Two Interesting Matrix/Operator Integral Theorems

1 Overview

Herein I present and prove two theorems (Theorem 1 and Theorem 2) as below. Letting $\mathbb{M}_n(\mathbb{C})$ be the set of $n \times n$ matrices with complex elements.

Theorem 1
Let $f : \mathbb{M}_n\big(\mathbb{C}\big) \to \mathbb{M}_n\big(\mathbb{C}\big)$ be a function with a Laurent series about its single singularity. Additionally let $A \in \mathbb{M}_n\big(\mathbb{C}\big)$ and γ be an arbitrary contour in the complex plane that stays within the radius of convergence of the aforementioned Laurent series. Then,

$$\oint_\gamma \left(f\big(z\hat{I}_n - \hat{A}\big)\right) dz = 2\pi i \alpha_{-1} \hat{P}_\gamma, \tag{1}$$

where $\hat{P}_\gamma$ is a projector onto the subspace spanned by the eigenvectors corresponding to the eigenvalues enclosed by γ.

Theorem 2
For a given matrix, $\hat{B}(z) \in \mathbb{M}_n(\mathbb{C})$, if there exists $\hat{A} \in \mathbb{M}_n(\mathbb{C})$ such that

$$\big[\hat{B}(z)\big]^{-1} + \hat{A} = z\hat{I}, \tag{2}$$

then

$$\oint_\gamma \left(\hat{B}(z) \right) dz = 2\pi i \hat{P}_\gamma, \tag{3}$$

where $\hat{P}_\gamma$ is a projector onto the subspace spanned by the eigenvectors corresponding to the eigenvalues - of $\hat{A}$ - enclosed by γ.

2 Projections and Projectors

Definition 1. Orthogonal Projection
Given a Hilbert space, $\mathcal{S}_0$, and a subspace of it, $\mathcal{S}_A \subseteq \mathcal{S}_0$, so the orthogonal projection of $|\psi\rangle \in \mathcal{S}_0$ is the state, $|\phi\rangle \in \mathcal{S}_A$, that minimises:

$$\big| |\psi\rangle - |\phi\rangle \big|, \tag{4}$$

where $\big| \cdot \big|$ denotes the Euclidean vector norm (and does so throughout this manuscript).

Lemma 1. *The orthogonal projection of a state onto a subspace may be computed from a basis decomposition by setting the coefficients of all basis-states outside the subspace to zero.*

Proof. Let $\mathcal{S}_0$ be any space such that $\{|\beta_j\rangle\}_j^N$ is a basis for $\mathcal{S}_0$, and $\mathcal{S}_A \subseteq \mathcal{S}_0$ such that $\{|\beta_j\rangle\}_j^M$ is a basis for $\mathcal{S}_A$. Any space and subspace has a basis meeting this requirement. Hence, if $|\psi\rangle$ and $|\phi\rangle$ are as in Def. 1, they may be expressed as:

$$|\psi\rangle = \sum_j^N \left(\alpha_j^\psi |\beta_j\rangle \right) \tag{5}$$

$$|\phi\rangle = \sum_j^M \left(\alpha_j^\phi |\beta_j\rangle \right) \tag{6}$$

So then, the value to be minimised, defining the orthogonal projection, $|\psi\rangle$, is:

$$\big||\psi\rangle - |\phi\rangle\big|^2 = \left| \sum_j^N \left(\alpha_j^\psi |\beta_j\rangle \right) - \sum_j^M \left(\alpha_j^\phi |\beta_j\rangle \right) \right| = \left| \sum_j^M \left(\alpha_j^\psi |\beta_j\rangle - \alpha_j^\phi |\beta_j\rangle \right) + \sum_{j=M}^N \left(\alpha_j^\psi |\beta_j\rangle \right) \right| \tag{7}$$

$$= \sum_j^M \left((\alpha_j^\psi - \alpha_j^\phi)^2 \right) + \sum_{j=M}^N \left((\alpha_j^\psi)^2 \right) \tag{8}$$

First note that minimizing $\big||\psi\rangle - |\phi\rangle\big|^2$ also minimizes $\big||\psi\rangle - |\phi\rangle\big|$, so we consider Eqn. 8 and minimize it as it is presented above. Then note that the second sum is independent of α_j^ϕ, hence no choice of α_j^ψ can give a value of $\big||\psi\rangle - |\phi\rangle\big|^2$ lower than the second sum (as the first sum is always positive). Hence any choice of α_j^ϕ eliminating the first sum is the required minimization. This can be achieved by setting $\alpha_j^\psi = \alpha_j^\phi$ for all $j \in \mathbb{N}^{\leq M}$. Then,

$$\big||\psi\rangle - |\phi\rangle\big|^2 = \sum_{j=M}^N \left((\alpha_j^\psi)^2 \right). \tag{9}$$

The remaining coefficients in Eqn. 9 are exactly the coefficients of all basis-states outside the subspace i.e. the coefficients that the lemma claims setting to zero would give the required projection onto the subspace. We can then see (in Eqn. 9) that setting these - remaining α_j^ψ - values to zero causes $\big||\psi\rangle - |\phi\rangle\big| = 0$ i.e. $|\psi\rangle = |\phi\rangle$, which shows that starting from the basis decomposition of a given state and setting the coefficients of all basis-states outside the subspace to zero results in the orthogonal projection of the given state. Hence, the lemma statement is true. □

Definition 2. Orthogonal Projectors
An orthogonal projector, with respect to a specified subspace, maps any state to its orthogonal projection onto the specified subspace.

Lemma 2. *If projector, $\hat{P}_\mathcal{S}$, projects onto a subspace, $\mathcal{S}$, spanned by a subset, β_s, of the basis, $\beta_\mathcal{L}$, of a larger space, $\mathcal{L}$, then $\hat{P}_\mathcal{S}$ can be expressed as:*

$$\hat{P}_\mathcal{S} = \sum_{|j\rangle \in \beta_s} \left(|j\rangle\langle j| \right) \tag{10}$$

Proof. Consider $|\psi\rangle \in \mathcal{L}$ with decomposition:

$$|\psi\rangle = \sum_{|k\rangle \in \beta_\mathcal{L}} \left(\alpha_k |k\rangle \right) \tag{11}$$

Hence,

$$\hat{P}_\mathcal{S}|\psi\rangle = \sum_{|j\rangle \in \beta_s} \left(|j\rangle\langle j| \right) \sum_{|k\rangle \in \beta_\mathcal{L}} \left(\alpha_k |k\rangle \right) = \sum_{|j\rangle \in \beta_s} \sum_{|k\rangle \in \beta_\mathcal{L}} \left(\alpha_k |j\rangle\langle j|k\rangle \right) \tag{12}$$

$$= \sum_{|j\rangle \in \beta_s} \left(\alpha_j |j\rangle \right). \tag{13}$$

Eqn. 13 is the orthogonal projection of $|\psi\rangle$ onto $\mathcal{S}$. Hence the proposed representation of $\hat{P}_\mathcal{S}$ holds. □

3 Eigenvalue and Eigenvector Lemmas

Lemma 3. *If λ is an eigenvalue of operator $\hat{A}$, with eigenvector $|\lambda\rangle$. Then λ^{-1} is an eigenvalue of $\hat{A}^{-1}$, with the same eigenvector.*

Proof. By the definition of $\hat{A}$, λ, and $|\lambda\rangle$:

$$\hat{A}|\lambda\rangle = \lambda|\lambda\rangle \Rightarrow \hat{A}^{-1}\hat{A}|\lambda\rangle = \lambda\hat{A}^{-1}|\lambda\rangle \Rightarrow \lambda^{-1}|\lambda\rangle = \hat{A}^{-1}|\lambda\rangle. \tag{14}$$

□

Lemma 4. *If λ is an eigenvalue of operator $\hat{A}$, with eigenvector $|\lambda\rangle$, and $\sigma \in \mathbb{C}$. Then $\lambda - \sigma$ is an eigenvalue of $(\hat{A} - \sigma\hat{I})$*

Proof.

$$\big(\hat{A} - \sigma\hat{I}\big)|\lambda\rangle = \hat{A}|\lambda\rangle - \sigma|\lambda\rangle = \lambda|\lambda\rangle - \sigma|\lambda\rangle = \big(\lambda - \sigma\big)|\lambda\rangle. \tag{15}$$

□

4 First Main Result

Theorem 1. *Let $f : \mathbb{M}_n\big(\mathbb{C}\big) \to \mathbb{M}_n\big(\mathbb{C}\big)$ be a function with a Laurent series about its single singularity. Additionally let $A \in \mathbb{M}_n\big(\mathbb{C}\big)$ and γ be an arbitrary contour in the complex plane that stays within the radius of convergence of the aforementioned Laurent series. Then,*

$$\oint_\gamma \Big(f\big(z\hat{I}_n - \hat{A}\big)\Big)dz = 2\pi i\alpha_{-1}\hat{P}_\gamma, \tag{16}$$

where $\hat{P}_\gamma$ is a projector onto the subspace spanned by the eigenvectors corresponding to the eigenvalues - of $\hat{A}$ - enclosed by γ.

Proof. Consider the Laurent series of f, which exists by assumption:

$$f\big(z\hat{I}_n - \hat{A}\big) = \sum_{n=-\infty}^{\infty}\Big(\alpha_n\big(z\hat{I}_n - \hat{A}\big)^n\Big), \tag{17}$$

where $\forall j \in \mathbb{N}$, $\alpha_j \in \mathbb{C}$. For all values of $n \in \mathbb{Z}$, the eigenvectors of $\big(z\hat{I}_n - \hat{A}\big)^n$ are identical and we label them $|\lambda_j\rangle$. Therefore,

$$\big(z\hat{I}_n - \hat{A}\big)^n|\lambda_j\rangle = \big(z\hat{I}_n - \lambda_j\big)^n|\lambda_j\rangle. \tag{18}$$

Hence the spectral decomposition of the Laurent series is:

$$\sum_{n=-\infty}^{\infty}\bigg(\alpha_n\sum_{\lambda_j\in\Lambda}\Big[\big(z-\lambda_j\big)^n|\lambda_j\rangle\langle\lambda_j|\Big]\bigg). \tag{19}$$

Then integrating Eqn. 19 with respect to z, over γ:

$$\oint_\gamma \Big(f\big(z\hat{I}_n - \hat{A}\big)\Big)dz = \oint_\gamma\bigg(\sum_{n=-\infty}^{\infty}\bigg(\alpha_n\sum_{\lambda_j\in\Lambda}\Big[\big(z-\lambda_j\big)^n|\lambda_j\rangle\langle\lambda_j|\Big]\bigg)\bigg)dz \tag{20}$$

$$= \sum_{\lambda_j\in\Lambda}\bigg[\sum_{n=-\infty}^{\infty}\bigg(\alpha_n\oint_\gamma\Big(\big(z-\lambda_j\big)^n\Big)dz|\lambda_j\rangle\langle\lambda_j|\bigg)\bigg]. \tag{21}$$

Using the residue theorem in Eqn 21:

$$\oint_\gamma \Big((z-\lambda_j)^n \Big) dz = \begin{cases} 0, \ \textit{if}\ n \neq -1 \ \textit{or}\ \lambda_j \ \textit{is outside of}\ \gamma \\ 2\pi i, \ \textit{otherwise} \end{cases} . \tag{22}$$

Therefore,

$$\oint_\gamma \Big(f(z\hat{I}_n - \hat{A}) \Big) dz = 2\pi i \alpha_{-1} \sum_{\lambda_j \in \Lambda_\gamma} \Big[|\lambda_j\rangle\langle\lambda_j| \Big]. \tag{23}$$

Due to Lemma 2, Eqn. 23 can be expressed as:

$$\oint_\gamma \Big(f(z\hat{I}_n - \hat{A}) \Big) dz = 2\pi i \alpha_{-1} \hat{P}_\gamma. \tag{24}$$

□

5 Second Main Result

Theorem 2. *For a given matrix, $\hat{B}(z) \in \mathbb{M}_n(\mathbb{C})$, and there exists $\hat{A} \in \mathbb{M}_n(\mathbb{C})$ such that*

$$\big[\hat{B}(z)\big]^{-1} + \hat{A} = z\hat{I}, \tag{25}$$

then

$$\oint_\gamma \Big(\hat{B}(z) \Big) dz = 2\pi i \hat{P}_\gamma, \tag{26}$$

where $\hat{P}_\gamma$ is a projector onto the subspace spanned by the eigenvectors corresponding to the eigenvalues - of $\hat{A}$ - enclosed by γ. Assuming the inverses of $\hat{B}(z)$ and $[z\hat{I} - \hat{A}]$ exist.

Proof.

$$\big[\hat{B}(z)\big]^{-1} + \hat{A} = z\hat{I} \Rightarrow \hat{B}(z) = \big[z\hat{I} - \hat{A}\big]^{-1} \tag{27}$$

Therefore, using Theorem 1, and setting $f(\hat{A}) = \hat{A}^{-1}$:

$$\oint_\gamma \Big(\hat{B}(z) \Big) dz = \oint_\gamma \Big((z\hat{I}_n - \hat{A})^{-1} \Big) dz = 2\pi i \hat{P}_\gamma. \tag{28}$$

□