Problem Set on Dynamic Programming

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Problem 1.

Consider the following optimal growth problem: Given initial capital $k_0 > 0$, choose consumption $\{c_t\}_{t=0}^{+\infty}$ to maximize utility

$$\sum_{t=0}^{\infty} \beta^t \cdot \ln(c_t)$$

subject to the resource constraint

$$k_{t+1} = A \cdot k_t^{\alpha} - c_t.$$

The parameters satisfy $0 < \beta < 1,\ A > 0,\ 0 < \alpha < 1$.

A. Derive the optimal law of motion of consumption c_t using a Lagrangian.

B. Identify the state variable and the control variable.

C. Write down the Bellman equation.

D. Derive the following Euler equation:

$$c_{t+1} = \beta \cdot \alpha \cdot A \cdot k_{t+1}^{\alpha-1} \cdot c_t.$$

E. Derive the first two value functions, $V_1(k)$ and $V_2(k)$, obtained by iteration on the Bellman equation starting with the value function $V_0(k) \equiv 0$.

F. The process of determining the value function by iterations using the Bellman equation is commonly used to solve dynamic programs numerically. The algorithm is called *value function iteration*. For this optimal growth problem, one can show show using value function iteration that the value function is

$$V(k) = \kappa + \frac{\ln(k^{\alpha})}{1 - \alpha \cdot \beta},$$

where κ is a constant. Using the Bellman equation, determine the policy function $k'(k)$ associated with this value function.

G. In light of these results, for which reasons would you prefer to use the dynamic-programming approach instead of the Lagrangian approach to solve the optimal growth problem? And for which reasons would you prefer to use the Lagrangian approach instead of the dynamic-programming approach?

Problem 2.

Consider the problem of choosing consumption $\{c_t\}_{t=0}^{+\infty}$ to maximize expected utility

$$\mathbb{E}_0 \sum_{t=0}^{+\infty} \beta^t \cdot u(c_t)$$

subject to the budget constraint

$$c_t + p_t \cdot s_{t+1} = \left(d_t + p_t\right) \cdot s_t.$$

d_t is the dividend paid out for one share of the asset, p_t is the price of one share of the asset, and s_t is the number of shares of the asset held at the beginning of period t. In equilibrium, the price p_t of one share is solely a function of dividends d_t. Dividends can only take two values d_l and d_h, with $0 < d_l < d_h$. Dividends follow a Markov process with transition probabilities

$$\mathbb{P}\left(d_{t+1} = d_l \mid d_t = d_l\right) = \mathbb{P}\left(d_{t+1} = d_h \mid d_t = d_h\right) = \rho$$

with $1 > \rho > 0.5$.

A. Identify state and control variables.

B. Write down the Bellman equation.

C. Derive the following Euler equation:

$$p_t \cdot u'(c_t) = \beta \cdot \mathbb{E}\left(\left(d_{t+1} + p_{t+1}\right) \cdot u'(c_{t+1}) \mid d_t\right).$$

D. Suppose that $u(c) = c$. Show that the asset price is higher when the current dividend is high.

Problem 3.

Consider the following optimal growth problem: Given initial capital $k_0 > 0$, choose consumption and labor $\{c_t, l_t\}_{t=0}^{+\infty}$ to maximize utility

$$\sum_{t=0}^{+\infty} \beta^t \cdot u(c_t, l_t)$$

subject to the law of motion of capital

$$k_{t+1} = A_t \cdot f(k_t, l_t) - c_t.$$

In addition, we impose $0 \leq l_t \leq 1$. The discount factor $\beta \in (0, 1)$. The function f is increasing and concave in both arguments. The function u is increasing and concave in c, decreasing and convex in l.

Deterministic case. First, suppose $A_t = 1$ for all t.

A. What are the state and control variables?

B. Write down the Bellman equation.

C. Derive the following optimality conditions:

$$\frac{\partial u(c_t, l_t)}{\partial c_t} = \beta \cdot \frac{\partial u(c_{t+1}, l_{t+1})}{\partial c_{t+1}} \cdot \frac{\partial f(k_{t+1}, l_{t+1})}{\partial k_{t+1}}$$
$$\frac{\partial u(c_t, l_t)}{\partial c_t} \cdot \frac{\partial f(k_t, l_t)}{\partial l_t} = -\frac{\partial u(c_t, l_t)}{\partial l_t}.$$

D. Suppose that the production function $f(k, l) = k^\alpha \cdot l^{1-\alpha}$. Determine the ratios c/k and l/k in steady state.

Stochastic case. Now, suppose A_t is a stochastic process that takes values A_1 and A_2 with the following probability:

$$\mathbb{P}(A_{t+1} = A_1 \mid A_t = A_1) = \mathbb{P}(A_{t+1} = A_2 \mid A_t = A_2) = \rho.$$

E. Write down the Bellman equation.

F. Derive the optimality conditions.