MA 2103 Assignment 05 1

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1. Ch. 7, Sec. 6, Problem 14 (page 358)

The given function is

$$f(x) = \begin{cases} 0, \text{ if } -\pi < x < 0 \\ x, \text{ if } 0 < x < \pi \end{cases}$$

The fourier expansion in sine-cosine terms are given as follows

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

Now,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi} xdx = \frac{1}{\pi}\frac{x^2}{2}\bigg|_0^{\pi} = \frac{\pi^2}{2\pi} = \frac{\pi}{2} = \frac{\pi}{2}$$

Now,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$
$$a_n = \frac{1}{\pi}\int_0^{\pi} x\cos nx dx$$
$$a_n = \frac{1}{\pi}\left[\frac{x}{n}\sin nx + \frac{1}{n^2}\cos nx\right]_0^{\pi}$$
$$a_n = \frac{1}{\pi n^2}\left[\cos nx - 1\right]$$
$$a_n = \begin{cases} 0, \text{when n is even} \\ \frac{-2}{n^2\pi}, \text{when n is odd} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$$
$$b_n = \frac{1}{\pi}\int_0^{\pi} x\sin nx dx$$
$$b_n = \frac{1}{\pi}\left[-\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx\right]_0^{\pi}$$
$$b_n = \frac{1}{\pi}\left[-\frac{\pi}{n}\cos n\pi + 0\right]$$
$$b_n = \begin{cases} -1/n, \text{when n is even} \\ 1/n, \text{when n is odd} \end{cases}$$

So we get,

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\sum_{\text{n is odd}} \frac{\cos nx}{n^2} + \sum_{\text{n is odd}} \frac{\sin nx}{n} - \sum_{\text{n is even}} \frac{\sin nx}{n}$$

By Dirichlet's Condition the series converges to 0 at x =0. Hence,

$$0 = \frac{\pi}{4} - \frac{2}{\pi} \sum_{\text{n is odd}} \frac{\cos(0)}{n^2}$$

$$\frac{\pi}{4} = \frac{2}{\pi} \sum_{\text{n is odd}} \frac{1}{n^2}$$

$$\sum_{\text{n is odd}} \frac{1}{n^2} = \frac{\pi^2}{8}$$

By Dirichlet's Condition the series converges to $\frac{\pi}{2}$ at x. $=\pi$.Hence

$$\frac{\pi}{2} = \frac{\pi}{4} - \frac{2}{\pi} \sum_{\text{n is odd}} \frac{\cos n\pi}{n^2}$$

$$\frac{\pi}{4} = \frac{2}{\pi} \sum_{\text{n is odd}} \frac{1}{n^2}$$

$$\sum_{\text{n is odd}} \frac{1}{n^2} = \frac{\pi^2}{8}$$

Hence we have showed/ Now to check what happens when x $=\frac{\pi}{2}$ By Dirichlet's Condition the series converges to $\frac{\pi}{2}$ at x $=\frac{\pi}{2}$.Hence we get

$$\frac{\pi}{2} = \frac{\pi}{4} - 0 + \sum_{\text{n is odd}} \frac{1}{n} \sin\frac{n\pi}{2} - \sum_{\text{n is even}} \frac{1}{n} \sin\frac{n\pi}{2}$$

Now the value of sin for even n is zero

$$\frac{\pi}{4} = \sum_{\text{n is odd}} \frac{1}{n} \sin\frac{n\pi}{2}$$

$$\sum_{\text{n is odd}} \frac{1}{n} = \frac{\pi}{4}$$

2. Ch. 7, Sec. 7, Problem 2 (corresponding to 5.2) (page 360)

The given function is

$$f(x) = \begin{cases} 0, \text{ if } -\pi < x < 0 \\ 1, \text{ if } 0 < x < \pi/2 \\ 0, \text{ if } \pi/2 < x < \pi \end{cases}$$

The fourier expansion in complex exponential terms is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-inx} \tag{1}$$

Now we will find the fouriers constants by the following expression

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

$$c_0 = \frac{1}{2\pi}\int_0^{\frac{\pi}{2}}(1)dx$$
$$c_0 = \frac{1}{2\pi}\left[x\right]_0^{\frac{\pi}{2}}$$
$$c_0 = \frac{1}{2\pi}\cdot\frac{\pi}{2}$$
$$c_0 = \frac{1}{4}$$

$$c_n = \frac{1}{2\pi}\int_0^{\frac{\pi}{2}}(1)e^{-inx}dx$$
$$= \frac{-1}{2\pi ni}e^{-inx}\Big|_0^{\frac{\pi}{2}}$$
$$= \frac{i}{2\pi n}\left[e^{-in\pi/2} - 1\right]$$
$$= \frac{i}{2\pi n}\left[\cos\frac{n\pi}{2} = i\sin\frac{n\pi}{2} - 1\right]$$
$$c_n = \begin{cases} \frac{-i}{n\pi}, \text{ when } n = \pm 2, \pm 6, \pm 10, ... \\ \frac{1-i}{2n\pi}, \text{ when } n = \pm 1, \pm 5, \pm 9, ... \\ \frac{-(1+i)}{2n\pi}, \text{ when } n = \pm 3, \pm 7, \pm 11, ... \\ 0, \text{ when } n = \pm 4, \pm 8, \pm 12, ... \end{cases}$$

So,

$$f(x) = \frac{1}{4} + \frac{1}{2\pi}[e^{ix}(1-i) + e^{-ix}(i+1)] - \frac{i}{2\pi}(e^{2ix} - e^{-2ix}) + \frac{1}{6\pi}[e^{3ix}(-1-i) + e^{-3ix}(i-1)] + \frac{1}{10\pi}[e^{5ix}(1-i) + e^{-5ix}(i+1)] + ...$$
$$f(x) = \frac{1}{4} + \frac{1}{2\pi}[e^{ix} - ie^{ix} + e^{-ix} + ie^{ix}] - \frac{i}{2\pi}[2i\sin 2x] + \frac{1}{6\pi}[-e^{i3x} - ie^{i3x} + e^{-i3x} - e^{-i3x}] + \frac{1}{10\pi}[e^{i5x} - ie^{i5x} + e^{-i5x} + ie^{-i5x}] + ...$$
$$f(x) = \frac{1}{4} + \frac{1}{\pi}[\cos x + \sin x] + \frac{1}{\pi}[\sin 2x] + \frac{1}{3\pi}[-cos3x + sin3x] + \frac{1}{5\pi}[cos5x + sin5x] + ...$$

Thus the final series is

$$f(x) = \frac{1}{4} + \frac{1}{\pi}\left[\cos x - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - ...\right] + \frac{1}{\pi}\left[\sin x + \sin 2x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + ...\right]$$

As we can see we get the same fourier series as provided in the solution

3. Ch. 7, Sec. 7, Problem 7 (corresponding to 5.7) (page 360)

The given function is

$$f(x) = \begin{cases} 0, \text{ if } -\pi < x < 0 \\ x, \text{ if } 0 < x < \pi \end{cases}$$

The fourier expansion in complex exponential terms is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-inx} \tag{2}$$

Now we will find the fouriers constants by the following expression

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$$

$$c_n = 0 + \frac{1}{2\pi}\int_{0}^{\pi} xe^{-inx}dx$$

$$c_0 = \frac{1}{2\pi}\int_0^{\pi} xdx = \frac{1}{2\pi}\frac{x^2}{2}\Big|_0^{\pi} = \frac{\pi^2}{4\pi} = \frac{\pi}{4}$$

$$c_0 = \frac{\pi}{4}$$

Now for n $\neq 0$

$$c_n = \frac{1}{2\pi}\int_0^{\pi} xe^{-inx}dx$$

$$= \frac{1}{2\pi}\left[-\frac{xe^{-inx}}{in} - \int_0^{\pi}\frac{e^{-inx}}{-in}dx\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{xie^{-inx}}{n} - \frac{ie^{-inx}}{-in^2}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{xie^{-inx}}{n} + \frac{e^{-inx}}{n^2}\right]_0^{\pi}$$

$$c_n = \frac{1}{2\pi}\left[\frac{\pi ie^{-in\pi}}{n} + \frac{e^{-in\pi}}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{1}{2\pi}\left[\frac{\pi i\cos n\pi}{n} + \frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right]$$

$$c_n = \begin{cases} i/2n, \text{ when n is even} \\ \frac{-(2+in\pi)}{2\pi n^2}, \text{ when n is odd} \end{cases}$$

So the series now is

$$f(x) = \frac{\pi}{4} + \frac{1}{2\pi}[(-2-i\pi)e^{ix} + (-2+i\pi)e^{-ix}] + \frac{i}{4}[e^{2ix} + e^{-2ix}] +$$
$$\frac{1}{2\pi 3^2}[(-2-i\pi)e^{3ix} + (-2+i\pi)e^{-3ix}] + \frac{i}{8}[e^{4ix} + e^{-4ix}] + ...$$

$$f(x) = \frac{\pi}{4} + \frac{1}{2\pi}[-2e^{ix} - i\pi e^{ix} + -2e^{3ix} + i\pi e^{-ix}] + \frac{i}{4}[2i\sin 2x] +$$
$$\frac{1}{2\pi 3^2}[-2e^{3ix} - i\pi e^{3ix} + -2e^{3ix} + i\pi e^{-3ix}] + \frac{i}{8}[2i\sin 4x] + ...$$

$$f(x) = \frac{\pi}{4} + \frac{1}{2\pi}[-4\cos x - 2\pi\sin x] - \frac{\sin 2x}{2} +$$
$$\frac{1}{2\pi 3^2}[-4\cos 3x - 6\pi\sin 3x] - \frac{\sin 4x}{4} + ...$$

Thus we get the following fourier series

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + ...\right] + \left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{+}...\right]$$

As we can see we get the same fourier series as provided in the solution.

4. Ch. 7, Sec. 7, Problem 11 (corresponding to 5.11) (page 360)

The given function is

$$f(x) = \begin{cases} 0, \text{ if } -\pi < x < 0 \\ \sin(x), \text{ if } 0 < x < \pi \end{cases}$$

The fourier expansion in complex exponential terms is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-inx} \tag{3}$$

Now we will find the fouriers constants by the following expression

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$$

$$c_n = 0 + \frac{1}{2\pi}\int_{0}^{\pi} xe^{-inx}dx$$

$$c_0 = \frac{1}{2\pi}\int_{0}^{\pi} \sin x dx = \frac{1}{2\pi} - \cos x\Big|_0^{\pi}$$

$$c_0 = \frac{-1}{2\pi}[\cos\pi - \cos 0] = \frac{-1}{2\pi}[-1-1]$$

$$c_0 = \frac{1}{\pi}$$

Now for n $\neq$ 0 and 1

$$c_n = \frac{1}{2\pi}\int_{0}^{\pi} \sin(x)e^{-inx}dx$$

$$c_n = \frac{1}{2\pi}\left[\frac{e^{-inx}}{1-n^2}(-in\sin x - \cos x)\right]_0^{\pi}$$

$$c_n = \frac{1}{2\pi(1-n^2)}[e^{-in\pi} + 1]$$

$$c_n = \frac{1}{2\pi(1-n^2)}[\cos n\pi + 1]$$

$$c_n = \begin{cases} 0, \text{when n is odd and n} \neq 1 \\ \frac{1}{\pi(1-n^2)}, \text{when n is even} \end{cases}$$

Now we need to find the coefficient for n = 1 i.e, c_1

$$c_1 = \frac{1}{2\pi}\int_0^{\pi} \sin(x)e^{-ix}dx$$
$$c_1 = \frac{1}{2\pi}\int_0^{\pi} \sin(x)[\cos x - i\sin x]dx$$
$$c_1 = \frac{1}{2\pi}\int_0^{\pi} \sin(x)\cos x dx - \frac{i}{2\pi}\int_0^{\pi}\sin^2 x dx$$
$$= \frac{1}{4\pi}\int_0^{\pi}\sin(2x)dx - \frac{i}{4\pi}\int_0^{\pi} 1-\cos 2x dx$$
$$= \frac{-1}{8\pi}[\cos\pi - \cos 0] - \frac{i}{4\pi}[x-(1/2)\sin 2x]_0^{\pi}$$
$$= 0 - \frac{i}{4\pi}[\pi - (1/2)\sin 2\pi]$$
$$= -\frac{i\pi}{4\pi}$$
$$= \frac{i}{4i}$$

Thus the Fourier series is found as

$$f(x) = \frac{1}{\pi} + \frac{i}{4i}(e^{ix} - e^{-ix}) + \sum_{-\infty(\text{for even n})}^{\infty} \frac{e^{inx}}{\pi(1-n^2)}$$
$$f(x) = \frac{1}{\pi} + \frac{i}{4i}(2i\sin x) + \frac{1}{\pi(1-2^2)}[e^{2ix}+e^{-2ix}] + \frac{1}{\pi(1-3^2)}[e^{3ix}+e^{-3ix}] + ...$$

Thus our fourier series is

$$f(x) = \frac{1}{\pi} + \frac{\sin x}{2i} - \frac{1}{\pi}\left[\frac{\cos 2x}{2^2-1} + \frac{\cos 3x}{3^2-1} + \frac{\cos 4x}{4^2-1} + ...\right]$$

As we can see we get the same fourier series as provided in the solution.

5. Ch. 7, Sec. 8, Problem 7 (corresponding to 5.7) (page 363)

The given function is

$$f(x) = \begin{cases} 0, \text{ if } -l < x < 0 \\ x, \text{ if } 0 < x < l \end{cases}$$

The fourier expansion in sine-cosine series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \tag{4}$$

Now we will find the fouriers constants as follows

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx$$

$$\begin{aligned} a_0 &= \frac{1}{l}\int_0^l x dx \\ &= \frac{1}{l}\left[\frac{x^2}{2}\right]_0^l \\ &= \frac{1}{l}\left[\frac{l^2}{2}\right] = \frac{l}{2} \end{aligned}$$

Now ,

$$\begin{aligned} a_n &= \frac{1}{l}\int_{-l}^l f(x)\cos\frac{n\pi x}{l}dx \\ a_n &= \frac{1}{l}\int_0^l x\cos\frac{n\pi x}{l}dx \\ a_n &= \frac{1}{l}\left[\frac{xl}{n\pi}\sin\frac{n\pi x}{l} + \frac{l^2}{(n\pi)^2}\cos\frac{n\pi x}{l}\right]_0^l \\ a_n &= \frac{1}{l}\left[\frac{l^2}{n\pi}\sin n\pi + \frac{l^2}{(n\pi)^2}\cos n\pi - \frac{l^2}{(n\pi)^2}\right] \\ a_n &= \frac{l}{(n\pi)^2}[\cos(n\pi) - 1] \end{aligned}$$

$$a_n = \begin{cases} \frac{-2l}{(n\pi)^2}, \text{ when n is odd} \\ 0, \text{ when n is even} \end{cases}$$

Similarly,

$$b_n = \frac{1}{l}\int_{-l}^l f(x)\sin\frac{n\pi x}{l}dx$$

Now $b_0 = 0$, Thus b_n

$$\begin{aligned} b_n &= \frac{1}{l}\int_0^l x\sin\frac{n\pi x}{l}dx \\ b_n &= \frac{1}{l}\left[-\frac{xl}{n\pi}\cos\frac{n\pi x}{l} + -\frac{l^2}{(n\pi)^2}\sin\frac{n\pi x}{l}\right]_0^l \\ b_n &= \frac{-l}{n\pi}[\cos n\pi] \\ b_n &= \begin{cases} l/n\pi, \text{ when n is odd} \\ -l/n\pi, \text{ when n is even} \end{cases} \end{aligned}$$

Thus the fourier series in sin-cosine is in

$$f(x) = \frac{l}{4} - \frac{2l}{\pi^2}\left[\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l} + ...\right] + \frac{l}{\pi}\left[\sin\frac{\pi x}{l} - \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} - ...\right]$$

The fourier expansion in complex exponential terms is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/l} \tag{5}$$

Now we will find the fouriers constants by the following expression

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-in\pi x/l}dx$$

$$c_n = 0 + \frac{1}{2l}\int_{0}^{l} xe^{-in\pi x/l}dx$$

$$c_0 = \frac{1}{2l}\int_0^l xdx = \frac{1}{2l}\frac{x^2}{2}\Big|_0^l = \frac{l^2}{4l} = \frac{l}{4}$$

$$c_0 = \frac{l}{4}$$

$$c_n = \frac{1}{2l}\int_0^l xe^{-in\pi x/l}dx$$

$$= \frac{1}{2l}\left[-\frac{xle^{-in\pi x/l}}{i\pi n} - \int_0^{\pi}\frac{le^{-in\pi x/l}}{-i\pi n}dx\right]_0^l$$

$$= \frac{1}{2l}\left[\frac{xlie^{-in\pi x/l}}{n\pi} - \frac{l^2ie^{-in\pi x/l}}{-i(n\pi)^2}\right]_0^l$$

$$= \frac{1}{2l}\left[\frac{xlie^{-inx\pi/l}}{n\pi} + \frac{l^2e^{-inx\pi/l}}{(n\pi)^2}\right]_0^l$$

$$c_n = \frac{1}{2l}\left[\frac{l^2ie^{-in\pi}}{n\pi} + \frac{l^2e^{-in\pi}}{(n\pi)^2} - \frac{l^2}{(n\pi)^2}\right]$$

$$= \frac{1}{2l}\left[\frac{l^2 i\cos n\pi}{n\pi} + \frac{l^2\cos n\pi}{(n\pi)^2} - \frac{l^2}{(n\pi)^2}\right]$$

$$c_n = \begin{cases} il/(2n\pi), & \text{when n is even} \\ \frac{-l(2+i\pi)}{2\pi^2n^2}, & \text{when n is odd} \end{cases}$$

So the series now is

$$f(x) = \frac{l}{4} + \frac{l}{2\pi^2}[(-2-i\pi)e^{ix\pi/l} + (-2+i\pi)e^{-ix\pi/l}] + \frac{il}{4}[e^{2ix\pi/l} + e^{-2ix\pi/l}] +$$
$$\frac{1}{2\pi^2 3^2}[(-2-i\pi)e^{3ix\pi/l} + (-2+i\pi)e^{-3ix\pi/l}] + \frac{il}{8}[e^{4ix\pi/l} + e^{-4ix\pi/l}] + \ldots$$

$$f(x) = \frac{\pi}{4} + \frac{1}{2\pi^2}[-2e^{ix\pi/l} - i\pi e^{ix\pi/l} + -2e^{ix\pi/l} + i\pi e^{-ix\pi/l}] + \frac{il}{4\pi}[2i\sin\frac{2x\pi}{l}] +$$
$$\frac{l}{2\pi^2 3^2}[-2e^{3ix\pi/l} - i\pi e^{3ix\pi/l} - 2e^{3ix\pi/l} + i\pi e^{-3ix\pi/l}] + \frac{il}{8\pi}[2i\sin\frac{8x\pi}{l}] + \ldots$$

$$f(x) = \frac{\pi}{4} + \frac{l}{2\pi^2}[-4\cos(\frac{x\pi}{l}) - 2\pi\sin(\frac{x\pi}{l})] - \frac{l}{2\pi}\sin\frac{2x\pi}{l} +$$
$$\frac{l}{2\pi^2 3^2}[-4\cos\frac{3x\pi}{l} - 6\pi\sin\frac{3x\pi}{l}] - \frac{l}{4\pi}\sin\frac{4x\pi}{l} + \ldots$$

$$f(x) = \frac{\pi}{4} - \frac{2l}{\pi^2}\left[\cos\frac{x\pi}{l} + \frac{1}{3^2}\cos\frac{3x\pi}{l} + \frac{1}{5^2}\cos\frac{5x\pi}{l} + \cdots\right] + \frac{l}{\pi}\left[\sin\frac{x\pi}{l} - \frac{1}{2}\sin\frac{2x\pi}{l} + \frac{1}{3}\sin\frac{3x\pi}{l} - \cdots\right]$$

We can see both sine-cosine and complex exponential expansions are same.