DYNAMIC BUSINESS-CYCLE MODEL

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Course material available at https://pascalmichaillat.org/c5/

OUTLINE

- present a dynamic model of business cycles, based on Michaillat, Saez (2022)
- solve the model
- study impact of aggregate demand and supply shocks
- study impact of monetary policy
  - contrast normal times and zero lower bound (ZLB)
  - review evidence on effectiveness of monetary policy

STRUCTURE OF THE MODEL

- dynamic, continuous-time model
- measure 1 of identical households are self-employed and produce services
  - no firms, and only one market for services markets
- households purchases and consume services produced by other households
- all services are traded on a matching market
  - trades are mediated by a Cobb-Douglas matching function
- government issues bonds, sets taxes, and sets nominal interest rate through central bank
- households save with government bonds, and they derive utility not only from consumption but also from their relative wealth

AGGREGATE SUPPLY

SUPPLY OF SERVICES

- size of the labour force is $l > 0$
- each worker has the capacity to produce $a > 0$ services per unit time
- services are sold through long-term worker-household relationships
  - after matching, a worker becomes a full-time employee of the household
- employees lose their jobs at rate $\lambda > 0$
- services are sold at a unit price $p(t)$
  - worker's income is $a\,p(t)$
  - inflation rate is $\pi(t) = \dot{p}(t)/p(t)$
- aggregate capacity = al : amount of services produced if all the labor force was employed

CAPACITY AND UNEMPLOYMENT

- because of the matching function, not all jobseekers find a job
  - ⇝ there is always some unemployment
- unemployment rate $u(t)$ = share of workers in the labour force who are not employed by any households
- number of employed workers:

$$n(t) = [1 - u(t)]l \,,$$

- aggregate output of services:

$$y(t) = an(t) = [1 - u(t)]al$$

- output $y(t) <$ capacity al because some workers are unemployed

MATCHING FUNCTION

- households advertise $v(t)$ vacancies
- $l - n(t) = u(t)l$ workers are unemployed
- Cobb-Douglas matching function determines the number of new employment relationships per unit time:

$$m(t) = \mu \cdot [l - n(t)]^{\eta} \cdot v(t)^{1-\eta}$$

- $\mu > 0$: matching efficacy
- $\eta \in (0, 1)$: matching elasticity
- market tightness $\theta(t)$ is the ratio of both arguments in matching function:

$$\theta(t) = \frac{v(t)}{l - n(t)}$$

MATCHING RATES

- each of the $l - n(t)$ unemployed workers finds a job at a rate

$$f(\theta(t)) = \frac{m(t)}{l - n(t)} = \mu\theta(t)^{1-\eta}$$

- each of the $v(t)$ vacancies is filled at a rate

$$q(\theta(t)) = \frac{m(t)}{v(t)} = \mu\theta(t)^{-\eta}$$

- since these are rates per unit time, they are not restricted to be in [0, 1]
  - probability to find a job in short time interval dt is $f(\theta(t))dt$
  - probability to fill a vacacny in short time interval dt is $q(\theta(t))dt$
- $f(0) = 0$, $f(\infty) = \infty$, $q(0) = \infty$, $q(\infty) = 0$

UNEMPLOYMENT DYNAMICS

- employment relationships evolves according to a differential equation:

$$\dot{n}(t) = f(\theta(t))\left[l - n(t)\right] - \lambda n(t)$$

- $f(\theta(t))\left[l - n(t)\right]$ = number of new relationships at time t
- $\lambda n(t)$: number of existing relationships dissolved at time t
- unemployment rate $u(t) = 1 - n(t)/l$ also follows a differential equation:

$$\dot{u}(t) = \lambda\left[1 - u(t)\right] - f(\theta(t))u(t)$$

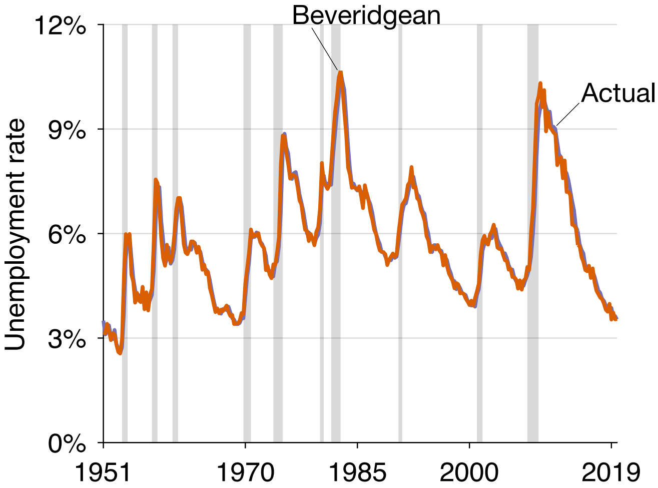
BEVERIDGE CURVE

- critical point of unemployment law of motion (u such as $\dot{u}$):

$$u = \frac{\lambda}{\lambda + f(\theta)}$$

- beveridge curve: negative relationship between unemployment rate and tightness
  - locus of points such that # new employment relationships created = # relationships dissolved at any point in time
  - unemployment rate is stable
- in practice in the US: unemployment always on Beveridge Curve
- technically: deviation between Beveridgean and actual unemployment rates decays at an exponential rate of 62% per month $\rightsquigarrow$ 90% deviation vanishes within a quarter
- assumption: unemployment rate is always on Beveridge curve

UNEMPLOYMENT: ALWAYS ON BEVERIDGE CURVE (MICHAILLAT, SAEZ 2021)



- accounting for unemployment dynamics does not add much descriptive power
  - because US labor market flows are so large
  - convergence to Beveridge curve is very fast
- using Beveridge curve instead of differential equation eliminates a state variables (u)

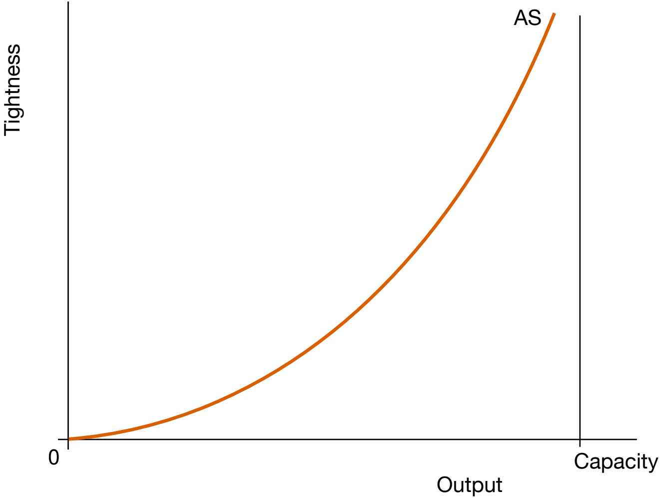
CONSTRUCTING THE AS CURVE

- AS curve gives the number of services sold at tightness θ given that unemployment is on the Beveridge curve:

$$y^s(\theta) = [1 - u(\theta)] \cdot al = \left[1 - \frac{\lambda}{\lambda + f(\theta)}\right] \cdot al = \frac{f(\theta)}{\lambda + f(\theta)} \cdot al$$

- properties of the AS are determined by $f(\theta)$:
  - $y^s(0) = 0$
  - $d\, y^s / d\theta > 0$
  - $d^2\, y^s / d\theta^2 < 0$
  - $\lim_{\theta \to \infty} y^s(\theta) = al$

PLOTTING THE AS CURVE



- diagram features tightness θ on y-axis, not price p or inflation π
- tightness is the central variable of the model:
  - determines all variables
  - responds to shocks

CONSUMPTION AND SAVING BY HOUSEHOLDS

RECRUITING COST

- each of $v(t)$ vacancy requires $\kappa > 0$ recruiters per unit of time
  - recruiters reading applications, interviewing candidates, and so on
- output = amount of services that households purchase = $y(t)$
- consumption = amount of services that provide utility = $c(t) < y(t)$
- $y(t) - c(t)$ = recruiting services = $\kappa v(t)$

COMPUTING THE RECRUITING WEDGE

- v vacancies give $q(\theta)v$ new employment relationships
- on Beveridge curve, $q(\theta)v$ = # relationships that separate at any point in time = λn
- sustaining employment n requires $v = \lambda n / q(\theta)$ vacancies
- vacancies are managed by $\kappa\lambda n/q(\theta)$ recruiters producing $a\kappa\lambda n/q(\theta) = \kappa\lambda\, y/q(\theta)$ services
- when y services are produced, # services actually consumed is:

$$c = \left[1 - \frac{\kappa\lambda}{q(\theta)}\right] y$$

- consumption and output are therefore related by

$$y = \left[1 + \tau(\theta)\right] c \quad \text{where} \quad \tau(\theta) = \frac{\kappa\lambda}{q(\theta) - \kappa\lambda}$$

PROPERTIES OF THE RECRUITING WEDGE

- recruiting wedge is:

$$\tau(\theta) = \frac{\kappa\lambda}{q(\theta) - \kappa\lambda}$$

- (same as in the static model except that $\kappa\lambda$ replaces κ)
- $\tau(0) = 0$
- $\tau(\theta)$ is increasing on $[0, \theta_\tau)$, where θ_τ is defined by $q(\theta_\tau) = \kappa\lambda$
- $\lim_{\theta \to \theta_\tau} \tau(\theta) = +\infty$
- when tightness is higher, a larger share of services are devoted to recruiting

NOMINAL BUDGET CONSTRAINT

$$\dot{b}(t) = i(t)b(t) + p(t)\left[1 - u(t)\right] al - p(t)\left[1 + \tau(\theta(t))\right] c(t) - T(t)$$

- $\dot{b}(t)$: change in nominal wealth (bond holdings)
- $i(t)b(t)$: interest income on wealth
- $p(t)\left[1 - u(t)\right] al$: labor income
- $p(t)\left[1 + \tau(\theta(t))\right] c(t)$: spending on services
  - $p(t)c(t)$: spending on consumption services
  - $p(t)\tau(\theta(t))c(t)$: spending on recruiting services
- $T(t)$: lump-sum tax/transfer that government uses to balance its budget

REAL BUDGET CONSTRAINT

- real stock of bonds: $w(t) = b(t)/p(t)$
- real interest rate: $r(t) = i(t) - \pi(t)$
- growth rates of the real and nominal stocks of bonds are related by:

$$\frac{\dot{w}(t)}{w(t)} = \frac{d\ln(w(t))}{dt} = \frac{d\ln(b(t))}{dt} - \frac{d\ln(p(t))}{dt} = \frac{\dot{b}(t)}{b(t)} - \frac{\dot{p}(t)}{p(t)} = \frac{\dot{b}(t)}{b(t)} - \pi(t)$$

- real stock of bonds evolves according to:

$$\dot{w}(t) = \frac{\dot{b}(t)}{p(t)} - \pi(t) \cdot w(t)$$

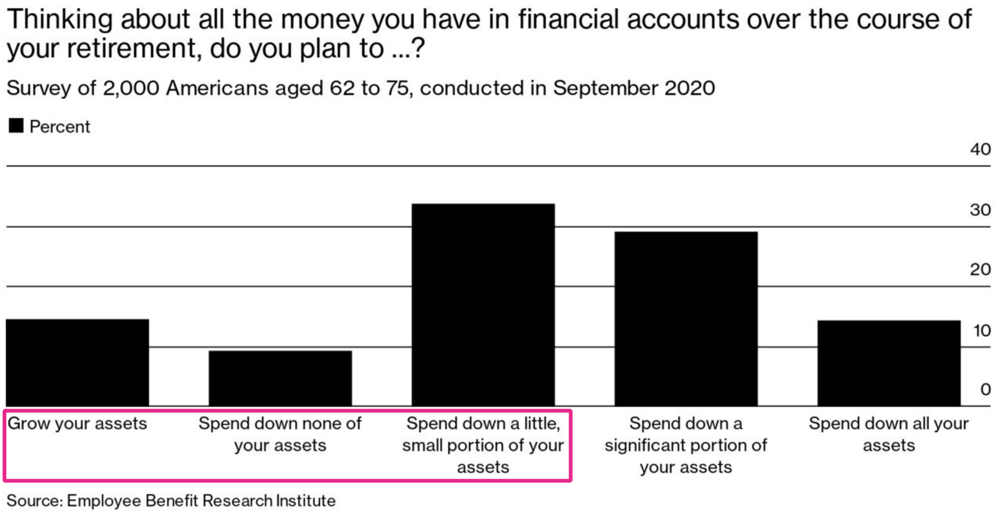
- which gives the flow real budget constraint:

$$\dot{w}(t) = r(t)w(t) + \left[1 - u(t)\right] al - \left[1 + \tau(\theta(t))\right] c(t) - \frac{T(t)}{p(t)}$$

UTILITY FUNCTION

- household consumes $c(t)$ services, which provide flow utility
$$\frac{\epsilon}{\epsilon - 1} c(t)^{(\epsilon-1)/\epsilon}$$
- $\epsilon > 1$: concavity of the utility function
- household's relative real wealth is $w(t) - \bar{w}(t)$
  - $w(t)$: real stock of bonds = real wealth
  - $\bar{w}(t)$: average real wealth across all households
- from its relative real wealth, the household enjoys flow utility:
$$\sigma(w(t) - \bar{w}(t))$$
- the function $\sigma : \mathbb{R} \to \mathbb{R}$ is increasing and strictly concave

RECENT, SURVEY EVIDENCE OF WEALTH IN UTILITY



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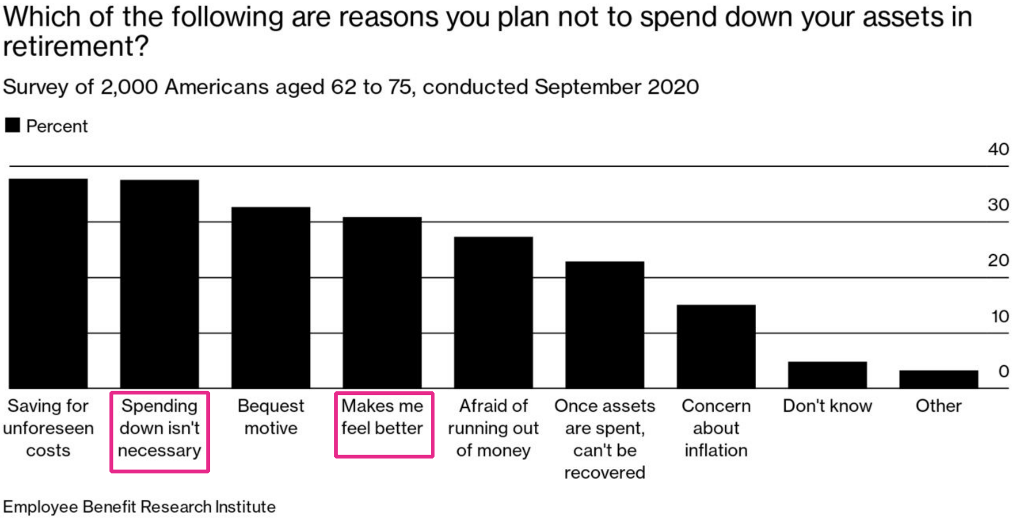

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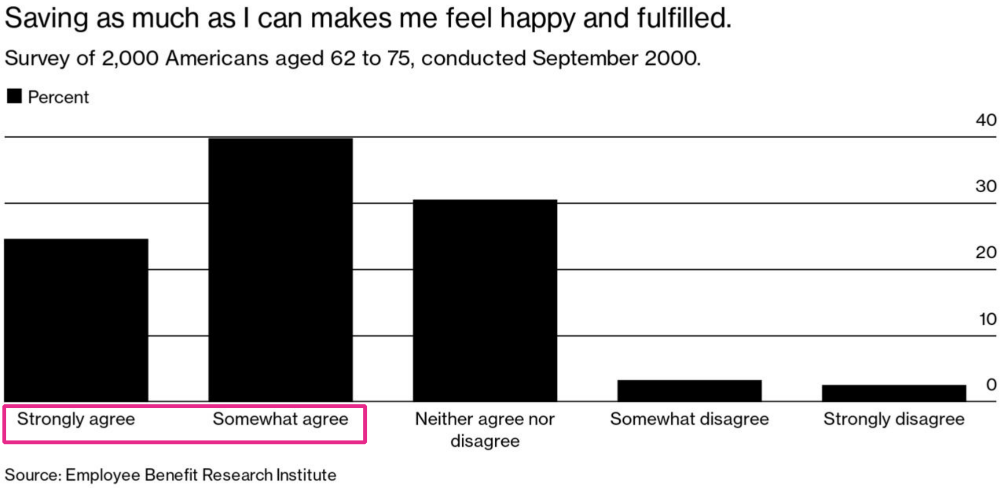


WEALTH IN UTILITY IMPROVES THE EULER EQUATION FOR CONSUMPTION/SAVING

- people report that saving makes them feel happy and fulfilled
  - as important as bequest motive and precautionary saving
- so adding wealth in the utility is realistic
- it also leads to better-behaved Euler equation (Michaillat, Saez 2021)
  - including better behavior at the zero lower bound
- utility for relative wealth is motivated by two reasons:
  - wealth as marker of social status
  - simpifies analysis since aggregate relative wealth is 0

HOUSEHOLD PROBLEM

- choose time paths for $c(t)$ and $w(t)$ to maximize the discounted sum of flow utilities:

$$\int_0^{\infty} e^{-\delta t}\left[\frac{\epsilon}{\epsilon-1}c(t)^{(\epsilon-1)/\epsilon}+\sigma(w(t)-\bar{w}(t))\right]dt,$$

- $\delta > 0$: time discount rate
- subject to the real budget constraint and to a borrowing constraint preventing Ponzi schemes
- takes as given paths of $\theta(t)$, $u(t)$, $p(t)$, $i(t)$, $T(t)$, and $\bar{w}(t)$
- takes as given initial real wealth $w(0)$

HAMILTONIAN

- Hamiltonian with control variable $c(t)$, state variable $w(t)$, and costate variable $\gamma(t)$:

$$\begin{aligned}\mathcal{H}(t, c(t), w(t)) = {} & \frac{\epsilon}{\epsilon - 1} c(t)^{(\epsilon-1)/\epsilon} + \sigma(w(t) - \bar{w}(t)) \\ & + \gamma(t) \left[r(t)w(t) + \left[1 - u(t)\right] al - \left[1 + \tau(\theta(t))\right] c(t) - \frac{T(t)}{p(t)} \right]\end{aligned}$$

- necessary conditions for an interior solution to the maximization problem:
  - $\partial\mathcal{H}/\partial c = 0$
  - $\partial\mathcal{H}/\partial w = \delta\gamma(t) - \dot{\gamma}(t)$
  - appropriate transversality condition
- since the utility function is strictly concave, interior paths of $c(t)$ and $w(t)$ that satisfy these conditions constitute the unique global maximum of the household's problem

OPTIMALITY CONDITIONS

- necessary conditions give:

$$c(t)^{-1/\epsilon} = \gamma(t)\left[1 + \tau(\theta(t))\right]$$

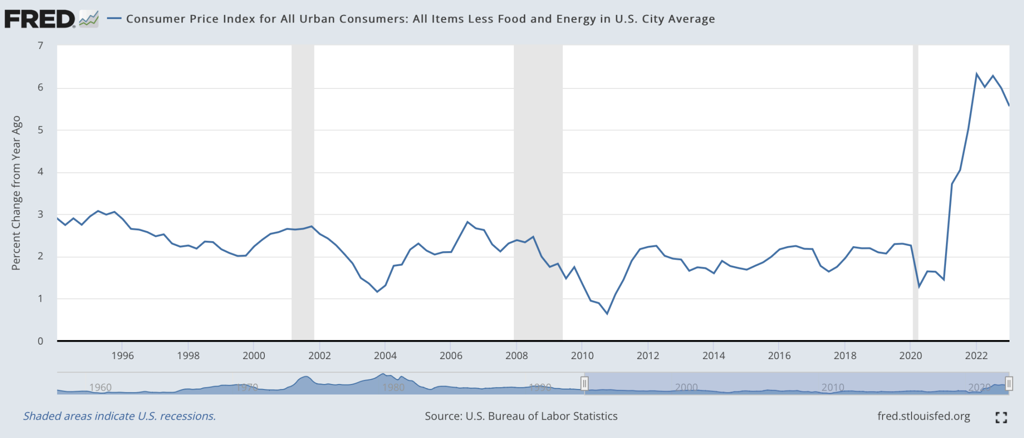
$$\dot{\gamma}(t) = \left[\delta - r(t)\right]\gamma(t) - \sigma'(w(t) - \bar{w}(t))$$

- without recruiting cost ($\tau = 0$) and no utility from wealth ($s' = 0$), the equations reduce to the standard continuous-time Euler equation:

$$\frac{\dot{c}(t)}{c(t)} = \epsilon[r(t) - \delta].$$

INFLATION AND MONETARY POLICY

US INFLATION ≈ 2% IN LAST 30 YEARS



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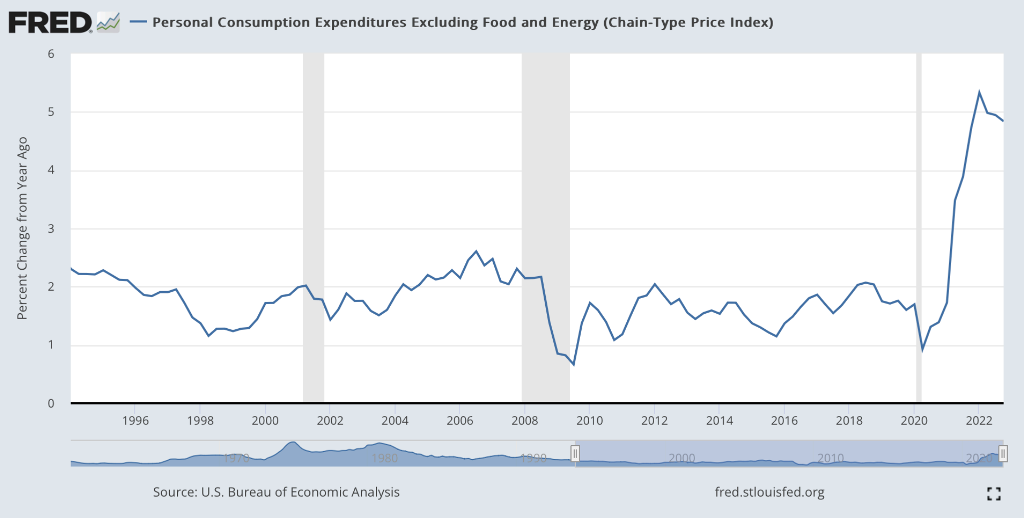
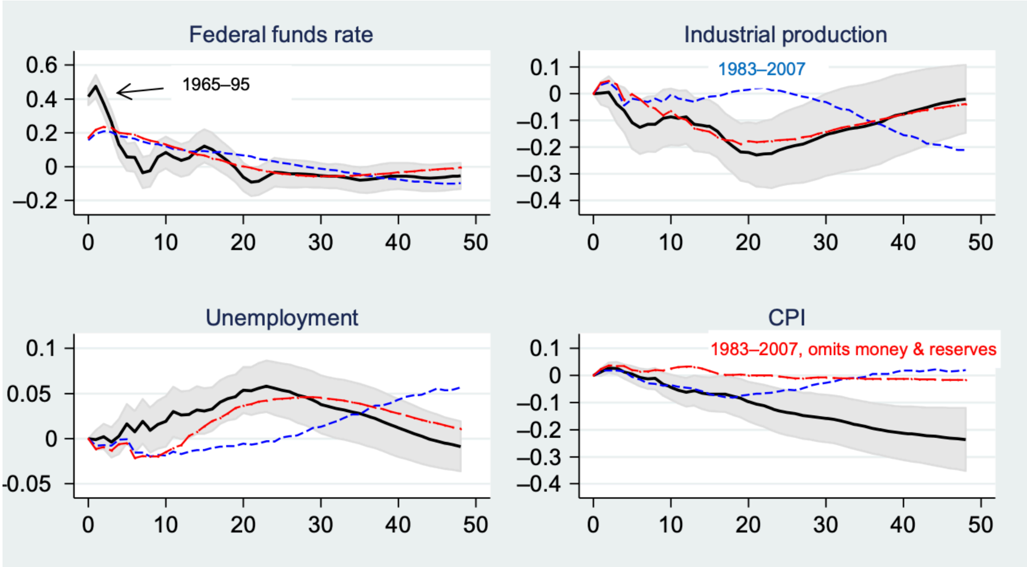
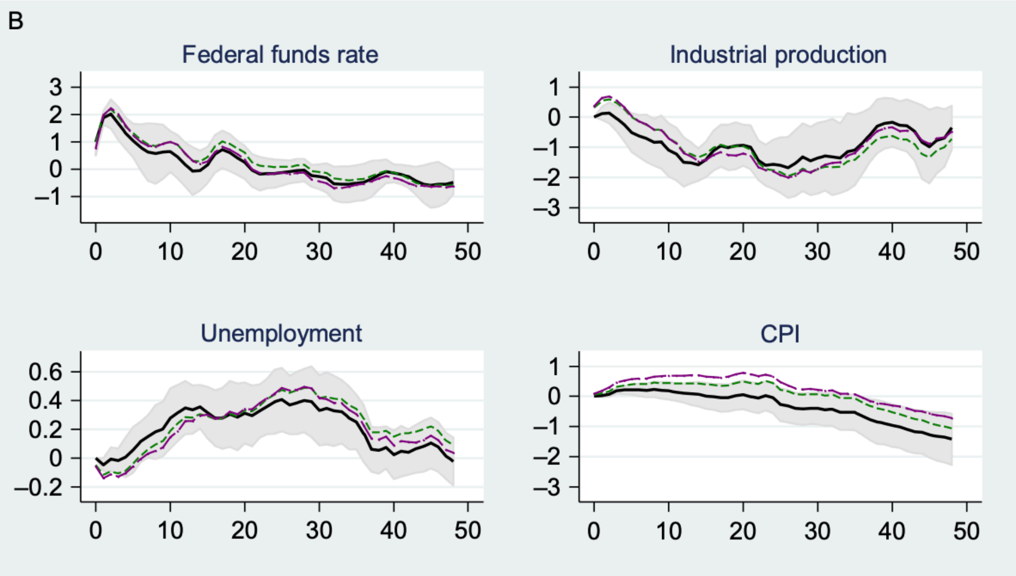


MONETARY POLICY AND INFLATION (RECURSIVE IDENTIFICATION, RAMEY 2016)



MONETARY POLICY AND INFLATION (NARRATIVE IDENTIFICATION, RAMEY 2016)



PRICE NORM: FIXED INFLATION

- any model with a matching function needs a price mechanism
- we assume that prices grow at a fixed rate of inflation π
  - interpretation: fixed inflation is a social norm (Hall 2005)
  - evolution of prices: $p(t) = e^{\pi t}$
- fixed inflation is realistic:
  - inflation does not respond to unemployment (Stock, Watson 2010, 2019)
  - inflation does not respond to monetary policy (Ramey 2016)
- fixed inflation does not create bilaterally inefficiencies:
  - buyers & sellers are happy to transact at the given price

MONETARY POLICY: INTEREST-RATE PEG

- central bank simply follows an interest-rate peg:

$$i(t) = i.$$

- ZLB constraint: $i \geq 0$
- since inflation rate and nominal interest rate are fixed, the real interest rate is fixed:

$$r = i - \pi.$$

- $r < \delta$ so the model has a solution
- thanks to wealth in the utility, solution will be unique (“determinate equilibrium”) despite the absence of Taylor rule

DYNAMICS OF THE MODEL AND AGGREGATE DEMAND

EULER EQUATION

- costate variable on budget constraint $\gamma(t)$ satisfies:

$$\dot{\gamma}(t) = \left[\delta - r(t)\right]\gamma(t) - \sigma'(w(t) - \bar{w}(t))$$

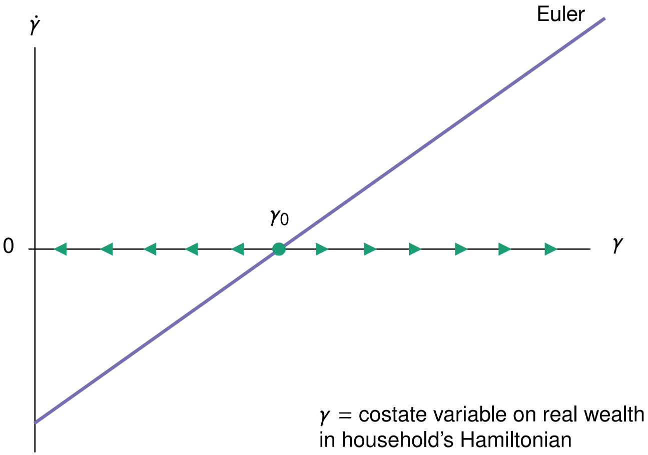
- but real interest rate is fixed to r
- all households are the same, so their relative wealth is 0
- so the Euler equation is an autonomous, first-order, linear differential equation:

$$\dot{\gamma}(t) = [\delta - r]\,\gamma(t) - \sigma'(0)$$

- Euler equation admits a unique critical point γ_0 (where $\dot{\gamma} = 0$):

$$\gamma_0 = \frac{\sigma'(0)}{\delta - r}$$

PHASE LINE FOR EULER EQUATION



- costate variable $\gamma(t)$ is nonpredetermined
- one constant solution: γ jumps to γ_0 at time 0 and remains there
- if γ jumps to another position, it diverges to $-\infty$ or $+\infty$ as $t \to \infty$
- interior path satisfiying optimality conditions $\rightsquigarrow$ unique global maximum to household's problem

OPTIMAL CONSUMPTION

- Euler equation does not induce any dynamics because costate variable γ jumps to the critical value γ_0 at time 0
- Using constant solution to costate equation, optimality conditions are

$$c(t)^{-1/\epsilon} = \gamma(t)\left[1 + \tau(\theta(t))\right]$$

$$\gamma(t) = \frac{\sigma'(0)}{\delta - r}$$

- consumption at time t is:

$$c(t) = \left[\frac{\delta - r}{\sigma'(0)} \cdot \frac{1}{1 + \tau(\theta(t))}\right]^{\epsilon}$$

$\rightsquigarrow$ consumption is just a function of tightness

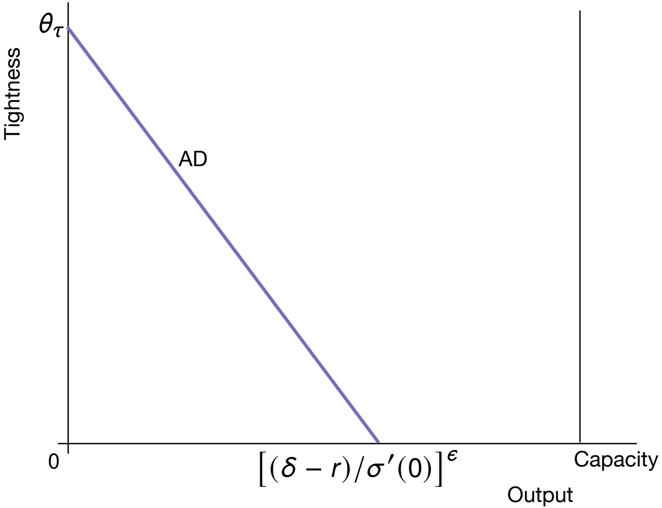
AD CURVE

- AD curve gives output demanded by households when households optimally consume and save over time:

$$y = \left[\frac{\delta - r}{\sigma'(0)}\right]^{\epsilon} \cdot \frac{1}{[1 + \tau(\theta)]^{\epsilon - 1}}$$

- properties of the AD are determined by $\tau(\theta)$:
  - $y^d(0) = \left[(\delta - r)/\sigma'(0)\right]^{\epsilon}$
  - $d\,y^d/d\theta > 0$
  - $\lim_{\theta \to \theta_\tau} y^d(\theta) = al$

PROPERTIES OF THE AD CURVE



- diagram features tightness θ on y-axis, not price p or inflation π
- tightness is the central variable of the model:
  - determines all variables
  - responds to shocks

SOLUTION OF THE MODEL

SOLUTION OF THE MODEL

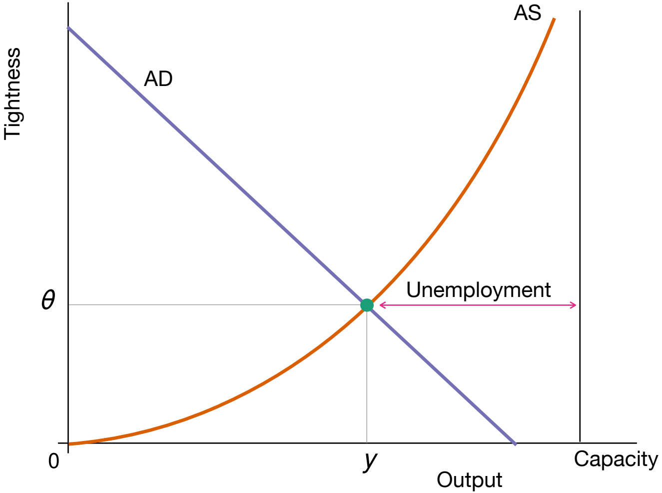
- Tightness x that solves the model is given by:

$$y^s(x) = y^d(x)$$

- then output can be read from the AS or AD curves
- $y^s(\theta)$ is increasing from 0 to al as θ is increasing from 0 to ∞
- $y^d(\theta)$ is decreasing from $\left[(\delta - r)/\sigma'(0)\right]^{\epsilon}$ to 0 as θ is increasing from 0 to θ_τ

$\rightsquigarrow$ Model always admits a unique solution, $\theta \in (0, \theta_\tau)$

GRAPHICAL REPRESENTATION OF THE SOLUTION

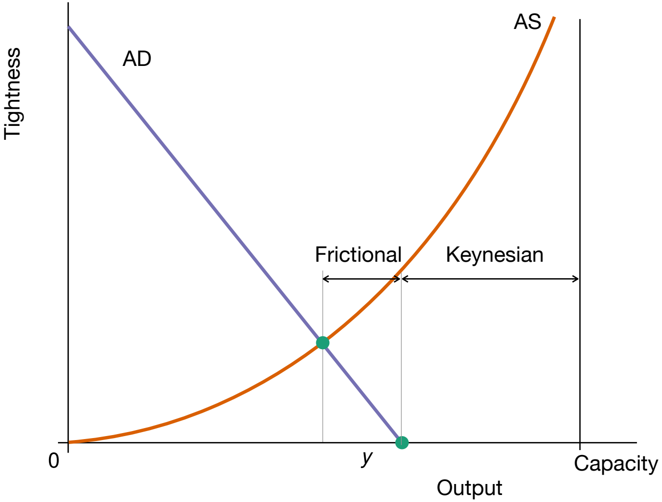


- market diagram features tightness θ on y-axis, not inflation π
- intersection of AD and AS curves gives y, θ that solve the model
- all other variables can be computed from θ
- model features unemployment $u(\theta) > 0$

COMPUTING AGGREGATE VARIABLES FROM TIGHTNESS

- aggregate output: $y = y^s(\theta) = y^d(\theta)$
- aggregate consumption: $c = y/[1 + \tau(\theta)]$
- rate of unemployment $= u(\theta)$
- aggregate employment: $n = [1 - u(\theta)]l = y/a$
- number of recruiters $= n/[1 + \tau(\theta)]$
- Aggregate number of vacancies: $v = \lambda \cdot n/q(\theta)$
- inflation π is given by pricing norm
- interest rates i and $r = i - \pi$ are set by monetary policy

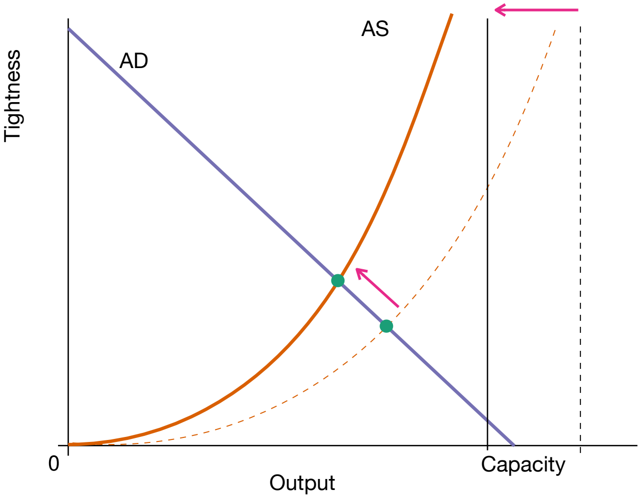
DECOMPOSITION OF UNEMPLOYMENT



- Keynesian unemployment caused by lack of demand
- frictional unemployment is additional unemployment caused by matching cost ($\kappa > 0$, $\tau > 0$)
- conceptually similar to unemployment decomposition in Michaillat (2012)

COMPARATIVE STATICS

AS SHOCKS



- change in productivity a or labor force participation l
- negative correlation between output and tightness
- movement along the AD curve

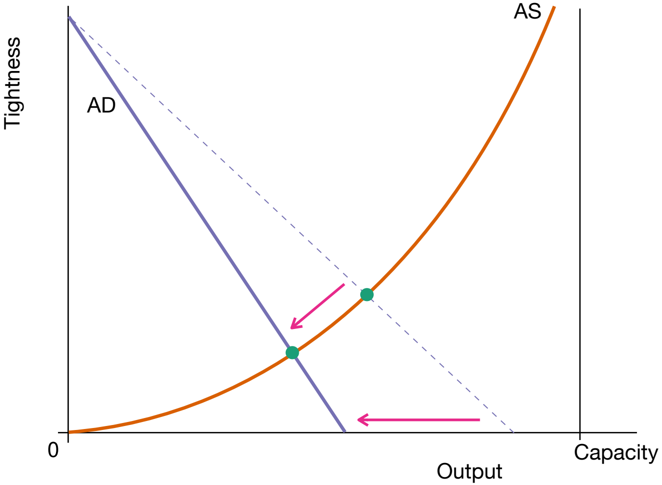
IMPACT OF NEGATIVE LABOR-FORCE PARTICIPATION SHOCK

- labor force participation l ↓
- tightness θ ↑
- output y ↓
- employment $n = y/a$ ↓
- unemployment rate $u(\theta)$ ↓
- shopping wedge $\tau(\theta)$ ↑
- consumption $c = y/[1 + \tau(\theta)]$ ↓
- rare since unemployment rate and output comove negatively (Okun's law)

IMPACT OF NEGATIVE PRODUCTIVITY SHOCK

- productivity $a \downarrow$
- tightness $\theta \uparrow$
- output $y \downarrow$
- unemployment rate $u(\theta) \downarrow$
- employment $n = (1 - u) \cdot l \uparrow$
- shopping wedge $\tau(\theta) \uparrow$
- consumption $c = y/[1 + \tau(\theta)] \downarrow$
- rare since unemployment rate and output comove negatively (Okun's law)
- consistent with evidence that higher productivity reduces employment (Basu, Fernald, Kimball 2006)

AD SHOCKS

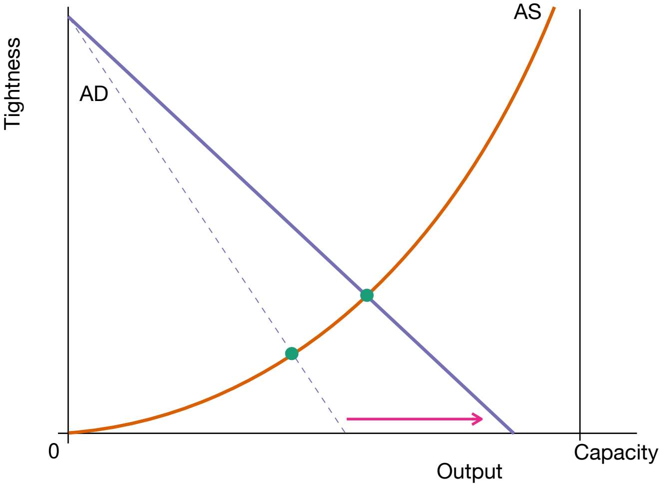


- change in desire to marginal utility from wealth $\sigma'(0)$
- positive correlation between output and tightness
- movement along the AS curve

IMPACT OF NEGATIVE AD SHOCK

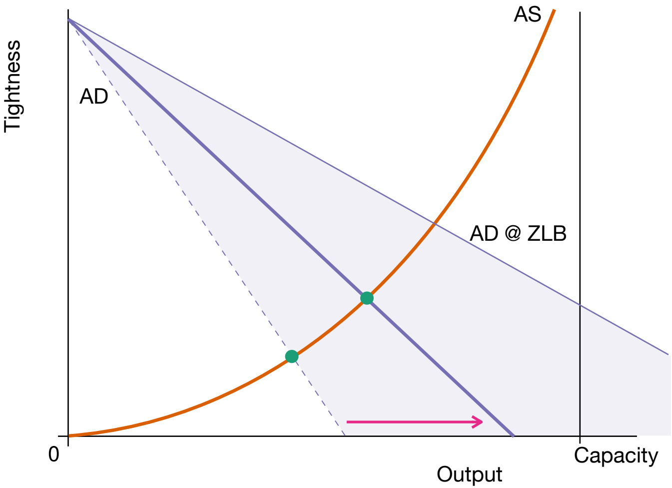
- marginal utility from wealth $\sigma'(0) \downarrow$
- tightness $\theta \downarrow$
- output $y \downarrow$
- employment $n = y/a \downarrow$
- unemployment rate $u(\theta) \uparrow$
- shopping wedge $\tau(\theta) \downarrow$
- most common shock since unemployment and output comove negatively (Okun's law)

MONETARY EXPANSION



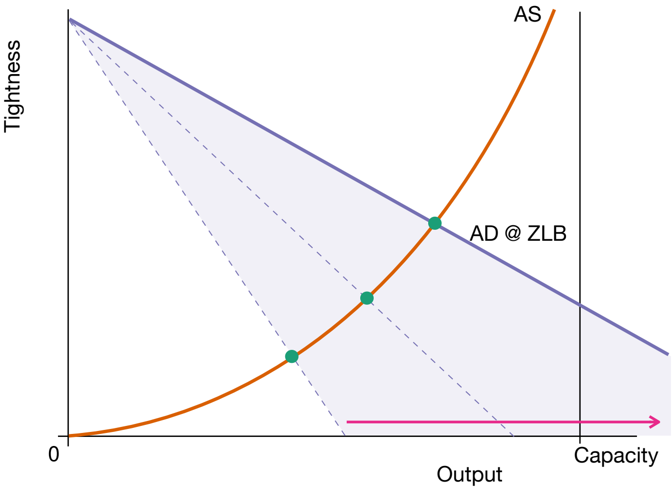
- $y^d(\theta) =$
  $\left[(\delta - i + \pi)/\sigma'(0)\right]^{\epsilon} \cdot [1 + \tau(\theta)]^{1-\epsilon}$
- decrease in *i* boosts aggregate demand
- which raises tightness and reduces unemployment

RANGE OF POSSIBLE EXPANSIONS



- $y^d(\theta) =$
  $\left[(\delta - i + \pi)/\sigma'(0)\right]^{\epsilon} \cdot [1 + \tau(\theta)]^{1-\epsilon}$
- monetary policy can stimulate demand as long as $i \geq 0$

MONETARY POLICY AT ZERO LOWER BOUND



- $y^d(\theta) = \left[(\delta+\pi)/\sigma'(0)\right]^{\epsilon} \cdot [1+\tau(\theta)]^{1-\epsilon}$
- monetary policy is most stimulative when $i = 0$
- highest tightness, lowest unemployment rate
- all model properties remain the same at ZLB: no strange behavior

SUMMARY

SUMMARY OF MODEL PROPERTIES

property	NK model	this model
AD relation	Euler equation	discounted Euler equation
AS relation	Phillips curve	Beveridge curve
inflation	fluctuating	fixed
unemployment	zero	fluctuating
ZLB world	topsy-turvy	normal
ZLB duration	must be short	can be permanent