CSC791/495

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When "All Natural" Doesn't Suffice

lots of problems ARE FPT wrt natural parameter

Thus far: we saw that k-coloring is NP-hard at $k=5$ $\Rightarrow$ not FPT wrt k.
(also not XP)

k-clique also does not have a $f(k)n^{O(1)}$ algorithm (unless ... bad things happen)

Known Problems:

Other (more practical) Issues: value of parameter may be prohibitively large.

good example: vertex cover

* your problem has already studied w/ natural parameter.

Alternative Medicine

Problem: Given a graph G and $k \in \mathbb{Z}^+$, does G contain a clique on k vertices?

Natural Parameter: k (clique is $W[1]$-hard $\Rightarrow$ not FPT)

Dual Parameter: $\ell = n-k$ Is there a k-clique? Use time $f(\ell)n^{O(1)}$.

observe: if YES $\Rightarrow$ every non-edge has ≥ 1 endpoint in a set of $\leq n-k$ vertices. [smells like VC]

Consider $\overline{G}$ (complement of G). If $\overline{G}$ has a vertex cover of size $\leq n-k \Rightarrow G$ has a k-clique. $\overset{?}{\Leftarrow}$

Win b/c VC is $2^{n-k} n^{O(1)} = 2^{\ell} n^{O(1)}$ ✓

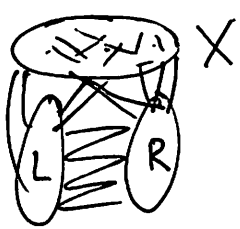
Structural Parameter: $d =$ max degree of G (Δ)

1. If $k > d \Rightarrow$ NO. So we can assume $k \leq d$.
2. If v is in a clique $\Rightarrow$ the whole clique lives in $N(v) \cup \{v\} = N[v]$. But $N[v] \leq d+1$

Brute force: $\binom{d+1}{k} \cdot n \rightsquigarrow (d+1)^k \cdot n \leq (d+1)^d \cdot n$ better bound: $O(2^{d+1})\, n^{O(1)}$

Distance Parameter: $b = |X|$ s.t. $G \setminus X$ is bipartite

observe: easy to solve clique in bipartite graphs. Given: $G, |X| = b, \& \ k$.



where can a clique "live"? ($k \geq 3$)

observe: it can have ≤ 1 vertex from each of L & R.

Pick $u \in L, v \in R$, look in $X \cup \{u\} \cup \{v\}$ for a k-clique. $\binom{b+2}{k}$

how many pairs matter? # edges in bipartite graph $\leq n^2$ again $k \leq b+2 \Rightarrow$ fpt wrt b.

Bounded De....

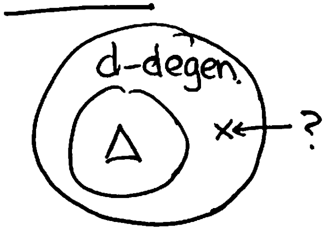
- we've had lots of success w/ Δ.

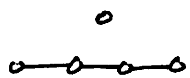
(induced) in the sub-graph

<u>Defn</u> A graph is <u>d-degenerate</u> if every subgraph has a vertex of $\boxed{\text{degree}} \le d$.

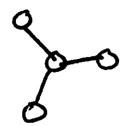
The ~~degeg~~ <u>degeneracy</u> of a graph is the minimum d so that G is d-degenerate.

<u>Examples:</u> Is this a weaker or stronger condition than bounded Δ? <u>weaker</u>

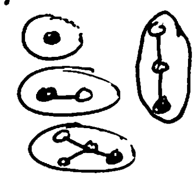


$d=2$: $\Delta = 2 \Rightarrow$  (cycles)  (paths) + disjoint unions

degen $\le 2 \Rightarrow$ can we have deg 3?



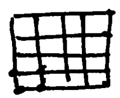
G



induced subgraphs

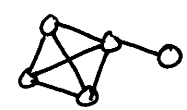
stars – degen 1

grids :– degen 2



<u>Q:</u> Can we calculate it? (in poly time) (yes in 2^n) in polynomial time?

suggestion: look @ min deg vertex. (If $> d \Rightarrow$ NO)
delete it & recurse. the max degree of a vertex <u>when chosen</u> is the degeneracy.

<u>claim</u> degen. $\geq$ min deg ✓
(not equal: )

<u>Useful observation:</u> the above algorithm implies every d-degenerate graph has an ordering $v_1, v_2, \ldots, v_n$ so that v_i has $\le d$ neighbors w/ higher index.

In-Class Exercise

Problem: Prove that k-Clique is FPT parameterized by degeneracy.

Algorithm ① get a $\underbrace{\text{degeneracy ordering}}_{\deg_{\geq}(v_i) \leq d}$ $v_1, \ldots, v_n$ (linear time)

② recall that any clique lives in $N(v)$ for some v.

So check $N(v_1)$ in time $\leq \binom{d+1}{k}$ ↗ if yes $\Rightarrow$ return
↘ if no $\Rightarrow$ safe to delete

Now $G \setminus v_1$ has $\deg(v_2) \leq d$, so we can repeat.

(iterate over the ordering)

$O(2^{d+1} \cdot n)$

Food for thought: ~~Max~~ Independent Set / Dominating Set ?

All the Parameters?

"natural parameter"

\# edges

\# components

degeneracy

\# vertices

max deg in $\overline{G}$

size of a (min) vertex cover

max degree

longest shortest path
(diameter)

(shortest induced ~~tone~~ cycle)
girth

min degree

Max independent set

genus
0 - planar
1 - toroidal graphs

\# of vertices deleted to become bipartite
(OCT #)

\#induced P_3's

treewidth

treedepth

pathwidth

Distance to Triviality

A useful perspective on parameterized complexity:

1. NP-hard problem
2. but it's easy/trivial (polynomial-time) on some class (subset of instances that share a property)
3. pick our parameter k to measure distance to this class.

What have we seen that fits this model?

Clique w/ OCT #

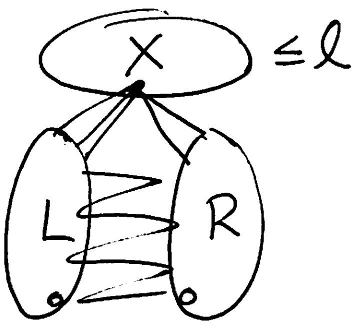
"trivial" could mean small ($n \le f(k)$ or k is very small)

then natural parameters are often "distance" to triviality.

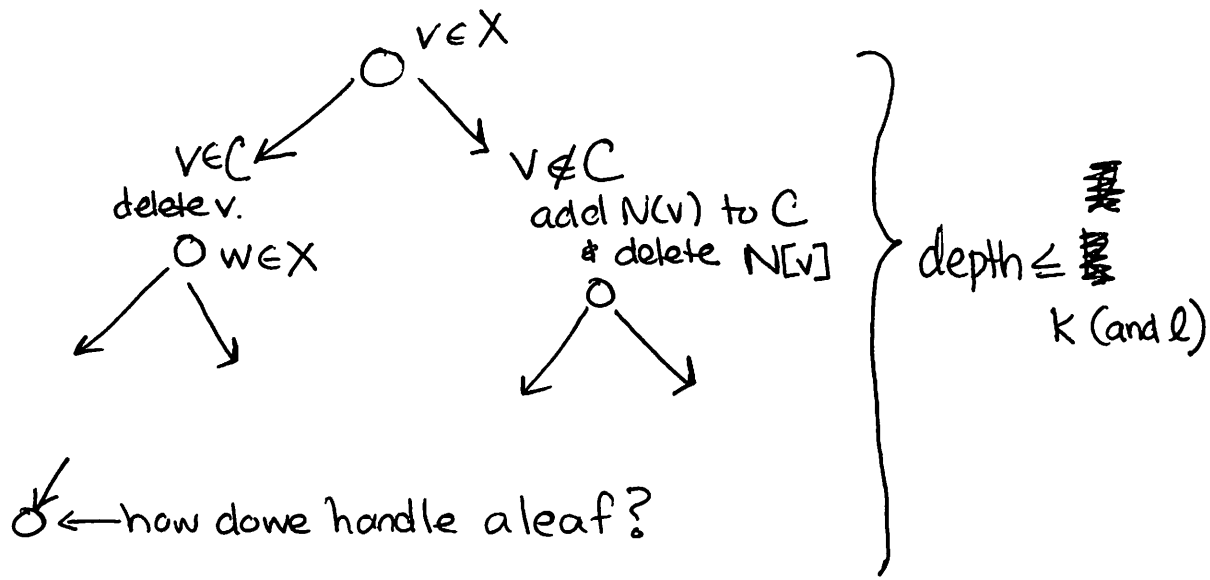
When more problems help

Problem: Given a graph G, $k \in \mathbb{Z}^+$ and an odd cycle transversal X of G with size at most ℓ. Does G have a vertex cover of size at most k?

Goal: Show this is FPT parameterized by ℓ.



If we're aiming at bounded search tree, X is an obvious candidate for branching.



at a leaf:

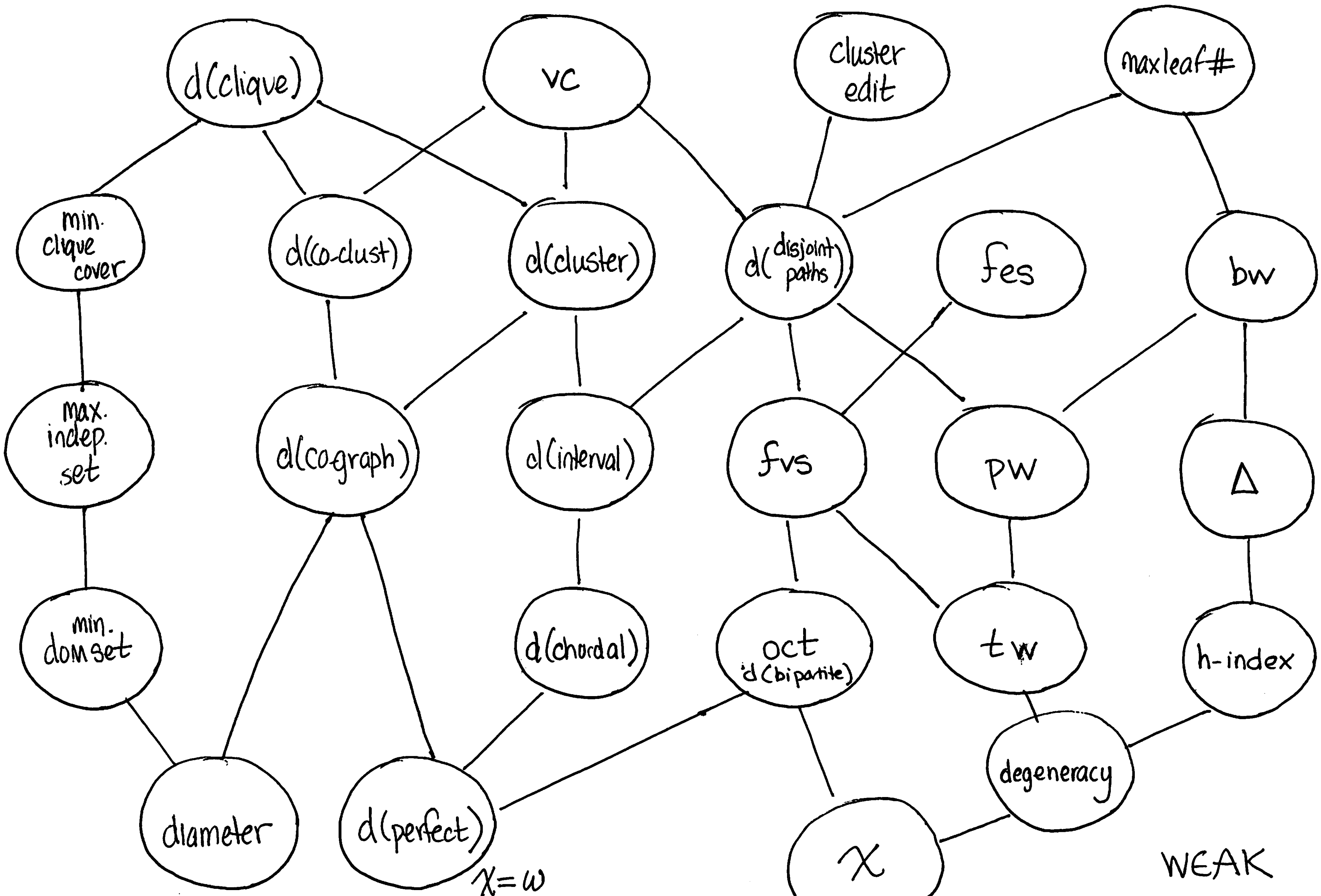
either: $k = 0$ → YES if no edges; → NO else

or: $\ell = 0 \Rightarrow G$ is bipartite

Lemma VC is poly-time on bipartite graphs.

Parameter Ecology

STRONG



Above Guarantees

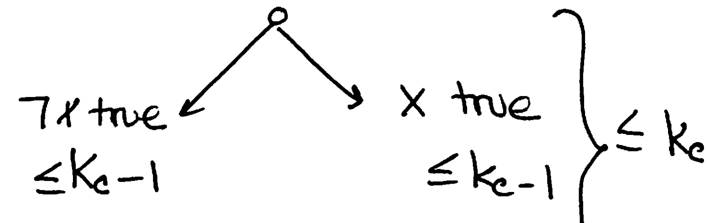
Typical CSP/SAT researcher: "SAT is trivially FPT wrt # variables; why care about PAC?"

<u>Thm</u>: 3SAT has $O^*(2^{k_v})$ and $O^*(2^{3k_c})$ algorithms where $k_v = \#$ vars, $k_c = \#$clauses.

- What about SAT parameterized by $\underline{\underline{k_c}}$? Try branching.

  If x and $\neg x$ appear in φ $\Rightarrow$ branch on x.

  either way ≥ 1 clause gets satisfied. Delete it.



  argue leaves in poly-time: $k_c = 0 \Rightarrow$ YES.
  run out of variables $\Rightarrow$ all remaining are <u>either</u> pos <u>or</u> neg. ✓

- If we consider MaxSAT (satisfy $\geq k$ clauses), is it more interesting?

  If $\geq 2k$ clauses $\Rightarrow$ WIN. random assignment satisfies $\lceil m/2 \rceil$ clauses

  If $< 2k$ clauses use SAT algorithm above :) $m = k_c$

- This motivates an "above guarantee" formulation: can we do k better than random?

  Is there an assignment satisfying $\lceil m/2 \rceil + \underbrace{k}_{\text{parameter}}$ clauses?

Today's Problems

① Give an FPT algorithm for k-coloring parameterized by vertex cover.

<u>Problem</u> Given a graph G and a vertex cover X of size ℓ, is G k-colorable?

② Give an FPT algorithm for (minimum) k-dominating set parameterized by feedback vertex set.

FVS $X \Rightarrow$
$G \setminus X$ a forest