Problem Set on Optimal Control

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Available at https://pascalmichaillat.org/c3/

Problem 1.

Consider the following optimal growth problem: Given initial capital $k_0 > 0$, choose a consumption path $\{c_t\}_{t\geq 0}$ to maximize utility

$$\int_0^\infty e^{-\rho\cdot t} \cdot \ln(c_t)dt$$

subject to the law of motion of capital

$$\dot{k}_t = f\left(k_t\right) - c_t - \delta \cdot k_t.$$

The discount factor $\rho > 0$, and the production function f satisfies

$$f\left(k\right) = A \cdot k^\alpha,$$

where $\alpha \in (0, 1)$ and $A > 0$.

A. Write down the present-value Hamiltonian.

B. Show that the Euler equation is

$$\frac{\dot{c}_t}{c_t} = \alpha \cdot A \cdot k_t^{\alpha-1} - (\delta + \rho) .$$

C. Solve for the steady state of the system.

Problem 2.

Consider the following investment problem: Given initial capital k_0, choose the investment path $\{i_t\}_{t\geq 0}$ to maximize profits

$$\int_0^{\infty} e^{-r\cdot t}\left[f\left(k_t\right) - i_t - \frac{\chi}{2}\cdot\left(\frac{i_t^2}{k_t}\right)\right] dt$$

subject to the law of motion of capital (we assume no capital depreciation)

$$\dot{k}_t = i_t.$$

The interest rate $r > 0$, the capital adjustment cost $\chi > 0$, and the production function f satisfies $f' > 0$ and $f'' < 0$.

A. Write down the current-value Hamiltonian.

B. Use the optimality conditions for the current-value Hamiltonian to derive the following differential equations:

$$\dot{k}_t = \left(\frac{q_t - 1}{\chi}\right)\cdot k_t$$

$$\dot{q}_t = r\cdot q_t - f'\left(k_t\right) - \frac{1}{2\cdot\chi}\left(q_t - 1\right)^2$$

C. Solve for the steady state.