CSC791/495

Dr. Blair D. Sullivan

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And you thought this class was hard before....

We've seen "lots" of techniques for designing FPT algorithms, but how do we know when they're unlikely to succeed?

Reasonable questions

- Can we prove a problem is NOT FPT?
- Is problem X as hard as problem Y?
- * Can we guarantee a problem X does NOT have an $2^{o(k)} n^{O(1)}$ algorithm?

Answering these requires two ingredients:

① a notion of reduction that is useful. (if $A \xrightarrow{R} B \Rightarrow B \in FPT \Rightarrow A \in FPT$)

② a hypothesis about distinct complexity classes (e.g. P vs NP)

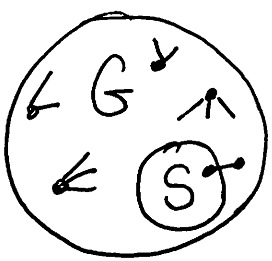
Reductions

Recall A polynomial many-one reduction (aka Karp reduction) is a function that maps instances of problem A to instances of problem B $f: x \to x'$ "from A to B"
s.t. ① f is poly-time computable
② x is a YES-instance of $A \iff f(x)$ is a YES-instance of B.

Implies: If $B \in P \Rightarrow A \in P$. If A is NP-hard $\Rightarrow$ B is NP-hard.

Example Give a reduction from k-IndSet to k'-Vertex Cover

observe: largest independent set is "inversely related" to the smallest vertex cover



$|S| = k$
S independent $\Rightarrow$ u ····· v is not an edge for all $u, v \in S$.

$\Rightarrow$ no edges inside S to cover $\Rightarrow$ we don't need any vertices of S in a cover: $G \setminus S$ is a vertex cover of size $n-k$.

- Is $G \setminus T$ independent for any VC T? YES.

$$\underset{IS}{(G, k)} \to \underset{VC}{(G, n-k)}$$

Parameterized Reductions

Definition A parameterized reduction from a problem A to a problem B is: a function that maps instances (x,k) of A to instances (x',k') of B s.t.

1. parameter preserving: $k' \le g(k)$ for a computable function $g: \mathbb{N} \to \mathbb{N}$
2. answer preserving: (x,k) yes for $A \iff (x',k')$ yes for B.
3. poly-time: runs in $f(k)|x|^{O(1)}$ where f is a computable function.
   $(|x'| \le f(k)|x|^{O(1)})$

Note: you may assume f, g are non-decreasing

Thm: If B is FPT $\Rightarrow$ A is FPT. If A is NOT FPT $\Rightarrow$ B is NOT FPT.

Example: k-IS has a param. reduction to k-Clique.

$(G,k) \longmapsto (\overline{G},k)$

complement

Multicolored Clique

Problem: Given a graph G and a vertex k-coloring, find k vertices so that all have distinct colors and they form a k-clique

Goal show this is exactly as hard as k-CLIQUE.

Observe every k-clique is necessarily multicolored (b/c color classes are indep. sets)

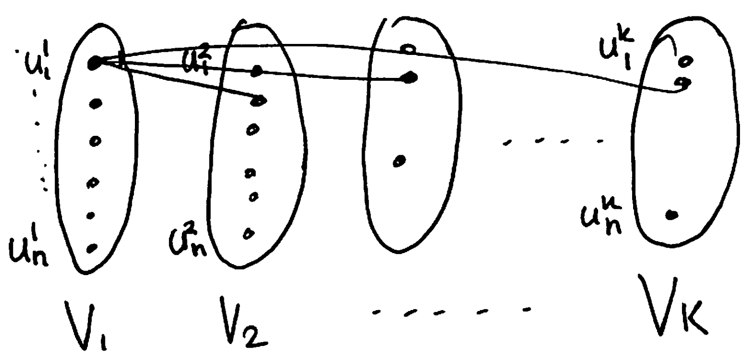
so given $\underbrace{(G, \{V_i\}_{i=1}^{k}, k)}_{MCC} \longmapsto \underbrace{(G, k)}_{C}$

1. $g(k) = k$ ✓
2. Yes by arg. above ✓
3. poly-time: linear ✓

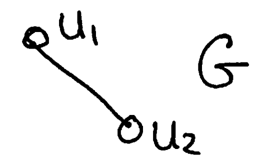
✓ MCC $\mapsto$ C

now, C $\longmapsto$ MCC? Given (G, k) need to produce $(G', \{V_i\}_{i=1}^{k}, k')$

$\tilde{G}$:



- need clique to use 1 vertex per set
- to represent all possible cliques in G need every vertex to be selectable from any set.



$V(G) = \{u_1, \dots, u_n\}$

add (u_ℓ^i, u_h^j) for $i \neq j$ and $(u_\ell, u_h) \in E(G)$

G has k-clique $\Rightarrow$ $\tilde{G}$ has a MC k-clique. Pick up j^{th} copy of the j^{th} vertex in clique ✓

$\tilde{G}$ has a MC-k-clique $\Rightarrow$ G has a k-clique. this works b/c u_j^i is not adj to u_j^ℓ for any ℓ.

In-Class Exercise

Prove that k-CLIQUE on regular graphs is at least as hard as kCLIQUE.

Need to map (G, k) to (G', k') so that G' is <u>degree-regular.</u>

Want any k-clique in G' to only use vertices / edges that correspond to those in G.

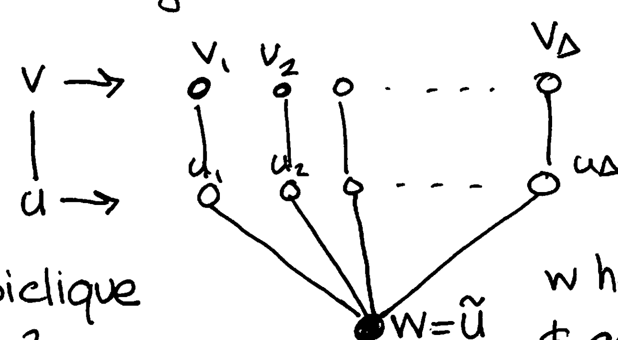
<u>Q1</u> What degree should G' have? Δ = max deg in G. $\leq n-1$

↑ don't have to delete edges ◡ (could choose something bigger)

need to start w/ G & add edges so all vertices get $\Delta - \deg(v)$ new edges <u>AND</u> any k-clique in G' <u>uses no new edges.</u>

suggestion: add vertices of deg 1 ← NOT regular.

<u>Trick</u> make Δ copies of G



k-clique in $G \Rightarrow$ one in each copy ✓

can a k-clique in $\hat{G}$ use a $\tilde{v}$ vertex? NO (unless $k=2$) <u>OK</u>

w has deg Δ & added 1 to $\deg(u_i)$ $\forall i$

If v has deg $d \Rightarrow$ add $\Delta - d$ new vertices $\tilde{v}_1, \ldots \tilde{v}_{\Delta-d}$ & create a biclique between $\{v_1, \ldots, v_\Delta\}$ and $\{\tilde{v}_1, \ldots, \tilde{v}_{\Delta-d}\}$. $\forall v \in G$

Replacing $P \neq NP$

NP = "non-deterministic polynomial" on Turing machines

- $P \longleftrightarrow FPT$

Definition: (short TM acceptance) Given (M, x, k),

input

parameter $\in \mathbb{N}$

(non-deterministic) TM

does M accept x (on some branch) in $\leq k$ steps?

(# states is bounded by $f(n)$ [$n = |\text{input}|$] $\Rightarrow$ in k steps, up to $f(n)^k$

XP

Theorists' Hypothesis: Short TM acceptance is not in FPT.

Engineers' Hypothesis: k-Clique is not in FPT

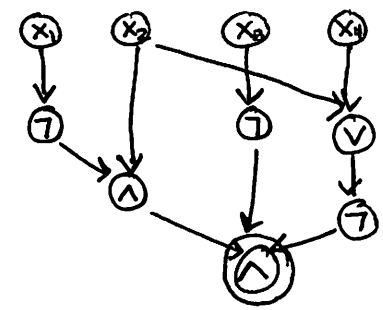
Thm: Theorists' Hyp. $\Longleftrightarrow$ Engineers' Hyp

Weighted Circuit SAT

Defn: a boolean circuit is a DAG w/ 3 types of nodes:

- nodes of indegree 0 (input)
- nodes of indegree 1 (negation)
- nodes of indegree >1 (and / or)

+ exactly one node of outdegree zero is labelled output



Problem: CIRCUIT SAT: Given a boolean circuit, does it have a satisfying assignment?

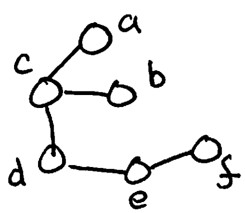
Problem: WEIGHTED CIRCUIT SAT: Is there a satisfying assignment of weight $\leq k$

↑ # TRUES.

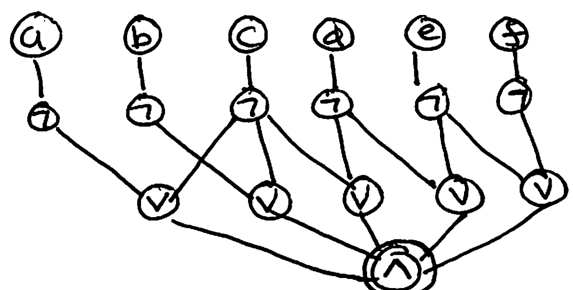
Example Reductions:

$(G, k) \rightarrow (C, k')$

vertices → inputs
(wt. k ⇒ set of k)

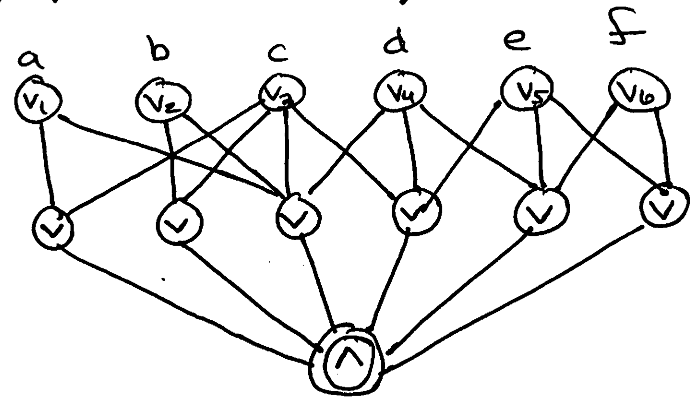


edge (u,v) ⇒ not u or not v



Independent Set

$(G, k) \longrightarrow (C, k)$



Dominating Set

Fine-grained Complexity

- The circuit for DOMSET was more "complex" than the one for INDSET, but how do we make this precise?

Defn The depth of a circuit is longest path from input to output

The weft of a circuit is max # of $\underbrace{\text{large}}_{>2 \text{ inputs (indegree)}}$ gates on a path from input to output

Examples: IS had depth 3, weft 1 ; DS had depth 2, weft 2

Defn the class $C[t,d]$ is the set of all boolean circuits of depth d, weft t

weft

cannot grow unbounded w/ instance size

The W Hierarchy P is the class $W[t]$ if $\exists d$ s.t. P is (param.) reducible to WCSAT on $C[t,d]$.

all circuits

$$FPT \subseteq W[1] \subseteq W[2] \subseteq \dots \subseteq W[t] \subseteq W[P] \subseteq XP$$

k-IS/Clique

k-DomSet (on general graphs)

Problems

1. Show K-DOMINATINGSET and K-HITTINGSET are equally hard.

   given $\mathcal{F} = \{S_1, \ldots, S_n\}$ $\exists$ k elements $x_1 \ldots x_k$ st. $\forall i$, $S_i \ni x_j$ for some j.

2. Prove K-MULTICOLORED-GRID is W[1]-hard.

Defn Given a graph G and a k^2-vertex coloring $\{V_{i,j}\}_{1 \le i,j \le k}$, find a $k \times k$ grid as a subgraph of G so that vertex $v_{i,j}$ (row i, column j) is in $V_{i,j}$.

3. Prove PARTIALVERTEXCOVER parameterized by k is W[1]-hard by reduction from K-INDEPENDENTSET on regular graphs.

Defn Given a graph G and integers k, s, is there a ~~indepe~~ set of k vertices that covers at least s edges?

Note: you get to pick s in the reduction!

Dominating Set

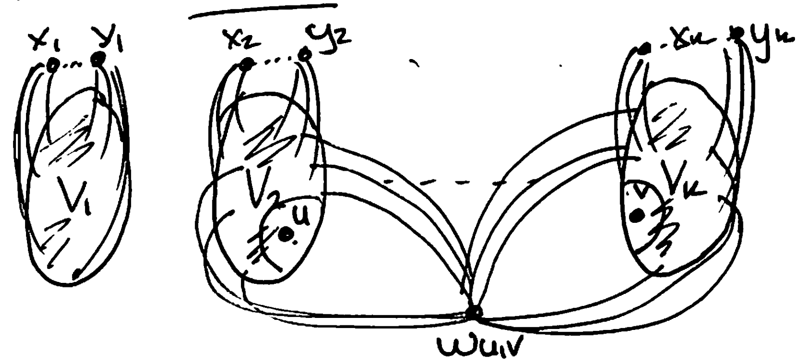
BONUS MATERIAL

Let's reduce from MULTICOLORED INDEPENDENT SET.

- Given (G, k) and $\{V_i\}_{i=1}^{k}$ a k-coloring $\Rightarrow$ produce (G', k') so G' has a k'-Domset $\Longleftrightarrow$ G had an indep set w/ 1 vertex in each V_i.

⓪ We'll let $k = k'$.

① Let's turn the V_i's into cliques (so we can pick one from each to dominate):



② We'll transform edges in G to vertices in G' so they can be dominated only if you didn't put both end points in your dom set. $(u,v) \rightarrow w_{uv}$ with $u \in V_i$, $v \in V_j$
w_{uv} is adj to all of $V_i \setminus \{u\}$ and $V_j \setminus \{v\}$. (so if you pick $u \in V_i$ and $v \in V_j$, $w_{u,v}$ is not dominated).

③ to make sure vertices $w_{u,v}$ aren't picked as dominators, we add two vertices x_i, y_i to G' for each i s.t. x_i is not a nbr. of y_i but x_i, y_i are adj to everything in V_i.
Then these aren't dominated by any w (or each other) $\Rightarrow$ must pick a vertex in each V_i.