What I did on my Holidays

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Abstract

A brief investigation into generalized behavior, entanglement, and state transfer in spin networks.

Part I
Introduction: What is a spin network?

Consider a node that has a spin associates with it. The node is pretty much a dot and the spin can be considered to be an arrow pointing either up or down, with it's tail at the dot. However spin is intrinsically quantum, so the spin can be in a state where it has a chance of being in either orientation. This is called a superposition.

Consider a set of these nodes. They are free to interact with each other, which affects the spin in each node. For our purposes we consider the single excitation subspace of the total Hilbert space of all the possible states of the spin network. This subspace is closed under time evolution due to the conservation of energy, so we are free to consider only it. Hence, the entire system can be considered as a superposition of all single excitation states of the system.

Part II
Mathematical Formulation

Consider the set of single excitation states of N nodes: $\{|\underline{i}\rangle\}$, where $|\underline{i}\rangle$ is a pure state where only the ith node is in an excited state and all other nodes are unexcited.

Because the single excitation pure states form a basis of the single excitation subspace, this means that any state of the system in the subspace can then be considered as a superposition of the pure states, this can be expressed mathematically as:

$$|\Psi\rangle = \sum_{j=1}^{N} \alpha_j |\underline{j}\rangle \tag{1}$$

Where $\alpha_j \in C, \forall j$

However, the system must be in one of the states when measured

$$\Rightarrow \sum_{j=1}^{N} |\alpha_j|^2 = 1 \tag{2}$$

Part III
Computational Methods

In this project, two different methods of simulating the behavior of spin networks have been used. One was based on calculating the eigenstates of the Hamiltonian and applying a time evolution, the other was based on solving the Schrödinger equation using the finite difference method.

1 Pre-computational maths

Consider the Schrödinger equation:

$$H|\Psi\rangle = -i\hbar\frac{\partial|\Psi\rangle}{\partial t} \tag{3}$$

Where H is the Hamiltonian of the system
For convenience, we set $\hbar = 1$. Therefore:

$$H|\Psi\rangle = -i\frac{\partial|\Psi\rangle}{\partial t} \tag{4}$$

As mentioned earlier, we can decompose the state to a linear combination of basis states:

$$\Rightarrow H\sum_{j=1}^{N}\alpha_j|\underline{j}\rangle = -i\frac{\partial\sum_{j=1}^{N}\alpha_j|\underline{j}\rangle}{\partial t} \tag{5}$$

$$\Rightarrow \sum_{j=1}^{N}\alpha_j H|\underline{j}\rangle = -i\sum_{j=1}^{N}\frac{\partial(\alpha_j|\underline{j}\rangle)}{\partial t} \tag{6}$$

Considering the partial derivative on the right hand side and applying the product rule:

$$\frac{\partial(\alpha_j|\underline{j}\rangle)}{\partial t} = \frac{\partial\alpha_j}{\partial t}|\underline{j}\rangle + \alpha_j\frac{\partial|\underline{j}\rangle}{\partial t} \tag{7}$$

But all single excitation pure states are constants so

$$\frac{\partial|\underline{j}\rangle}{\partial t} = 0; \qquad \forall j \tag{8}$$

Substituting the partial derivative back into it's original position:

$$\sum_{j=1}^{N}\alpha_j H|\underline{j}\rangle = -i\sum_{j=1}^{N}\frac{\partial\alpha_j}{\partial t}|\underline{j}\rangle \tag{9}$$

$$\Rightarrow \langle\underline{k}|\sum_{j=1}^{N}\alpha_j H|\underline{j}\rangle = -i\langle\underline{k}|\sum_{j=1}^{N}\frac{\partial\alpha_j}{\partial t}|\underline{j}\rangle \tag{10}$$

$$\Rightarrow \sum_{j=1}^{N}\alpha_j\langle\underline{k}|H|\underline{j}\rangle = -i\sum_{j=1}^{N}\frac{\partial\alpha_j}{\partial t}\langle\underline{k}|\underline{j}\rangle \tag{11}$$

As the set of single excitation pure states is a basis for the single excitation subspace:

$$\langle\underline{k}|\underline{j}\rangle = \begin{cases}1 & if k = j\\ 0 & if k \neq j\end{cases} \tag{12}$$

From the matrix formulation of quantum mechanics, we know that:

$$\langle\underline{k}|H|\underline{j}\rangle = \mathbf{H}_{k,j} \tag{13}$$

Where **H** is the matrix representation of H
Applying all of this to equation 11:

$$\sum_{j=1}^{N}\alpha_j\mathbf{H}_{k,j} = -i\frac{\partial\alpha_k}{\partial t} \tag{14}$$

As there are N single excitation pure states, there are N possible values of $\langle\underline{k}|$ that can be used in this process, generating a series of N coupled first order differential equations. We will call these the coefficient equations.

How the different programs go about solving these equations is where they differ.

2 Eigenstates Method

Theorem 2.1. *The system of coupled differential equations:*

$$\bar{\alpha'} = \sum_{x=1}^{N} \boldsymbol{F}\bar{\alpha} \tag{15}$$

Where $\boldsymbol{F}$ is a matrix, $\bar{\alpha}$ is a vector of all the variables, and $\bar{\alpha'}$ is a vector of the derivatives of the variables. Can be satisfied by:

$$\bar{\alpha} = \sum_{x=1}^{N} A_x \bar{V}_x e^{\lambda_x t} \tag{16}$$

Where $\bar{V}_x$ is the xth eigenvector of $\boldsymbol{F}\lambda_x$ is the corresponding eigenvalue, and A_x are constants that can take any value (Used to meet initial conditions)

Proof. Calculating the derivative of the ansatz, the left hand side of equation 15:

$$\frac{\partial}{\partial t}(\bar{\alpha}) = \sum_{x=1}^{N} A_x \bar{V}_x \lambda_x e^{\lambda_x t} \tag{17}$$

Applying $\mathbf{F}$ to the ansatz, to get the right hand side:

$$\mathbf{F}\bar{\alpha} = \mathbf{F}\sum_{x=1}^{N} A_x \bar{V}_x e^{\lambda_x t} = \sum_{x=1}^{N} A_x (\mathbf{F}\bar{V}_x) e^{\lambda_x t} = \sum_{x=1}^{N} A_x (\lambda_x \bar{V}_x) e^{\lambda_x t} \tag{18}$$

$\Rightarrow$ The ansatz satisfies all the differential equations □

Theorem 2.2. *The Schrödinger equation, for a particular system, is satisfied by:*

$$\bar{\alpha} = \sum_{x=1}^{N} A_x \bar{V}_x e^{i\lambda_x t} \tag{19}$$

Where $\bar{V}_x$ is the xth eigenvector of the Hamiltonian of the system λ_x is the corresponding eigenvalue, and A_x are constants

Proof. Consider the Schrödinger equation:

$$H|\Psi\rangle = -i\frac{\partial|\Psi\rangle}{\partial t} \tag{20}$$

This can be rearranged to:

$$iH|\Psi\rangle = \frac{\partial|\Psi\rangle}{\partial t} \tag{21}$$

Let G = iH

The question is then: how do the eigenvectors and eigenvalues of G and H relate to each other.

Clearly, they share all eigenvectors. As for eigenvalues:

Let $\bar{V}$ be a shared eigenvector. When H operates on $\bar{V}$, the eigenvalue is λ and when G operates on $\bar{V}$, the eigenvalue is $\lambda\prime$:

$$G\bar{V} = \lambda\prime\bar{V} \tag{22}$$

and

$$iH\bar{V} = i\lambda\bar{V} \tag{23}$$

$$\Rightarrow \lambda\prime = i\lambda \tag{24}$$

If we now consider Schrödinger's equation using G:

$$G|\Psi\rangle = \frac{\partial|\Psi\rangle}{\partial t} \tag{25}$$

Then we can apply Theorem 2.1:

$$\bar{\alpha}(t) = \sum_{x=1}^{N} A_x \bar{V}_x \exp(\lambda\prime_x t) \tag{26}$$

We can then get the solutions for H by exchanging $\lambda\prime_x$ for $i\lambda_x$:

$$\bar{\alpha}(t) = \sum_{x=1}^{N} A_x \bar{V}_x \exp(i\lambda_x t) \tag{27}$$

□

Now that it has been shown that any function of the form given above satisfies Schrödinger's equation, all that remains is to select the values of the A_xs such that the initial conditions are met.

Theorem 2.3. *Let $\bar{A}$ be a vector of the values of the A_xs that make the solutions fit the initial conditions. If $\boldsymbol{M}$ is a matrix where the xth column is the normalized xth eigenvector of H, and $\bar{I}$ is a vector representing the initial conditions of the system. Then:*

$$\boldsymbol{M}^{-1}\bar{I} = \bar{A} \tag{28}$$

Assuming $\boldsymbol{M}^{-1}$ exists

Proof. As shown previously:

$$\bar{\alpha}(t) = \sum_{x=1}^{N} A_x \bar{V}_x e^{i\lambda_x t} \tag{29}$$

Setting t = 0 means that by the definition of I:

$$\bar{\alpha}(0) = \sum_{x=1}^{N} A_x \bar{V}_x = \bar{I} \tag{30}$$

By consideration of the operation of matrices on vectors, it can been seen that:

$$\mathbf{M}\bar{A} = \bar{I} \tag{31}$$

$$\Rightarrow \mathbf{M}^{-1}\bar{I} = \bar{A} \tag{32}$$

This uniquely defines $\bar{A}$ □

Hence we have now rigorously devised a method for deriving the equations of a quantum system.

3 Iterative Method

The iterative method was more numeric. Starting from Schrödinger's equation (with $\hbar = 1$):

$$H|\Psi\rangle = -i\frac{\partial|\Psi\rangle}{\partial t} \tag{33}$$

A computer can only deal with a finite number of data points, so it's concept of a function can only be the values it takes at specific points. If a function is not continuous at a point then it cannot be differentiable at that point. At any of the data points, an arbitrarily small change in the input variable results in a input where the function is not defined. Therefore, the data points are not continuous at any point so are not differentiable at any point. This all means that there is no way for a computer to properly evaluate a derivative. The way around this is a method called the finite difference, you calculate the average gradient over a small, but finite, step δt:

$$\frac{\partial f(t)}{\partial t} \approx \frac{f(t+\delta t) - f(t)}{\delta t} \tag{34}$$

Substituting this approximation into the coefficient equations (equation 15):

$$\sum_{j=1}^{N} \alpha_j \mathbf{H}_{k,j} \approx -i \left(\frac{\alpha_j(t+\delta t) - \alpha_j(t)}{\delta t} \right) \tag{35}$$

This can be rearranged to:

$$\alpha_j(t+\delta t) \approx \alpha_j(t) + i\delta t \left(\sum_{j=1}^{N} \alpha_j \mathbf{H}_{k,j} \right) \tag{36}$$

This can be used to iteratively evaluate the solution to Schrödinger's equation.

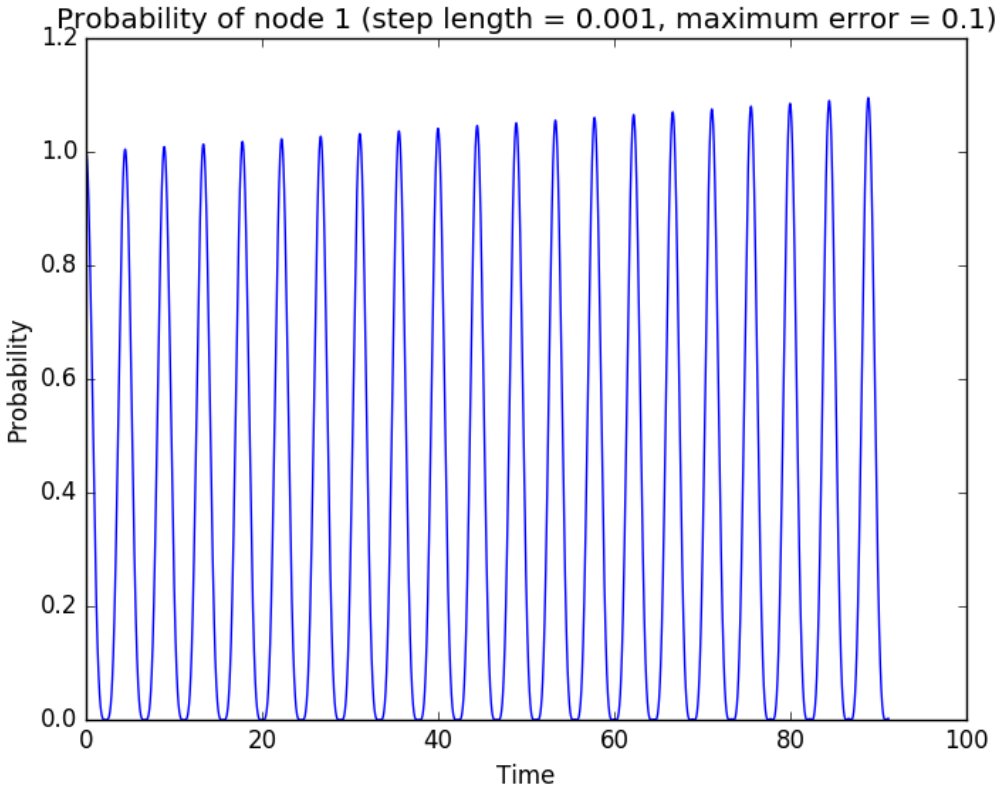
4 Comparison of the two methods

Each computational method takes a very different approach to solving the coefficient equations. The eigenvector method finds an analytic solution with the parameters of the equation derived numerically, while the iterative method applied the effects of the differential equations to the initial conditions, and then all subsequent states.

The main problem with the eigenvector method, is that calculating the constants ($\bar{(}A)$) relies on being able to invert the Hamiltonian matrix and this is not possible for singular matrices.

In particular, I looked at a lot of systems in this project where the on-site energies of the nodes were disregarded. This meant that the diagonal elements of their Hamiltonians were all zeros, which made the Hamiltonian singular. So this method failed.

The iterative method doesn't have this problem, and works regardless of the Hamiltonian and initial conditions, it doesn't provide an analytic solution, but rather provides the values of the solution. Each value depends entirely on the values at the previous time (as we only have discrete values of time). This means that errors compound very quickly and the solution provided has a tendency to be stretched vertically as time increases (as can be seen below).



5 Programs used

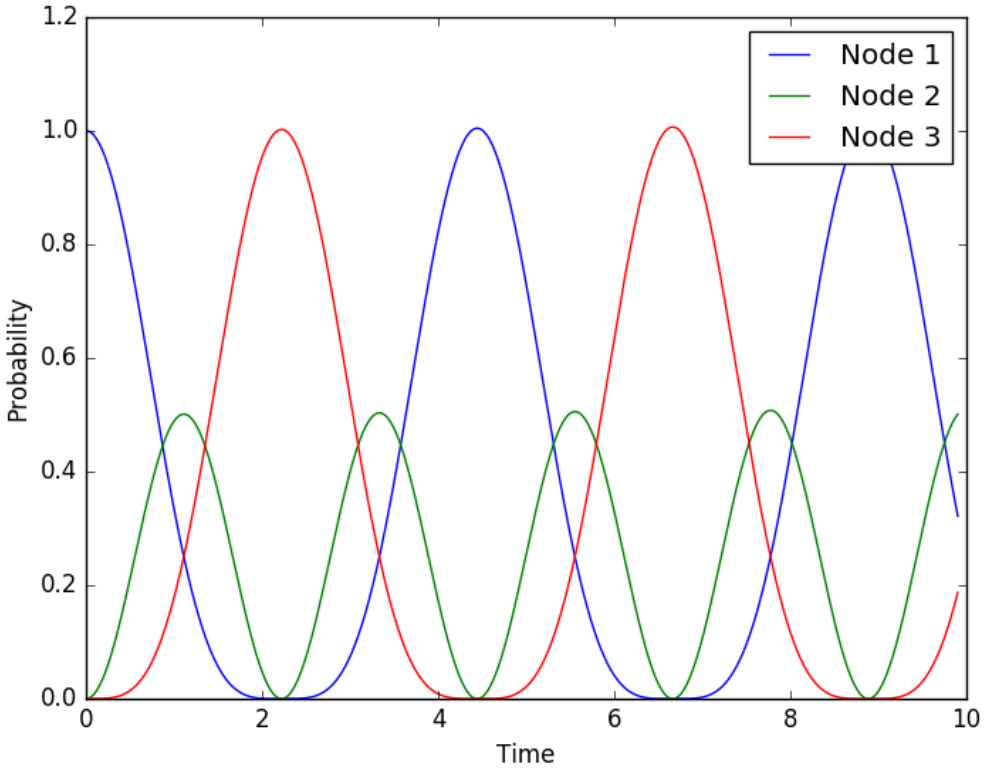
For the purposes of this report, 3 programs were written. One carried out the eigenstate method mentioned above, using numerical approximations of the inverse, eigenvalues,and eigenvectors of the Hamiltonian, to derive analytic solutions. There were then two programs using the iterative method. One was written in python, and used the method to calculate plots of the state of the superposition, the probability of the excitation being in each node, the error in it's own predictions, the entanglement between the nodes in the system. Unless stated otherwise, this program is used in this report. With the allowed error set to 0.01, and the step length as 0.001.

The other program was written in Fortran, and would only calculate the coefficient of a single node in the schmidt decomposition of the superposition. This was to allow the program to calculate the node's behavior with much smaller step lengths and/or with much larger maximum allowed error, in a reasonable amount of time.

Part IV
Testing the program

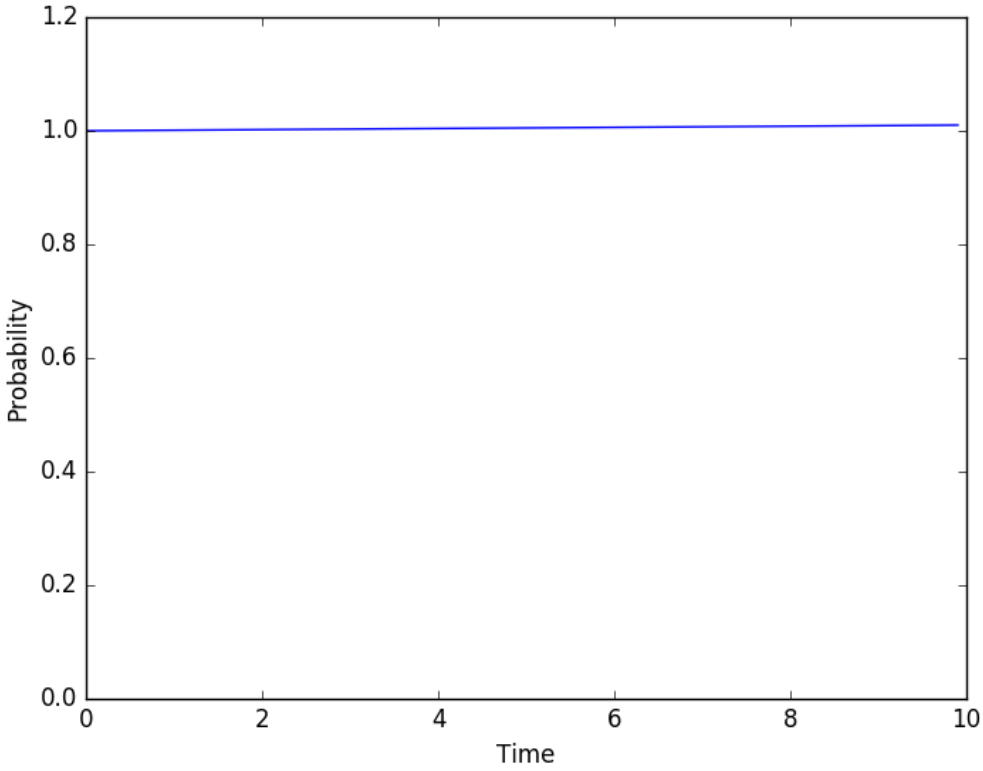
6 Trimer

One of the simplest spin networks we can look at is a trimer initialized with the excitation in the first node:



An interesting point to note is the periodicity of the graph. This tells us that the system undergoes a cyclic time evolution, with no noticeable change in the cycle. It's also interesting to note that the middle node (node 2) never has a probability greater than 0.5.

By plotting the sum of the probabilities of the excitation being in each node, we can consider how accurate the simulation is:



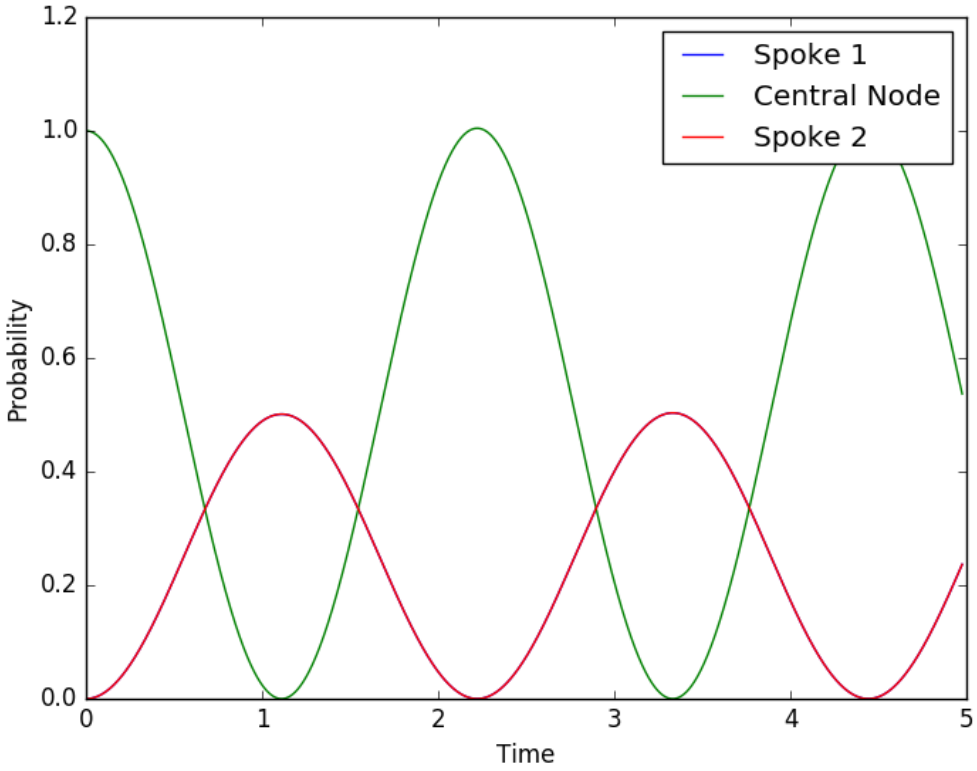
It can clearly be seen that the total probability in the system is always roughly equal to 1 in the time frame being examined,so the simulation appears to be a good approximation across the time frame being examined. we can also have confidence in the simulation over greater time spans as the gradient of the total probability is very low

Part V
Dynamics of N-spoke systems

7 Central node with N spokes, initialized with the excitation in the central node

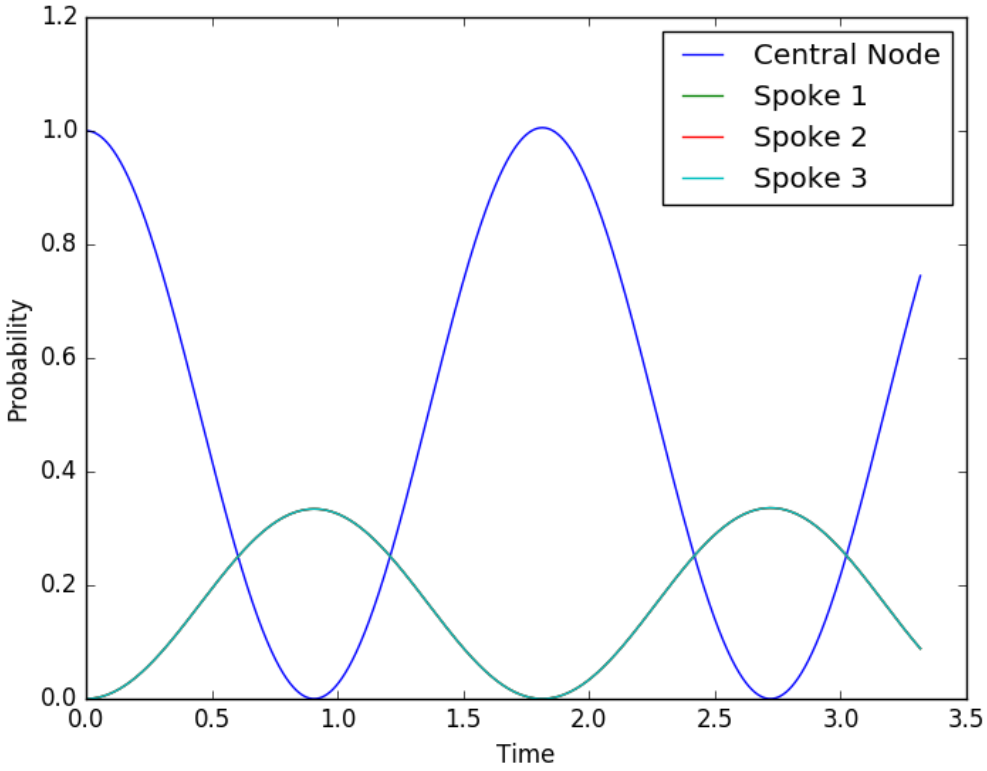
The first thing we can say about these networks is that, because of the symmetry of the system, the equations to describe each node at the end of a spoke is the same.

The simplest example in this category is actually just the trimer: it's a central node with 2 spokes:

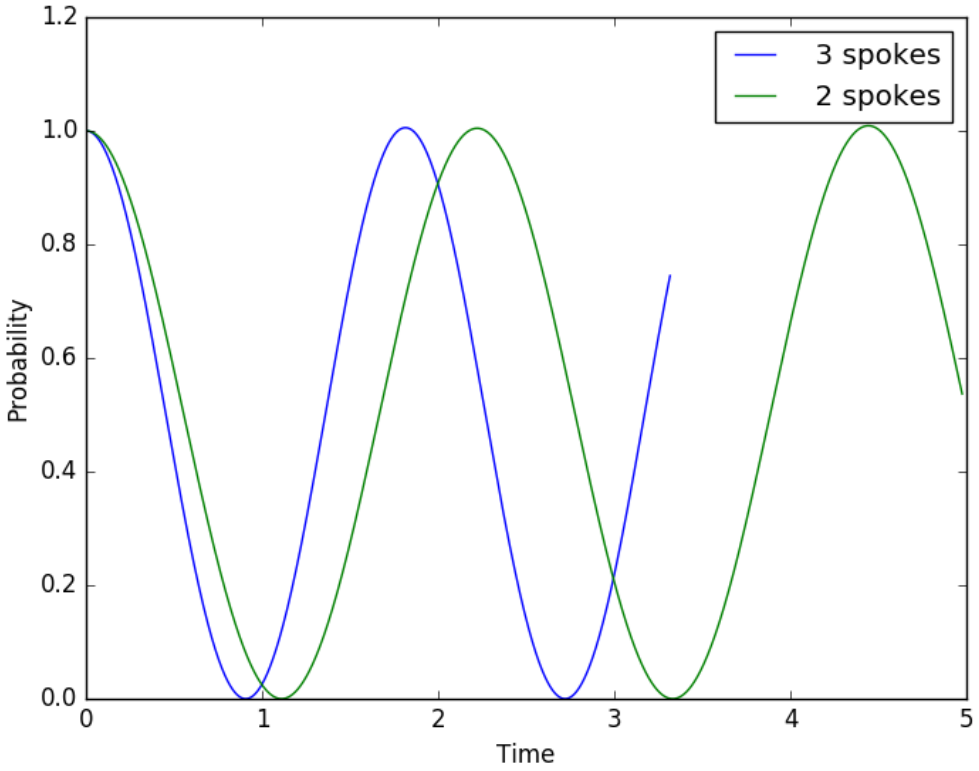


It is immediately noticeable that, in accordance with the symmetry prediction, the probability of the excitation being in each spoke is exactly the same for both of the spokes. It can also be seen that the maximum probability for the central node coincides with the minimum probability for the spokes, and vice versa. This happening cyclically indicates that the the probability flows from the central node to the spokes completely and equally, and then back again, in a loop.

The next step would be then to test whether these are general patterns for this category of spin networks by simulating the same system but with an extra spoke. We can then also consider the differences that the extra spoke causes.

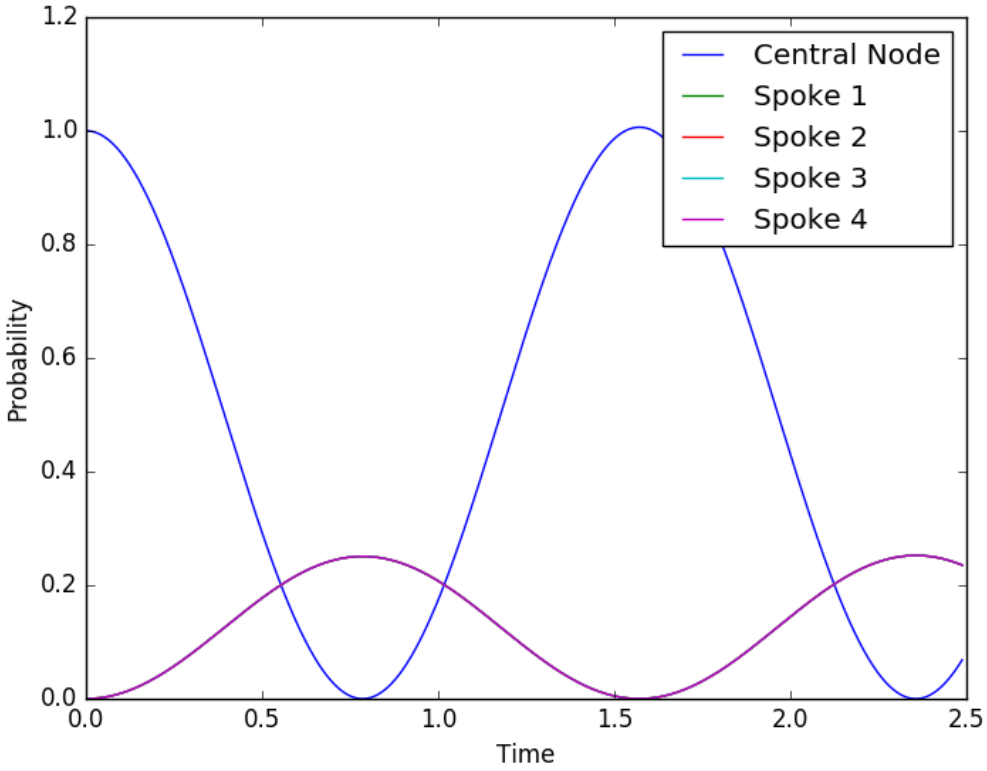


Having plotted the graph, we can see the that all the observations from the last simulations hold, so we can infer that similar dynamics are happening. In addition, we can see that the maximum values of the spokes' probability has decreased. Looking at two separate graphs makes it hard to see more subtle differences. If we plot the central node of the two networks on the same graph, we can investigate these differences:



It is easily visible, that the two curves start at the same points but the difference between them increases with time. This indicates that the 3 spoke system takes less time to complete one cycle than the 2 spoke system. This suggests that it takes longer for probability to flow into 2 spokes than it does to flow into 3 spokes and raises questions as to whether there is a maximum capacity of probability that can flow through an interaction, in a finite time.

Finally, it's worth plotting a 4 spoke system, in the hope that we will then have enough data points to start quantifying the patterns mentioned above. A 4 spoke system looks like:



We then read off the values of the period and maximum value of the spokes' probability from each simulation:

no. of spokes	period	max spoke probability
2	4.4	0.5
3	3.6	0.33
3	3.1	0.25

The simplest equations that fit these values are:

$$\text{The period} = \frac{2\pi}{\sqrt[2]{N}} \tag{37}$$

and, for a given spoke:

$$\text{Max probability of a spoke} = \frac{1}{N} \tag{38}$$

All that remains is to test this hypothesis.

From our previous simulations, we can predict that the behavior will be periodic, and that the probability will flow from all being in the central node to being equally distributed amongst the spokes and then flow back again.

But what would our equations predict that the period and the maximum probability of a given spoke is?

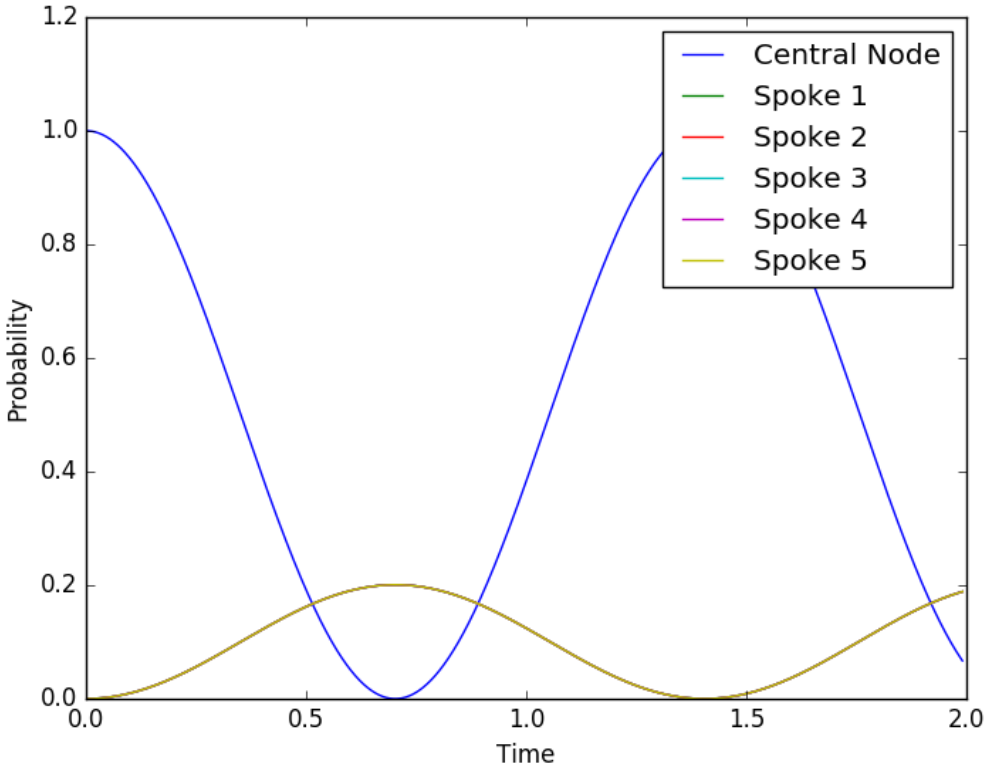
If N = 5,

$$\text{The period} = \frac{2\pi}{\sqrt[2]{N}} = \frac{2\pi}{\sqrt[2]{5}} \approx 2.81 \tag{39}$$

and

$$\text{Max probability of a spoke} = \frac{1}{N} = \frac{1}{5} = 0.2 \tag{40}$$

When the simulation runs it produces:



This exactly confirms the predictions of the equations, so we can be pretty confident in them. From an analytic perspective, it is possible to derive
Hypothesis

$$\alpha_1(t) = \frac{e^{\sqrt[2]{N}Jit} + e^{-\sqrt[2]{N}Jit}}{2} \tag{41}$$

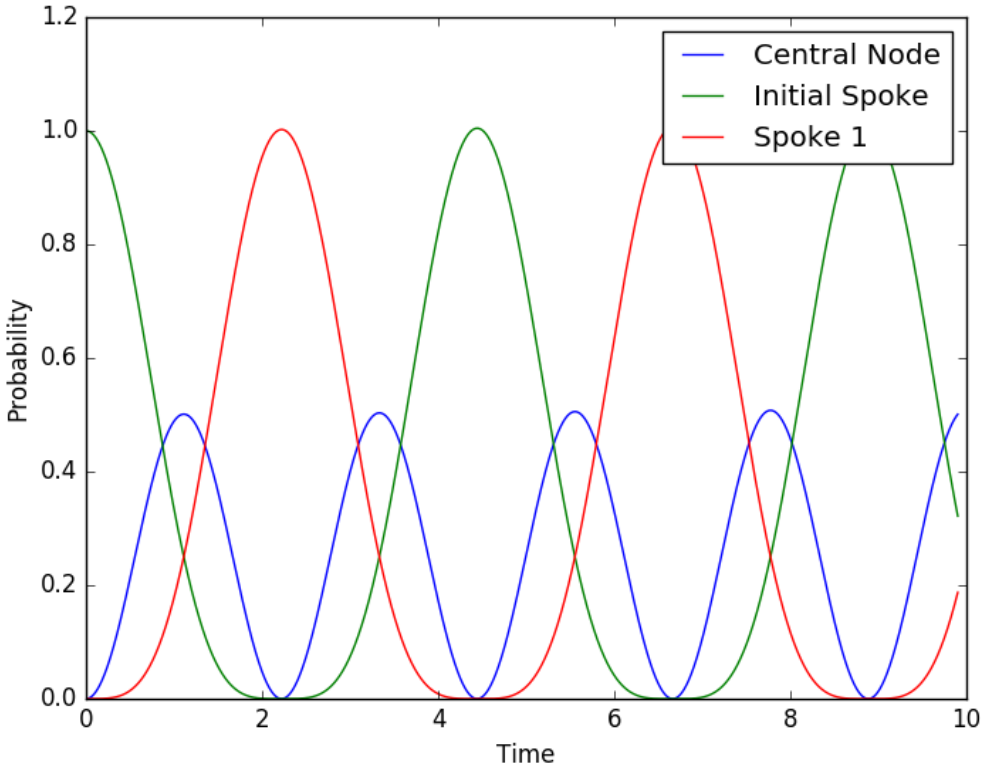
and

$$\alpha_{2,3,\cdots,N}(t) = \frac{e^{\sqrt[2]{N}Jit} - e^{-\sqrt[2]{N}Jit}}{2\sqrt[2]{N}} \tag{42}$$

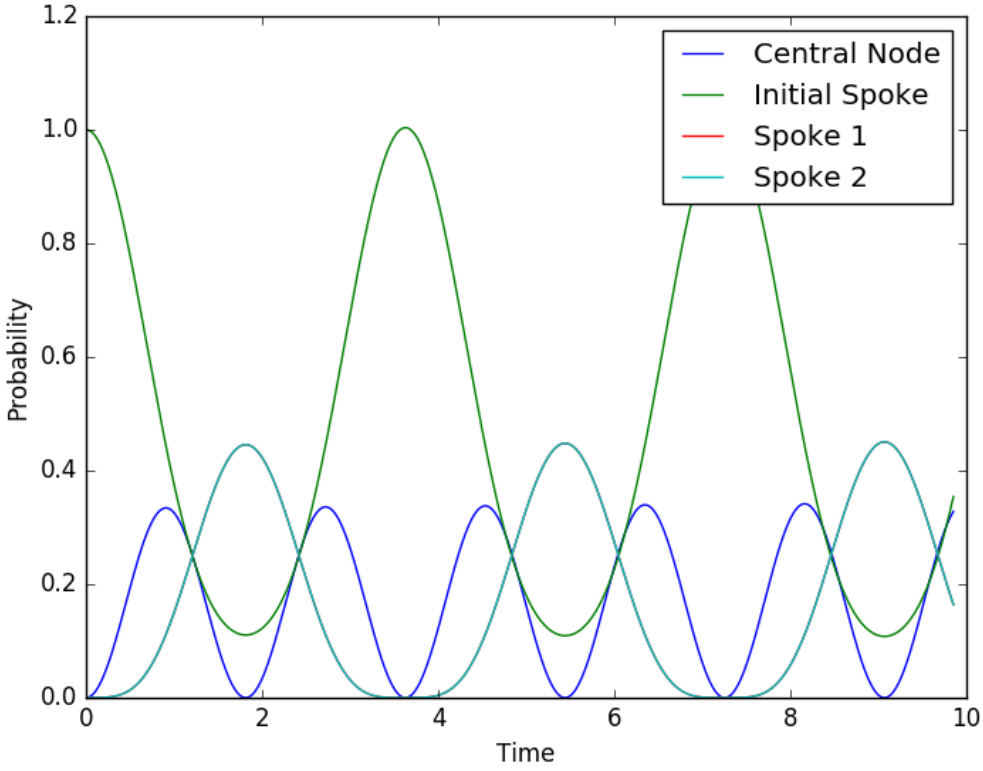
Where node 1 is the central node

8 Central node with N spokes, initialized with the excitation in a spoke

We again simulate the systems with 2, 3,and 4 spokes but this time the excitation is initialized in a spoke. We call this spoke the initial spoke.

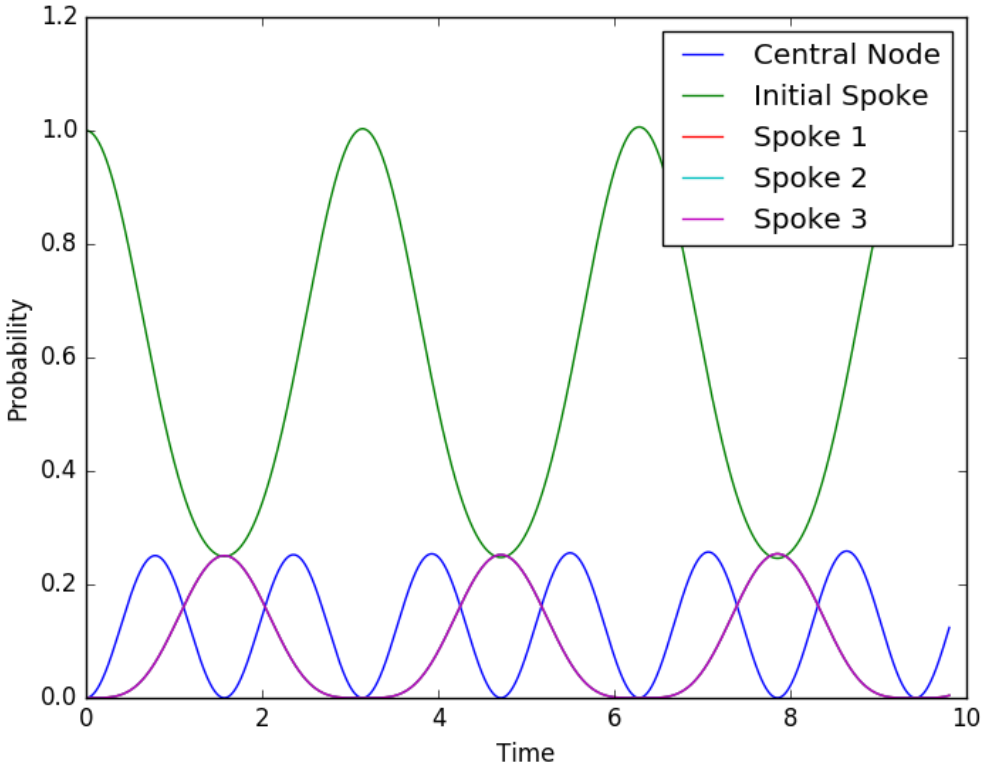


The first system, a two spoke system, is exactly the same, physically, as the trimer (Initialized with the excitation at one end) that we considered before, and the graph reflects this.



The second system, this time with 3 spokes, sees the maximum value of the spokes without excitations initially decrease dramatically as the probability must now be shared more.

Additionally, the initially excited spoke's curve begins to lift up off of the x axis. This means that at any time, there is always a chance that the excitation will be found in the initially excited spoke.



When we look at the graph for a system with 4 spokes, the thing that really jumps out is the central node's maximum probability and the initially unexcited nodes' maximum probability are equal, and this is also equal to the minimum probability of the initially excited nodes.

Hypothesis

WLOG say that the excitation is initialized at node 2 and that node 1 is the central node

$$\alpha_1(t) = \frac{e^{\sqrt[2]{N}Jit} - e^{-\sqrt[2]{N}Jit}}{2\sqrt[2]{N}} \tag{43}$$

$$\alpha_2(t) = \frac{e^{\sqrt[2]{N}Jit} + e^{-\sqrt[2]{N}Jit}}{2N} + \frac{N-1}{N} \tag{44}$$

$$\alpha_{3,4,\cdots,N}(t) = \frac{e^{\sqrt[2]{N}Jit} + e^{-\sqrt[2]{N}Jit}}{2N} - \frac{1}{N} \tag{45}$$

It's interesting to note that the equation for the central node's coefficient in this case is the same as the equation for a node on a spoke in the case where the excitation is initialized in the central node.

It's also worth noting that the equation for the spoke where the excitation is initialized is the equation for the other spokes + 1

But finally as N $\to \infty$

$$\alpha_2(t) \to 1 \tag{46}$$

Part VI
Entanglement

9 What is entanglement?

Entanglement is a measure of how holistic a system is. To say two particles are entangled is to say that to study either of them separately would provide a very incomplete understanding it: The system must be studied as a whole.

An example of this would be a system where measuring the spin of one particle guarantees the results of measuring the spin of the other.

This would be true if the system were in a bell state:

$$|\Phi^+\rangle = \frac{1}{\sqrt[2]{2}}(|00\rangle + (|11\rangle) \tag{47}$$

$$|\Phi^-\rangle = \frac{1}{\sqrt[2]{2}}(|00\rangle - |11\rangle) \tag{48}$$

$$|\Psi^+\rangle = \frac{1}{\sqrt[2]{2}}(|01\rangle + |10\rangle) \tag{49}$$

$$|\Psi^-\rangle = \frac{1}{\sqrt[2]{2}}(|01\rangle - |10\rangle) \tag{50}$$

The bell states are the states with the most entanglement. So any measure of entanglement must be maximized by the bell states.

Example

Assume the system is in state $|\Phi^+\rangle$, and the first qubit has been measured to be a 1

If we were looking only at the second qubit, with no idea about the existence of the first qubit, then we would have no idea how a measurement of the second qubit will turn out and would assume the outcome to be probabilistic.

However, if we look at the system as a whole:

$$|\Phi^+\rangle = \frac{1}{\sqrt[2]{2}}(|00\rangle + (|11\rangle) \tag{51}$$

The measurement of the first qubit as a 1 makes the $|00\rangle$ state impossible. So we can be sure that the overall state of the system is $|11\rangle$, which means that without measuring it, we can be sure that the second qubit is in the state: $|1\rangle$

10 Derivation of a measurement of entanglement

Any 2 particle state can be Schmidt decomposed into:

$$|\Psi^+\rangle = A|00\rangle + B|01\rangle + C|10\rangle + D|11\rangle \tag{52}$$

The state above is written in the j basis: $|00\rangle, |10\rangle, |01\rangle, |11\rangle$. However, from a quick inspection, we can see that the bell states are mutually orthogonal and, because they are all normalized, the bell states are also orthonormal. There are 4 bell states and (as there are 4 basis states in the j basis) the 2 qubit vector space is 4 dimensional. Any N orthonormal vectors in a N dimensional vector space form a basis for that vector space. So we can see that the bell states form a basis. Call this the bell basis, B, and in this basis:

$$|\Psi\rangle = \alpha|\Phi^+\rangle + \beta|\Phi^-\rangle + \gamma|\Psi^+\rangle + \delta|\Psi^-\rangle \tag{53}$$

And the change of basis matrix, C_j^B, is given by:

$$C_j^B = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \tag{54}$$

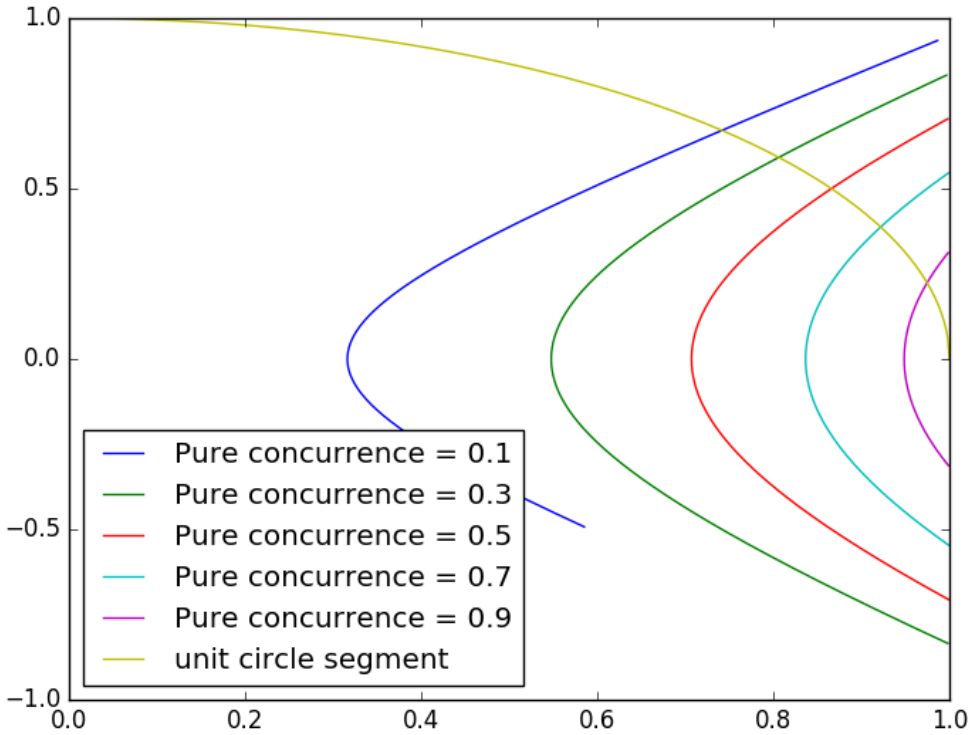
$$\Rightarrow C_j^B \bar{A} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} \tag{55}$$

$$\Rightarrow \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} A+D \\ A-D \\ B+C \\ B-C \end{bmatrix} \tag{56}$$

In the bell basis, the problem of figuring out how entangled a state is has been simplified to how close that state is to a basis state or an anti-basis state (a basis state multiplied by -1) as multiplying the state by -1 still gives the same probability amplitudes.

To consider how we might measure the "closeness to a basis or anti-basis," we first look at a lower dimensional example. In R**2, we look for functions roughly surrounding the axis's. A good choice is the hyperbola:

$$|x^2 - y^2| = \eta \tag{57}$$



Any point on the curve $x^2 + y^2 = 1$, the unit sphere, intersects a hyperbola for some value of η. We can see that if the value of is increased, then the hyperbolic curves all move away from the origin and the points at which the hyperbola intersects the unit sphere move closer to the basis's. So the value of η can be seen as a measure of entanglement.

The simplest way to expand this up to 4 dimensions is to use the 2 dimension case twice:

$$|\left(\alpha^2 - \beta^2\right) + \left(\delta^2 - \gamma^2\right)| = \eta \tag{58}$$

We can then use the relations we derived earlier relating the coordinates from the two basis':

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} A+D \\ A-D \\ B+C \\ B-C \end{bmatrix} \tag{59}$$

Substituting this into the measure of entanglement

$$\eta = \left(\frac{\sqrt{2}}{2}\right)^2 |(A+D)^2 - (A-D)^2 - (B+C)^2 + (B-C)^2| \tag{60}$$

$$\Rightarrow \eta = \left(\frac{1}{2}\right) |A^2 + 2AD + D^2 - A^2 + 2AD - D^2 - B^2 - 2BC - C^2 + B^2 - 2BC + D^2| \tag{61}$$

$$\Rightarrow \eta = \left(\frac{1}{2}\right) |4AD - 4BC| \tag{62}$$

$$\Rightarrow \eta = 2|AD - BC| \tag{63}$$

This is called the pure concurrence.

In much of this project we will be looking at the single excitation subspace. This means that the state $|11\rangle$ is impossible and so the value of D will always be zero. Therefore, the pure concurrence in the single excitation subspace is given by:

$$\eta = 2|BC| \tag{64}$$

11 Properties of pure concurrence

The first priority in establishing the properties of the pure concurrence is to question what it's maximum and minimum values are. We know that the Bell states are maximally entangled. Calculating the pure concurrence for the $|\Phi^+\rangle$ state:

$$|\Phi^+\rangle = \frac{1}{\sqrt[2]{2}}(|00\rangle + (|11\rangle) \tag{65}$$

$$\Rightarrow PC(|\Phi^+\rangle) = 2|\frac{1}{\sqrt[2]{2}}^2| \tag{66}$$

$$PC(|\Phi^+\rangle) = 1 \tag{67}$$

So the maximum value of pure concurrence is 1 If we then engineer a state with minimum entanglement

$$|\Psi\rangle = \frac{1}{\sqrt[2]{2}}(|10\rangle + |11\rangle) \tag{68}$$

We can see that this state is minimally entangled as if we measure the first qubit it's guaranteed to be in the $|1\rangle$ state but this tells us nothing about the state of the second qubit.

$$PC(|\Psi\rangle) = 0 \tag{69}$$

So we know that the minimum value of the pure concurrence is zero.

Theorem 11.1. *For any value of the pure concurrence, it is possible to select a state of two qubits, where all the coefficients are real, with that pure concurrence.*

Proof. Let η be the given pure concurrence.

$$\Rightarrow 0 \leq \eta \leq 1 \tag{70}$$

We can disregard the modulus sign as

$$BC - AD = \eta \Rightarrow |BC - AD| = \eta \tag{71}$$

We aim for:

$$BC = (1+g)\eta \tag{72}$$

$$AD = g\eta \tag{73}$$

Where $g \in R$
We then set

$$A = Jg\eta \tag{74}$$

$$D = \frac{g\eta}{J} \tag{75}$$

$$B = h(1+g)\eta \tag{76}$$

$$C = \frac{(1+g)\eta}{h} \tag{77}$$

Where J, h $\in$ R

This all guarantees that any solution we can find has the desired pure concurrence. We also require:

$$A^2 + B^2 + C^2 + D^2 = 1 \tag{78}$$

Substituting in the equations above:

$$J^2g^2\eta^2 + h^2(1+g)^2\eta^2 + \frac{(1+g)^2\eta^2}{h^2} + \frac{g^2\eta^2}{J^2} = 1 \tag{79}$$

$$\Rightarrow g^2\left(J^2 + \frac{1}{J^2}\right) + (1+g)^2\left(h^2 + \frac{1}{h^2}\right) = \frac{1}{\eta^2} \tag{80}$$

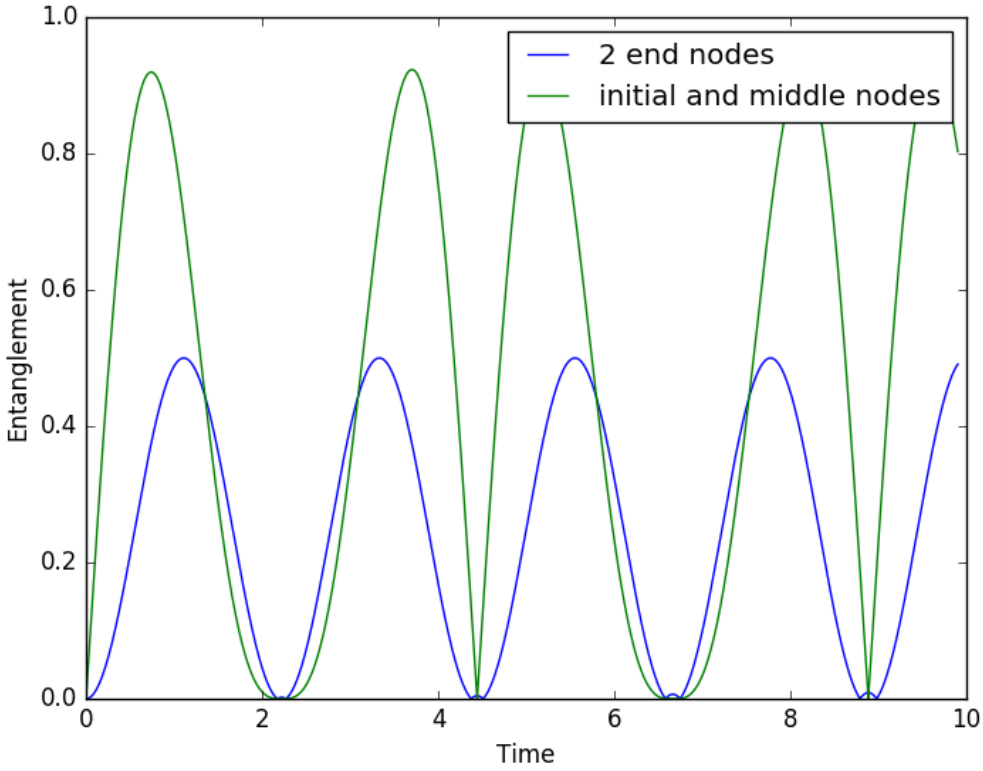
Then we need to show which values of η this equation can be satisfied for. We can calculate which values it is possible for $\frac{1}{\eta^2}$ to take

$$0 \leq \eta \leq 1 \Rightarrow 1 \leq \frac{1}{\eta^2} \leq \infty \tag{81}$$

This has been shown, in the appendix, to be surjective □

12 Trimer

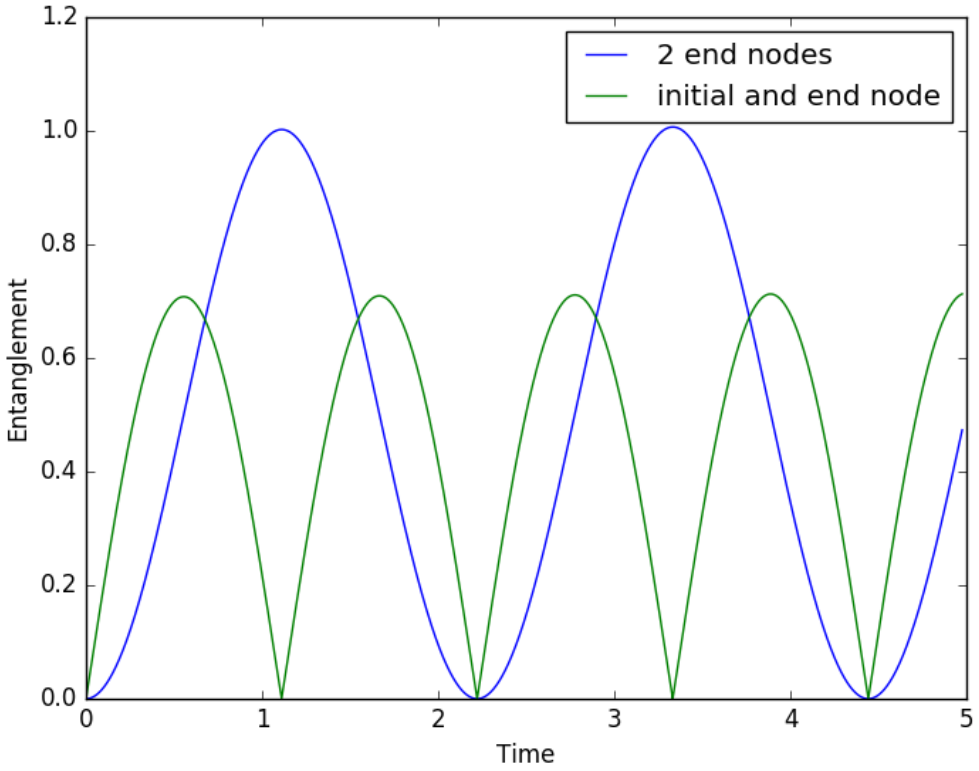
We again start with the trimer to get a baseline and to see what interesting trends we can find. However there are now more graphs that it's possible to plot as you must choose 2 nodes to measure the entanglement between. This means that for a N node network there are C_2^N possible possible graphs to plot, for each initial condition. To investigate which nodes it is most useful to measure the entanglement between we plot the entanglement between various pairs of nodes:



We don't plot the entanglement between the middle node and the edge node not initially excited, as we know from earlier that it would only the the graph from the middle node and the initially excited node shifted in time.

Looking at the graph, the entanglement between the two end nodes never gets above 0.5 but the entanglement between the initial node and the middle node can be much higher but it doesn't reach 1 either.

We can also initialize the system with the excitation in the middle nodes. By symmetry, we know that the middle node's entanglement with either end node is exactly the same, so we only plot one:

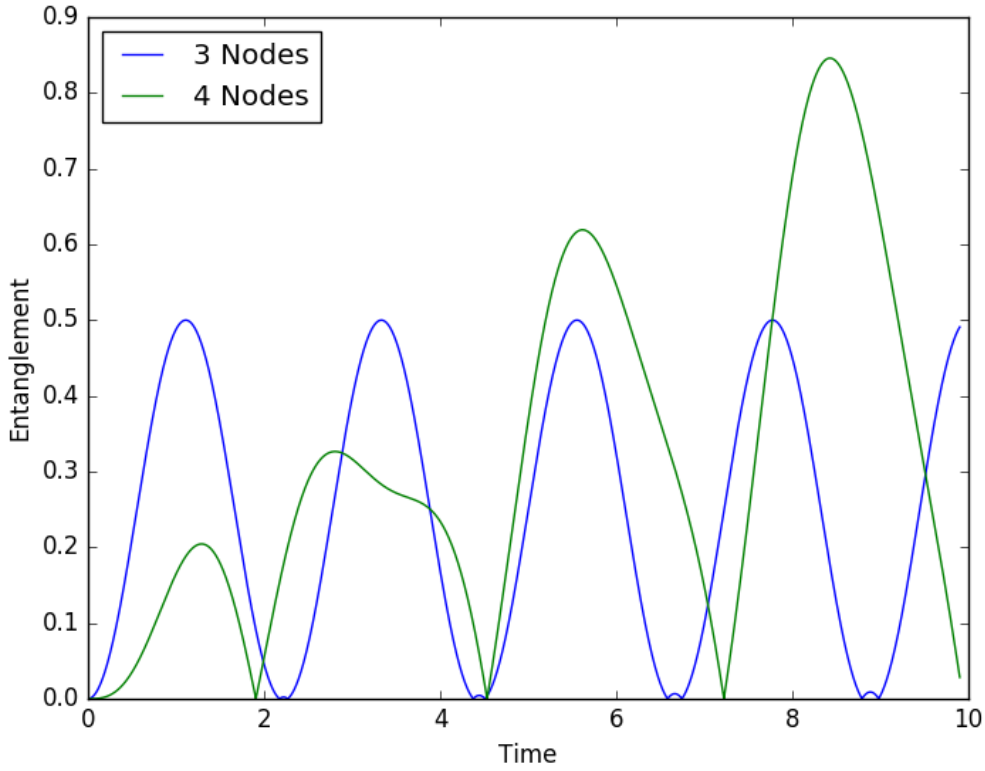


This has similar patterns to the previous graphs, however the difference in maximum probability is not as marked.

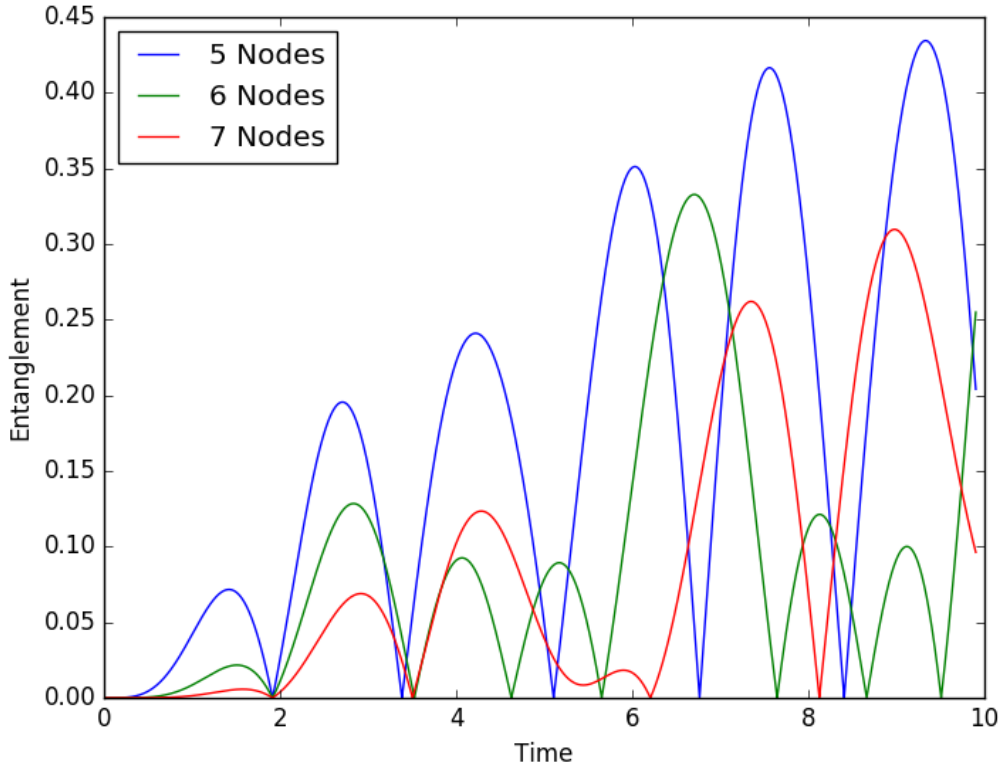
13 Entanglement between the ends of linear chains

A question that immediately comes to mind is how is the entanglement between the end nodes is affected by having a longer spin chain.

Comparing the entanglement of the end nodes for 3 and 4 node systems:

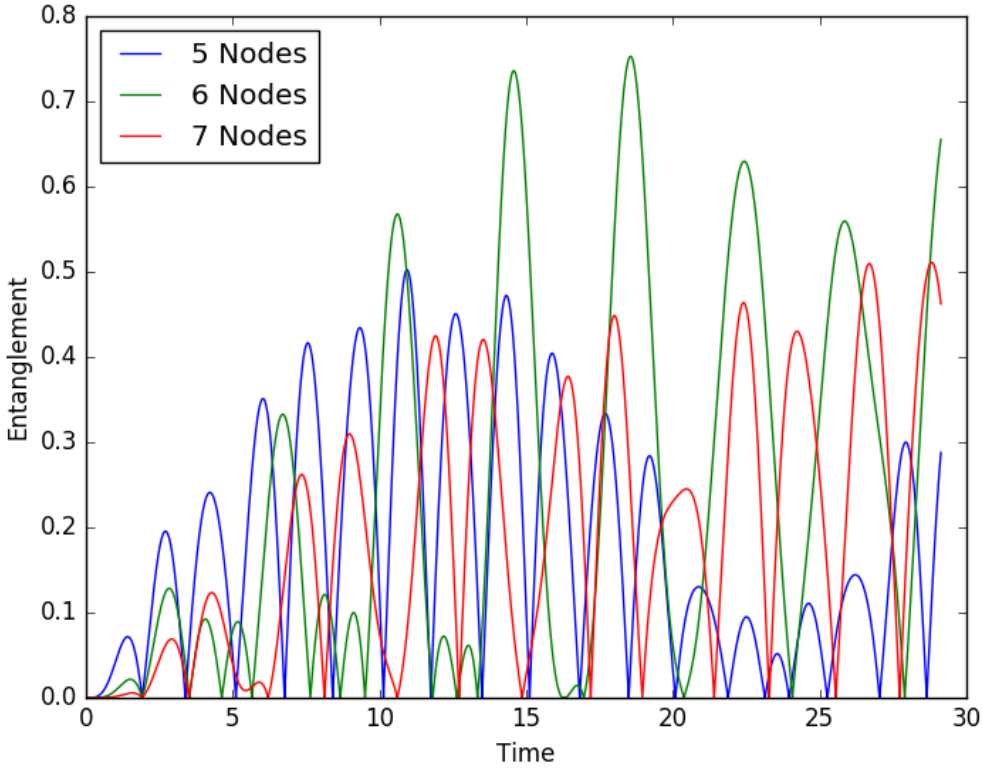


We can see that a 4 node system has a much higher maximum entanglement. To see if the trend continues we then plot the graphs of chains with 5, 6, and 7 nodes:



We can then see that as the length of the chain increases, then the maximum entanglement between the two ends of the chain decreases. This could only be true within the timeframe that

we've been looking at, so extending the timeframe by allowing 3 times the error:

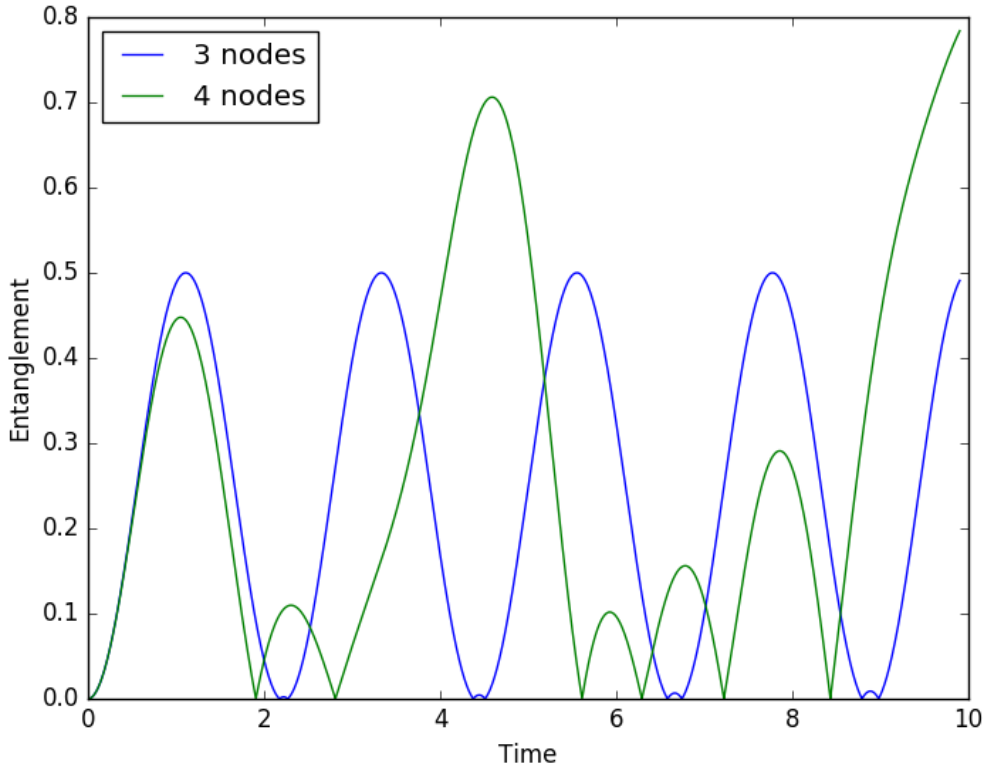


We now see that in this plot that the 7 node system does obtain an equal maximum to the 5 node system, albeit at a much later time, but the 6th node system obtains a lot more entanglement than either of them.

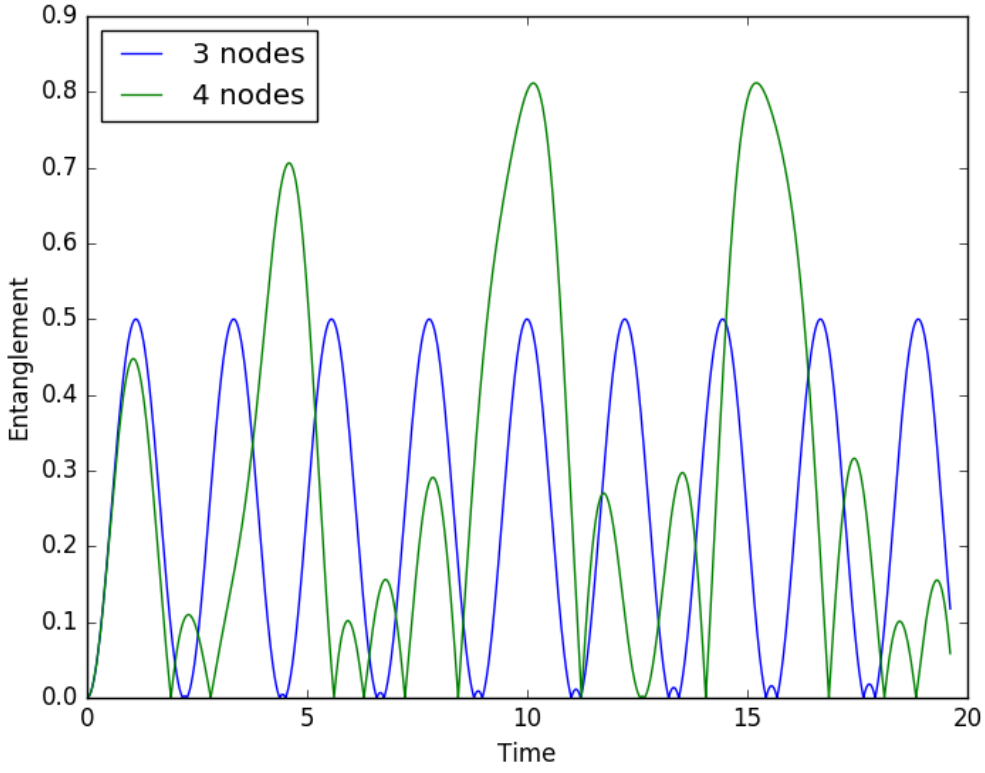
It's worth noting that the two chain lengths bucking the trend that a longer chain takes longer to get to a certain value of entanglement are both even (4 and 6). So we say that while it is generally true that a longer chain means that more entanglement in a time frame, chains with an even number of nodes produce extra entanglement.

14 Entanglement with a fixed point in the chain

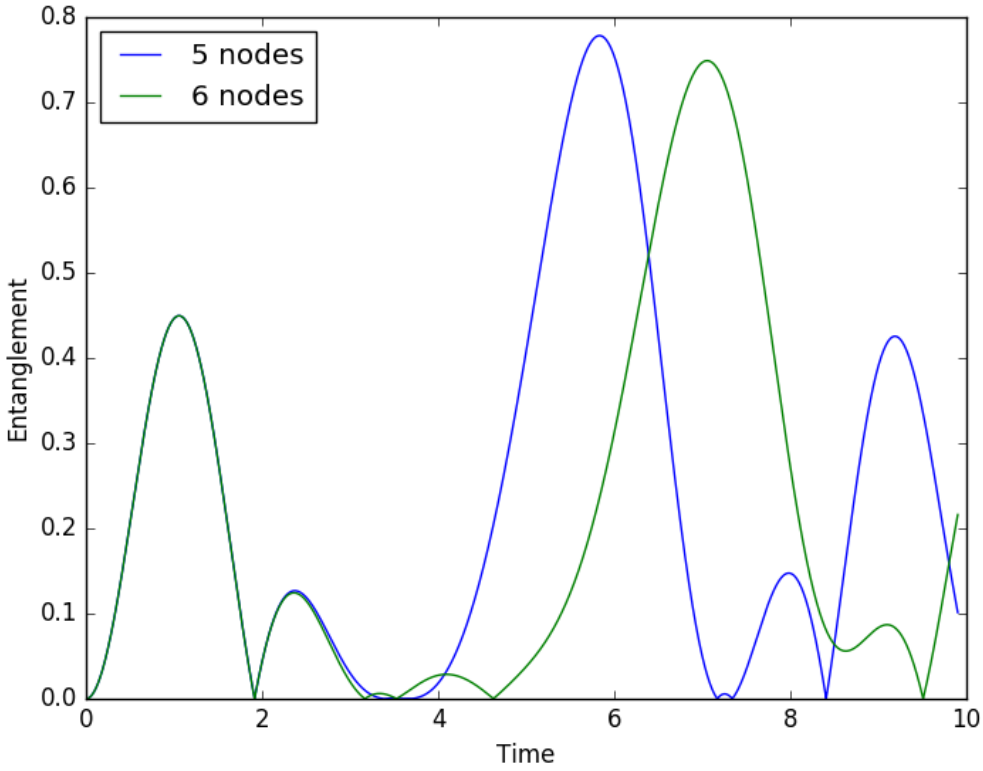
Another question is how the entanglement between a node a fixed length from the initially excited end, and the initially excited node, is affected by lengthening the chain. In the examples below we look at the node 2 along from the initially excited node. In the trimer, this is the other end. Plotting the trimer, with the excitation initialized at an end:



Increasing the length of the spin chain can be seen to increase the complexity of the system but there are those peaks that rise far above the maximum value of the trimer's entanglement. Plotting again but allowing for more error (and hence getting more data) we can see that this is not a rarity.

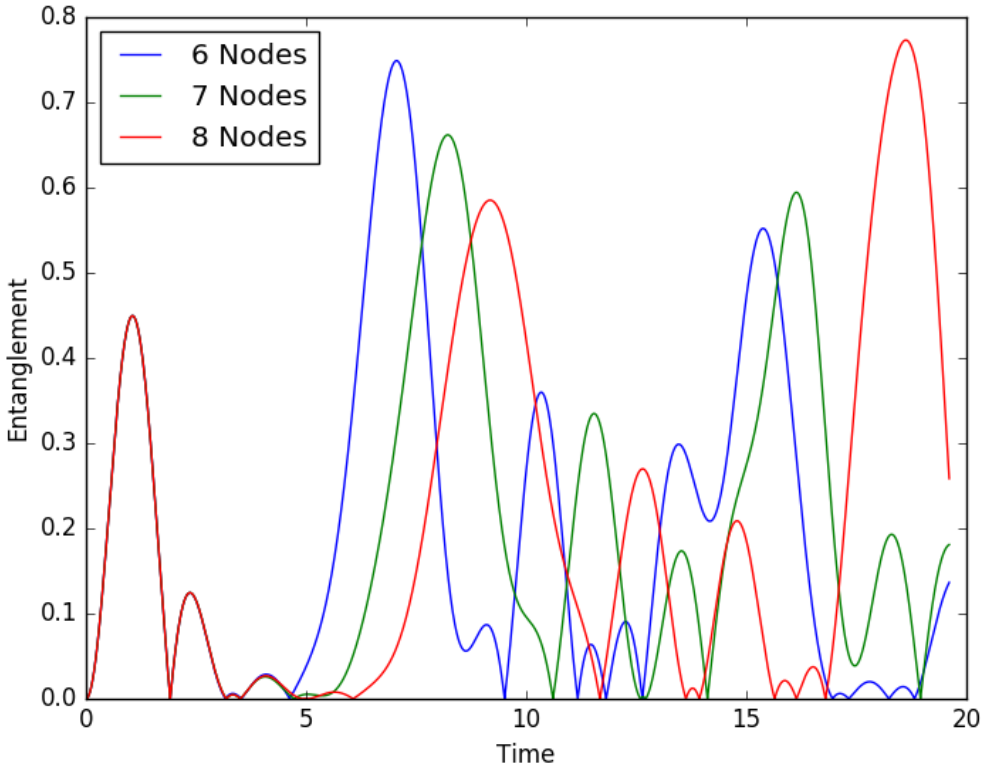


It then makes sense to plot even longer chains:



But this does not conform to the pattern of bigger chains get more entanglement. The 5 node chain does get more entanglement than the 4 node chain but, for some reason, the 6 node chain gets less than the 5 node chain.

To try to figure out the pattern, we need to get more data. So we look at the 7 and 8 node chains:

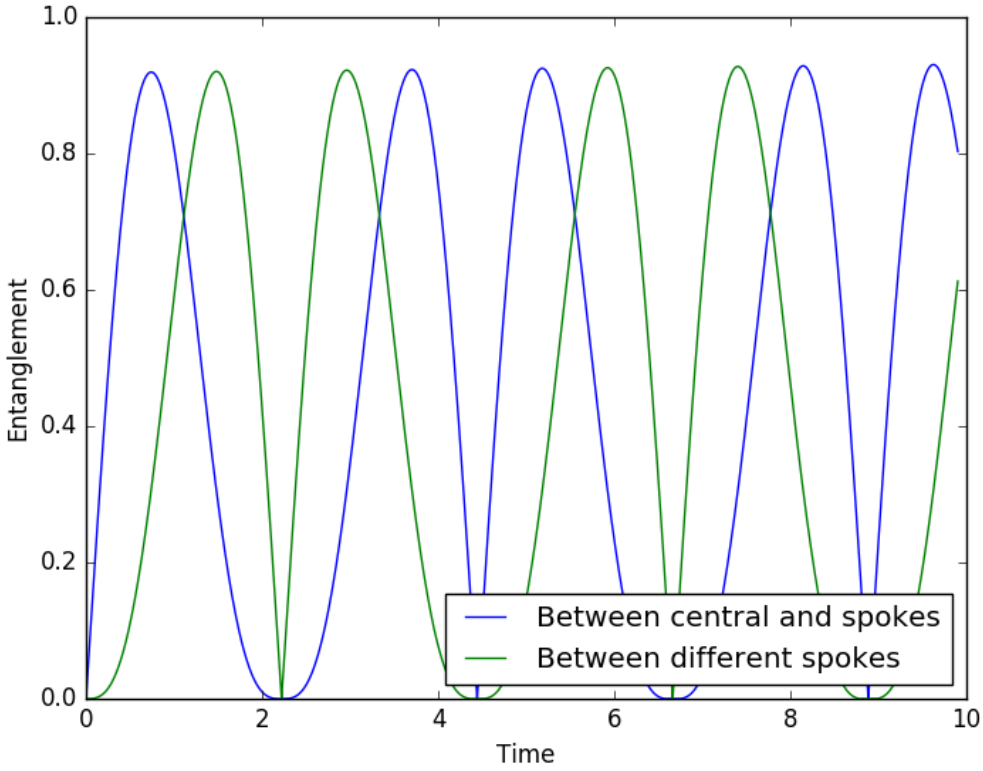


Here we can see that 8 node chains get the most entanglement, but this is very closely followed by the 6 node chain and the 7 node chain gets the most entanglement.

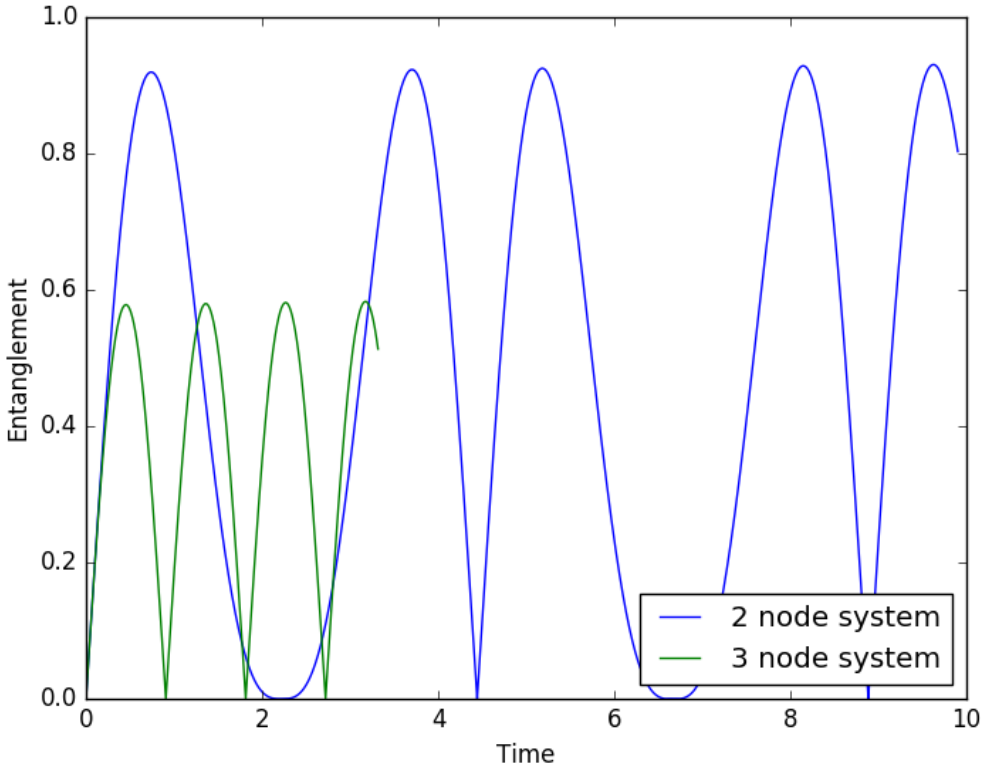
We are now in a position to hazard a guess at what the pattern might be. Similarly to before: as a vague and general rule, a longer chain produces more entanglement but this produces diminishing returns. Superimposed on this is the effect that spin chains with an even number of nodes seem to produce more entanglement.

15 N-spokes

We now consider the N-spoke, as mentioned before. Using another symmetry argument we know that every spoke is equally entangled, both with other spokes, and with the central node. Plotting the two spoke system:

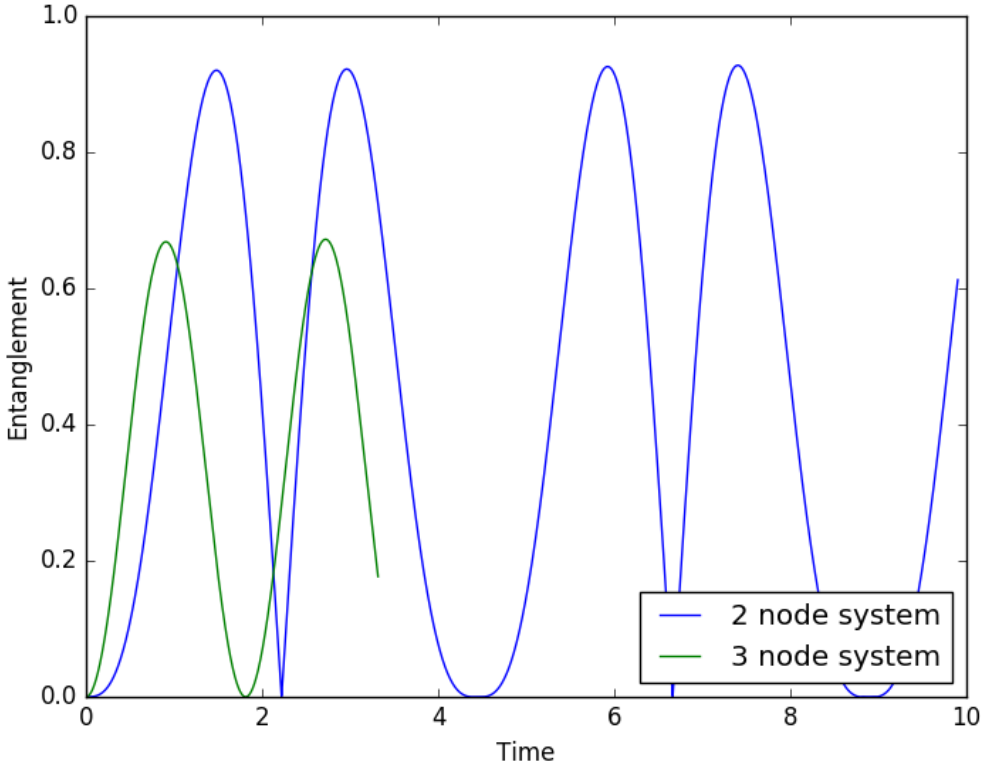


We can see that the entanglement between the central node and spokes, attains the same maximum value as the the entanglement between the different nodes and the two curves seem to be the same but shifted horizontally. We then plot the 2 and 3 spoke system's entanglement between the central and spoke nodes, on the same graphs:



We can see that the 3 spoke system evolves much faster than the 2 spoke system (as we saw earlier) but also that it achieves much less entanglement.

And then looking at the entanglement between the spokes:



This sees the same trends as entanglement with the central node.

Part VII
State Transfer

16 What is state transfer?

The ultimate aim of state transfer is to move a state from one place, the initial position, to another, pre-determined, position, the target position, in a finite amount of time, the transfer time.

There are two important terms in considering state transfer:

Perfect state transfer (PST): when the state of the initial position is exactly replicated at the second position, at some later time.

Pretty good state transfer (PGST): when the state of the initial position is approximated at the second position, at some later time.

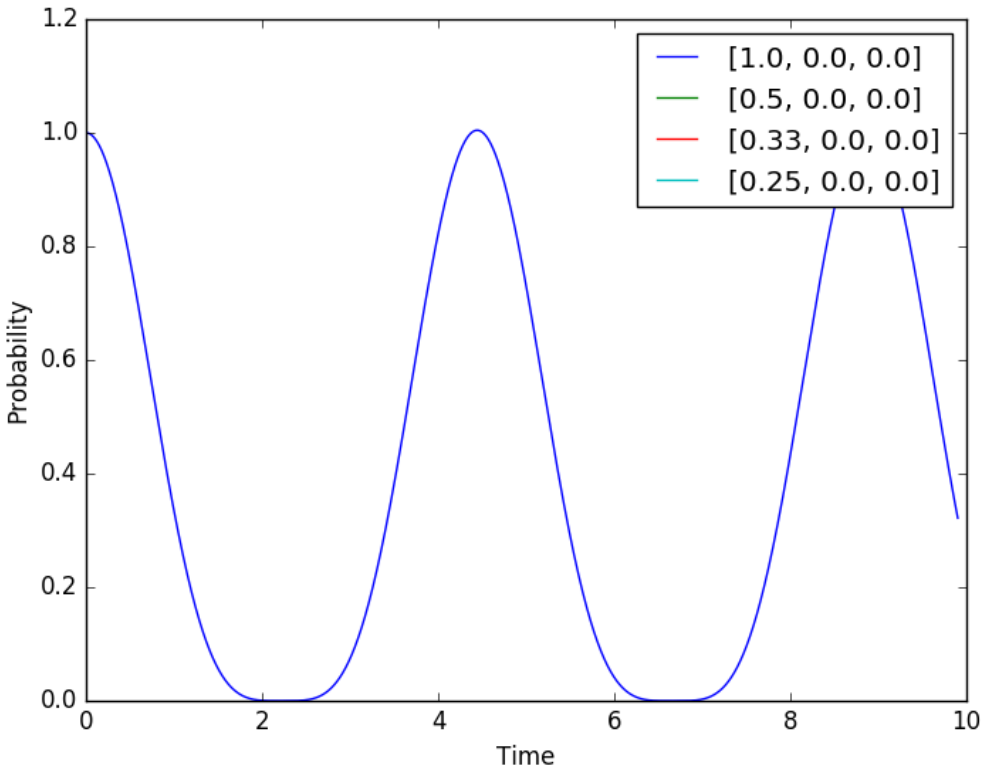
Aiming to attain either PST or PGST is important as if we are aiming to send a state from one place to another, it's pretty important that the state we pick up at the target position is at least a decent approximation of the state we placed in the initial position

The first step in the analysis of state transfer is to find a simpler method of examining every possible state that could be placed in the inital position.

Like in all qubits, the hilbert space of the possible states of the inital position at time, t=0, has a basis: $|0\rangle, |1\rangle$. So any state of the qubit can be expressed as:

$$|\Psi\rangle = \alpha|1\rangle + \beta|0\rangle \tag{82}$$

But as we know $\alpha^2 + \beta^2 = 1$, a qubit can be characterized by just α. We consider a trimer with the first qubit in the $|\Psi\rangle$ state. We call this $\alpha|\underline{1}\rangle$. Looking at how this system evolves for different values of α

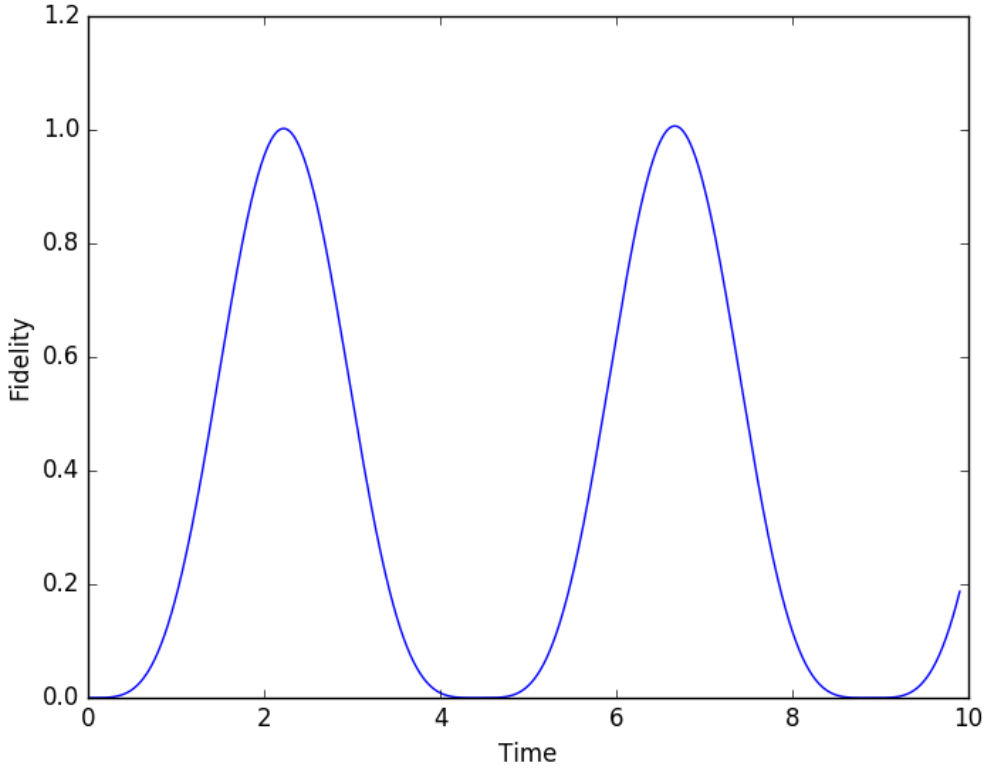


There is only a single curve visible which indicates that the motion of any state placed in the initial position is exactly the same, regardless of exactly what the initial state is.

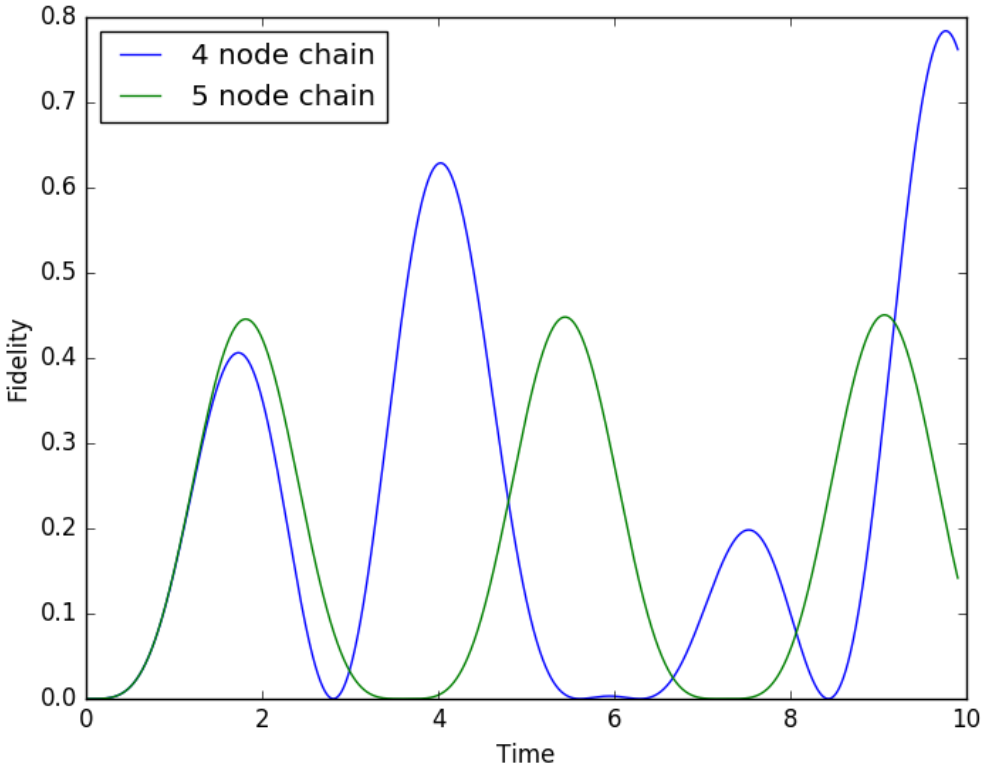
This is useful as it allow us to say general things about state transfer without considering every possible initial state, by only considering the full excitation transfer. That is, the state where $\alpha = 1$.

17 Simple linear chains

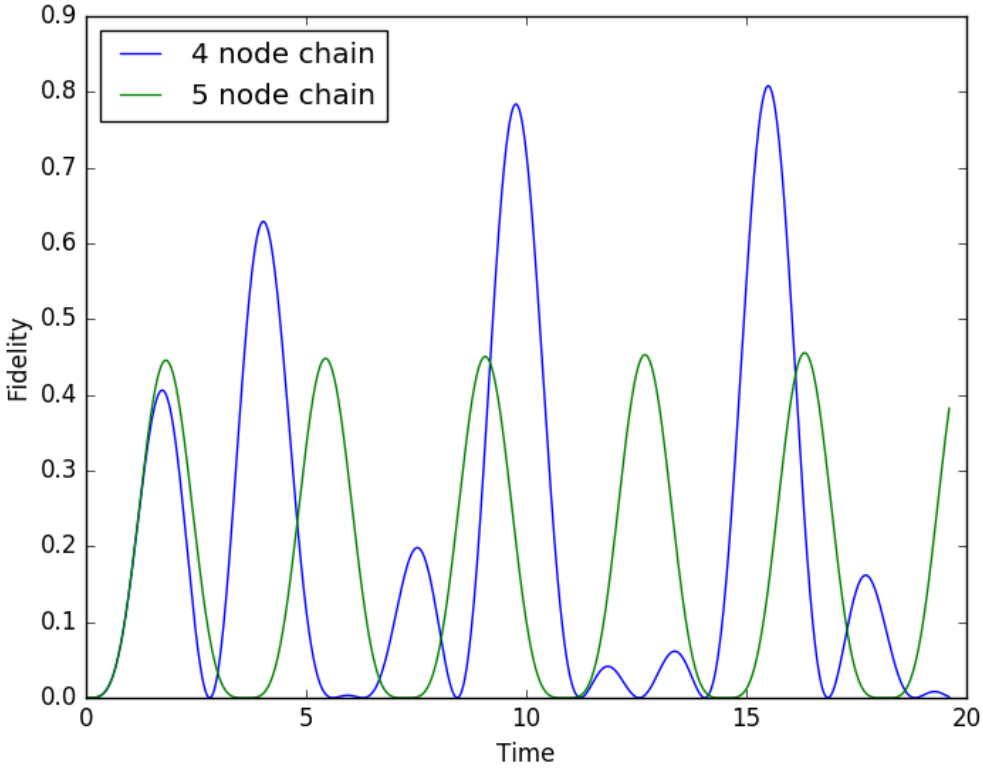
It makes sense to, again, start with the trimer:



Here, we can see that at very specific points perfect state transfer is achieved, and close to these points the state transfer is still pretty good. We then look at longer chains to see how the pattern develops:



Here we see that for both systems, perfect state transfer is no longer achieved and the maximum fidelity has been reduced to , at least within the time frame that we are looking at. Increasing the allowed error threefold.



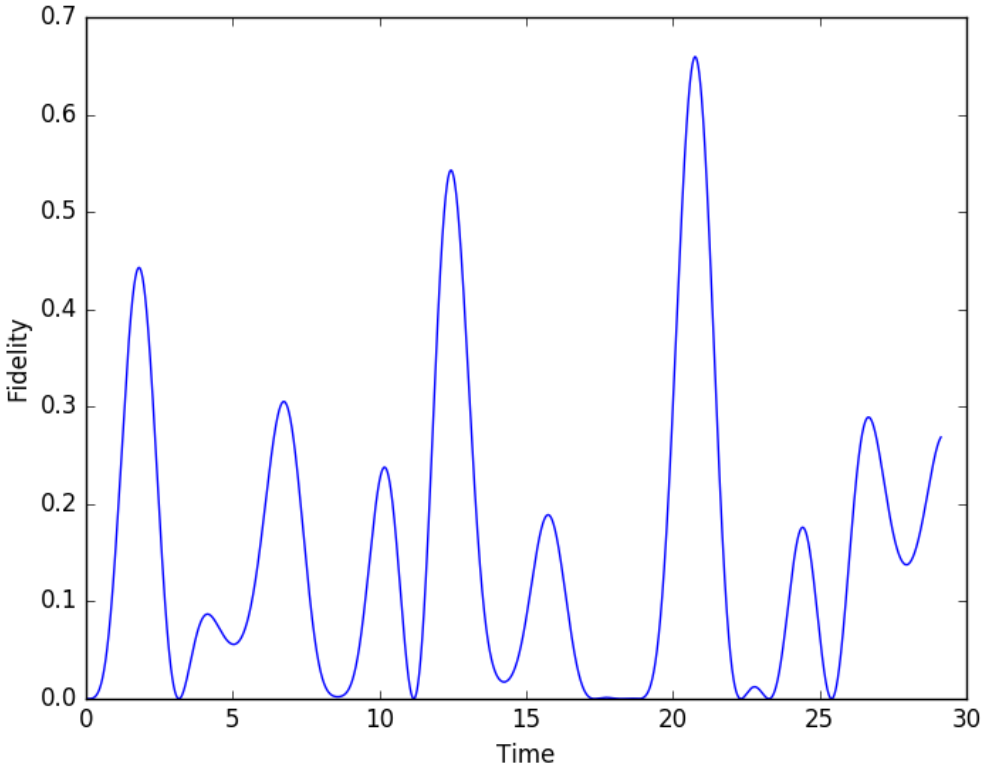
You can see that the fidelity does rally and gets better after a while. However, you do pay a cost in that you have to wait longer and, if the system is not isolated from the world, then this extra

time means a greater chance of decohering.

This leads us to suspect, if we wait long enough we can achieve a state transfer arbitarily close to perfect state transfer. Mathematically, we say

$$Let F_T = \text{Maximum(F) between t=0 and t=T}$$
$$\text{As t} \to \infty, \qquad F_T \to 1 \tag{83}$$

To test this, let's massively increase the allowed error and see how long it takes a 5 node chain to reach the same fidelity as the second peak of the 4 node chain in the first plot (the maximum of the 4 chain takes so long to reach that the error makes it meaningless):



This makes it clear that the 5 node chain takes around 5 times as long to reach the specified value. But the value is reached, supporting the hypothesis.

Part VIII
Appendix

Lemma 17.1. ??

$$f(g,J,h) = g^2\left(J^2 + \frac{1}{J^2}\right) + (1+g)^2\left(h^2 + \frac{1}{h^2}\right) \tag{84}$$

is surjective into z s.t. $1 \leq z \leq \infty$.

Proof First we show that $x^2 + \frac{1}{x^2}$ *is surjective into y s.t.* $2 \leq y \leq \infty$

$$x^2 + \frac{1}{x^2} = y \tag{85}$$

$$\Rightarrow x^4 - yx^2 + 1 = 0 \tag{86}$$

Let $u = x^2$

$$\Rightarrow u^2 - yu + 1 = 0 \tag{87}$$

This is satisfied by:

$$u = \frac{y \pm \sqrt[2]{y^2 - 4}}{2} \tag{88}$$

This gives real solutions for u if and only if $y \geq 2$. Assuming this gives us $y \geq 0$, so at least 1 solution for u is guaranteed to be positive. So there is guaranteed to be a real solution for x. $\Rightarrow$ The equation is surjective into $2 \leq y \leq \infty$

We then split the co-domain into $2 \leq z \leq \infty$ and $1 \leq z \leq 2$

For $2 \leq z \leq \infty$, set $g = 0$

$$f(0, J, h) = \left(h^2 \frac{1}{h^2}\right) \tag{89}$$

And as shown previously, this is surjective into $2 \leq z \leq \infty$

For $1 \leq z \leq 2$ set:

$$h^2 \frac{1}{h^2} = J^2 \frac{1}{J^2} = 2 \tag{90}$$

Taking a short cut, we then plot the graph of:

$$f(g, 1, 1) = 2\left(g^2 + \left(1 + g^2\right)\right) \tag{91}$$

It is immediately obvious that the function is surjective into $1 \leq z \leq 2$