

Problem Set on Differential Equations

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Problem 1.

Find the solution of the initial value problem

$$\dot{a}(t) = r \cdot a(t) + s$$

$$a(0) = a_0$$

where both r and s are known constant.

Problem 2.

Find the solution of the initial value problem

$$\dot{a}(t) = r(t) \cdot a(t) + s(t)$$

$$a(0) = a_0$$

where both $r(t)$ and $s(t)$ are known functions of t .

Problem 3.

Consider the linear system of differential equations given by

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}(t).$$

- A. Find the general solution of the system.
- B. What would you need to find a specific solution of the system?
- C. Draw the trajectories of the system.

Problem 4.

Consider the initial value problem

$$\dot{k}(t) = s \cdot f(k(t)) - \delta \cdot k(t)$$

$$k(0) = k_0$$

where the saving rate $s \in (0, 1)$, the capital depreciation rate $\delta \in (0, 1)$, and the production function f satisfies the *Inada conditions*. That is, f is continuously differentiable and

$$f(0) = 0$$

$$f'(x) > 0$$

$$f''(x) < 0$$

$$\lim_{x \rightarrow 0} f'(x) = +\infty$$

$$\lim_{x \rightarrow +\infty} f'(x) = 0.$$

- A. Give a production function f that satisfies the Inada conditions.
- B. Find the steady state of the system.
- C. Draw the dynamic path of $k(t)$ and show that it converges to the steady state.

Problem 5.

The solution of the problem studied in Problem 4 is characterized by a system of two nonlinear first-order differential equations:

$$\begin{aligned}\dot{k}_t &= f(k_t) - c_t - \delta \cdot k_t \\ \frac{\dot{c}_t}{c_t} &= \alpha \cdot A \cdot k_t^{\alpha-1} - (\delta + \rho) .\end{aligned}$$

The first differential equation is the law of motion of capital. The second differential equation is the Euler equation, which describes the optimal path of consumption over time.

- A. Draw the phase diagram of the system.
- B. Linearize the system around its steady state.
- C. Show that the steady state is a saddle point locally.
- D. Suppose the economy is in steady state at time t_0 and there is an unanticipated decrease in the discount factor ρ . Show on your phase diagram the transition dynamics of the model.

Problem 6.

The solution of the investment problem studied in Problem 5 is characterized by a system of two nonlinear first-order differential equations:

$$\begin{aligned}\dot{k}_t &= \left(\frac{q_t - 1}{\chi} \right) \cdot k_t \\ \dot{q}_t &= r \cdot q_t - f'(k_t) - \frac{1}{2 \cdot \chi} (q_t - 1)^2.\end{aligned}$$

The first differential equation is the law of motion of capital k_t . The second differential equation is the law of motion of the co-state variable q_t .

- A. Draw the phase diagram.
- B. Show that the steady state is a saddle point locally.

Problem 7.

Consider a discrete time version of the typical growth model:

$$\begin{aligned}k(t+1) &= f(k(t)) - c(t) + (1 - \delta) \cdot k(t) \\c(t+1) &= \beta \cdot [1 + f'(k(t)) - \delta] \cdot c(t).\end{aligned}$$

The discount factor $\beta \in (0, 1)$, the rate of depreciation of capital $\delta \in (0, 1)$, initial capital k_0 is given, and the production function f satisfies the Inada conditions. These two equations are a system of first-order difference equations. Whereas a system of first-order differential equations relates $\dot{\mathbf{x}}(t)$ to $\mathbf{x}(t)$, a system of first-order difference equations relate $\mathbf{x}(t+1)$ to $\mathbf{x}(t)$.

We will see that we can study a system of first-order difference equations with the tools that we used to study systems of first-order differential equations. In particular, we can use phase diagrams to understand the dynamics of the system.

A. Construct a phase diagram for the system. First, define

$$\begin{aligned}\Delta k &\equiv k(t+1) - k(t), \\ \Delta c &\equiv c(t+1) - c(t).\end{aligned}$$

Second, draw the $\Delta k = 0$ locus and the $\Delta c = 0$ locus on the (k, c) plane. Finally, find the steady state as the intersection of the $\Delta k = 0$ locus and the $\Delta c = 0$ locus.

B. Show that the steady state is a saddle point in the phase diagram.

Problem 8.

We consider the following optimal growth problem. Given initial human capital h_0 and initial physical capital k_0 , choose consumption $c(t)$ and labor $l(t)$ to maximize utility

$$\int_0^{\infty} e^{-\rho \cdot t} \cdot \ln(c) dt$$

subject to

$$\begin{aligned}\dot{k}_t &= y_t - c_t - \delta \cdot k_t \\ \dot{h}_t &= B \cdot (1 - l_t) \cdot h_t.\end{aligned}$$

Output y_t is defined by

$$y_t \equiv A \cdot k_t^{\alpha} \cdot (l_t \cdot h_t)^{\beta}.$$

We also impose that $0 \leq l_t \leq 1$. The discount factor $\rho > 0$, the rate of depreciation of physical capital $\delta > 0$, the constants $A > 0$ and $B > 0$, and the production function parameters $\alpha \in (0, 1)$ and $\beta \in (0, 1)$.

- A. Give state and control variables.
- B. Write down the present-value Hamiltonian for this problem.
- C. Derive the optimality conditions.
- D. Show that the growth rate of consumption $c(t)$ is

$$\frac{\dot{c}}{c} = \frac{\alpha \cdot y}{k} - (\delta + \rho).$$

- E. From now on, we assume that $B = 0$. Show that $l = 1$.
- F. Draw the phase diagram in the (k, c) plane.
- G. Show on the diagram that the steady state of the system is a saddle point.
- H. Derive the Jacobian of the system.
- I. Show that the steady state of the system is a saddle point.