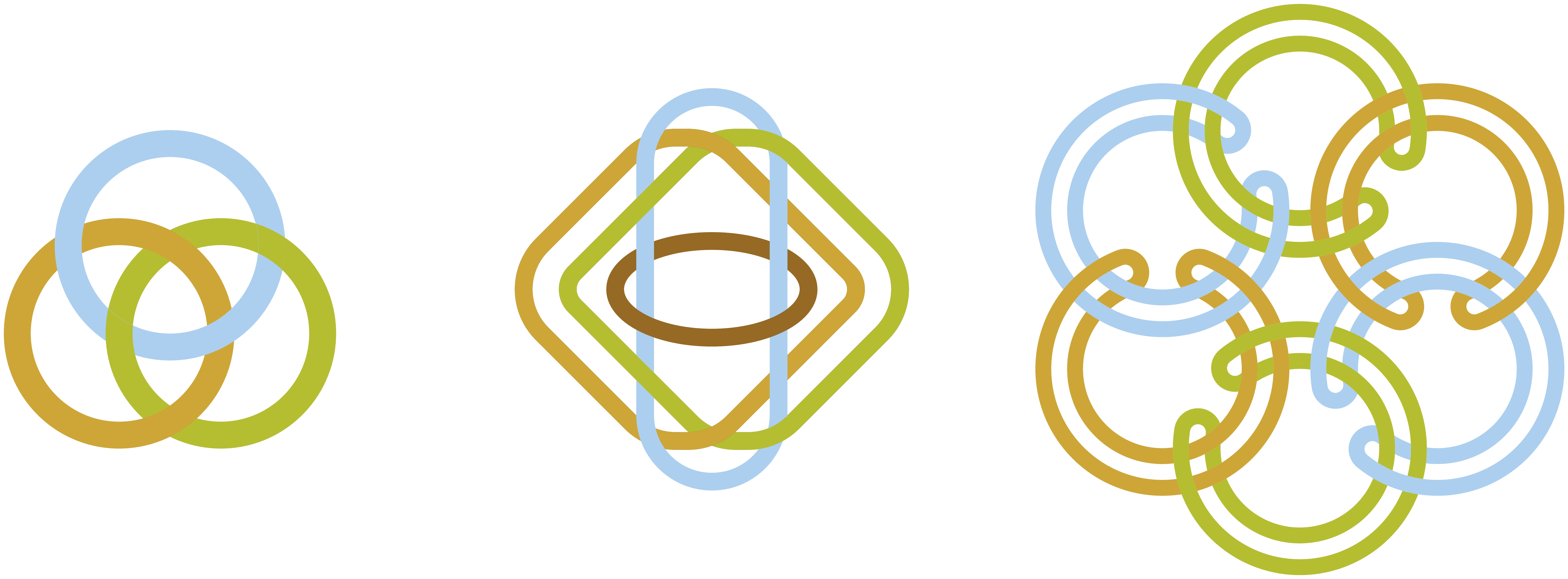
From Macro to Micro with Möbius

The *Mereology* of Higher-Order Interactions

Abel Jansma — DIEP seminar — 27.02.25

Brunnian links



- ‘Higher-order’ structure is relative to a decomposition.
- **Aim:** to make this **precise** and **useful** as a new way to study complex systems.

Overview

1. Decomposing complex systems
    - *Mereologies*

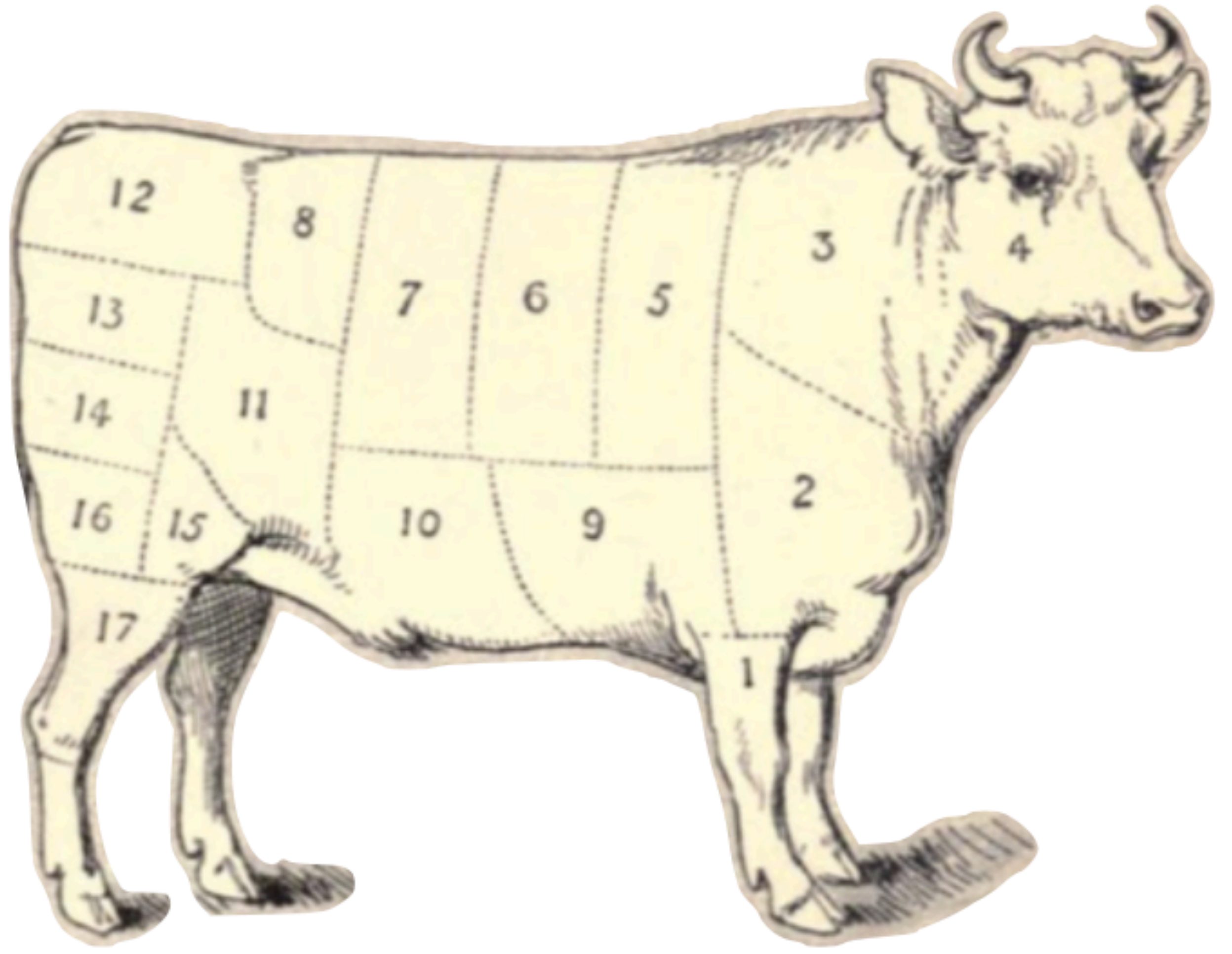
2. Calculus on mereologies
    - Incidence algebras & the Möbius inversion theorem

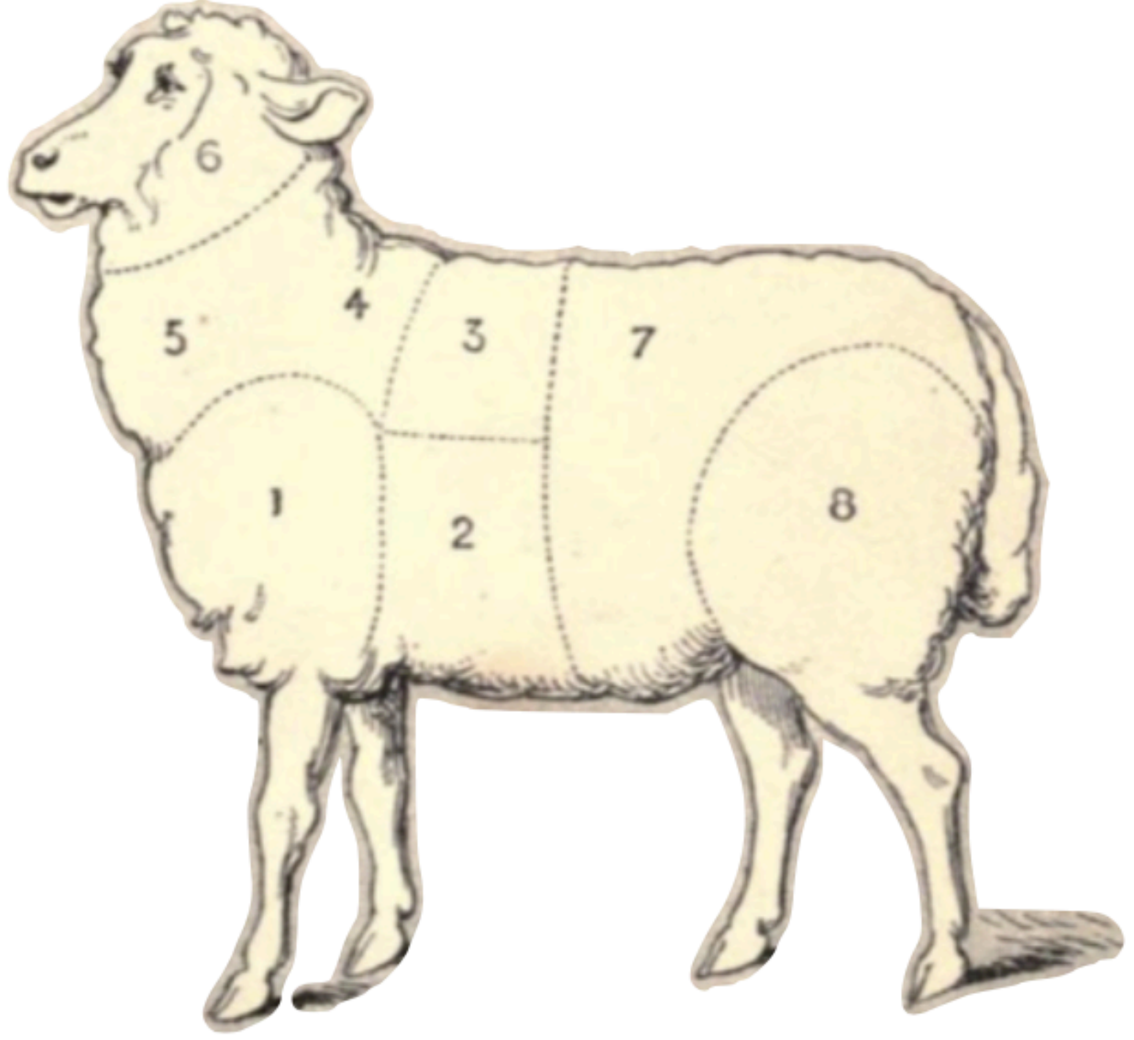
3. Examples
    - Information theory, Biology, Physics, Game theory

4. New applications
    - Information Decomposition
    - Decomposing KL-divergence and causal effects
    - *"Renormalised couplings are Möbius inversions over a Galois connection"*

Part 1

Decomposing Complex Systems





Decomposing complex systems

- **Example.** Height decomposed over 3 genetic variants $G = \{g_1, g_2, g_3\}$

$$
\begin{aligned}
H(G) &= \sum_{g \in G} h(g) \\
&= h(g_1) + h(g_2) + h(g_3)
\end{aligned}
$$

- This is boring! Genes interact!

$$
\begin{aligned}
H(G) &= \sum_{g \in \mathscr{P}(G)} h(g) \\
&= h(g_1) + h(g_2) + h(g_3) + h(g_1, g_2) + h(g_2, g_3) + h(g_1, g_3) \\
&\quad + h(g_1, g_2, g_3) + h(\varnothing)
\end{aligned}
$$

- Interactions emerge from the powerset decomposition.

Decomposing complex systems

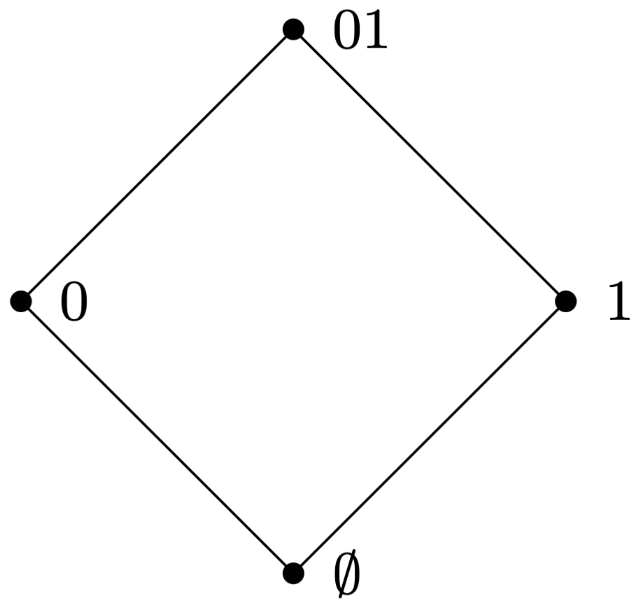
- Quantity Q as a **decomposition** over S:

$$Q(S) = \sum_{t \in \mathscr{D}(S)} q(t)$$

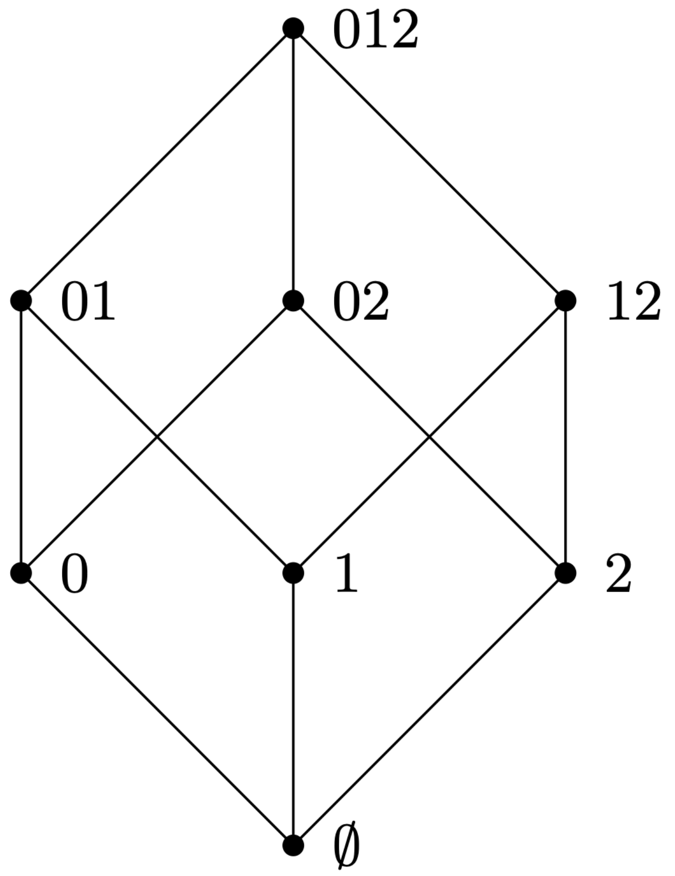
- *Q: "macro"* *q: "micro"*
- Complexity of model is hidden in decomposition (*cf.* design matrix in regression)
- **Forward problem**: $q \implies Q$.
- **Inverse problem:** $Q \implies q$
- I will show: inverse problem can be solved by exploiting structure of $\mathscr{D}(S)$
- Hierarchy of parts $\implies$ partial order on $\mathscr{D}(S)$

Higher than what?

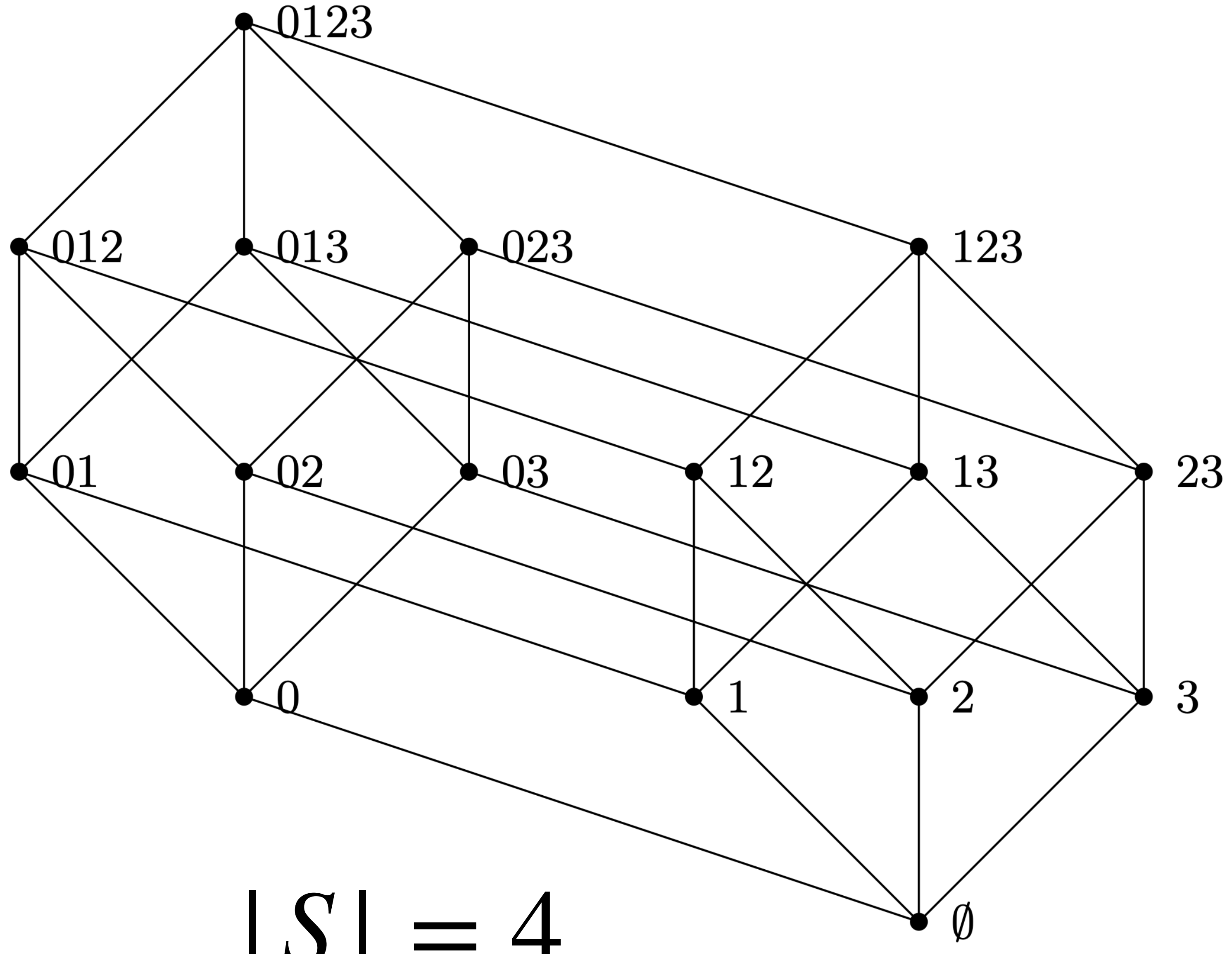
- **Definition.** *A partial order on a set S is a binary relation $\leq$ that is $\forall a, b, c \in S$:*
    - ***1)*** *Reflexive:* $a \leq a$
    - ***2)*** *Transitive:* $a \leq b$ *and* $b \leq c \implies a \leq c$
    - ***3)*** *Antisymmetric:* $a \leq b$ *and* $b \leq a \implies a = b$
- **Example.** Powerset $\mathscr{P}(S)$ ordered by inclusion: $a \leq b \iff a \subseteq b$



$|S| = 2$



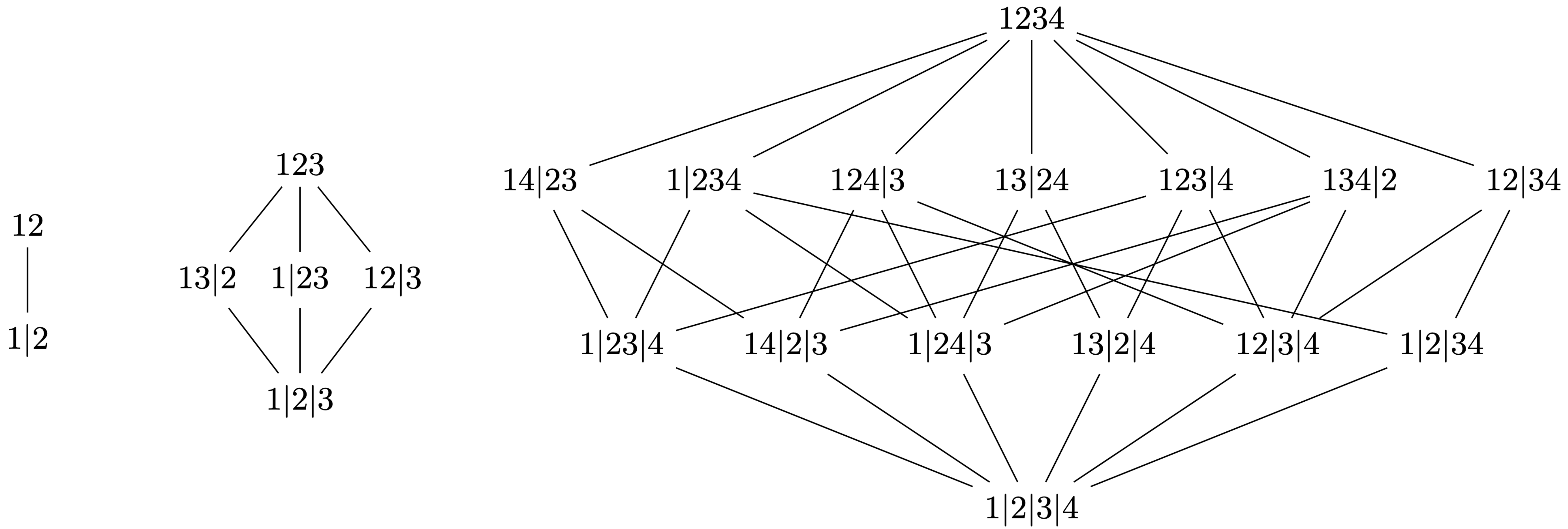
$|S| = 3$



$|S| = 4$

Higher than what?

- **Example.** Partitions $\Pi(S)$ ordered by refinement:



$|S| = 2$ $\quad$ $|S| = 3$ $\quad$ $|S| = 4$

Mereology

- **Mereology:** The study of parthood—the relationship between parts and wholes

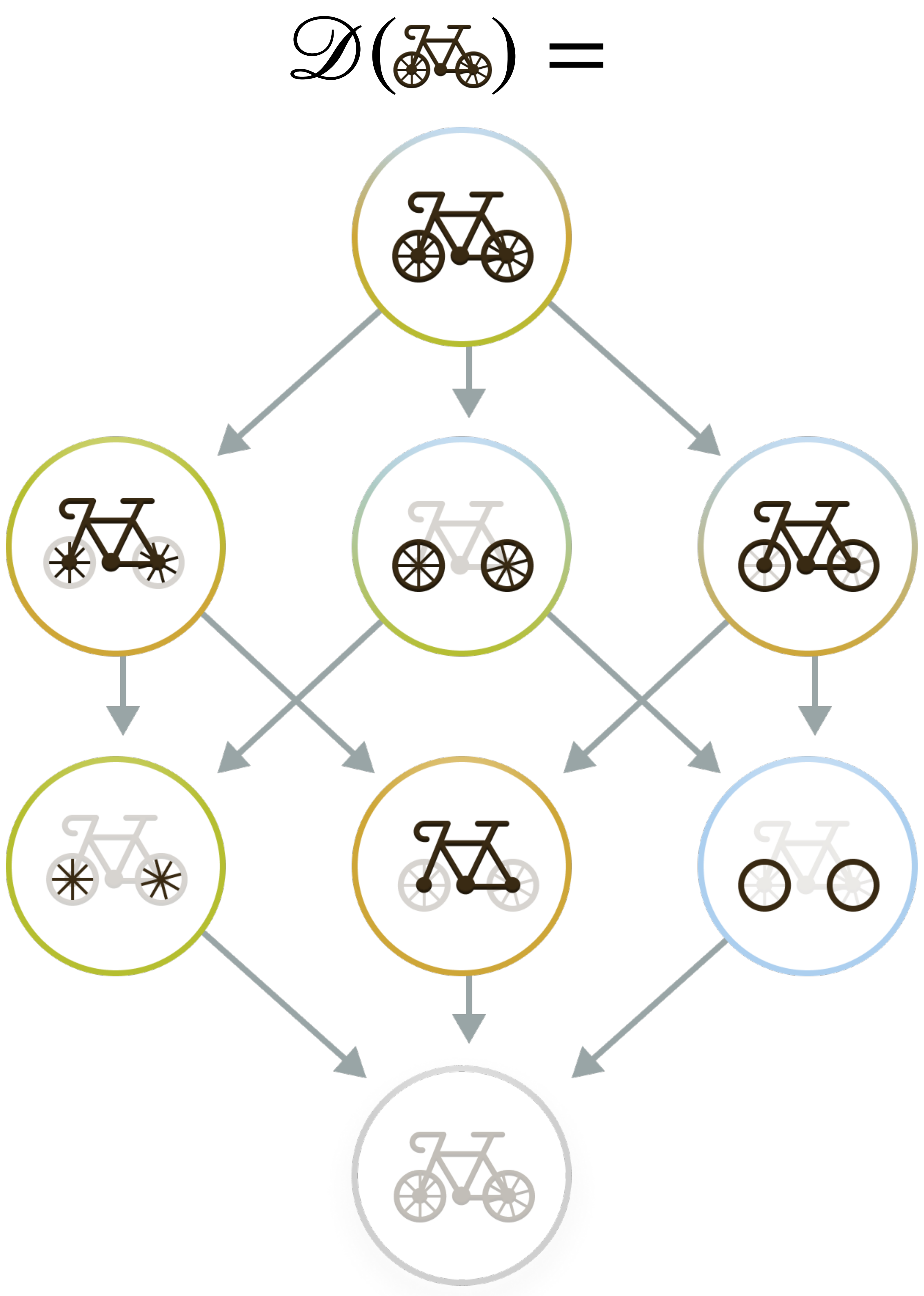
- **Definition.** *A mereology $\mathscr{D}(S)$ on a system S is a locally finite co-rooted poset with S as its unique largest element.*

    - Locally finite: all intervals $[a, b] = \{x \,|\, a \leq x \leq b\}$ are finite.
    - Co-rooted: $\mathscr{D}(S)$ has a unique largest element.

- **NB:** every finite topology is a mereology under subset inclusion, but not *vice versa.*

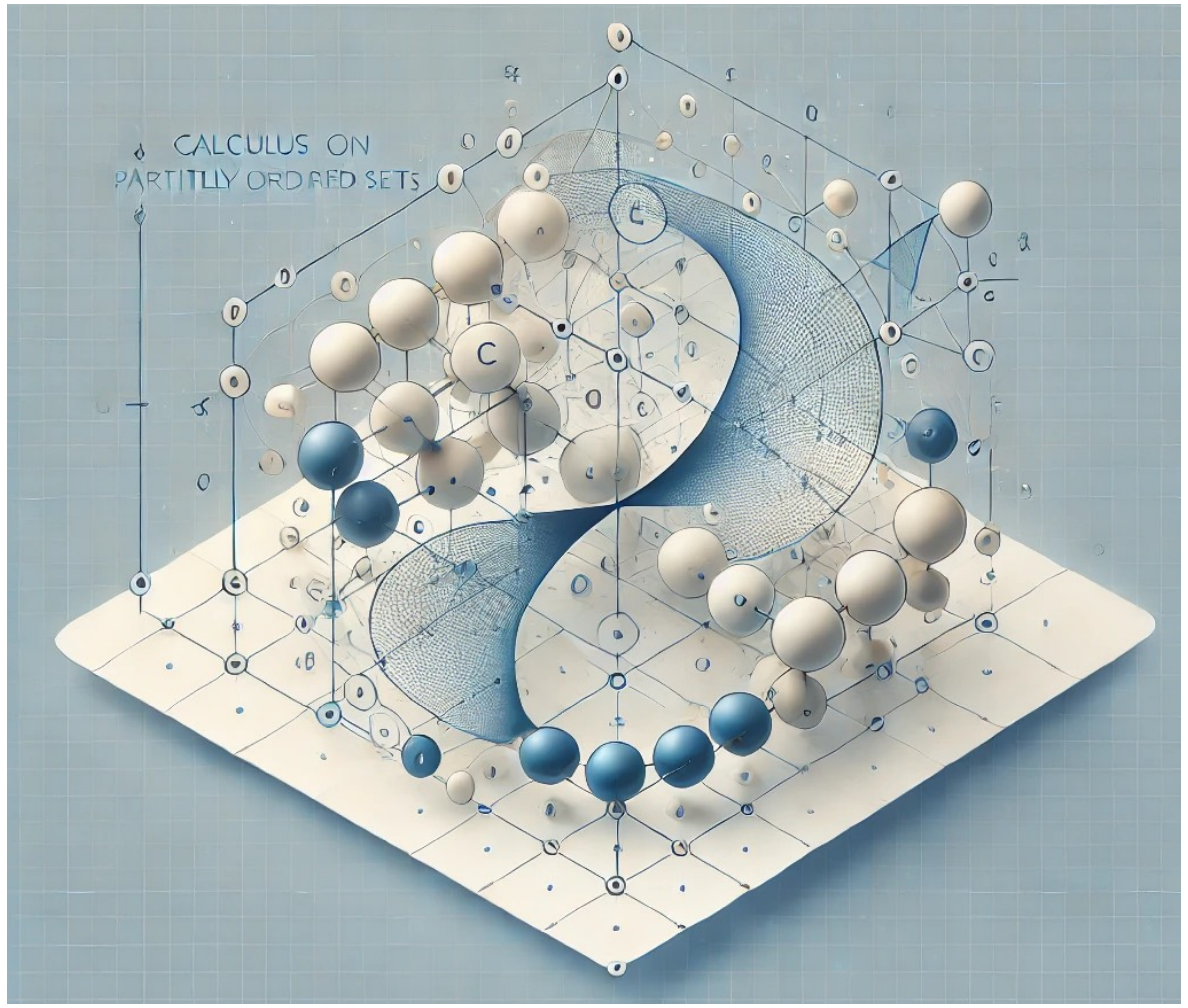
- When $\mathscr{D}(S)$ is a mereology under $\leq$, then we can write:

$$Q(S) = \sum_{t \leq S} q(t)$$



Part 2

Calculus on Mereologies



Calculus on posets

- Decompositions are functions on posets $\implies$ we should study these functions…
- *Incidence algebra $(P, \mathbb{R}, \star)$:*
    - Locally finite poset P
    - $\mathbb{R}$-vector space of functions $\{f \mid f : P \times P \to \mathbb{R}\}$
    - Bilinear product $\star$ such that $(f \star g)(a, b) = \sum_{a \leq x \leq b} f(a, x) g(x, b)$
- Special elements:
    - $\delta_P(a, b) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}$
    - $\zeta_P(a, b) = \begin{cases} 1 & \text{if } a \leq b \\ 0 & \text{otherwise} \end{cases}$

Calculus on posets

$$(f \star g)(a,b) = \sum_{a \le x \le b} f(a,x) g(x,b)$$

$$\delta_P(a,b) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}$$

$$\zeta_P(a,b) = \begin{cases} 1 & \text{if } a \le b \\ 0 & \text{otherwise} \end{cases}$$

- δ_P is multiplicative unit: $f \star \delta_P = \delta_P \star f = f$
- ζ_P is 'integrator' on *P*: $(\zeta \star f)(a,b) = \sum_{a \le x \le b} f(x)$ for functions *f(a, b) = f(a)*
- **Question:** Does ζ_P have a $\star$-inverse that 'undoes' integration?
- **Answer: Yes!** This is called the Möbius function μ_P such that $\mu_P \star \zeta_P = \delta_P$

Calculus on posets

- **Definition.** The Möbius function of a poset P is given by

$$\mu_P(a,b) = \begin{cases} 1 & \text{if } a = b \\ -\sum_{a \le x < b} \mu_P(a,x) & \text{if } a < b \\ 0 & \text{otherwise} \end{cases}$$

- **Möbius Inversion Theorem** (Rota, 1964). *Let P be a locally finite poset with Möbius function μ_P, then for any $f : P \to \mathbb{R}$*

$$f(b) = \sum_{a \le b} g(a) \quad \Longleftrightarrow \quad g(b) = \sum_{a \le b} \mu_P(a,b) f(a)$$

- Sums over mereologies can be inverted!

$$Q(S) = \sum_{t \le S} q(t) \quad \Longleftrightarrow \quad q(S) = \sum_{t \le S} \mu_{\mathscr{D}(S)}(t,S) Q(t)$$

From Macro to Micro with Möbius

- Sums over mereologies can be inverted

$$Q(S) = \sum_{t \leq S} q(t) \qquad \Longleftrightarrow \qquad q(S) = \sum_{t \leq S} \mu_{\mathscr{D}(S)}(t, S) Q(t)$$

  - **If you know the Möbius function.**
  - **If you can observe Q on $\mathscr{D}(S)$.**

- **Example.** Height as a function of 2 genetic variants.

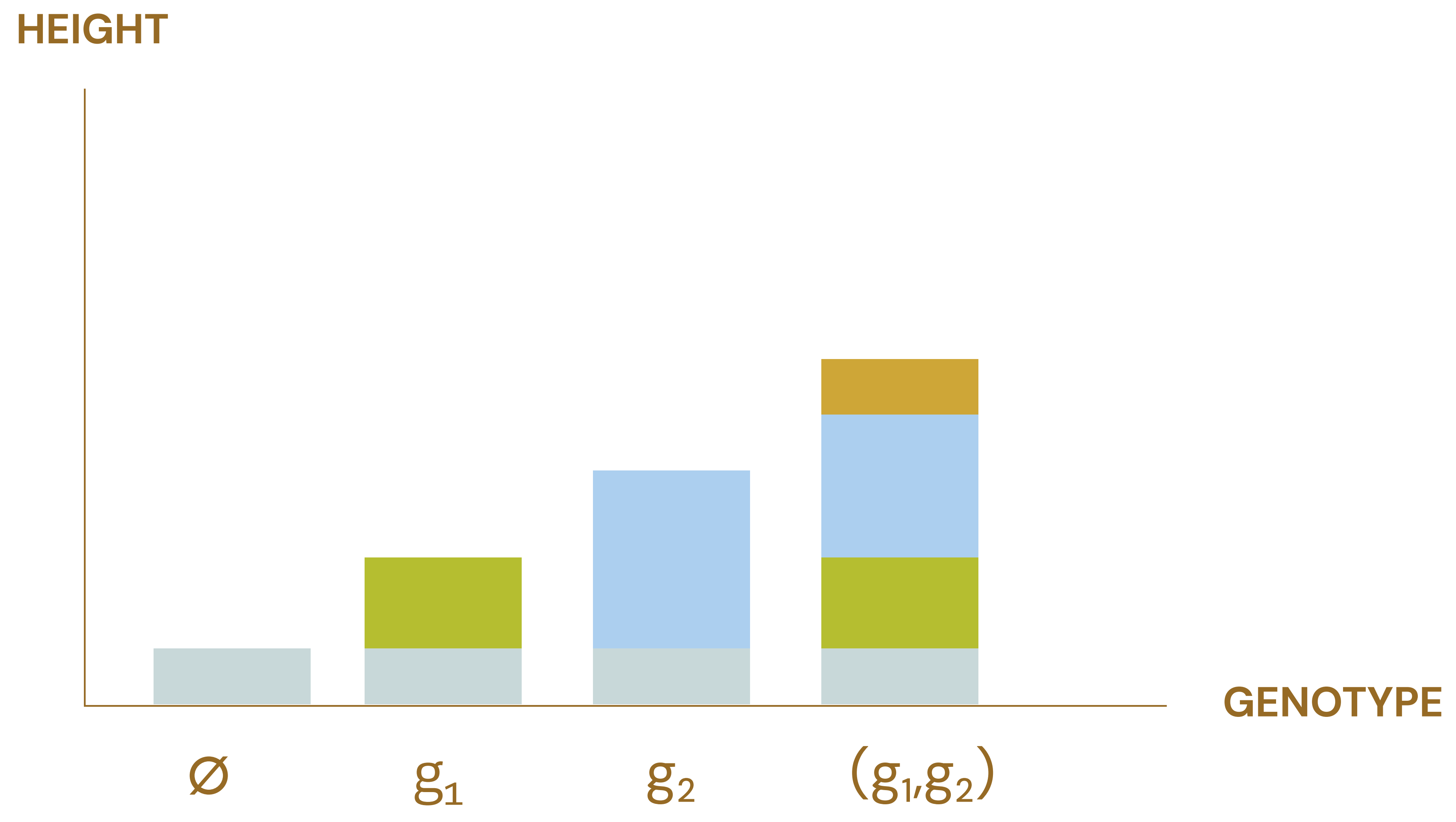
$$H(G) = \sum_{s \subseteq \{g_1, g_2\}} h(s) = h(\varnothing) + h(g_1) + h(g_2) + h(g_1, g_2)$$

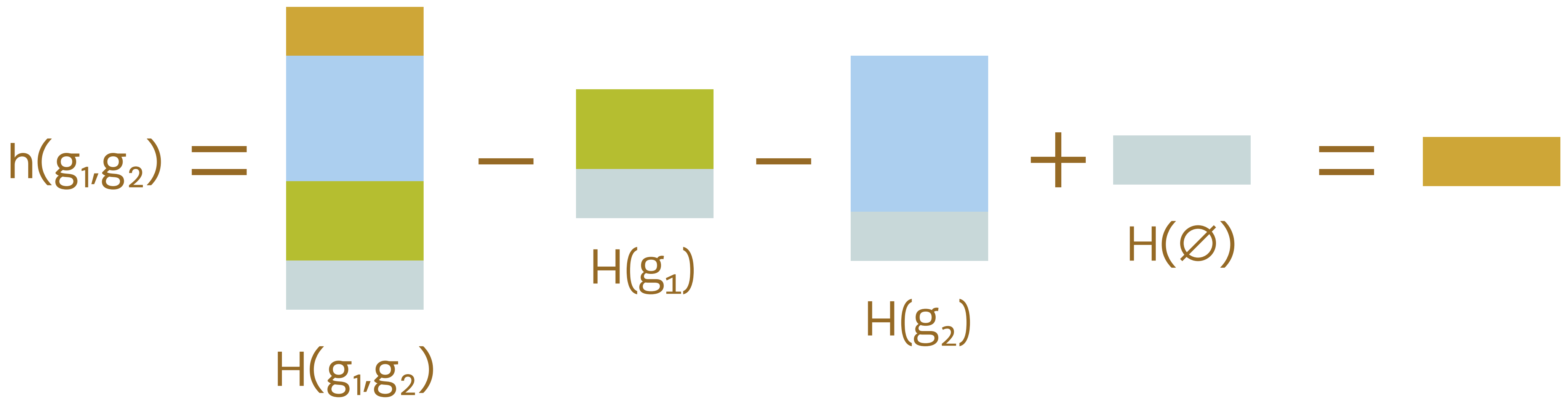
$$\mu_{\mathscr{P}(S)}(a, b) = (-1)^{|b| - |a|}$$

$$\Longrightarrow \; h(g_1) = H(g_1) - H(\varnothing)$$

$$\Longrightarrow \; h(g_1, g_2) = H(g_1, g_2) - H(g_1) - H(g_2) + H(\varnothing)$$

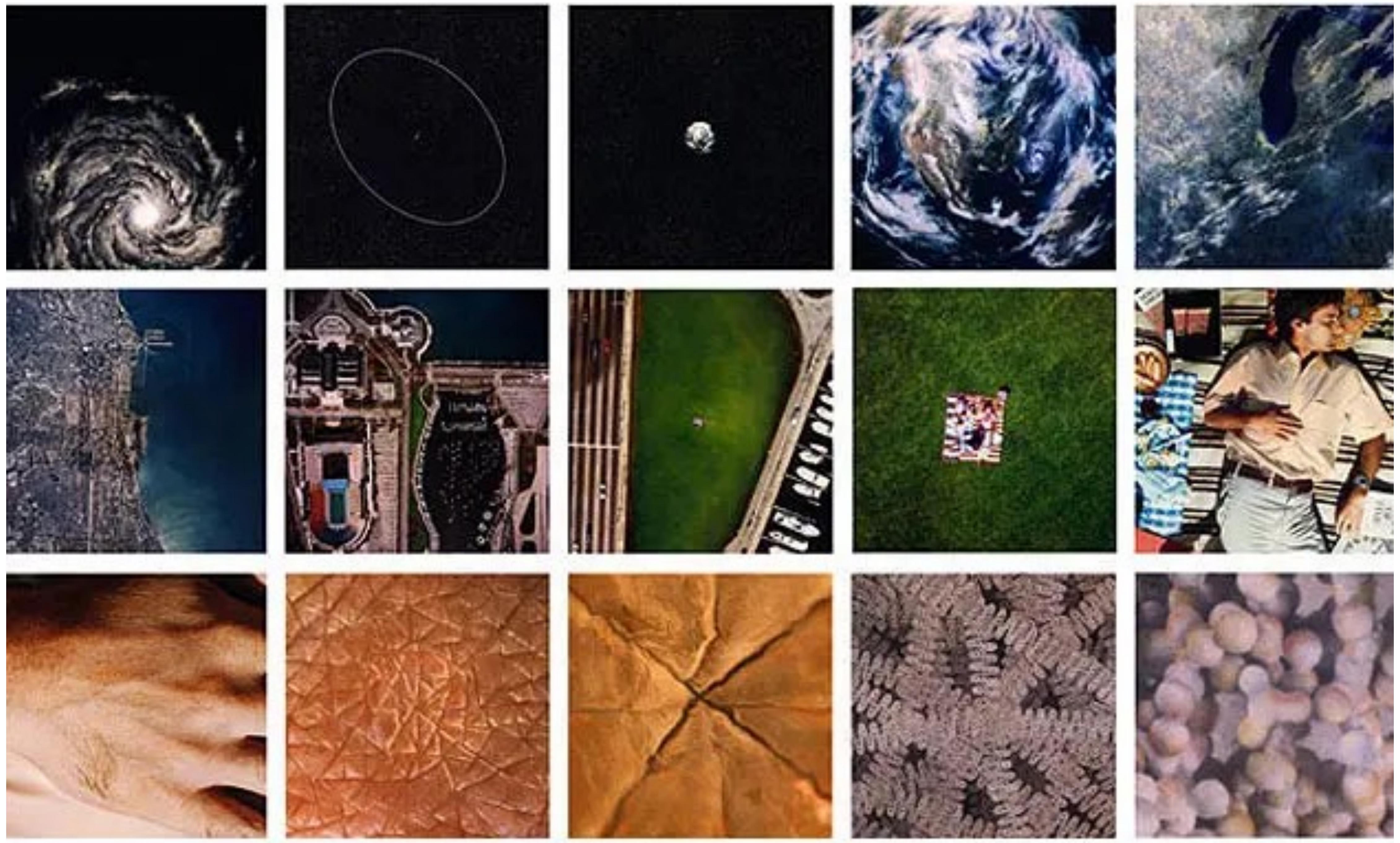
- “Calculus on mereologies”





Part 3

Möbius inversions across scales



Name of the game:

- Choose ‘macroscopic’ observable to decompose.

$$Q(S)$$

- Choose a mereology.

$$Q(S) = \sum_{t \in \mathscr{D}(S)} q(t)$$

- Derive ‘microscopic’ theory.

$$q(t) = \sum_{u \leq t} \mu_{\mathscr{D}(S)} Q(u)$$

Example: Information theory

- Entropy of a set S of random variables:

$$H(S) = -\sum_{s} p(S=s)\log p(S=s)$$

- Let's impose the powerset mereology, so that

$$H(S) = \sum_{t \in \mathcal{P}(S)} I(t) \implies I(S) = \sum_{T \subseteq S} \mu_{\mathcal{P}}(T,S)H(T) = \sum_{T \subseteq S} (-1)^{|S|-|T|}H(T)$$

$$\implies I(X,Y) = H(X,Y) - H(X) - H(Y) + H(\varnothing)$$

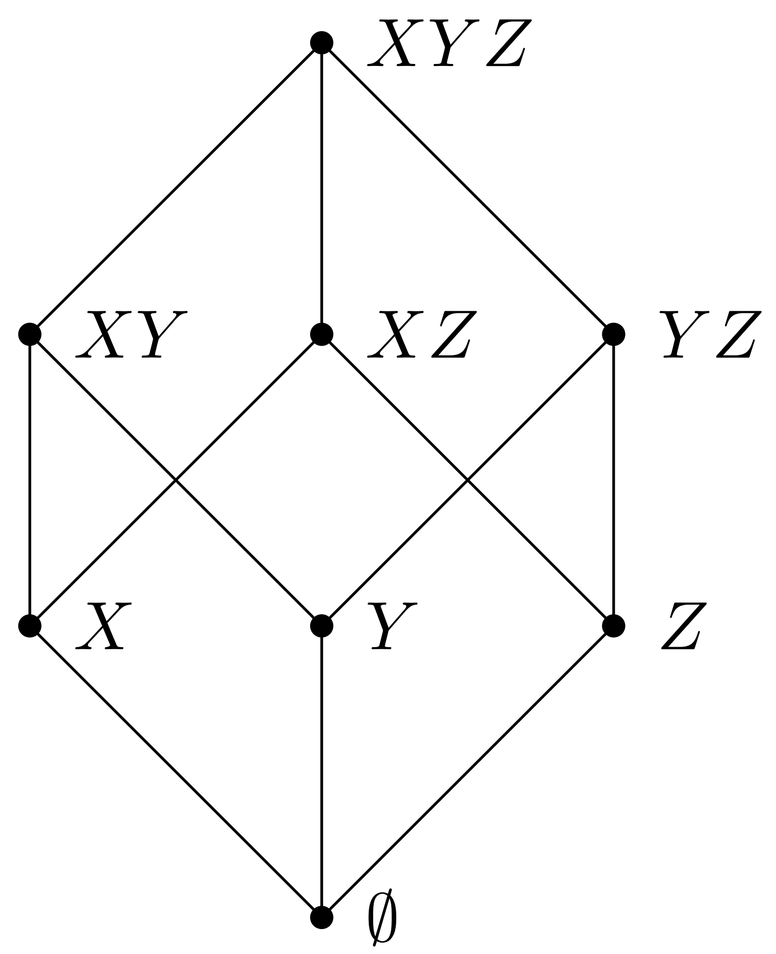
$$= -\sum_{x,y} p(x,y)\log\frac{p(x,y)}{p(x)p(y)}$$

- *Mutual information is the $\mathcal{P}$-Möbius inverse of entropy (idem for pointwise).*

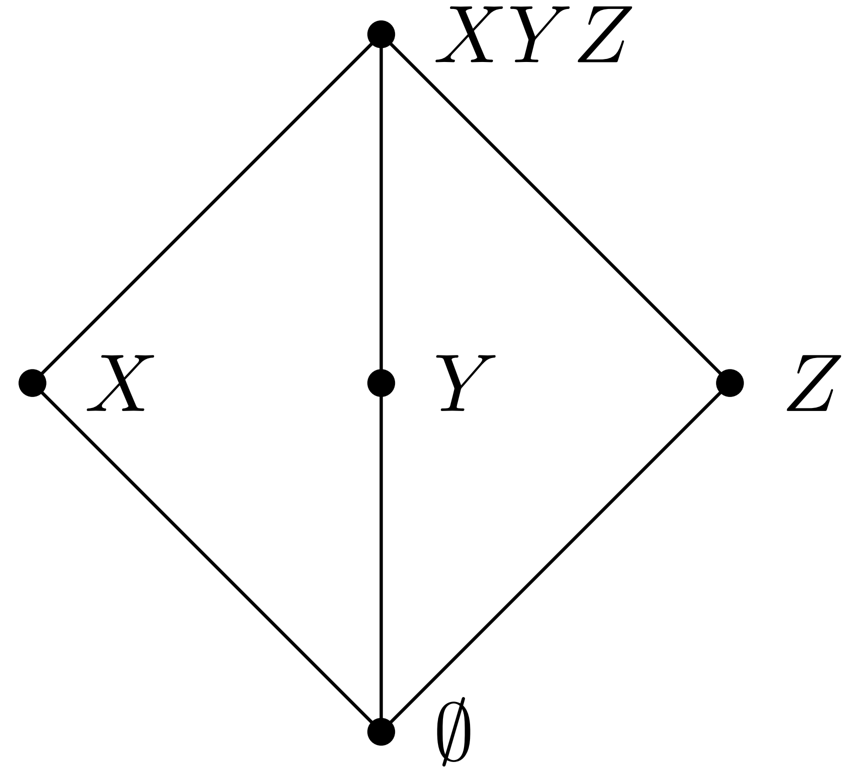
Example: Information theory

- Mutual/interaction information vs total correlation:

$$I(X, Y, Z) = \sum_{x,y,z} p(x, y, z) \log \frac{p(x, y, z)p(x)p(y)p(z)}{p(x, y)p(y, z)p(x, z)}$$

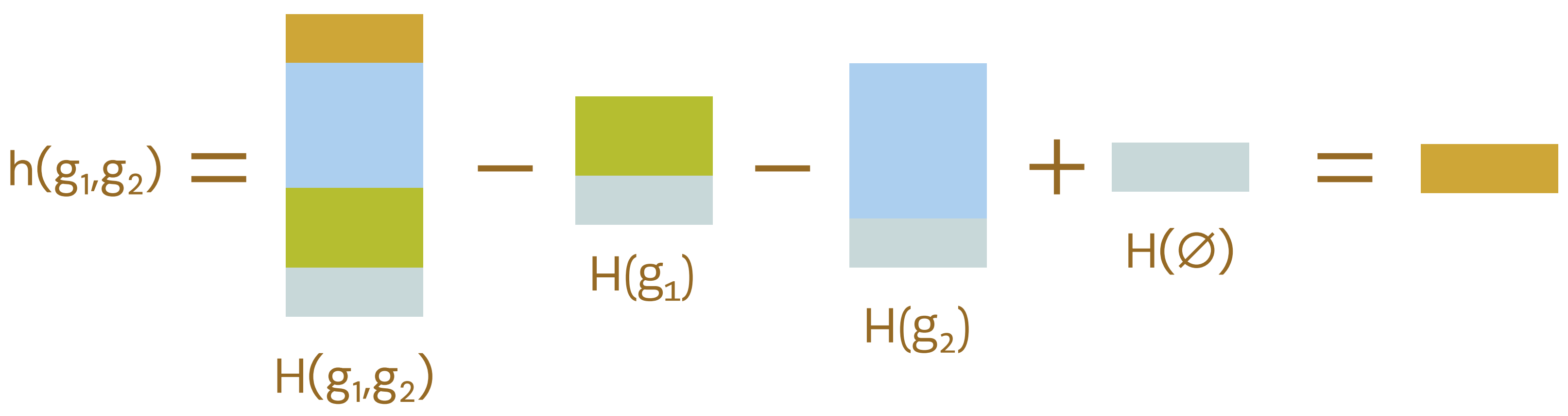


$$TC(X, Y, Z) = \sum_{x,y,z} p(x, y, z) \log \frac{p(x, y, z)}{p(x)p(y)p(z)}$$



Example: Biology

- (Generalisation of the height example)



- Phenotype F decomposed over a genotype G :

$$F(g=\vec{1}, G\backslash g=\vec{0}) = \sum_{s\in\mathscr{P}(g)} I(s) \quad \Longleftrightarrow \quad I(g) = \sum_{s\subseteq g} (-1)^{|s|-|g|} F(s=\vec{1}, G\backslash s=\vec{0})$$

$$I_{g_1g_2g_3} = F(1,1,1,\vec{0}) - F(1,1,0,\vec{0}) - F(1,0,1,\vec{0}) - F(0,1,1,\vec{0})$$

$$+F(1,0,0,\vec{0}) + F(0,1,0,\vec{0}) + F(0,0,1,\vec{0}) - F(0,0,0,\vec{0})$$

- Used in e.g.
  - Beerenwinkel et al. (2007), Epistasis and shape of fitness landscapes, Stat. Sin.
  - Gould et al. (2018). Microbiome interactions shape host fitness, PNAS
  - Eble et al. (2023), Master regulators of biological systems in higher dimensions, PNAS

Example: Physics

- Ising Model $p(s) = Z^{-1}e^{-E(s)}$, and $s \in \{0,1\}^{|s|}$

$$E(S) = \sum_{i,j} J_{ij}s_i s_j \to \sum_{T \in \mathcal{P}(S)} J_T \prod_{t \in T} t = \sum_i J_i s_i + \sum_{ij} J_{ij}s_i s_j + \sum_{ijk} J_{ijk}s_i s_j s_s + \dots$$

- Note that for any $T \subseteq S$

$$E(T=1, S \backslash T = 0) = \sum_{U \subseteq T} J_U \quad \Longrightarrow \quad J_T = \sum_{U \subseteq T} (-1)^{|S|-|U|} \log p(U=1, S \backslash U = 0)$$

- Write $p_{ab} = p(s_i = a, s_j = b, S \backslash \{s_i, s_j\}) = \vec{0})$, then the inverse Ising problem is solved by:

$$J_{ij} = \log \frac{p_{11}p_{00}}{p_{01}p_{10}} \qquad\qquad J_{ijk} = \log \frac{p_{111}p_{100}p_{010}p_{001}}{p_{000}p_{011}p_{101}p_{110}}$$

- Used in e.g.
  - Kennedy (2010), Renormalization group maps for Ising models in lattice-gas variables, Journal of Statistical Physics
  - Beentjes & Khamseh (2020), Higher-order interactions in statistical physics and machine learning, PRE
  - Jansma (2023), Higher-Order Interactions and Their Duals Reveal Synergy and Logical Dependence beyond Shannon-Information, Entropy
  - My previous DIEP talk (2023)!
  - Jansma, Yao, et al. (2025), High order expression dependencies finely resolve cryptic states and subtypes in single cell data, Molecular Systems Biology

Example: Physics

- What if you can only measure correlation functions/moments?

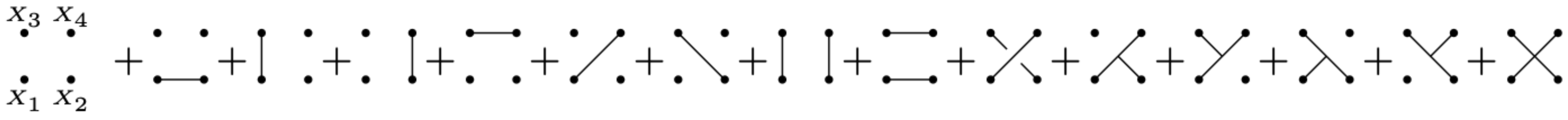
$$\langle X_1 X_2 \rangle = \sum_{x_1, x_2} p(X_1 = x_1, X_2 = x_2) x_1 x_2$$

- Powerset mereology $\implies$ central moments.

- Partition mereology $\mu_{\Pi(X)}(a, X) = (-1)^{|a|-1}(|a| - 1)!$

$$\langle X \rangle = \sum_{\pi \in \Pi(X)} u(\pi)$$

$$\langle X_1 X_2 X_3 X_4 \rangle =$$



- Möbius inverse: **Ursell functions**

$$= u(\{X_1\}) = \langle X_1 \rangle$$

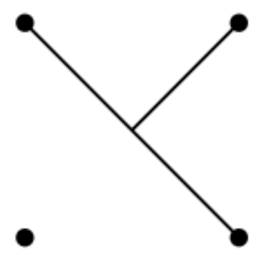
$$= u(\{X_1, X_2\}) = \langle X_1 X_2 \rangle - \langle X_1 \rangle \langle X_2 \rangle$$

$$= u(\{X_1, X_2, X_3\}) = \langle X_1 X_2 X_3 \rangle - \langle X_1 X_2 \rangle \langle X_3 \rangle - \langle X_1 X_3 \rangle \langle X_2 \rangle - \langle X_2 X_3 \rangle \langle X_1 \rangle + 2\langle X_1 \rangle \langle X_2 \rangle \langle X_3 \rangle$$

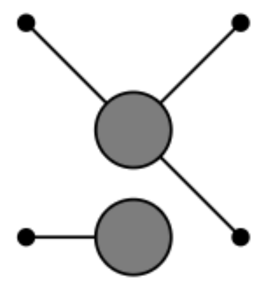
Example: Physics

- Quantum field theory: $\langle X_1 X_2 X_3 \rangle \to G(x_1, x_2, x_3) = \langle \Omega \,|\, T\phi(x_1)\phi(x_2)\phi(x_3) \,|\, \Omega \rangle$

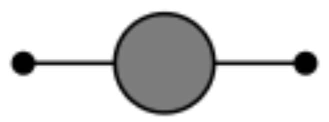
Statistical Mechanics:



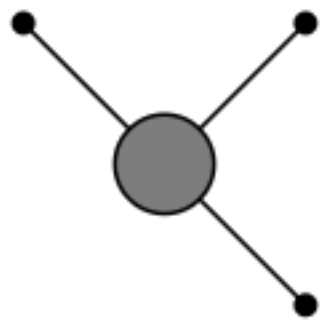
Quantum Field Theory:



- Möbius inverse over partition mereology: connected correlation functions.



$$= \langle \Omega | \, T\phi(x_1)\phi(x_2) \, | \Omega \rangle - \langle \Omega | \, \phi(x_1) \, | \Omega \rangle \, \langle \Omega | \, \phi(x_2) \, | \Omega \rangle - 1$$



$$\begin{aligned}
&= \langle \Omega | \, T\phi(x_1)\phi(x_2)\phi(x_3) \, | \Omega \rangle - \langle \Omega | \, \phi(x_1) \, | \Omega \rangle \, \langle \Omega | \, \phi(x_2)\phi(x_3) \, | \Omega \rangle \\
&- \langle \Omega | \, \phi(x_2) \, | \Omega \rangle \, \langle \Omega | \, \phi(x_1)\phi(x_3) \, | \Omega \rangle - \langle \Omega | \, \phi(x_3) \, | \Omega \rangle \, \langle \Omega | \, \phi(x_1)\phi(x_2) \, | \Omega \rangle \\
&+ 2 \, \langle \Omega | \, \phi(x_1) \, | \Omega \rangle \, \langle \Omega | \, \phi(x_2) \, | \Omega \rangle \, \langle \Omega | \, \phi(x_3) \, | \Omega \rangle
\end{aligned}$$

Example: Game Theory

- **Coalitional game theory:** set of players S works together to get payoff $v(S)$.
- Some coalitions might be *synergistic* $\Longrightarrow$ powerset mereology

$$v(S) = \sum_{T \subseteq S} w(T) \qquad \Longleftrightarrow \qquad w(T) = \sum_{U \subseteq T} (-1)^{|T|-|U|} v(U)$$

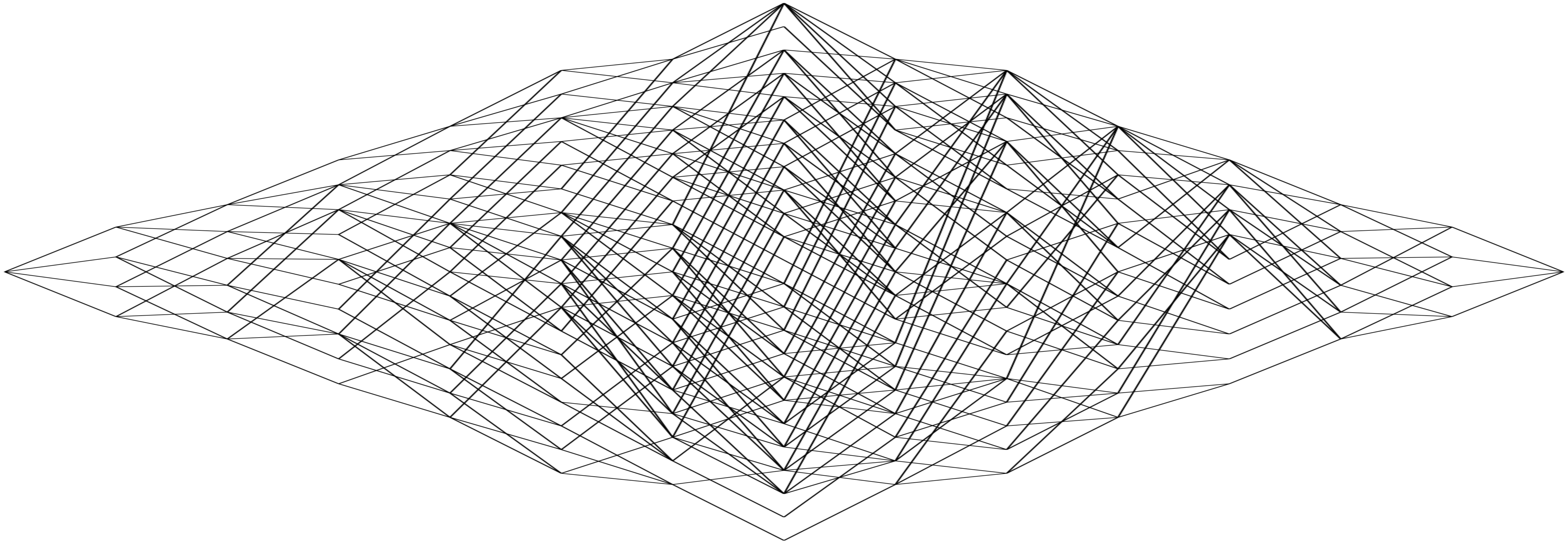
- $w(T)$: *synergy* of coalition T
- How should a coalition S divide the payoff?
- *Shapley value* for player i :

$$\phi_i(v) = \sum_{i \subseteq T \subseteq S} \frac{w(T)}{|T|}$$

Field of Study	Macro Quantity	Mereology	Micro Quantity/Interactions
Statistics	Moments	Powerset	Central moments
	Moments	Partitions	Cumulants
	Free moments	Non-crossing partitions	Free cumulants
	Path signature moments	Ordered partitions	Path signature cumulants
Information Theory	Entropy	Powerset	Mutual information
	Surprisal	Powerset	Pointwise mutual information
	Joint Surprisal	Powerset	Conditional interactions
	Mutual Information	Antichains	Synergy/redundancy atoms
Biology	Pheno- & Genotype	Powerset	Epistasis
	Gene expression profile	Powerset	Genetic interactions
	Population statistics	Powerset	Synergistic treatment effects
Physics	Ensemble energies	Powerset	Ising interactions
	Correlation functions	Partitions	Ursell functions
	Quantum corr. functions	Partitions	Scattering amplitudes
Chemistry	Molecular property	Subgraphs	Fragment contributions
	Molecular property	Reaction poset	Cluster contributions
Game Theory	Coalition value	Powerset	Coalition synergy
	Shapley value	Powerset	Normalised coalition synergy
Artificial Intelligence	Generative model probabilities	Powerset	Feature interactions
	Predictive model predictions	Powerset	Feature contributions
	Dempster-Shafer Belief	Distributive	Evidence weight
	$D_{KL}(p\|q)$	Powerset	$\Delta_{p\|q}$ (See Sec. IV)

Part 4

New Applications



Name of the game:

- Choose ‘macroscopic’ observable to decompose.

$$Q(S)$$

- Choose a mereology.

$$Q(S) = \sum_{t \in \mathscr{D}(S)} q(t)$$

- Derive ‘microscopic’ theory.

$$q(t) = \sum_{u \leq t} \mu_{\mathscr{D}(S)} Q(u)$$

1. Decomposing information

- Assume: $I(X_1, X_2; Y) = \mathrm{Un}(X_1; Y) + \mathrm{Un}(X_2; Y) + \mathrm{Red}(X_1, X_2; Y) + \mathrm{Syn}(X_1, X_2; Y)$
- More general:

$$I(S; Y) = \sum_{\alpha \in \mathscr{A}(S)} I_\partial(\alpha; Y) \qquad \mathscr{A}(S) : \text{incomparable subsets of } \mathscr{P}(S) \text{ (antichains)}$$

- Some elements of $\mathscr{A}(\{X_1, X_2, X_3\})$:
  - $\{\{X_1, X_2, X_3\}\} \rightarrow$ synergy among X_1, X_2, X_3
  - $\{\{X_1\}, \{X_2\}, \{X_3\}\} \rightarrow$ redundancy among X_1, X_2, X_3
  - $\{\{X_1, X_2\}, \{X_3\}\} \rightarrow$ redundancy among $\{X_1, X_2\}$ and X_3
- Not an element of $\mathscr{A}(\{X_1, X_2, X_3\}) : \{\{X_1, X_2\}, \{X_2\}\}$
- (This was already proposed in Williams & Beer (2010), arXiv:1004.2515

1. Decomposing information

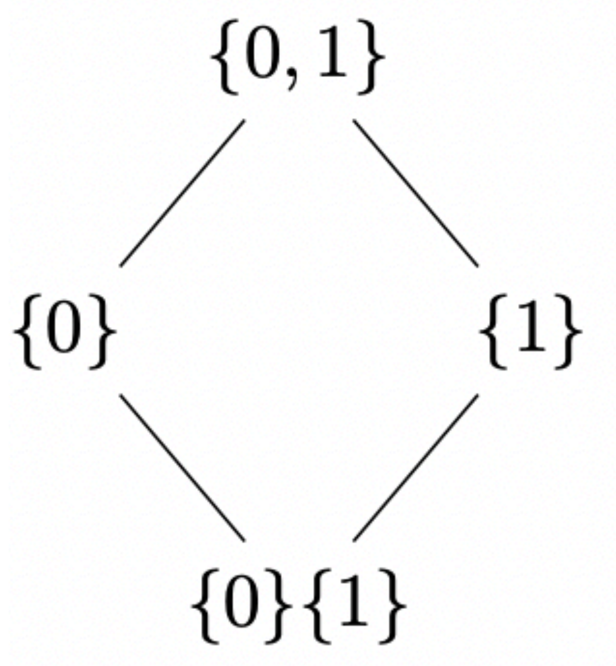
- Antichains can be ordered along *redundancy* as
  - $\alpha \leq \beta \iff \forall B \in \beta \exists A \in \alpha : A \subseteq B$
- Jansma, Mediano, Rosas (2024), arXiv:2410.06224 :

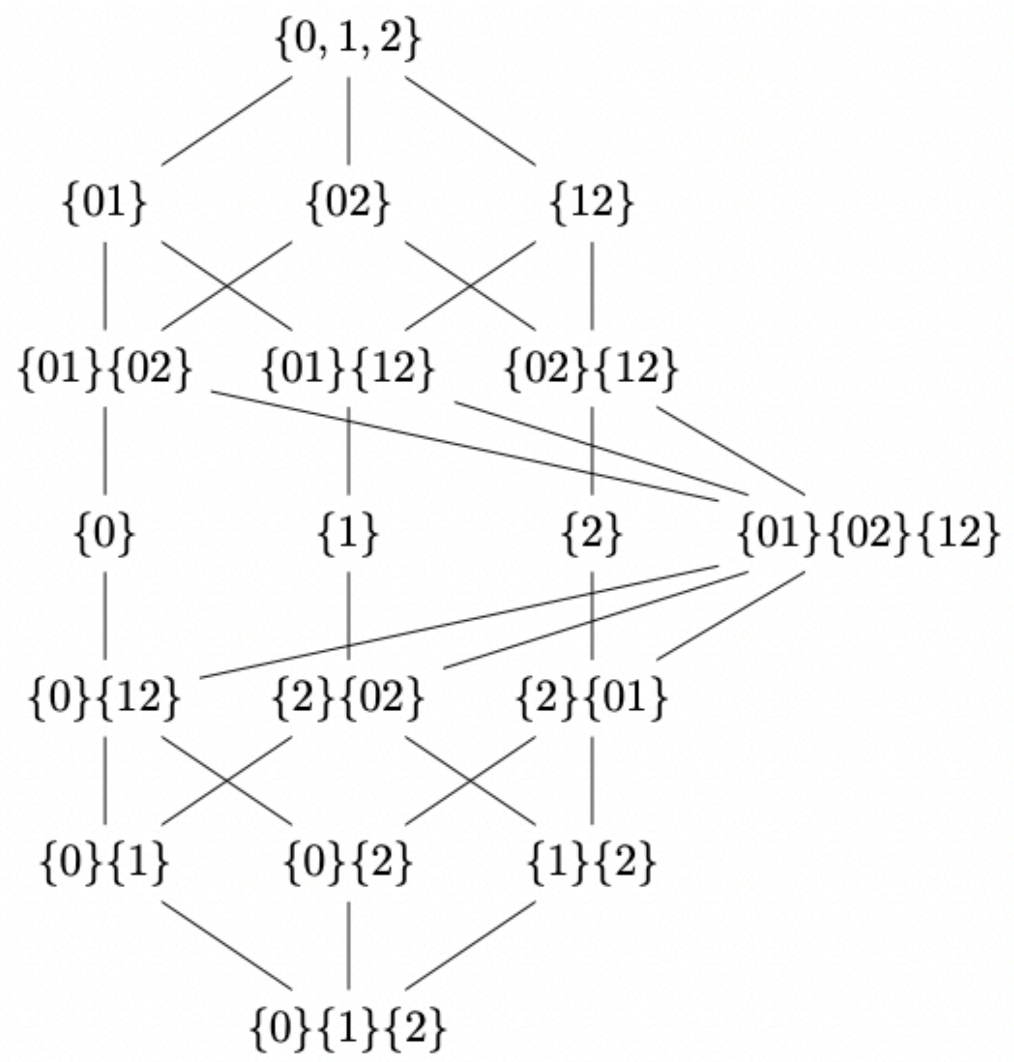
$$\mu_{\mathscr{A}(X)}(\alpha, \beta) = \begin{cases} (-1)^{|I_{\alpha^*} \backslash I_{\beta^*}|} & \text{if } I_{\alpha^*} \backslash I_{\beta^*} \text{ is an antichain of } \mathscr{P}(X) \\ 0 & \text{otherwise} \end{cases}$$

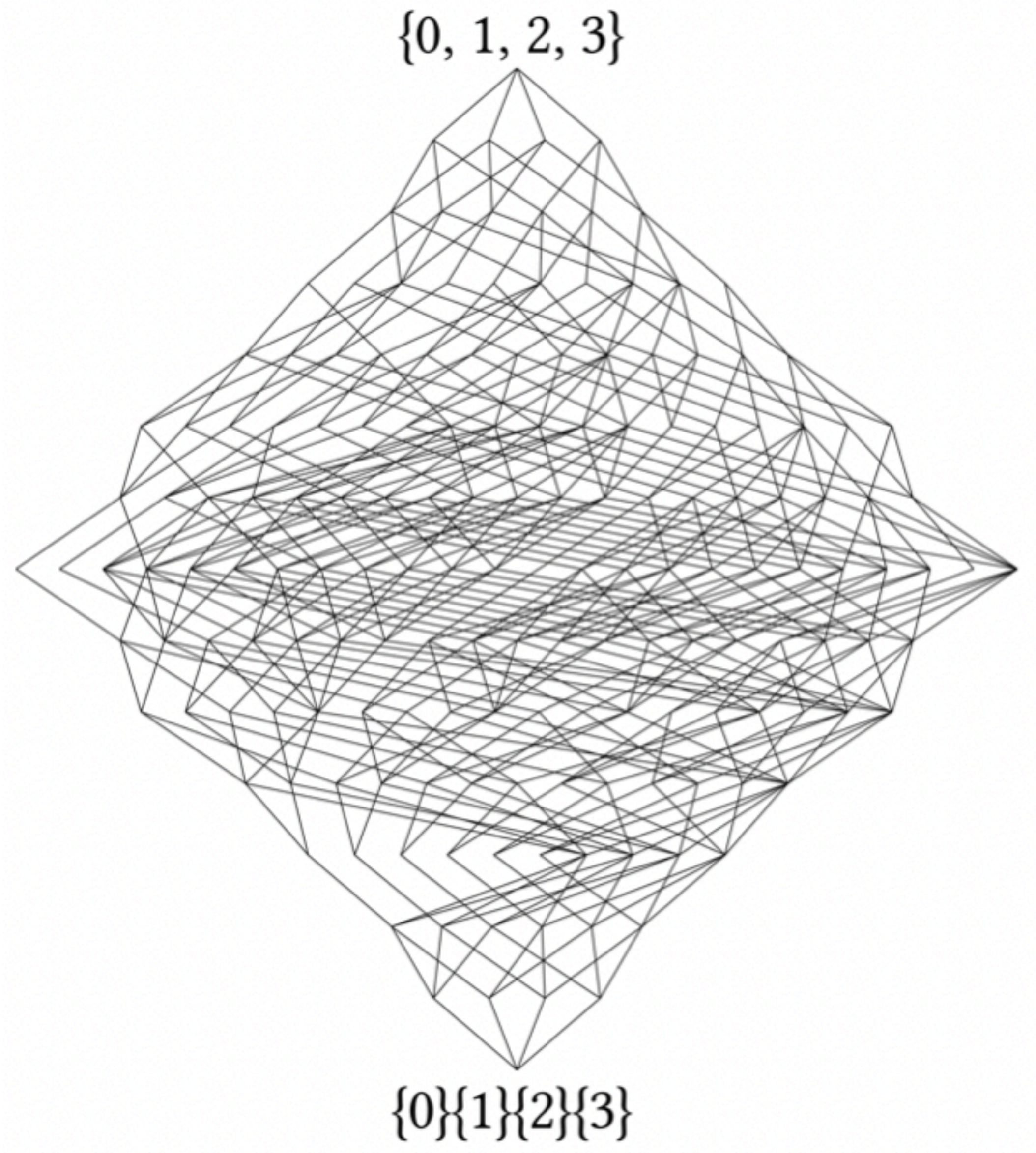
- Partial information (redundancy, synergy, etc) can be calculated:

$$I_\partial(\alpha, Y) = \sum_{\beta \leq \alpha} \mu_{\mathscr{A}(X)}(\beta, \alpha) I(\beta; Y)$$

- **Double exponential speedup!**

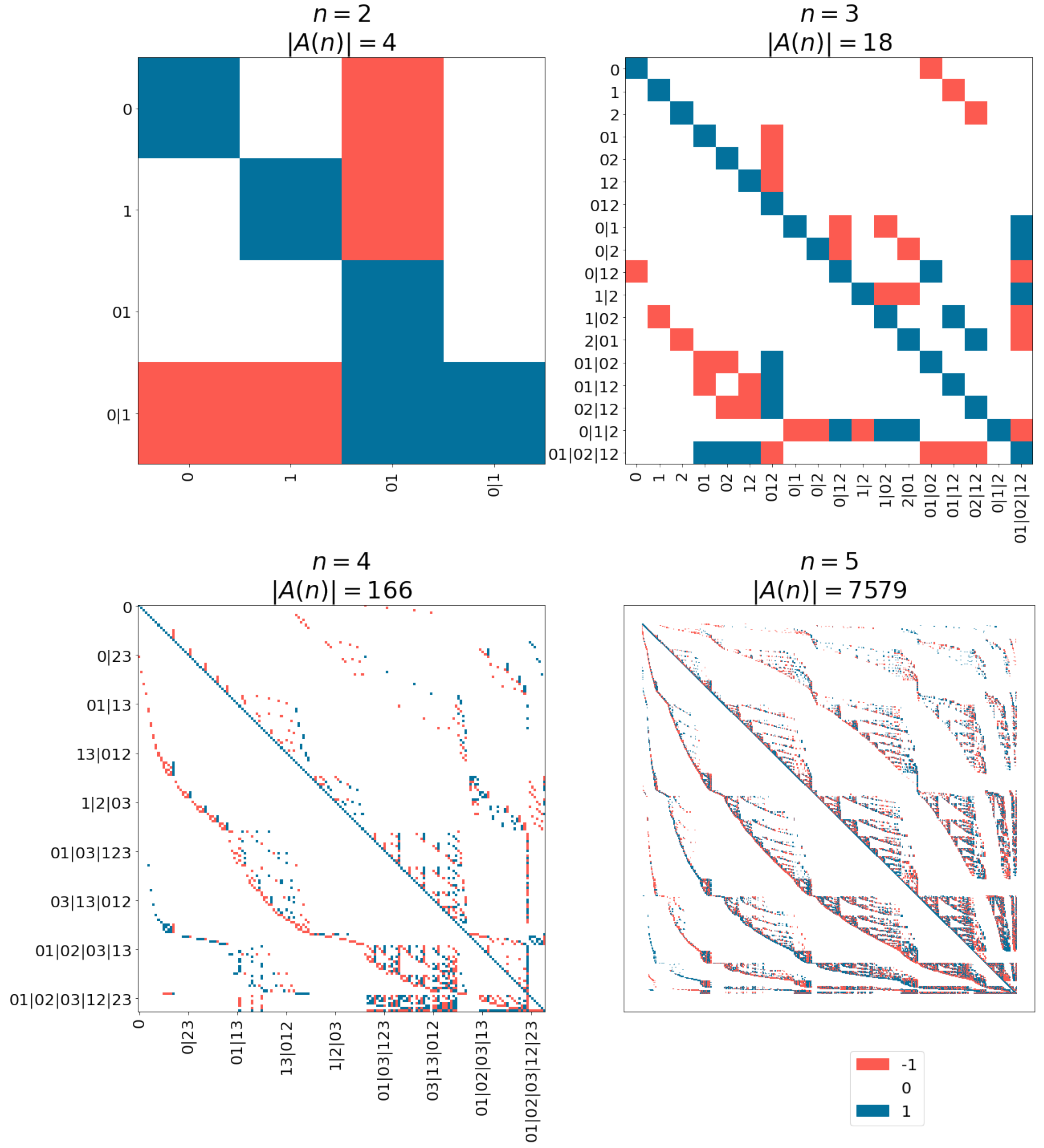






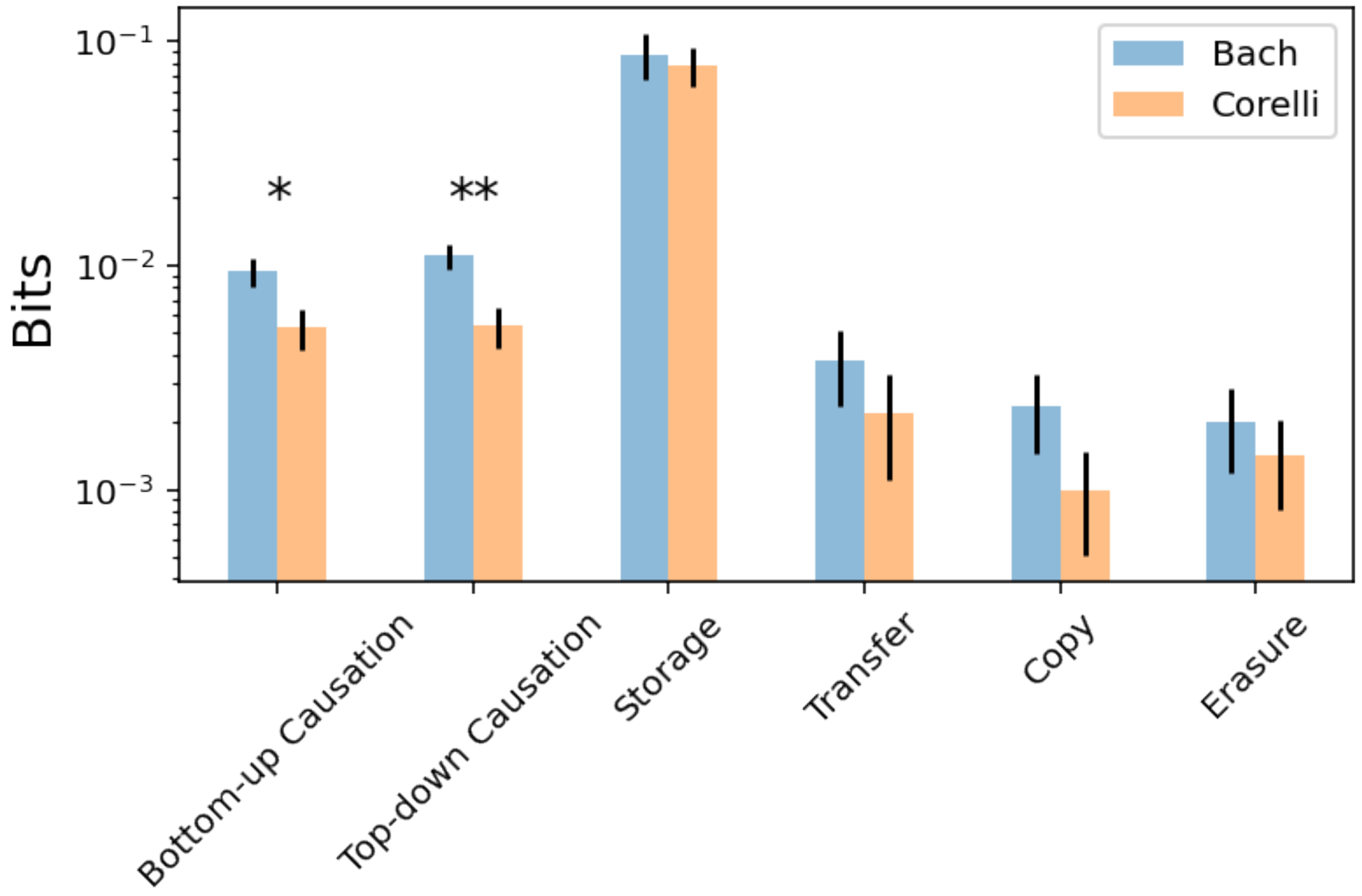
1. Decomposing information

$\mu_{\mathscr{A}(X)}$ as a matrix



https://github.com/AJnsm/algebraicPID

Decomposed Baroque music (4 voices)



arXiv:2410.06224

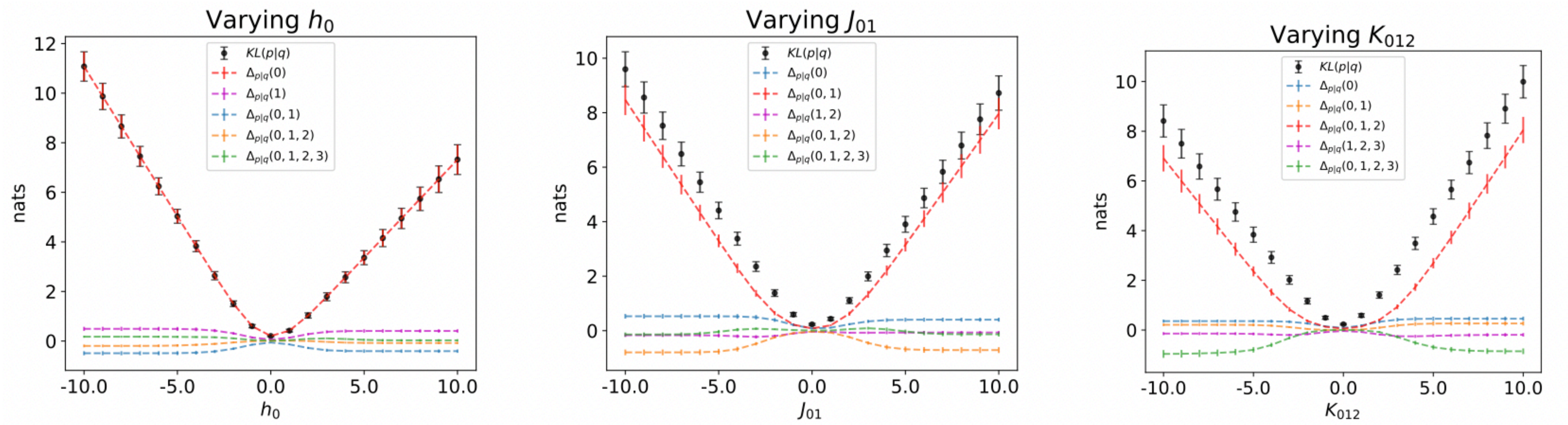
2. Decomposing KL-divergence

KL-divergence: $D_{KL}(p\,|\,q; X) = \sum_{x} p(X=x) \log \frac{p(X=x)}{q(X=x)}$

$$= \sum_{S \subseteq X} \Delta_{p|q}(S) \quad \Longleftrightarrow \quad \Delta_{p|q}(S) = \sum_{T \subseteq S} (-1)^{|S|-|T|} D_{KL}(p\,|\,q; T)$$

marginalised

- The Δ's *disentangle* the discrepancy.
- **Test case:**
    - p is Ising model with Gaussian $\leq$ 3-order couplings
    - q is copy of p, but with one varying coupling.



3. Decomposing causality ([Jansma 2025, arXiv:2501.11447])

- **Recall.** $\mathscr{A}(S)$ corresponds to a redundancy mereology.
- What about redundant vs synergistic causality?
- **Definition.** Maximum average causal effect: causal power of X on Y

$$\mathrm{MACE}(X;Y) = \max_{x,x' \in \mathcal{X}} \left(\mathbb{E}[Y \mid do(X=x)] - \mathbb{E}[Y \mid do(X=x')]\right)$$

- Decompose over $\mathscr{A}(X)$ mereology:

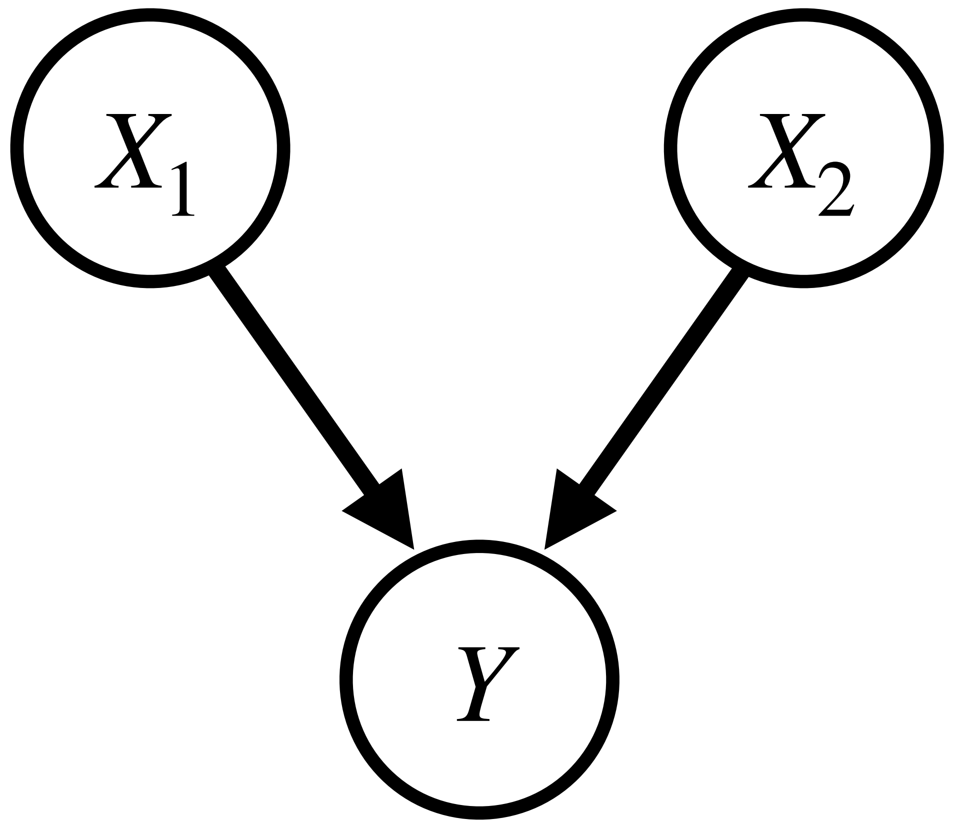
$$\mathrm{MACE}(X;Y) = \sum_{\alpha \in \mathscr{A}(X)} C(\alpha;Y) \quad \iff \quad C(\alpha;Y) = \sum_{\beta \leq \alpha} \mu_{\mathscr{A}(X)}(\beta,\alpha)\mathrm{MACE}(\beta;Y)$$

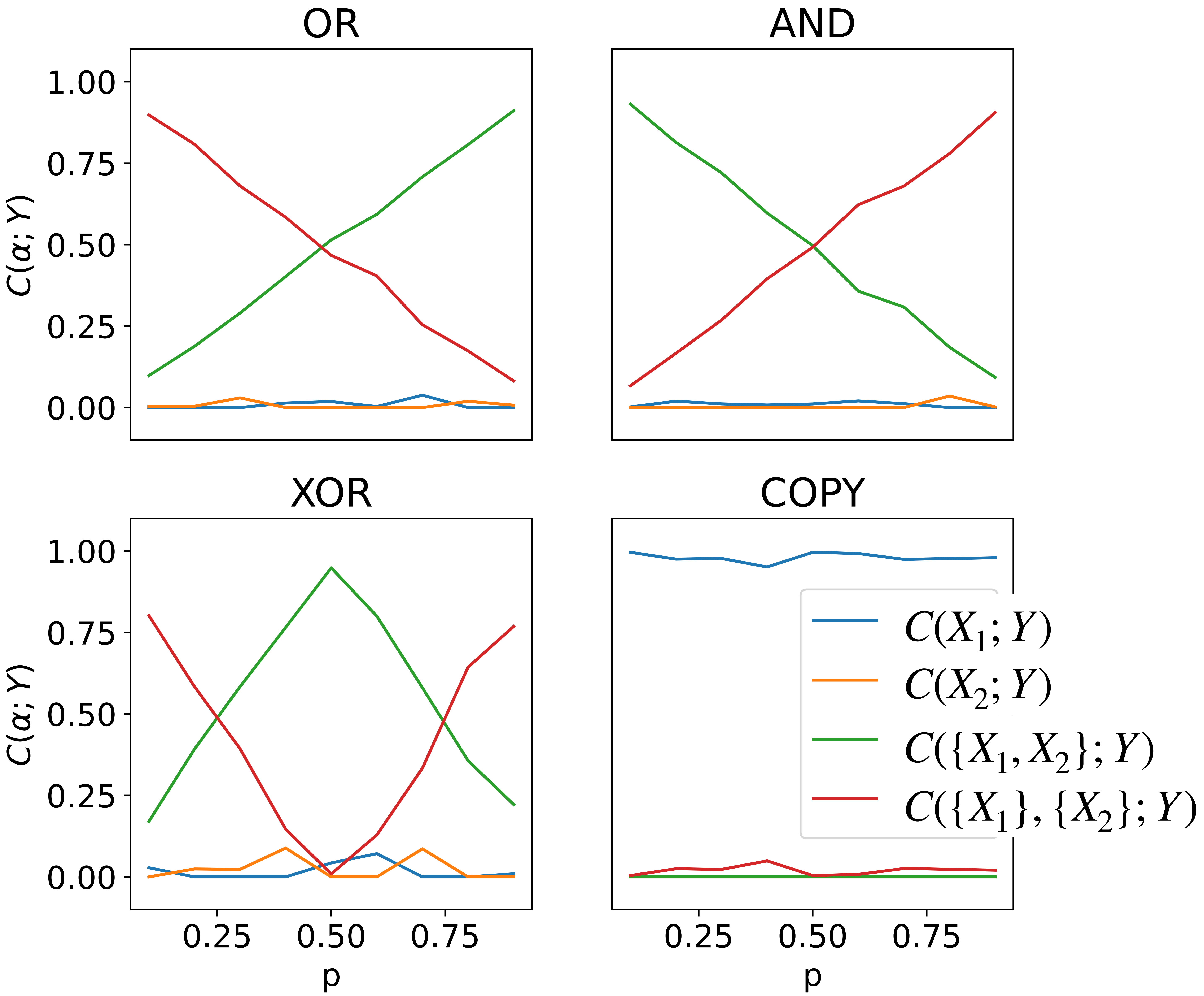
- **Definition.**

$$\mathrm{MACE}(\alpha;Y) = \min_{A \in \alpha} \mathrm{MACE}(A;Y)$$

3. Decomposing causality ([Jansma 2025, arXiv:2501.11447])

- **Test case:** Logic gates
- Inputs: $X_1, X_2 \sim \mathrm{Binom}(p)$
- Causal graph: collider





3. Decomposing causality ([Jansma 2025, arXiv:2501.11447])

- **Test case:** chemical network to produce molecule Y

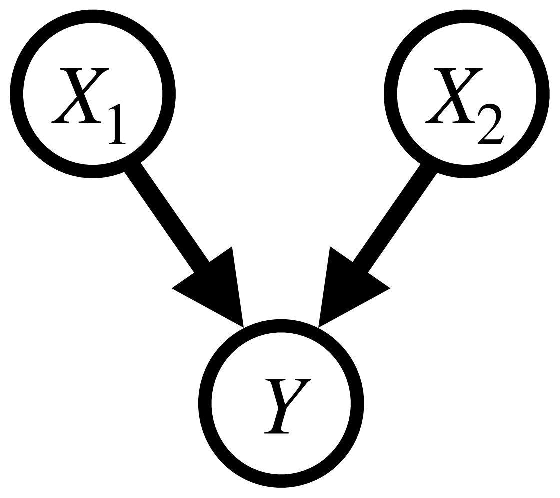
$$\frac{d[Y]}{dt} = k_1[X_1] + k_2[X_2] + k_3[X_1][X_2] - k_4[Y]$$

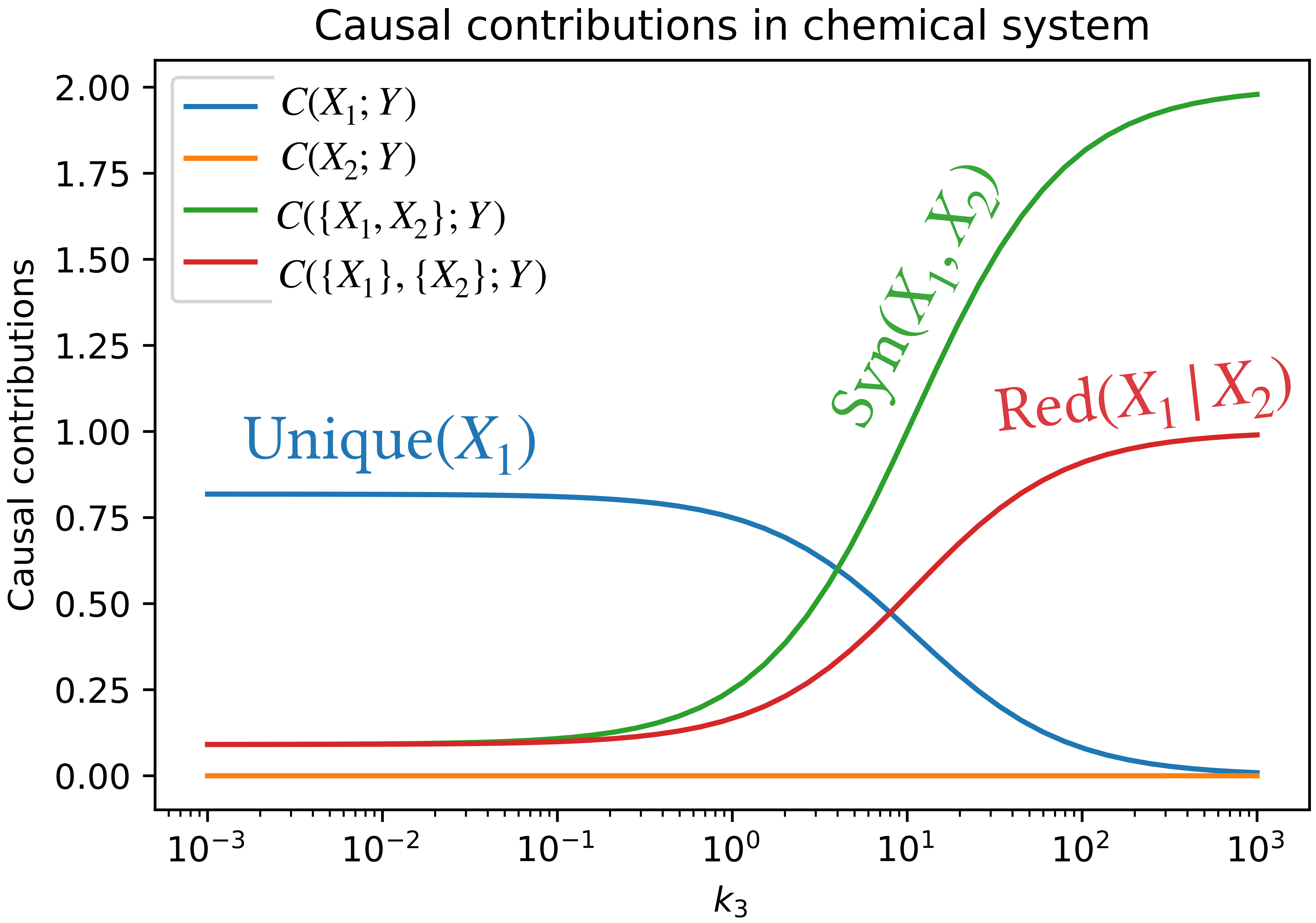
$$[Y]_{ss}(X1,X2) = \frac{k_1[X_1] + k_2[X_2] + k_3[X_1][X_2]}{k_4}$$

- Intervention: add some molecules

$$E(\hat{Y} | do(\delta_1 = \epsilon)) = \frac{[Y]_{ss}(X_1 + \epsilon, X_2)}{[Y]_{ss}(X_1, X_2)}$$

- Causal graph: collider





[For nontrivial causal graphs, see: Jansma 2025, arXiv:2501.11447]

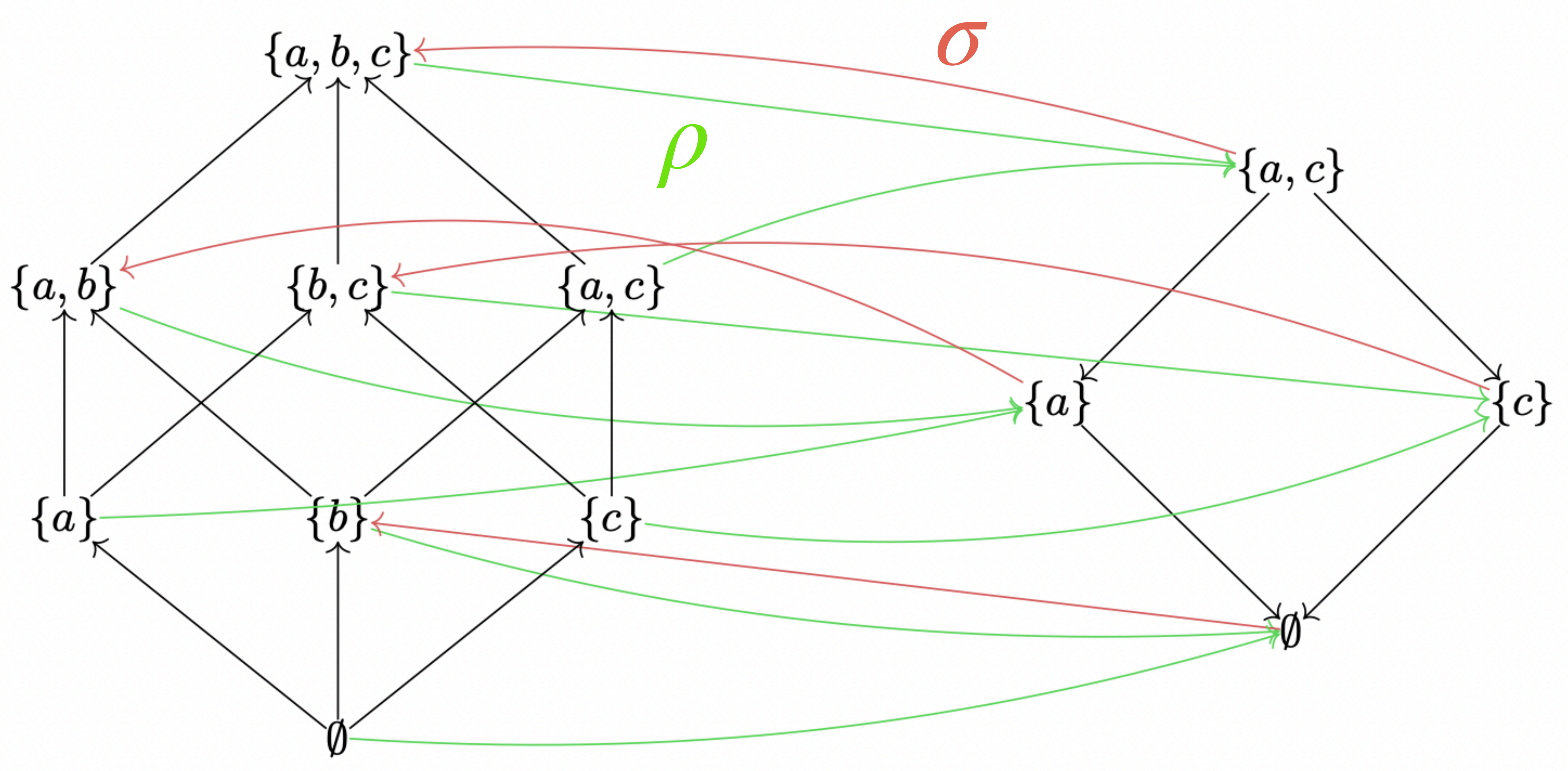
4. Renormalisation

- **Note:** Coarse-graining is a **very specific** change in mereology.
- Namely: one half of a *Galois connection*
- **Definition.** A Galois connection between two posets P and Q is given by a pair of monotone functions $f : P \to Q$ and $g : Q \to P$ such that for all $p \in P$, $q \in Q$:

$$p \leq g(f(p)) \quad \text{and} \quad f(g(q)) \leq q$$

- **Claim.** These form a Galois connection:

$$\rho : \mathcal{P}(S) \to \mathcal{P}(\tilde{S}) \qquad \sigma : \mathcal{P}(\tilde{S}) \to \mathcal{P}(S)$$
$$A \mapsto A \setminus U \qquad\qquad A \mapsto A \cup U$$



4. Renormalisation

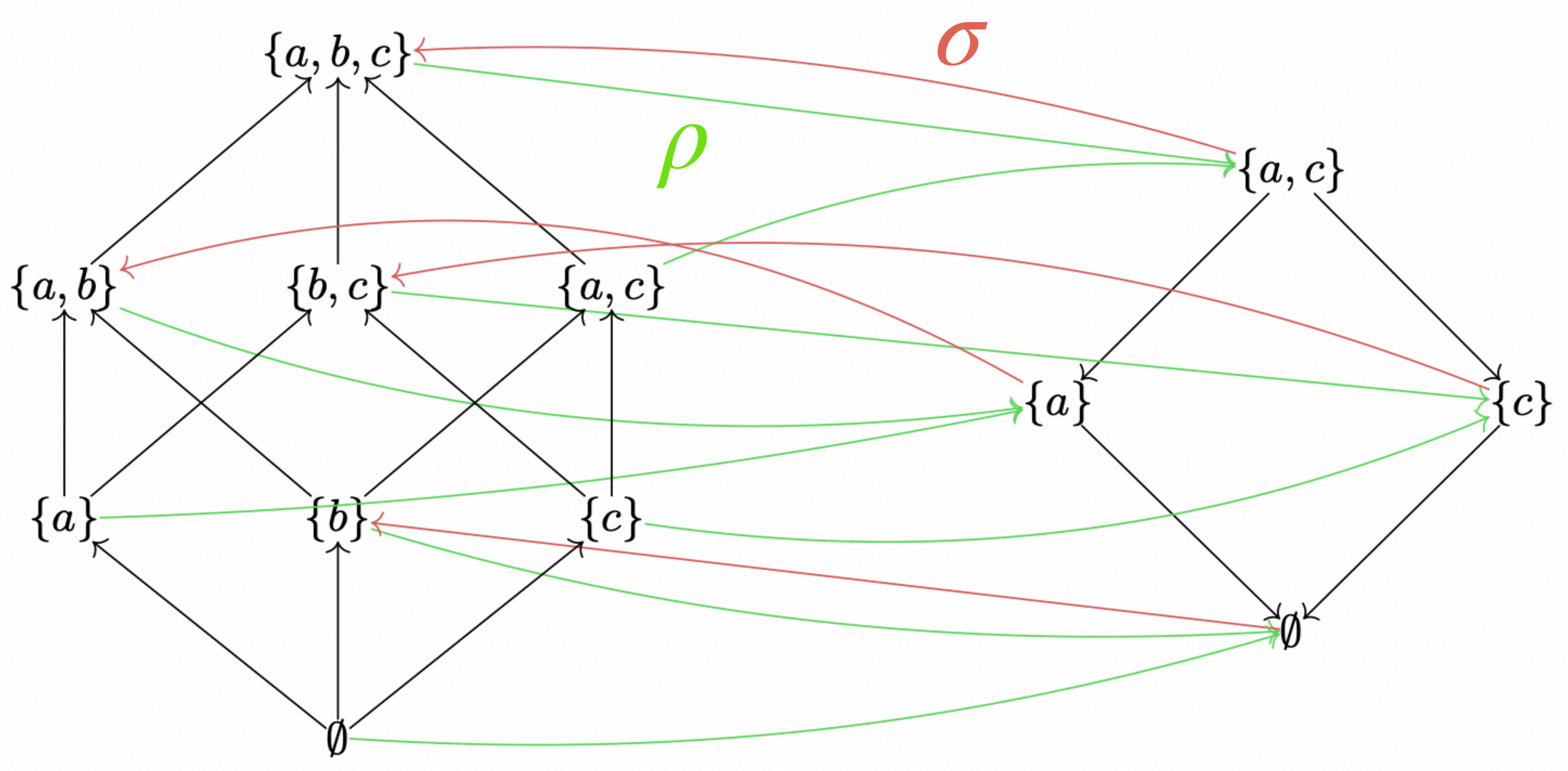
- **Why is this useful???**
- **Theorem** (Rota, 1964) Let $\rho : P \to Q$ and $\sigma : Q \to P$ form a Galois connection. Then

$$\sum_{\substack{u \in P \\ \rho(u) = y}} \mu_P(x, u) = \sum_{\substack{v \in Q \\ \sigma(v) = x}} \mu_Q(v, y)$$

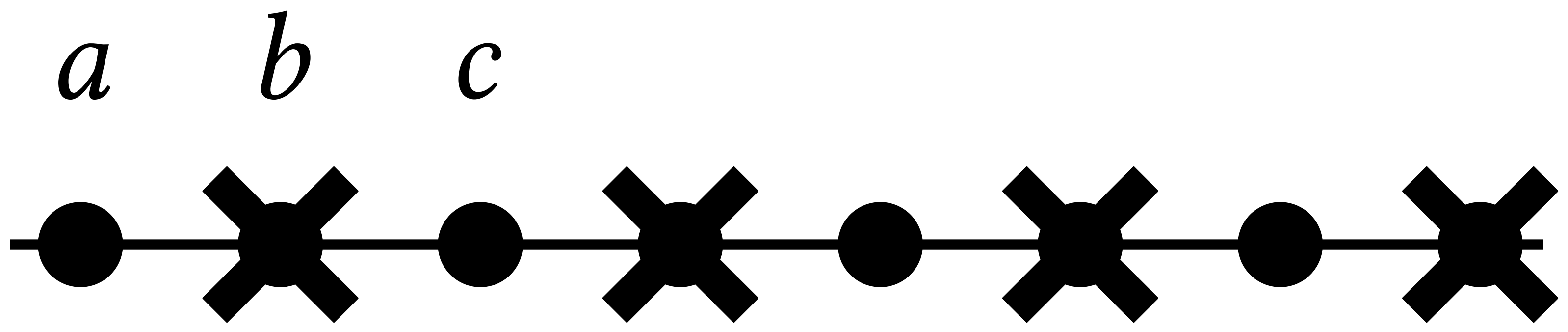
- Interactions after coarse graining:

$$\tilde{q}(\tilde{b}) = \sum_{\tilde{a} \le \tilde{b}} \mu_{\mathscr{P}(\tilde{S})} \tilde{Q}(\tilde{a})$$

$$= \sum_{\tilde{a} \le \tilde{b}} \left(\sum_{\substack{u \in \mathscr{P}(S) \\ \rho(u) = \tilde{b}}} \mu_{\mathscr{P}(S)}(\sigma(\tilde{a}), u) \right) \tilde{Q}(\tilde{a})$$



4. Renormalisation



- **Test case:** 1D Ising model, decimation of odd spins.
- Renormalised interactions: $\tilde{J}(\tilde{b}) = \sum_{\tilde{a} \leq \tilde{b}} \mu_{\mathscr{P}(\tilde{S})} \tilde{Q}(\tilde{a})$
- $\mu_{\mathscr{P}(\tilde{S})}$ given by Galois Connection Thm. $\tilde{Q}(\tilde{a})$ given by marginalisation:

$$\tilde{Q}(\tilde{b}) = \log\left(\sum_{r \in \{0,1\}^{|U|}} p(b = 1, U = r, S \backslash (b \cup U) = 0) \right)$$

$$\implies \tilde{I}_{ac} = -\log \frac{(p_{101} + p_{111})(p_{000} + p_{010})}{(p_{100} + p_{110})(p_{001} + p_{011})} := \tilde{J}$$

- Impose symmetries of 1D Ising without field:

$$\begin{aligned} p_{100} = p_{010} = p_{001} &\coloneqq a \quad &&\text{(translation invariance)} \\ p_{110} = p_{011} &\coloneqq b \quad &&\text{(translation invariance)} \\ p_{101} = p_{000} = p_{100} &= a \quad &&\text{(no field)} \\ p_{111} &\coloneqq b^2 \end{aligned}$$

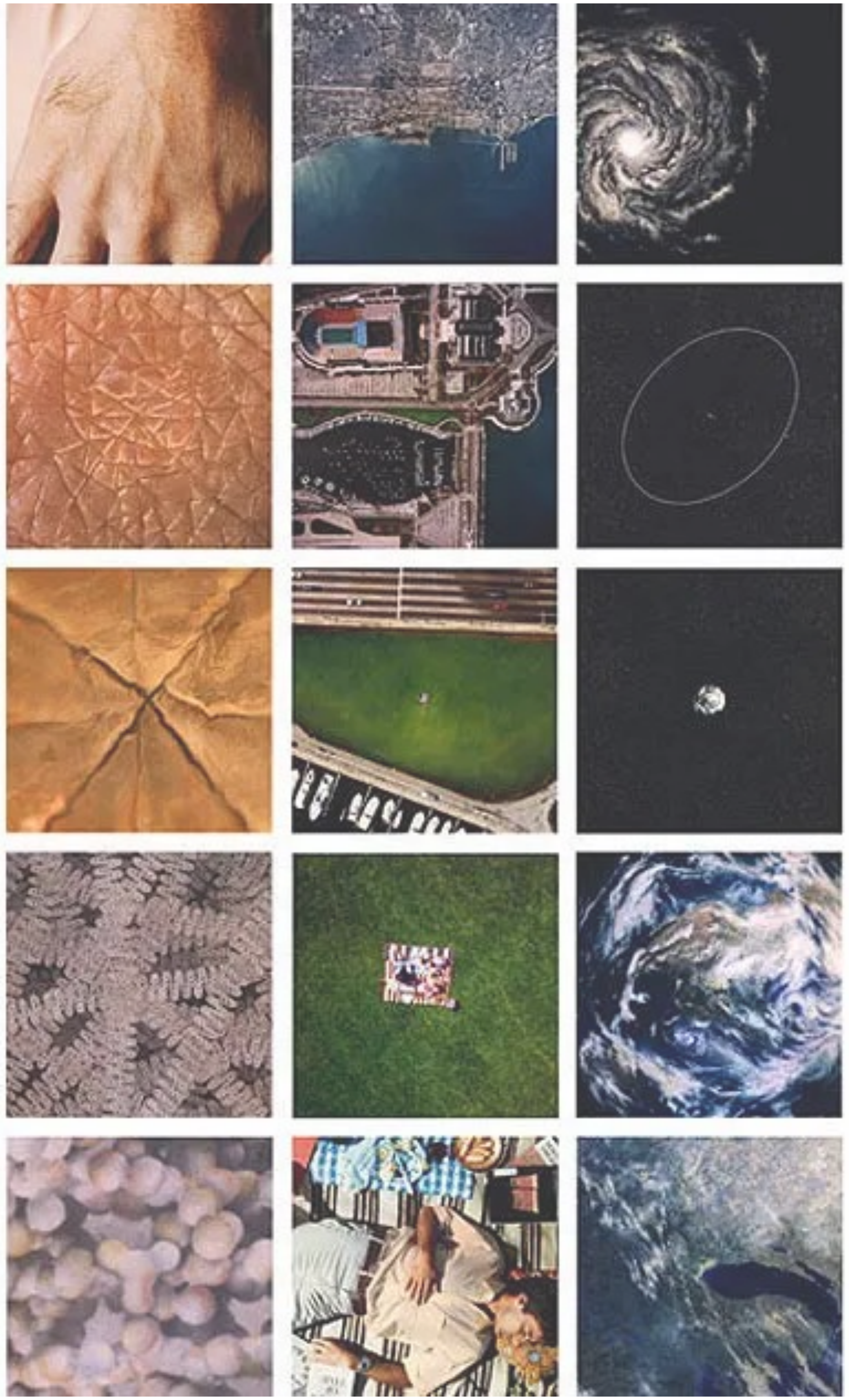
- Substitute $J = -\log b$

$$\implies \tilde{J} = \log\left(\frac{1}{2}\operatorname{sech}(J) + \frac{1}{2}\right)$$

Part 5

Conclusion

Recap



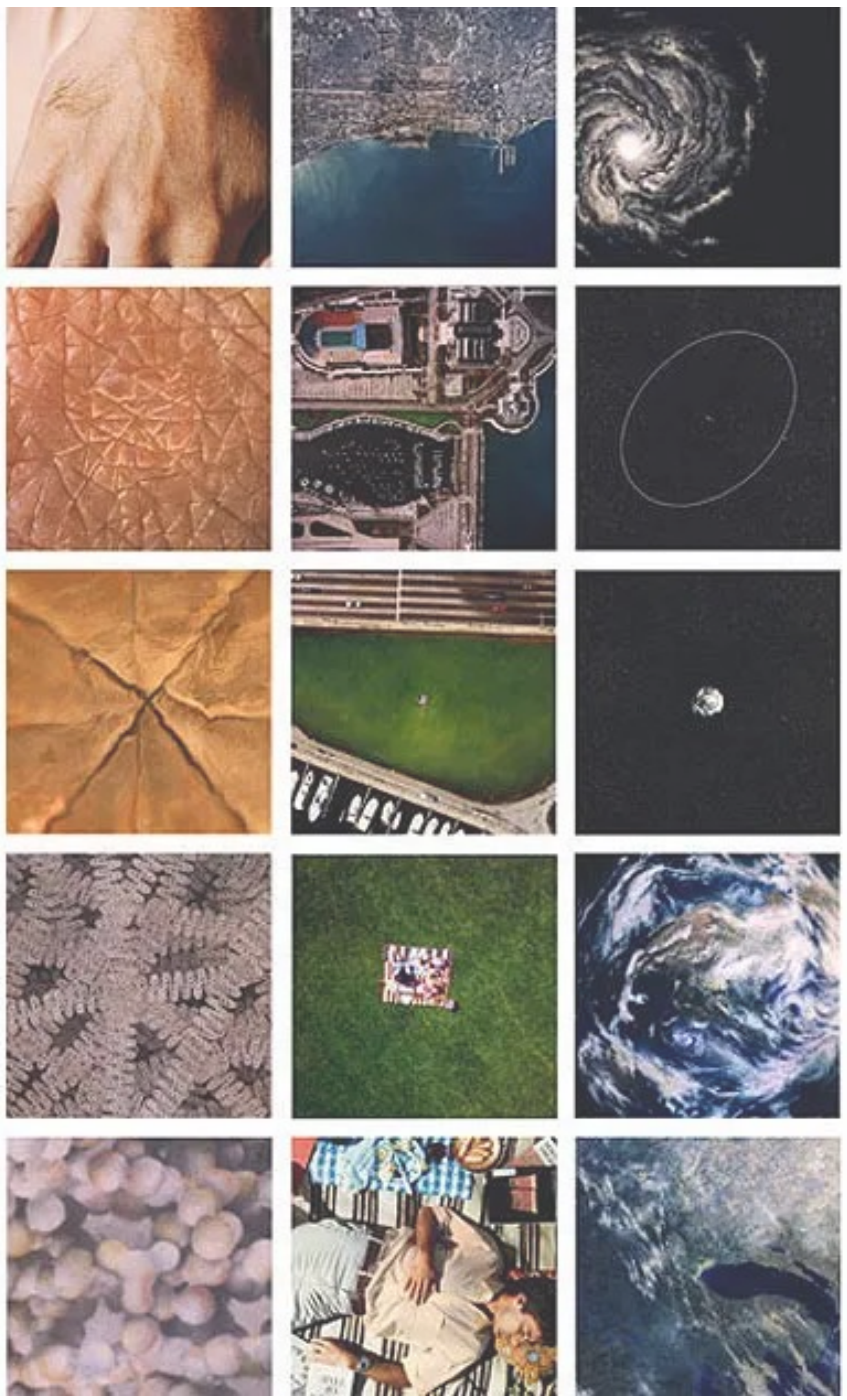
- When you study a complex system, consider how the parts come together to form the whole
    - Summarised as a *mereology.*

- Fixing the mereology fixes the microscopic theory.
    - Meaningful mereology $\iff$ meaningful interactions — justification is inherited
    - Higher-order $\iff$ higher in the mereology

Recap

- **New applications:**
    - Double exponential speedup for information decomposition.
    - Context-dependent decomposition of causal effects.
    - Decomposition of the KL-divergence.
    - New way to calculate renormalised interactions (?).
    - ? ? ?

- **Much to do!**
    - Generalised Shapley values?
    - Critical temperature 2D Ising?
    - Redundancy mereology : Classical logic :: ? : Quantum logic
    - Dual of rooted mereologies $\implies$ dual interactions?
    - Search through mereologies?



- **Claim.**

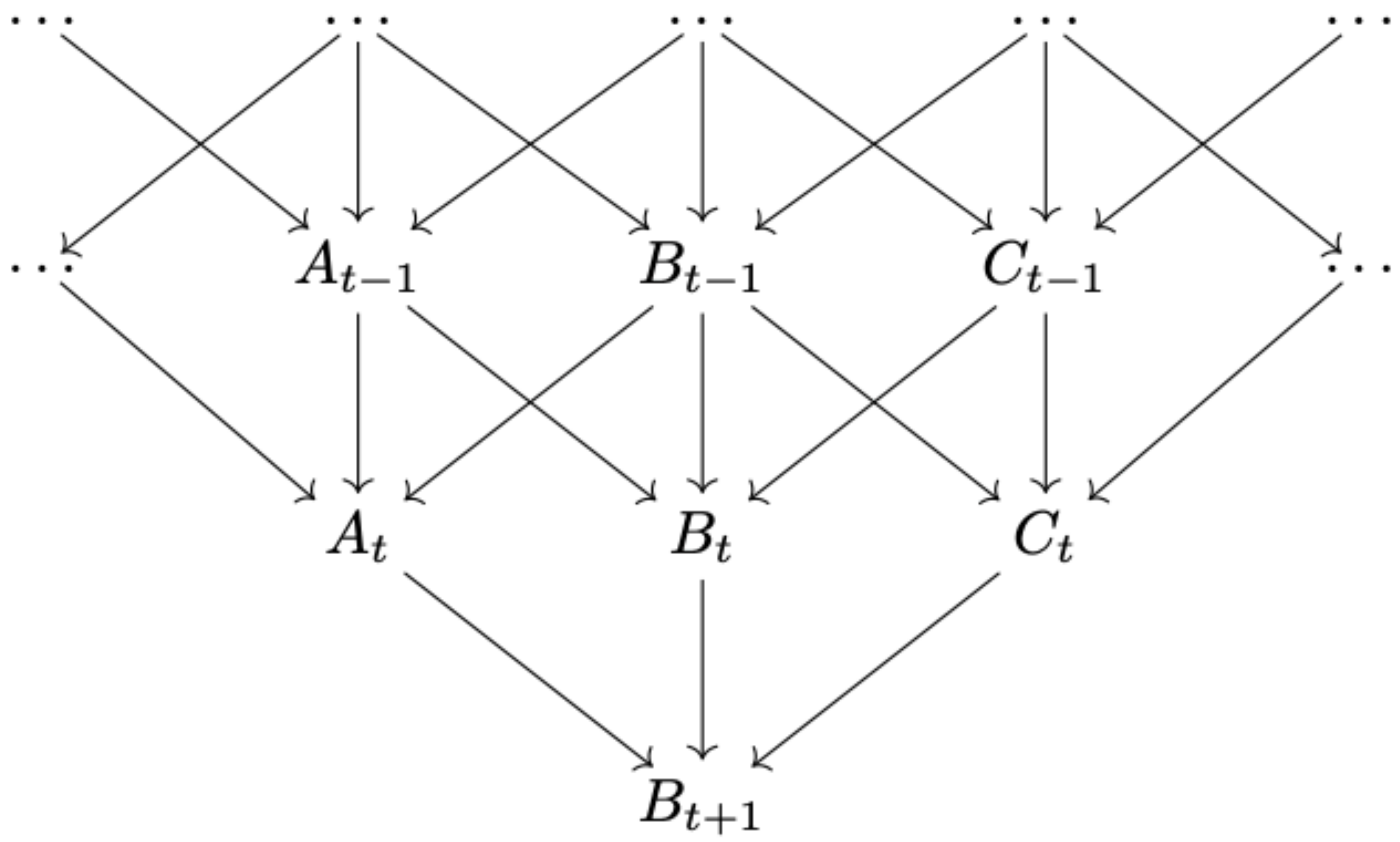
  i) This pattern is *ubiquitous!*

  ii) This pattern is *insightful!*

  iii) This pattern is *useful!*

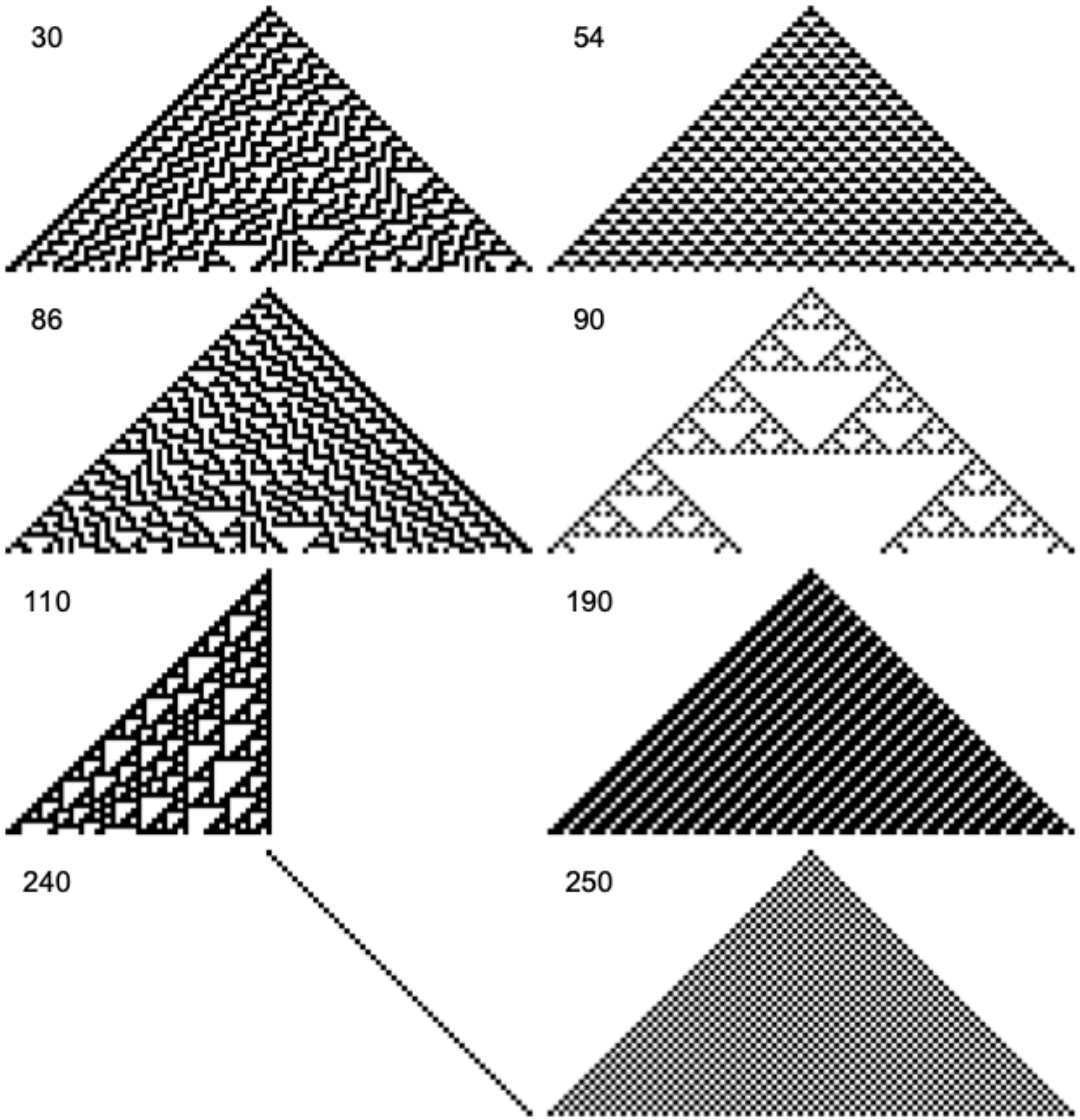
Thank you

Context-dependent causality in CAs

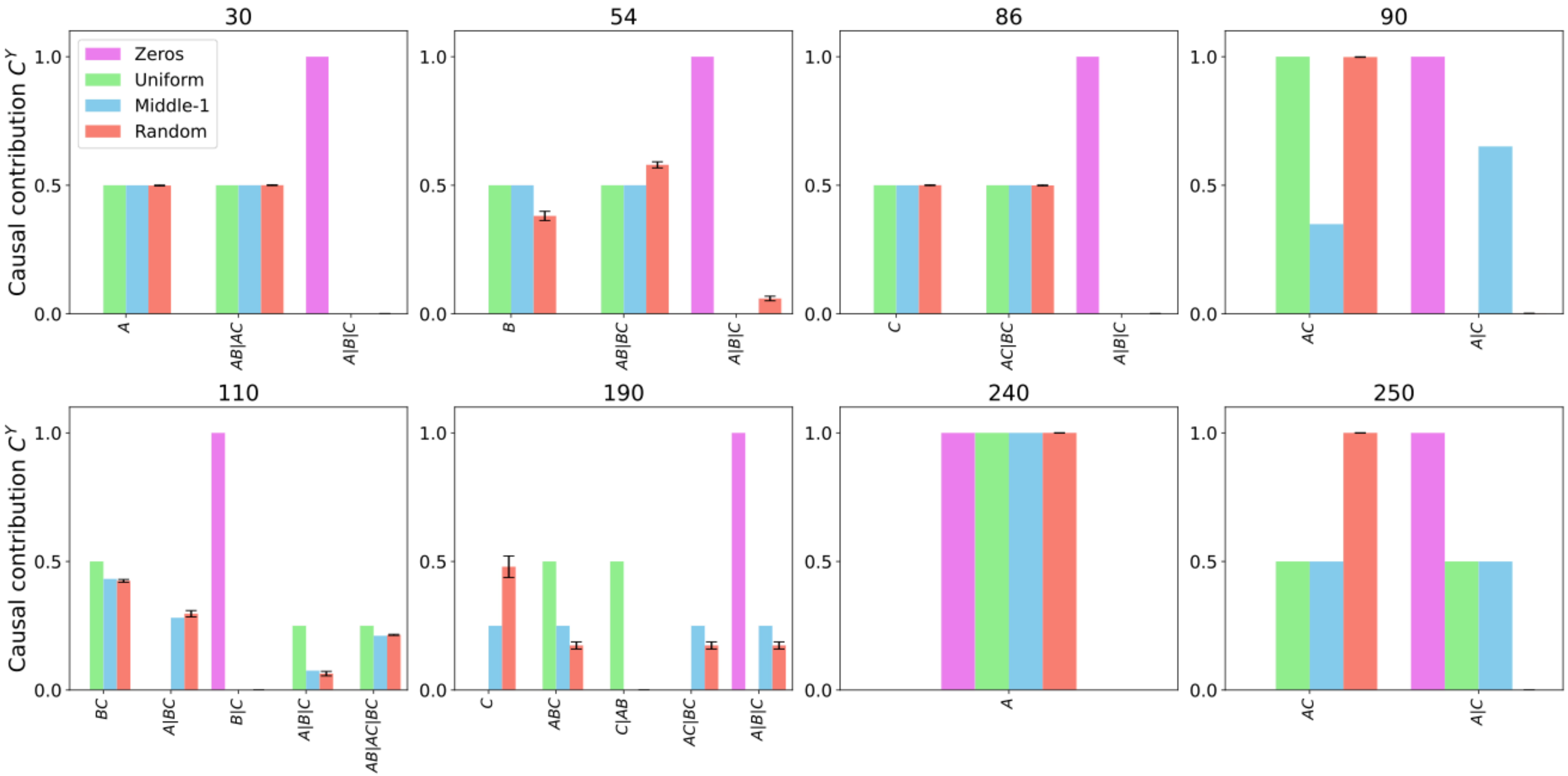


The MACE of A can be written as

$$\begin{aligned}&\mathrm{MACE}(A_t; B_{t+1})\\ &= \max_{s,s'} \left(\mathbb{E}[B_{t+1}|do(A_t = s)] - \mathbb{E}[B_{t+1}|do(A_t = s')]\right)\end{aligned}$$



Context-dependent causality in CAs



Decomposition of 5 brain frequencies