CSC791/495-011

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August 18, 2017

An Academic Problem

- n faculty members, known conflicts between some pairs.
- how many guests (max) can we invite w/no fistfights? give an algorithm to find a solution.

<u>Suggestion:</u> model problem as graph V = faculty E = conflicts $n = 1000$

<u>Try!</u> Look at all possible guest lists & pick one with largest size & no fights

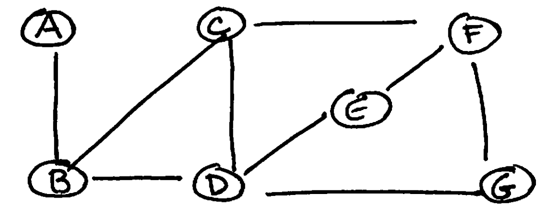
$2^n \underbrace{\cancel{n!}}_{\text{sets}} \cdot n^2 \quad \sim 10^{300}$

<u>Reality</u> I can't afford to not invite more than $\underline{\underline{k}}$ faculty members
↑ very small

try brute force on finding non-invited faculty: $K = 10$

consider all subsets of size k: $O\left(\binom{n}{k} \cdot (n-k)^2\right) \sim 10^{23}$

Resolving Conflicts



$n = 7$

q: $k = 3$

Strategy: make easy decisions 1st

① if there's a faculty member w/no conflicts, invite them (k)

② if a faculty member has $\geq k+1$ conflicts, add to black list ($k \to k-1$)

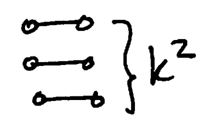
claim ① & ② are "safe" operations — if there is a solution w/ $\leq k$ on the blacklist

$\Rightarrow$ performing these doesn't (can't) change YES/NO answer.

Alg: iteratively apply ① & ② until you can't anymore. What can we say about remaining instance - how many faculty could be on undecided list?

If this remainder has too many conflicts $\Rightarrow$ it must be a NO-instance.

? How many conflicts can be resolved by excluding k people? Each person is in $\leq k$ conflicts $\Rightarrow$ $\leq k^2$ conflicts.



③ If $> k^2$ conflicts $\Rightarrow$ answer NO. (applys only after ② is done)

Now $\leq k^2$ conflicts & every remaining faculty is in ≥ 1 conflict $\Rightarrow \leq 2k^2$ faculty.

Brute force: $\binom{2k^2}{k} \underline{\underline{(k^2)^2}}_{n}$ overall: exercise $\sim 10^{16}$

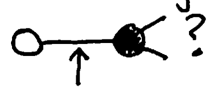
Resolving Conflicts, II

① remove vertices of degree 0 (invite them; k stays same)

② remove vertices of degree $\geq k+1$ (blacklist them; $k \to k-1$)

③ remove vertices of degree 1 (invite them; blacklist their neighbor; $k \to k-1$)

"Safe" b/c nbr also has deg ≥ 1 & all conflicts must be resolved



• must blacklist ≥ 1 vertex from each edge.

④ reject if $> k^2$ conflicts (b/c all degrees are $\leq k$)

<u>Claim</u> at most k^2 faculty still need decisions.

<u>Pf</u> If $\leq k^2$ conflicts & every remaining vertex has deg ≥ 2

$\Rightarrow \leq k^2$ faculty remaining.

BF runtime $O\left(\binom{k^2}{k}\right)$ $\sim 10^{13}$

An Egalitarian Approach

Algorithm 1 Pick a vertex uniformly at random & black list it ($k \to k-1$).

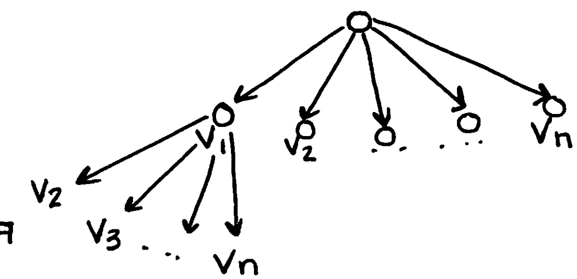
Apply recursively until either $k = 0$ $\begin{cases} \text{if } |E| = 0 \Rightarrow \text{YES} \\ \text{else} \Rightarrow \text{NO} \end{cases}$

or $|V| = 0 \Rightarrow$ YES

Claim $O(n^k)$

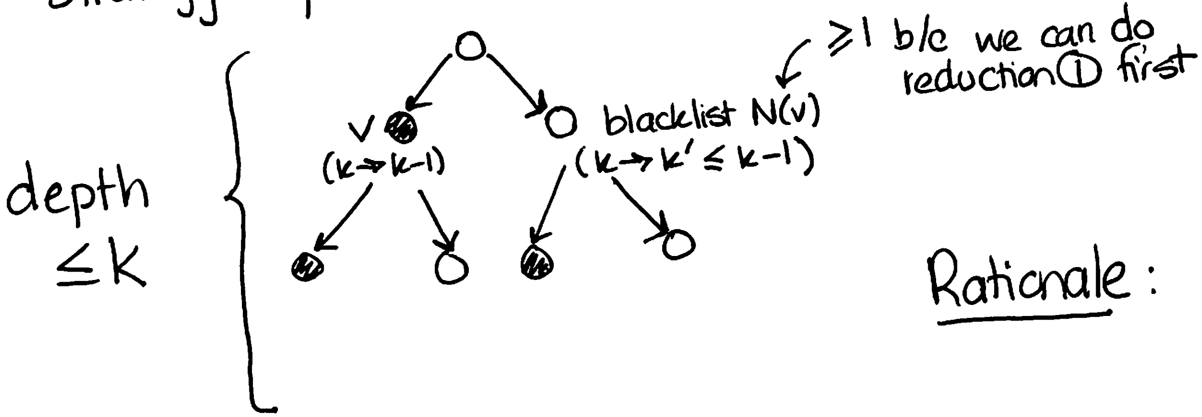
Rationale Apply reduction ① first. Then we have

$n(n-1) \cdots (n-(k-1)) \leq n^k$ leaves in oursearch tree



Algorithm 2 Observe that if v is invited, its neighbors in G cannot be.

Strategy: pick a vertex v (uniformly @ random). Either blacklist v OR blacklist all neighbors of v.



Claim $O(2^k \cdot n \cdot k) \sim 10^7$

Rationale: $\leq 2^k$ leaves in search tree

Lesson(s) Learned

1. model problem as a graph (abstraction)
2. we gave algorithms for an NP-hard problem (VERTEX COVER) with running times of the form $n^{f(k)}$ and $f(k)\,n^c$

   w/c constant

   these are parameterized algorithms w/ parameter k ← relevant measure of the input
3. it might be important to reduce $f(k)$ &/or n^c

Course Highlights

git, latex & slack will be your friends!

→ all about solving (NP-hard) problems – *efficiently!*

→ two major projects

① how to identify an interesting (& approachable) open problem

② actual research – start to finish: lit review, positive & negative results, Collaboration, & communication (paper & pres.)

→ big focus on communication

* weekly homework is writing up a polished proof for a problem worked on in class. Clarity counts more than correctness!
* proof review exercises (hands-on NOW!)
* weekly research log required

→ special topics will be based on interest & relevance to ongoing research projects.

Not a Panacea

- Can't afford to alienate other faculty. So I rent k rooms. Now, I need to group all n faculty into $\leq k$ sets so no fistfights occur in each room.

- again model w/ conflict graph. This now asks for a partition of V into $\leq k$ <u>independent sets</u> (a proper k-coloring)

- k-coloring is NP-hard, but I could hope for $n^{f(k)}$ (or $f(k)n^c$)

<u>Problem:</u> 5-coloring is NP-hard.

<u>Thm</u> k-coloring has no FPT (XP) algorithm unless P=NP.

<u>Proof</u> (sketch) Assume not - that is, $\exists$ an algorithm to solve k-coloring w/ $O(f(k)n^c)$ running time for some constant c. Then we can solve 5-coloring in $O(f(5)\cdot n^c)$, but $f(5)$ and c are constants, so this implies that 5-coloring $\in$ P. If P $\neq$ NP, $\rightarrow\leftarrow$. $\square$

Hardness is Subtle

- bouncer problem : n people, need to know if any cliques of size $\geq k$.

Can we give an XP algorithm?

Approach: look at all subsets of size k.

$$O\left(\binom{n}{k} \cdot k^2\right) \sim n^k$$

→ unknown if there is an FPT algorithm

NP-hardness fails to give us insight b/c k-CLIQUE is not NP-hard for fixed k.

Need more refined notion of complexity & hardness.

W[1]-hierarchy