(x) = f(0) + \frac{x^1}{1}f'(0) + \frac{J[x,1]}{1} \tag{6}$$
$$= f(0) + xf'(0) + \frac{x^2}{2}f''(0) + \frac{J[x,2]}{2} \tag{7}$$
$$= f(0) + xf'(0) + \frac{x^2}{2}f''(0) + \frac{x^3}{6}f'''(0) + \frac{J[x,3]}{6} \tag{8}$$
$$= f(0) + xf'(0) + \frac{x^2}{2}f''(0) + \frac{x^3}{6}f'''(0) + \frac{x^4}{24}f''''(0) + \frac{J[x,4]}{24} \tag{9}$$

Hopefully you can see this is starting to produce the Taylor series. Via this method, a Taylor series can be seen as a continual application of integration by parts.

4 Taylor Series from Maclaurin Series

Define a new function in terms of the previously established general function:

$$g(x) = f(x - z) \tag{10}$$

Where $z \in \mathbb{R}$ and can be chosen as needed.
For any chosen g a corresponding f exists, and for any chosen, f a corresponding g exists. Then,

$$g(z) = f(0) \tag{11}$$
$$g'(z) = f'(0) \tag{12}$$
$$g''(z) = f''(0) \tag{13}$$
$$g'''(z) = f'''(0) \tag{14}$$
$$g''''(z) = f''''(0) \tag{15}$$
$$g'''''(z) = f'''''(0) \tag{16}$$

and so on. Then, based on Equation 9:

$$g(x+z) = g(z) + (x-z)g'(z) + \frac{(x-z)^2}{2}g''(z) + \frac{(x-z)^3}{6}g'''(z) + \frac{J[x,3]}{6} \tag{17}$$

This gives the Taylor series of g about $x = z$. To calculate this, an appropriate f can always be found.