Definition. *A graph is* Eulerian *if it has a circuit (closed walk) containing every edge exactly once.*

Theorem. *A graph G is Eulerian if and only if it has at most one nontrivial component and its vertices all have even degree.*

Proof. We show the two halves of the if and only if:

Necessity. Suppose that G is Eulerian circuit C. Each passage of C through a vertex uses two incident edges, and the first edge is paired with the last at the first vertex. Hence every vertex has even degree. Also, two edges can be in the same walk only when they lie in the same component, so there is at most one nontrivial component.

Sufficiency. Assuming that the condition holds, we obtain an Eulerian circuit using induction on the number of edges, m.

Basis step: $m = 0$. A closed walk consistenting of one vertex suffices.

Induction step: $m > 0$. With even degrees, each vertex in the nontrivial component of G has degree at least 2. We want to show that G contains a cycle we can delete so that we may use the induction hypothesis.

To construct the cycle C, pick a maximal path P in G and let u be one of its endpoints. Since P is maximal and has degree at least 2, u has two neighbors v_1, v_2 in P. We therefore let C be the union of the path between v_1 and v_2, and the edges (u, v_1) and (u, v_2).

Let G' be the graph obtained from G by deleting the edges of C. Since C has 0 or 2 edges at each vertex, each component of G' is also an even graph. Since each component is also connected and has fewer than m edges, we can apply the induction hypothesis to conclude that each component of G' has an Eulerian circuit. To combine these into an Eulerian circuit of G, we traverse C, but when a component of G' is entered for the first time we detour along an Eulerian circuit of that component. This circuit ends at the vertex where we began the detour. When we complete the traversal of C, we have completed an Eulerian circuit of G. □

Definition. *A graph G is* bipartite *if its vertices can be partitioned into two sets L and R such that every edge in G has one endpoint in L and one in R.*

Theorem. *A graph is bipartite if and only if it has no odd cycle.*

Proof. We have a bipartite graph. Every walk around the graph takes an even number of steps if it ends in the same set it begins, so the graph has no odd cycle. Now suppose the graph has no odd cycle. We can start at a vertex and put every vertex in one partite set if it is an odd number of steps from the vertex, and in the other partite set if it is an even number of steps. An edge in a partition means that we have an odd cycle, which contradicts our claim. □

Problem 3

Theorem. *All students at NC State have the same major.*

Proof. We proceed by induction:

Base case: $n = 1$. One NC State student has the same major, so the claim is true.

Induction step: Suppose the claim is true for k students, we will show that it is also true for $k + 1$ students. Label our students $S = \{x_1, x_2, \ldots, x_{k+1}\}$ and split the students into two subsets: $S_1 = S \setminus \{x_2\}$ and $S_2 = S \setminus \{x_3\}$. By the induction hypothesis, all students in S_1 have the same major, and all students in S_2 have the same major. But x_1 is in both sets, therefore all students have the same major. $\square$