MA 2103 Assignment 02 1

Abhay Kshirsagar, 19MS172 ()

September 13, 2020

1. Problem 4 Pg. 632

The temperature Distribution u(x,t) is given by $u(x,t) = X(x)T(t)$ which satisfies the Heat/Diffusion Equation gives as

$$\nabla^2 u(x,t) = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}$$

The possible solutions for the above PDE is

$$u(x,t) = \begin{cases} e^{-k^2\alpha^2 t}\sin kx, \\ e^{-k^2\alpha^2 t}\cos kx \end{cases} \tag{1}$$

The given boundary conditions at t=0 are given as

$$\begin{cases} 0, & x \in (0,5) \\ 20, & x \in (5,10) \end{cases} \tag{2}$$

It is given that the surfaces are kept at 0. i.e u(0,t) = u(10,t) = 0
At x = 0, u = 0 , thus we can discard all the $\cos kx$ solutions. At x = 10, u = 0 , thus $\sin 10k = 0$

$$\begin{aligned} \sin 10k &= 0 \\ \text{we get,}10k &= n\pi \\ k &= \frac{n\pi}{10} \end{aligned}$$

Now our general solution is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{10} e^{-[(n\pi\alpha)/10]^2 t} \tag{3}$$

We will consider the initial condition(t = 0) to find the Fourier coefficient
u(x, t = 0) = u_0

$$u_0 = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{10} = \begin{cases} 0, & x \in (0,5) \\ 20, & x \in (5,10) \end{cases} \tag{4}$$

We will use the Fourier trick to find the coefficient

$$\int_0^5 (0)\sin\frac{n\pi x}{10}dx + \int_5^{10} 20\sin\frac{n\pi x}{10}dx = \int_0^{10}\sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{10}\sin\frac{m\pi x}{10}dx$$

$$\begin{aligned} \frac{-200}{n\pi}\cos\frac{n\pi x}{10}\bigg|_5^{10} &= a_n\frac{10}{2} = 5a_n \\ a_n &= \frac{40}{n\pi}\left[\cos n\pi - \cos n\frac{\pi}{2}\right] \end{aligned}$$

$$a_n = \begin{cases} \frac{0}{,} & n = 4k \\ \frac{40}{n\pi}, & n = 2k+1 \\ \frac{-80}{n\pi}, & n = 4k+2 \end{cases}$$

Now our final general solution is given as follows:

$$u(x,t) = \frac{40}{\pi} \sum_{\text{n is odd}}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{10} e^{-[(n\pi\alpha)/10]^2 t} - \frac{80}{\pi} \sum_{\text{n= 2k + 2}}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{10} e^{-[(n\pi\alpha)/10]^2 t}$$

where, k ∈ ℕ

2. Problem 6 Pg. 632

The given Boundary conditions is given as

$$u(0,0) = 20 \qquad u(L,0) = 150 \tag{5}$$
$$u(0,t) = 20 \qquad u(l,t) = 50 \tag{6}$$

The temperature distribution pf the bar of length L is given by Eq(3.15) given as follows

$$u(x,t) = \frac{200}{\pi} \sum_{\text{n = 1}}^{\infty} \frac{(-1)^{n-1}}{n} e^{-[(n\pi\alpha)/L]^2 t} \sin n\pi x \tag{7}$$

This equation satisfies the Laplace's condition $\nabla u = 0$ Hence we get u = ax + b. We will find the coefficients a and b at initial steady state distribution.
Thus at x = 0, u_0 = 20, we get b = 20. Now at x = L, u_0 = 150, we get $\frac{130}{L}$.

$$u_0 = \frac{130}{L} x + 20 \tag{8}$$

The final steady state distribution is given by u_f. Thus at x = 0, u_f = 20, we get b = 20. Now at x = L, u_f = 50, we get $\frac{130}{L}$.

$$u_f = \frac{30}{L} x + 20 \tag{9}$$

The temperature distribution for both ends kept at 0 is given by

$$u(x,t) = \sum_{\text{n = 1}}^{\infty} a_n e^{-[(n\pi\alpha)/L]^2 t} \sin n\pi x \tag{10}$$

Now if we add a linear question to a Fourier series it leaves the Fourier series unchanged. Thus,

$$u(x,t) = u_f + \sum_{\text{n = 1}}^{\infty} a_n e^{-(\frac{n\pi\alpha}{L})^2 t} \sin n\pi x \tag{11}$$

At t = 0,

$$u(x,0) = u_f + \sum_{\text{n = 1}}^{\infty} a_n e^{-(\frac{n\pi\alpha}{L})^2 t} \sin n\pi x$$
$$\frac{130}{L} x + 20 = \frac{30}{L} x + 20 + \sum_{\text{n = 1}}^{\infty} a_n \sin n\pi x$$
$$\sum_{\text{n = 1}}^{\infty} a_n \sin n\pi x = \frac{100}{L} x$$

Now we will find the Fourier coefficient a_n.

$$a_n = \frac{2}{L}\int_0^L \frac{100x}{L}\sin\frac{n\pi x}{L}dx$$
$$a_n = \frac{200}{L^2}\left[\frac{-Lx}{n\pi}\cos\frac{n\pi x}{L} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}\right]$$
$$a_n = \frac{200}{n\pi}(-1)^{n-1}$$

Thus we get the final solution as follows

$$u(x,t) = u_f + \frac{200}{\pi}\sum_{\text{n = 1}}^{\infty}(-1)^{n-1}e^{-(\frac{n\pi\alpha}{L})^2 t}\sin n\pi x \tag{12}$$

3. Problem 7 Pg. 633

Both the ends of the bar of length L are insulated thus by Neumann condition.

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0$$

It is given that Initially (t = 0) the temperature distribution is given as by

$$u(x,0) = x \tag{13}$$

Our required heat equation is given as u(x,t) = F(x)T(t) which satisfies the heat/ diffusion equation. The solutions are given as follows

$$u(x,t) = \begin{cases} e^{-k^2\alpha^2 t}\sin kx, \\ e^{-k^2\alpha^2 t}\cos kx \end{cases} \tag{14}$$

As $\frac{\partial u}{\partial x} = 0$ at x = 0. Thus we can discard the $\sin kx$ solution. As $\frac{\partial u}{\partial x} = 0$ at x = L we get $\frac{\partial \cos kx}{\partial x}$ = sin(kL) = 0, thus we get $k = \frac{n\pi}{L}$

The solution satisfies the diffusion equation given below now putting the value u(x,t) = F(x)T(t)

$$\nabla^2 u(x,t) = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}$$
$$\frac{1}{F(x)}\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2 T(x)}\frac{\partial u}{\partial t} = 0$$
$$\frac{\partial^2 F(x)}{\partial x^2} = -k^2 \qquad \frac{\partial T(t)}{\partial t} = 0$$

For k = 0

$$\frac{\partial^2 F(x)}{\partial x^2} = 0 \qquad \frac{\partial u}{\partial t} = 0$$

Lets solve the left PDE we get the solution

$F(x) = ax + b$Now as per Neumann condition

$$\left.\frac{\partial F(x)}{\partial x}\right|_{x=0} = 0 \qquad \left.\frac{\partial T(t)}{\partial t}\right|_{x=0} = 0$$

Thus we get a = 0 and T(t) = d = constant. Thus we get $u(x,t)|_{k=0}$ = b * d = constant. Thus the value for k =0 of u(x,t) is a constant which we will be added later in the heat distribution
For k > 0, we got before $k = \frac{n\pi}{L}, n \in \mathbb{N}$. To include the constant for k = 0 in the general solution is given by

$$u(x,t) = \sum_{n=0}^{\infty} b_n e^{[(n\pi\alpha)/L]^2 t} \cos\frac{n\pi x}{L} \tag{15}$$

For initial condition t = 0, u(x, 0) = u_0 = x

$$u_0 = \sum_{\text{n = 0}}^{\infty} b_n \cos\frac{n\pi x}{L}$$
$$u_0 = b_0 + \sum_{\text{n = 1}}^{\infty} a_n \cos\frac{n\pi x}{L}$$

Lets find b_0 and b_n as follows,

$$b_0 = \frac{1}{L}\int_0^L u(x,0)dx$$
$$b_0 = \frac{1}{L}\int_0^L xdx$$
$$b_0 = \frac{L}{2}$$

$$b_n = \frac{2}{L}\int_0^L x\cos\frac{n\pi x}{L}dx$$
$$b_n = \frac{2}{L}\frac{L}{n\pi}\left[x\sin\frac{n\pi x}{L}\Big|_0^L - \int_0^L \cos\frac{n\pi x}{L}dx\right]$$
$$b_n = \frac{2L}{n^2\pi^2}\left[\cos n\pi - 1\right]$$
$$b_n = \frac{2L}{n^2\pi^2}\left[(-1)^n - 1\right]$$

If n is odd $b_n = \frac{-4L}{n^2\pi^2}$
If n is even $b_n = 0$
Thus the general solution is

$$u(x,t) = \frac{L}{2} - \frac{4L}{\pi^2}\sum_{\text{odd n}}^{\infty} b_n e^{-[(n\pi\alpha)/L]^2 t}\cos\frac{n\pi x}{L} \qquad n \in \mathbb{N}$$

4. Problem 9 Pg. 633

The bar is insulated at x = 0, hence we can apply Neumann condition $\frac{\partial u}{\partial x}\big|_{x=0} = 0$. The heat distribution is given by u(x,t) = F(x)T(t) which satisfies the heat/diffusion equation. The solution of the distribution is given as

$$u(x,t) = \begin{cases} e^{-k^2\alpha^2 t}\sin kx, \\ e^{-k^2\alpha^2 t}\cos kx \end{cases} \tag{16}$$

Now applying the Neumann condition $\frac{\partial u}{\partial x}\Big|_{x=0} = 0$, we can see that the sin(kx) terms can be eliminated. At x = 2, u = 100.

$$u = 100 = e^{-k^2\alpha^2 t}\cos 2k$$

Thus we will have to add 100 to u(x,t).

$$u(x,t) = 100 + e^{-k^2\alpha^2 t}\cos kx$$

Now, $\cos 2k = 0$, i.e $k = \frac{2n-1}{2}\frac{\pi}{2} = \frac{2n-1}{4}\pi$ The general solution is given as follows.

$$u(x,t) = 100 + \sum_{n=1}^{\infty} b_n e^{-[(2n-1)\pi\alpha/4L]^2 t}\cos\frac{2n-1}{4}\frac{\pi x}{L}$$

Initially(t=0) u(x,0) = 0

$$-100 = \sum_{n=1}^{\infty} b_n \cos\frac{2n-1}{4}\frac{\pi x}{L}$$

Lets find the fourier coefficient b_n.

$$b_n = \frac{2}{2}(-100)\int_0^2 \cos\frac{2n-1}{4}\pi x dx$$

$$b_n = \frac{-400}{(2n-1)\pi}\sin\frac{2n-1}{4}\pi x\Bigg|_0^2$$

$$b_n = \frac{-400(-1)^{n+1}}{(2n-1)\pi}$$

For convenience we will fit our summation from n = 0 to $n = \infty$. This converts our $k = \frac{2n-1}{4}$ to $k = \frac{2n+1}{4}$ and similarly $b_n = \frac{-400(-1)^n}{(2n+1)\pi}$.Thus the final solution we obtain is.

$$u(x,t) = 100 - \frac{400}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}e^{-[(2n+1)\pi\alpha/4L]^2 t}\cos\frac{2n+1}{4}\frac{\pi x}{L}$$

5. Problem 12 Pg. 633

Given : $\Psi(x,0) = \sin^2 \pi x$ on the interval (0, 1).
The Schrödinger's equation is given by

$$-\frac{\hbar^2}{2m\Psi}\nabla^2\Psi + V = i\hbar\frac{1}{T}\frac{\partial T}{\partial t} \tag{17}$$

For a particle in a box problem we take V(x) = 0 for x $\in (0,1)$ Thus we get

$$-\frac{\hbar^2}{2m\Psi}\nabla^2\Psi = i\hbar\frac{1}{T}\frac{\partial T}{\partial t} \tag{18}$$

By solving the Schrödinger's equation we get the following solution

$$\Psi = \begin{cases} e^{-[iEt]/\hbar} \sin kx \\ e^{-[iEt]/\hbar} \cos kx \end{cases} \tag{19}$$

where k = $\frac{\sqrt{2mE}}{\hbar}$
At x = 0, $\Psi = 0$ so we can discard all the cos(kx) solutions.
At x = L, $\Psi = 0$ sin(k) = 0, hence we get k = $n\pi = \frac{\sqrt{2mE}}{\hbar}$, which gives us $E_n = \frac{\hbar^2 n^2 \pi^2}{2m}$. Now our general solution is ,

$$\Psi(x,t) = \sum_{n=1}^{\infty} a_n \sin{(n\pi x)} e^{-[iE_n t]/\hbar}$$

At t = 0,

$$\Psi(x, t=0) = \sum_{n=1}^{\infty} a_n \sin{(n\pi x)} = \sin^2(\pi x)$$

We will now find the Fourier coefficients as follows,

$$a_n = 2\int_0^1 \sin^2{(n\pi)} \sin{(n\pi x)} dx$$
$$a_n = 2\int_0^1 \frac{[1-\cos{(2n\pi)}]}{2} \sin{(n\pi x)} dx$$
$$a_n = \int_0^1 \sin{(n\pi x)} dx - \int_0^1 \cos{(2n\pi)} \sin{(n\pi x)} dx$$
$$a_n = \frac{1}{n\pi}[1-\cos n\pi] + \frac{1}{2}\left[\frac{\cos(n+2)\pi}{(n+2)\pi} - \frac{1}{n+2} + \frac{\cos(n-2)\pi}{(n-2)\pi} - \frac{1}{n+2}\right]$$

$$a_n = \begin{cases} 0, & \text{for even n} \\ \frac{8}{n\pi(4-n^2)}, & \text{for odd n} \end{cases} \tag{20}$$

Thus the general solution is

$$u(x,t) = \frac{8}{\pi} \sum_{Oddn}^{\infty} \frac{\sin(n\pi x)}{n(4-n^2)} e^{-[iE_n t]/\hbar} \qquad \text{where } E_n = \frac{n^2\pi^2\hbar^2}{2m}$$