General Notes on General Topology

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Contents

1 Basic Definitions

1.1 Topological Spaces

Definition 1. A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

1. $\varnothing$ and X are in $\mathcal{T}$.
2. The union of elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.
3. The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

A set X for which a topology $\mathcal{T}$ has been specified is called a **topological space**. Topological spaces may be denoted by $(X, \mathcal{T})$, or just by X if it's sufficiently clear.

Definition 2. If X is a topological space with topology $\mathcal{T}$, a subset $U \subset X$ is an **open set** of X if U belongs to the collection $\mathcal{T}$. The complement of an open set is said to be **closed**, and the complement of a closed set is open.

We can also define topologies in terms of $\mathfrak{T}$ is also a topology if

1. $\varnothing$ and X are closed.
2. Finite unions of closed sets are closed.

3. Arbitrary intersections of closed sets are closed.

Example: Let X be a set. Let $\mathcal{T}_c$ be the collection of all subsets $U \subset X$ such that $X - U$ is either countable or all of X. Then $\mathcal{T}_c$ is a topology on X (called the finite-complement topology).

Definition 3. Suppose $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a set X. If $\mathcal{T} \subset \mathcal{T}'$, then $\mathcal{T}'$ is **finer** than $\mathcal{T}$, and $\mathcal{T}$ is **coarser** than $\mathcal{T}$;. If $\mathcal{T}'$ properly contains $\mathcal{T}$, we say add the prefix "strictly" to each. If $\mathcal{T} \subset \mathcal{T}'$ or vice versa, we say $\mathcal{T}$ is **comparable** with $\mathcal{T}'$.

1.2 Basis

Definition 4. If X is a set, a **basis** for a topology on X is a collection $\mathcal{B}$ of subsets of X (called basis elements) such that

1. For each $x \in X$, there is at least one basis element B containing x.
2. If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these two conditions, then we define the topology $\mathcal{T}$ generated by $\mathcal{B}$ as follows: A subset U of X is said to be open in X if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$. Each basis element is an element of $\mathcal{T}$. Another way to generate $\mathcal{T}$ from $\mathcal{B}$ is to take all possible unions of elements of $\mathcal{B}$.

We define the basis this way for several reasons. Namely, it's often easier to define a topology in terms of its basis, as we shall soon see, and since the basis is usually smaller than the topology it generates and has simpler elements, it can make it easier to prove certain theorems.

Example: If X is any set, the collection of all one-point subsets of X is a basis for the discrete topology on X (the topology in which all subsets of X are open).

Definition 5. A **subbasis** $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X. The topology generated by the subbasis of $\mathcal{S}$ is defined to be the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$.
In other words, we generate $\mathcal{B}$ by taking the set of all finite intersections of $\mathcal{S}$, and then generate $\mathcal{T}$ by taking the set of all unions of $\mathcal{B}$.

The subbasis is in some sense an even simpler object than the basis, and is, like the basis, helpful in defining certain things (such as the product topology, which we'll soon discuss) and proving certain theorems.

1.3 Example: Topologies on $\mathbb{R}$

Definition 6. Let $\mathcal{B}$ be the collection of all open intervals in the real line,

$$(a, b) = \{x | a < x < b\}.$$

The topology generated by $\mathcal{B}$ is called the **standard topology** on $\mathbb{R}$.

Definition 7. Let $\mathcal{B}'$ be the collection of all half-open intervals,

$$[a, b) = \{x | a \leq x < b\}.$$

The topology generated by $\mathcal{B}'$ is called the **lower limit topology** on $\mathbb{R}$, and is denoted by $\mathbb{R}_\ell$.

Definition 8. Let K denote the set of all numbers of the form $1/n$ for $n \in \mathbb{Z}_+$, and let $\mathcal{B}''$ be the collection of all open intervals (a, b) along with all sets $(a, b) - K$. The topology generated by $\mathcal{B}''$ is called the **K-topology** on $\mathbb{R}$, and is denoted by $\mathbb{R}_K$.

$\mathbb{R}_\ell$ and $\mathbb{R}_K$ are strictly finer than the standard topology on $\mathbb{R}$, but are not comparable.

1.4 Motivation for the definition of a topology

Originally, general topology was developed as a generalization of the study of metric spaces and their properties, which are of great interest in real analysis. As it turns out, many of the notions related to metric spaces, such as continuity and connectedness, can be precisely defined in terms of open sets, and so the simplest way to define topologies is then entirely in terms of open sets.

Next, why is it that only finite unions of open sets are open? We can motivate this condition by observing what happens in standard euclidean space $\mathbb{R}^n$ (which is a metric space!).

Consider an arbitrary intersection of an infinite number of open sets in the real line, starting with $(-1, 1)$, and with each subsequent interval containing zero and itself being contained in the previous interval (such as the intersection of all $(-1/n, 1/n)$ for $n \in \mathbb{Z}_+$). Such a set would have only one element, zero, and would therefore be closed (the boundaries of the set are contained in the set). The definition of a topology helps us generalize this notion to arbitrary topological spaces.

Similarly, suppose we took an infinite union of closed sets (closed intervals), where each closed set would be strictly contained in the subsequent closed set (such as the union of all $[-n, n]$ for $n \in \mathbb{Z}_+$). The definition of a topology (in terms of closed sets) doesn't permit this, because such a union could not 'contain its own boundaries' in the real line, since, for the boundary of any particular closed set we choose, there will always be numbers 'beyond' that boundary in the union (greater than the upper boundary and less than the lower boundary).Once again, this definition extends to arbitrary topological spaces.

2 Continuous Functions

2.1 Closed Sets and Limit Points

Definition 9. Given a subet A of a topological space X, the **interior** of A is the union of all open sets contained in A, and the **closure** of A is defined as the intersection of all closed sets containing A.

Definition 10. Given $A \subset X$ and $x \in X$, x is a **limit point** of A if it belongs to the closure of $A - \{x\}$, or, equivalently, if every neighbour of x intersects A in some point other than x itself.

Definition 11. A topological space X is called a **Hausdorff space** if for each pair x_1, x_2 of distinct points of X, there exist neighbourhoods U_1 and U_2 of x_1 and x_2, respectively, that are disjoint.

Definition 12. In some space X, a sequence $x_1, x_2, \ldots$ of X **converges** to the point x if, for each neighbourhood U of x, there is a positive integer N such that $x_n \in U$ for all $n \geq N$.

2.2 Continuity

Definition 13. Let X and Y be topological spaces. $f : X \to Y$ is **continuous** if for each open subset V of Y, $f^{-1}(V)$ is an open subset of X.

Definition 14. Given topological spaces X and Y and a bijection $f : X \to Y$, if f and f^{-1} are continuous, f is called a **homeomorphism**.

Signifance of the homeomorphism: If f is a homeomorphism, any property of X that can be entirely expressed in terms of the topology on X yields the corresponding property on Y via f. Such a property is known as a topological property.

Definition 15. Given an injective continuous map $f : X \to Y$ between two topological spaces, X and Y, let $Z = f(X)$ be a subspace of Y. Then $f' : X \to Z$ is bijective, and if f' is a homeomorphism from X to Z, f is called an **imbedding** of X in Y.

Rules for constructing continuous functions. Let X, Y, and Z be topological spaces.

1. If $f : X \to Y$ maps all of X into the single point y_0 of Y, then f is continuous.

2. If A is a subspace of X, the inclusion $j : A \to X$ is continuous.

3. If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.

4. If $f : X \to Y$ is continuous and if A is a subspace of X, then the restricted function $f|A : A \to Y$ is continuous.

5. Let $f : X \to Y$ be continuous. If Z is a subspace of Y containing the image set $f(X)$, then the function $g : X \to Z$ obtained by restricting the range of f is continuous. If Z is a space having Y as a subspace, then $h : X \to Z$ obtained by expanding the range of f is continuous.

6. The map $f : X \to Y$ is continuous if X can be written as the union of open sets U_α such that $f|U_\alpha$ is continuous for each α.

Although the way we defined continuity may seem a bit arbitrary, we can attempt to motivate it by seeing how it's equivalent to the more familiar definition of continuity that's used in analysis or calculus, known as the $\epsilon - \delta$ definition.

Definition 16. A function $f : \mathbb{R} \to \mathbb{R}$ is **continuous** if for all x in the domain,

$$\forall \epsilon > 0, \exists\ \delta > 0 \text{ such that } |y - x| < \delta \implies |f(y) - f(x)| < \epsilon$$

Theorem 1. *The $\epsilon - \delta$ definition of continuity (1) is equivalent to the open set definition (2) (Definition 13).*

Proof. (1) $\implies$ (2). For some $x_0 \in \mathbb{R}$ and some $\epsilon > 0$, let $V = (f(x_0) - \epsilon, f(x_0) + \epsilon)$. We show that $f^{-1}(V)$ is also open. By the $\epsilon - \delta$ definition, for any $x \in f^{-1}(V)$, there exists a $\delta > 0$ such that for any $y \in (x - \delta, x + \delta)$, $f(y) \in V$. This means that $(x - \delta, x + \delta)$ must be in $f^{-1}(V)$. Since there is such an open set for every $x \in f^{-1}(V)$, $f^{-1}(V)$ is a union of open sets and is therefore open. Since any open set in the codomain is going to be the union of intervals like V, this applies to any open set in the codomain.

(2) $\implies$ (1). For some $x \in \mathbb{R}$, choose an open interval in the image of f, $U = (f(x) - \epsilon, f(x) + \epsilon)$. By (2), $f^{-1}(U)$ is an open set, meaning it's a union of open intervals. Since x is in $f^{-1}(U)$, it must then be contained in some open interval (a, b). Let. $\delta = \min(|x - a|, |x - b|)$. Then for any y such that $|x - y| < \delta$, it must be the case that $f(y) \in U$, meaning that $|f(x) - f(y)| < \epsilon$. ∎

3 Methods For Constructing Topological Spaces

Definition 17. Let X be a set with a simple order relation; assume X has more than one element. Let $\mathcal{B}$ be the collection of all sets of the following types:

1. All open intervals (a, b) in X.

2. All intervals of the form $[a_0, b)$, where a_0 is the smallest element (if there is one) of X.

3. All intervals of the form $(a, b_0]$, where b_0 is the largest element (if there is one) of X.

The collection $\mathcal{B}$ is a basis for the **order topology** on X.
If X has no smallest element, there are no sets of type 2, and if X has no largest element, there are no sets of type 3. The order topology gives us a way to generalize the 'standard topology' on $\mathbb{R}$.

Definition 18. Let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X, the collection

$$\mathcal{T}_Y = \{Y \cap U \mid U \subset \mathcal{T}\}$$

is a topology on Y called the **subspace topology**. Y is then a **subspace** of X, whose open sets consist of all intersections of open sets of X with Y.

At first it's perhaps a little bizarre that the subspace topology is defined this way, in terms of intersections of open sets of X with Y. Wouldn't it make more sense to define it as the set of open sets of X that are contained in Y? Well, the problem with defining it this way is that it doesn't provide a natural way to define an inclusion map $Y \hookrightarrow X$, whereas the standard definition does. Check to see that this is true.

3.1 The Product Topology

Definition 19. Let $\pi_1 : X \times Y \to X$ be defined by

$$\pi_1(x, y) = x.$$

And let $\pi_2 : X \times Y \to Y$ be defined by

$$\pi_2(x, y) = y.$$

Then π_1 and π_2 are called **projections** of $X \times Y$. If U is open in X, then

$$\pi_1^{-1}(U) = U \times Y$$

or the set of all (x, y) such that $x = U$.

Definition 20. Given $X_1, X_2, \ldots, X_n$, the **box topology** is the topology generated by basis elements of the form $U_1 \times U_2 \times \cdots \times U_n$ where U_i is open in X_i.

Definition 21. Let $\mathcal{S}_\beta$ denote the collection

$$\mathcal{S}_\beta = \{\pi_\beta^{-1}(U_\beta) \mid U_\beta \text{ open in } X_\beta\}.$$

And let $\mathcal{S}$ denote the union

$$\mathcal{S} = \bigcup_{\beta \in J} \mathcal{S}_\beta.$$

The topology generated by $\mathcal{S}$ as subbasis is called the **product topology**. In this topology, $\prod_{\alpha \in J} X_\alpha$ is called the **product space**.

The box topology on $\prod x_\alpha$ has as basis all sets of the form $\prod U_\alpha$ where U_α is open in X_α for each α. The product topology on $\prod x_\alpha$ has as basis all sets of the form $\prod U_\alpha$ where U_α is open in X_α for each α and $U_\alpha = X_\alpha$ for all but finitely many values of α.

Properties of the box/product topologies

- If $\mathcal{B}$ is a basis for the topology of X and $\mathcal{C}$ is a basis for the topology on Y, then the collection

  $$\mathcal{D} = \{B \times C \mid B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$$

  is a basis for the topology of $X \times Y$

3.2 The Metric Topology

Definition 22. A **metric** on a set X is a function

$$d : X \times X \to R$$

that has the following properties:

1. $d(x, y) \geq 0$ for all $x, y \in X$; equality holds if and only if $x = y$.
2. $d(x, y) = d(y, x)$ for all $x, y \in X$.

3. $d(x,y) + d(y,z) \geq d(x,z)$ for all $x, y, z \in X$ (triangle inequality).

Given $\epsilon > 0$, the set

$$B_d(x,\epsilon) = \{y | d(x,y) < \epsilon\}$$

of all points y whose distance from x is less than ϵ is called the **ϵ-ball centered at x**.

Definition 23. If d is a metric on a set X, then the collection of all ϵ-balls $B_d(x,\epsilon)$ for $x \in X$ and $\epsilon > 0$ is a basis for a topology on X, called the **metric topology** induced by d.
In other words, a set U is open in the metric topology induced by d if and only if for each $y \in U$, there is a $\delta > 0$ such that $B_d(y,\delta) \subset U$.

Definition 24. If X is a topological space, X is said to be **metrizable** if there exists a metric d on the set X that induces the topology of X. A **metric space** is a metrizable space X together with a metric d that gives the topology of X.

Definition 25. Let X be a metric space with metric d. A subset $A \subset X$ is **bounded** if there is some number M such that

$$d(a_1, a_2) \leq M$$

for every a_1, a_2 of A. if A is bounded and nonempty, the **diameter** of A is defined to be the number

$$\operatorname{diam} A = \sup\{d(a_1, a_2) \mid a_1, a_2 \in A\}$$

The boundedness of a set is not a topological property (example: every open interval in the real line is homeomorphic to the real line itself)

Definition 26. Given $\mathbf{x} = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$, we define the **square metric** ρ by the equation

$$\rho(\mathbf{x},\ \mathbf{y}) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}.$$

Definition 27. Given an index set J, and given points $\mathbf{x} = (x_\alpha)_{\alpha \in J}$ and $\mathbf{y} = (y_\alpha)_{\alpha \in J}$ of $\mathbb{R}^J$, we define a metric $\overline{\rho}$ on $\mathbb{R}^J$ by the equation

$$\overline{\rho}(\mathbf{x},\ \mathbf{y}) = \sup\{\overline{d}(x_\alpha, y_\alpha) | \alpha \in J\}$$

where $\overline{d}$ is the standard bounded metric on $\mathbb{R}$. $\overline{\rho}$ is called the **uniform metric** on $\mathbb{R}^J$, and the topology it induces is called the **uniform topology**. The uniform topology on $\mathbb{R}^J$ is finer than the product topology and coarser than the box topology, and strictly so if J is infinite.

3.3 The Quotient Topology

Definition 28. Let X and Y be topological spaces; let $p : X \to Y$ be a surjective map. p is said to be a **quotient map** if a subset $U \subset Y$ is open if and only if $p^{-1}(U)$ is open in X.

A subset $C \subset X$ is **saturated** with respect to p if C contains every set $p^{-1}(\{y\})$ that it intersects. Equivalently, C is saturated if it contains the *complete inverse image* of a subset of Y.

Definition 29. If X is a space and A is a set and if $p : X \to A$ is a surjective map, then there exists exactly one topology $\mathcal{T}$ on A relative to which p is a quotient map. It is called the **quotient topology** induced by p. This topology consists of the subsets $U \subset A$ such that $p^{-1}(U)$ is open in X.

Definition 30. Let X be a topological space and let X^* be a partition of X into disjoint subsets whose union is X. Let $p : X \to X^*$ be the surjective map that carries each point in X to the element of X^* (which will be a subset) containing it. In the quotient topology induced by p, the space X^* is called a **quotient space** of X.

Since X^* is a partition of X, each of its elements are equivalence classes of some equivalence relation. X^* is often called the identification space or decomposition space of the space X. A subset U of X^* is a collection of equivalence classes, and $p^{-1}(U)$ is the union of the equivalence classes belonging to U.

Why are quotient spaces defined this way?
This is actually a really elegant way to specify which points in a space you'd like to 'glue' together under a quotient map. Any elements in the same equivalence class in X^* will essentially be 'glued together' under a quotient map. Take the example given in the book of the homeomorphism from a quotient space of the unit disk to S^2. By defining an equivalence relation on the disk such that all points inside the disk in their own unique equivalence classes and the border of the disk is in just one equivalence, we can define a homeomorphism to S^2 where all the points along the border are glued to one point on the sphere and the rest are mapped to other points.

4 Connectedness and Compactness

4.1 Connectedness and Path-Connectedness

Definition 31. Let X be a topological space. A **separation** of X is a pair U, V of disjoint nonempty open sets of X whose union is X. X is **connected** if no such separation of it exists.

Definition 32. Let X be a space. A **path** from $x \in X$ to $y \in X$ is a continuous function $f : [0,1] \to X$ where $f(0) = x$ and $f(1) = y$. A **path-component** is an equivalence class of X wherein $x, y \in X$ are equivalent if there is a path between them. A space is **path-connected** if there exactly one path-component. Path connectedness is stronger than continuity.

What's the difference between connectedness and path-connectedness?
Path-connectedness is stronger than connectedness. If X can be separated into disjoint open sets U, V, then continuous paths from elements in U to elements in V cannot exist. Intuitively, I like to think of the difference as being that connectedness means the space 'as a whole' is 'together', while path-connectedness means that the space is 'uniform'. For example, the real line $\mathbb{R}$ with the K-topology defined in section 1.3 is an example of a topological space that is connected but not path-connected.

4.2 Compactness

Definition 33. A collection $\mathcal{A}$ of subsets of a space X **covers** X if the union of elements of $\mathcal{A}$ is equal to X. It's called an **open covering** of X if its elements are open subsets of X.

Definition 34. A space X is **compact** if every open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X.

Examples

- The closed interval $[0,1]$ in $\mathbb{R}$ is compact.
- The open interval $(0,1)$ in $\mathbb{R}$ is not compact.
- $\mathbb{R}$ is not compact.
- All spheres S^n in $\mathbb{R}^n$ are compact (except the infinite-dimensional sphere S^∞).

What does compactness actually mean?
To my understanding, the definition of compactness generalizes the notion of closed and bounded sets in euclidean space (this is useful when, say, determining if a function has. Another way I've heard compactness explained is that it measures whether a space is "big" or "small". That is, if a space can be completely "described" by a finite number of its subsets no matter how you cover it, then it is in some sense finite. This is perhaps made more clear when observing some of the properties of familiar non-compact spaces. For example, any open interval in $\mathbb{R}$ is homeomorphic to $\mathbb{R}$, any open disk in $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}^2$, and so on.

5 References

I followed James Munkres's wonderful book titled 'Topology' while taking these notes.