

A Appendix

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A.1 Risk in perspective

Background information

- A *bond* is an instrument of indebtedness. The issuer owes the bond holder a debt and is obliged to pay at *maturity T* the principal and a *coupon* (interest; typically paid at fixed time points).
- *Netting* refers to the compensation of long versus short positions on the same underlying.
- A *derivative* is a financial instrument *derived from an underlying asset*, e.g. stocks, bonds, commodities, currencies, interest rates etc. Examples:
 - ▶ *Options* (*right, but not the obligation*, to buy (*call*) or sell (*put*) an asset at an agreed-upon price (the *strike price K*) during a predetermined period (*American*) or date (*exercise date T* ; *European*);
 - ▶ *Futures* (*obligation* for the buyer (seller) to purchase (sell) an asset at a predetermined date and price);

- ▶ *Swaps* (any exchange of an asset for another to change the maturity (e.g. of a bond) or because investment objectives have changed; include currency swaps, interest rate swaps).
- A *credit default swap (CDS)* is a credit derivative which allows the (protection) buyer (who pays premiums) to transfer credit risk inherent in a reference entity to a seller (investor; pays in case of default).
- A *CDS spread* is the annual amount the protection buyer must pay the protection seller over $[0, T]$, expressed as a fraction (often in 1 *basis point* = 0.01%) of the notional amount.
- The *Fundamental Theorems of Asset Pricing*:
 - ▶ A (model for) a market is *arbitrage free* if and only if there exists a risk-neutral probability measure Q equivalent to \mathbb{P} ;
 - ▶ A market is *complete* (i.e. every contingent claim can be replicated) if and only if Q is unique.

QRM beyond finance

- Some of the earliest applications of QRM are to be found in the **manufacturing industry**, where similar concepts and tools exist under names like **reliability** or **total quality control**. Industrial companies have recognized the **risks associated with bringing faulty products to the market**.
- QRM techniques have been adopted in the **transport and energy industries** (cost of storage and transport of electricity).
- There is an interest in the **transfer of risks between industries**; this process is known as **alternative risk transfer (ART)**, e.g. the risk transfer between the **insurance and banking industries**.
- QRM methodology also applies to **individuals**, e.g. via the **risk of unemployment**, **depreciation in the housing market** or the investment in the **education of children**.

A.2 Basics concepts in risk management

Background information

- A *balance sheet* is a financial statement showing *assets* (investments) and *liabilities* (obligations; show how funds have been raised)
- (X_t) is a (discrete) *martingale* with respect to the filtration (\mathcal{F}_t) if
 - ▶ $X_t \in \mathcal{F}_t$ for all $t \in \mathbb{N}_0$ (*adapted*);
 - ▶ $\mathbb{E}X_t < \infty$ for all $t \in \mathbb{N}_0$;
 - ▶ $\mathbb{E}(X_{t+1} | \mathcal{F}_t) = X_t$ for all $t \in \mathbb{N}_0$.

Physical (\mathbb{P}) vs risk-neutral (\mathbb{Q}) measure: An example

- Consider a defaultable bond with principal 1 and maturity $T = 1y$. In case of a default (real world probability $p = 0.01$), the recovery rate is $R = 60\%$. The risk-free interest rate is $r = 0.05$. Moreover, assume the bond's current price to be $V_0 = 0.941$ ($t = 0$).
- The **expected discounted value** of the bond is

$$\frac{1}{1+r}(1 \cdot (1-p) + R \cdot p) = \frac{1}{1.05}(0.99 + 0.6p) = 0.949$$

which is $> V_0$ since investors demand a **premium** for bearing the bond's **default risk**.

- Here, \mathbb{Q} is determined by specifying a q such that

$$\frac{1}{1+r}(1 \cdot (1-q) + R \cdot q) = V_0.$$

This implies $q = 0.03$ which is greater than $p = 0.01$; the larger value reflects the risk premium.

Expected shortfall under continuity

Elicitability explained in words

We follow Kou and Peng (2014, Sections 1 and 2.2) and McNeil et al. (2005, Chapter 9).

- Computing a (one-period ahead) risk measure $\varrho(L) =: \varrho(F_L)$ is a point forecasting problem because F_L is unknown and one has to find an estimate \hat{F}_L of it and forecast the unknown true $\varrho(F_L)$ via the point forecast $\varrho(\hat{F}_L)$.
- As different \hat{F}_L can be used to forecast the risk measure, it is desirable to be able to evaluate which of them gives a better point forecast.
- Suppose we want to forecast L (or F_L) by a point y . The *forecasting error* is

$$\mathbb{E}(S(y, L)) = \int_{\mathbb{R}} S(y, l) dF_L(l),$$

where $S(y, l)$ is a *scoring* (i.e. forecasting objective) *function*.

- Two point forecasting methods can be compared via their forecasting errors. For a given S , the **optimal point forecast** is

$$\varrho^*(F_L) = \operatorname{arginf}_y \mathbb{E}(S(y, L)) \quad (\text{minimizing the forecast error}).$$

For example, for $S(y, l) = (y - l)^2$ and $S(y, l) = |y - l|$, the optimal point forecasts are the mean and median of F_L , respectively.

- Elicitable risk measures** (or: statistical functionals) **are risk measures ϱ which minimize $\mathbb{E}(S(y, L))$ of some scoring function S** ; hence that S can be used to compare different point forecasting procedures for ϱ (“the smaller the forecasting error, the better” makes sense).
- If ϱ is not elicitable, one cannot find such an S and thus the minimization of the forecasting error does not yield the true value $\varrho(F_L)$ for any S .** Hence, for two competing point forecasts of $\varrho(F_L)$, one cannot tell which performs the best by comparing their forecasting error, no matter what S is used.

The (nonparametric) bootstrap

- Suppose $X_1, \dots, X_n \stackrel{\text{ind.}}{\sim} F$ (F unknown) and we are interested in estimating $\theta = \theta(F)$. If we could at least sample from F , we would estimate θ by $\hat{\theta}_n = \theta(\hat{F}_n)$. But even then, we would only get a point estimate $\hat{\theta}_n$ of θ . How do we get a $(1 - \beta)$ -confidence interval for θ ? How $\text{var}(\hat{\theta}_n)$? How the distribution of $\hat{\theta}_n$?
- The (nonparametric) bootstrap treats \hat{F}_n as the true df F (first approximation) and samples from that (i.e. resamples X_1, \dots, X_n) to approximate the distribution of $\hat{\theta}_n$ (second approximation).
- Advantages: Applicable if F or the df of θ is unknown; applicable if n is small (unlike the CLT); applicable if F is skewed (CIs based on the CLT are always centered around the sample mean); easy to implement.

Algorithm A.1 (Nonparametric bootstrap)

- 1) Fix a large $B \in \mathbb{N}$.

2) For $b \in \{1, \dots, B\}$, do:

2.1) Randomly sample $X_{b,1}, \dots, X_{b,n}$ from X_1, \dots, X_n with replacement.

2.2) Compute the bootstrap estimator $\hat{\theta}_{b,n} = \theta(\hat{F}_{b,n})$ where $\hat{F}_{b,n}(x) = \frac{1}{n} \sum_{i=1}^n I_{\{X_{b,i} \leq x\}}$ (informally: compute $\hat{\theta}_{b,n}$ from $X_{b,1}, \dots, X_{b,n}$).

3) Use the *bootstrap sample* $\hat{\theta}_{b,n}$, $b \in \{1, \dots, B\}$, to approximate the distribution of $\hat{\theta}_n$.

Examples for Step 3):

- $\hat{\mu}_{B,n} = \frac{1}{B} \sum_{b=1}^B \hat{\theta}_{b,n} \approx \mathbb{E}(\hat{\theta}_n)$; $\hat{\sigma}_{B,n}^2 = \frac{1}{B-1} \sum_{b=1}^B (\hat{\theta}_{b,n} - \hat{\mu}_{B,n})^2 \approx \text{var}(\hat{\theta}_n)$; the empirical df based on the $\hat{\theta}_{b,n}$'s approximates the df of $\hat{\theta}_n$.
- An *bootstrapped $(1 - \beta)$ -confidence interval* for θ based on $\hat{\theta}_{(1),n} \leq \dots \leq \hat{\theta}_{(B),n}$ is

$$\left[\hat{\theta}_{(\max\{\lfloor \frac{\beta}{2} B \rfloor, 1\}),n}, \hat{\theta}_{(\lceil (1-\frac{\beta}{2}) B \rceil),n} \right].$$

A.3 Empirical properties of financial data

Non-normality and heavy tails

Justification for P-P and Q-Q plots:

- 1) Glivenko–Cantelli: $\sup_{x \in \mathbb{R}} |\hat{F}_n(x) - F(x)| \xrightarrow[n \uparrow \infty]{\text{a.s.}} 0$
- 2) $\hat{F}_n(x) \xrightarrow[n \rightarrow \infty]{} F(x) \quad \forall x \in C(F) \Leftrightarrow \hat{F}_n^{\leftarrow}(u) \xrightarrow[n \rightarrow \infty]{} F^{\leftarrow}(u) \quad \forall u \in C(F^{\leftarrow});$
see van der Vaart (2000, Lemma 21.2)

By 1), the first (and thus the 2nd) part of 2) holds. Hence, for the true underlying F , $x_{(i)} = \hat{F}_n^{\leftarrow}(i/n) \approx \hat{F}_n^{\leftarrow}(p_i) \underset{2)}{\approx} F^{\leftarrow}(p_i)$ (a justification for both P-P and Q-Q plots).

A.4 Financial time series

Conditional expectations

Definition A.2 (Conditional expectation, conditional probability)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathbf{X} \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ – i.e. $\mathbf{X} : \Omega \rightarrow \mathbb{R}^d$, \mathbf{X} is \mathcal{F} -measurable (i.e. $\mathbf{X}^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{B}(\mathbb{R}^d)$) and $\mathbb{E}|\mathbf{X}| < \infty$ – and $\mathcal{G} \subseteq \mathcal{F}$ be a σ -algebra. Then any rv \mathbf{Y} such that

- 1) $\mathbf{Y} \in \mathcal{G}$ (\mathbf{Y} is \mathcal{G} -measurable);
 - 2) $\mathbb{E}|\mathbf{Y}| < \infty$; and
 - 3) $\mathbb{E}(\mathbf{Y} I_G) = \int_G \mathbf{Y} d\mathbb{P} = \int_G \mathbf{X} d\mathbb{P} = \mathbb{E}(\mathbf{X} I_G)$ for all $G \in \mathcal{G}$
- is called *conditional expectation of \mathbf{X} given \mathcal{G}* and denoted by $\mathbb{E}(\mathbf{X} | \mathcal{G})$.
 $\mathbb{P}(A | \mathcal{G}) = \mathbb{E}(I_A | \mathcal{G})$ is called *conditional probability of A given \mathcal{G}* .

The following property of conditional expectations is used frequently and known as *tower property*.

Lemma A.3 (Tower property; the smallest σ -algebra remains)

If $\mathcal{G} \subseteq \mathcal{F}$, then $\mathbb{E}(\mathbb{E}(X | \mathcal{G}) | \mathcal{F}) = \mathbb{E}(X | \mathcal{G}) = \mathbb{E}(\mathbb{E}(X | \mathcal{F}) | \mathcal{G})$.

Idea of proof. Let $G \in \mathcal{G} \subseteq \mathcal{F}$. Applying Definition A.2 Part 3) to $\mathbb{E}(\mathbb{E}(X | \mathcal{G}) | \mathcal{F})$ and then to $\mathbb{E}(X | \mathcal{G})$ implies that $\mathbb{E}(\mathbb{E}[\mathbb{E}[X | \mathcal{G}] | \mathcal{F}]I_G) = \mathbb{E}(\mathbb{E}[X | \mathcal{G}]I_G) = \mathbb{E}(XI_G)$. \square

On partial autocorrelation in stationary time series

For introducing it, we need some tools.

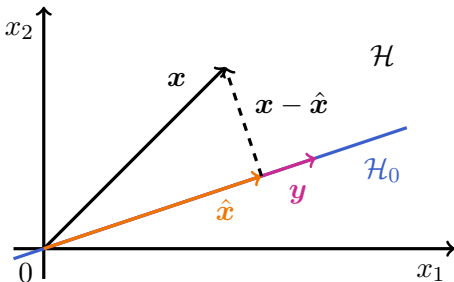
- *Hilbert's Projection Theorem* (see Brockwell and Davis (1991, p. 51)):
If \mathcal{H}_0 is a closed subspace of the Hilbert space \mathcal{H} and $x \in \mathcal{H}$, then:
 - i) There exists a unique $\hat{x} \in \mathcal{H}_0 : \|x - \hat{x}\| = \inf_{y \in \mathcal{H}_0} \|x - y\|$;
 - ii) $\hat{x} \in \mathcal{H}_0$, $\|x - \hat{x}\| = \inf_{y \in \mathcal{H}_0} \|x - y\|$ if and only if $\hat{x} \in \mathcal{H}_0$, $x - \hat{x} \in \mathcal{H}_0^\perp = \{x \in \mathcal{H} : \langle x, y \rangle = 0 \text{ for all } y \in \mathcal{H}_0\}$.

Note:

- ▶ \hat{x} is the (orthogonal) projection of x onto \mathcal{H}_0 , denoted by $P_{\mathcal{H}_0}x$.
- ▶ $\hat{x} = P_{\mathcal{H}_0}x$ is the unique element: $\langle x - \hat{x}, y \rangle = 0 \ \forall y \in \mathcal{H}_0$ (prediction equations; $P_{\mathcal{H}_0}x$ is the best approximation/prediction of x in \mathcal{H}_0).

Example A.4

$$x \in \mathcal{H} = \mathbb{R}^2, \mathcal{H}_0 = \text{span}\{y\}$$



- **Yule–Walker equations.** Let X_1, \dots, X_{n-1}, X_n be elements of a stationary time series $(X_t)_{t \in \mathbb{Z}}$ with $\mu(t) = 0$, $t \in \mathbb{Z}$. Suppose we would like to find $\hat{X}_n = \sum_{k=1}^{n-1} \phi_{n-1,k} X_{n-k}$ such that

$$\mathbb{E}((X_n - \hat{X}_n)^2) \rightarrow \min_{(\phi_{n-1,k})_{k=1}^{n-1}}.$$

$\mathcal{H} = L^2(\Omega, \mathcal{F}, \mathbb{P})$ is a Hilbert space with $\langle X, Y \rangle = \mathbb{E}(XY)$ and $\mathcal{H}_{n-1} = \text{span}\{X_1, \dots, X_{n-1}\} = \{\sum_{k=1}^{n-1} \alpha_k X_{n-k} : \alpha_1, \dots, \alpha_{n-1} \in \mathbb{R}\}$ is a subspace. Therefore, $\hat{X}_n = P_{\mathcal{H}_{n-1}} X_n$ satisfies the prediction equations

$$\begin{aligned} \langle X_n - \hat{X}_n, Y \rangle &= 0, \quad \forall Y \in \mathcal{H}_{n-1} \\ \Leftrightarrow \langle X_n - \hat{X}_n, \sum_{k=1}^{n-1} \alpha_k X_{n-k} \rangle &= 0, \quad \forall \alpha_1, \dots, \alpha_{n-1} \in \mathbb{R} \\ &= \underbrace{\sum_{k=1}^{n-1} \alpha_k \langle X_n - \hat{X}_n, X_{n-k} \rangle} \end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow \underbrace{\langle X_n - \hat{X}_n, X_l \rangle}_{=0} = 0, \quad \forall l \in \{1, \dots, n-1\} \\
&= \mathbb{E}((X_n - \sum_{k=1}^{n-1} \phi_{n-1,k} X_{n-k}) X_l) \\
&= \mathbb{E}(X_n X_l) - \sum_{k=1}^{n-1} \phi_{n-1,k} \mathbb{E}(X_{n-k} X_l) \\
&\Leftrightarrow \gamma(n-l) = \sum_{k=1}^{n-1} \gamma(n-k-l) \phi_{n-1,k} \\
&\stackrel{\text{station.}}{\Leftrightarrow} \gamma(h) = \sum_{k=1}^{n-1} \gamma(h-k) \phi_{n-1,k}, \quad \forall h \in \{1, \dots, n-1\} \\
&\Leftrightarrow \Gamma_{n-1} \phi_{n-1} = \gamma_{n-1}, \quad (\text{Yule-Walker equations})
\end{aligned}$$

where

$$\begin{aligned}
\phi_{n-1} &= (\phi_{n-1,1}, \dots, \phi_{n-1,n-1}), \\
\gamma_{n-1} &= (\gamma(1), \dots, \gamma(n-1)), \\
\Gamma_{n-1} &= (\gamma(|i-j|))_{i,j=1}^{n-1}.
\end{aligned}$$

Hilbert's Projection Theorem ii) \Rightarrow there exists at least one solution ϕ_{n-1} and all of them lead to the same \hat{X}_n (unique by i)). If Γ_{n-1}

is regular (invertible), ϕ_{n-1} is unique. This holds, e.g. if $\gamma(0) > 0$, $\gamma(h) \rightarrow 0$ ($h \rightarrow \infty$); see Brockwell and Davis (1991, p. 167).

- ϕ_n can be computed with the *Durbin–Levinson algorithm*: Let $(X_t)_{t \in \mathbb{Z}}$ be stationary with $\mu(t) = 0$, $t \in \mathbb{Z}$, $\gamma(0) > 0$, $\gamma(h) \rightarrow 0$ ($h \rightarrow \infty$). Then, for all $n \in \mathbb{N}$,

$$\begin{aligned}\phi_{n,n} &= \frac{\gamma(n) - \sum_{k=1}^{n-1} \gamma(n-k)\phi_{n-1,k}}{(*) \gamma(0) - \sum_{k=1}^{n-1} \gamma(n-k)\phi_{n-1,n-k}} \\ &= \frac{\rho(n) - \sum_{k=1}^{n-1} \rho(n-k)\phi_{n-1,k}}{1 - \sum_{k=1}^{n-1} \rho(n-k)\phi_{n-1,n-k}},\end{aligned}$$

$$\begin{pmatrix} \phi_{n,1} \\ \vdots \\ \phi_{n,n-1} \end{pmatrix} \stackrel{(**)}{=} \begin{pmatrix} \phi_{n-1,1} \\ \vdots \\ \phi_{n-1,n-1} \end{pmatrix} - \phi_{n,n} \begin{pmatrix} \phi_{n-1,n-1} \\ \vdots \\ \phi_{n-1,1} \end{pmatrix}.$$

Proof. The Yule–Walker equations hold if and only if

$$\begin{pmatrix} \gamma(0) & \cdots & \gamma(n-2) & \gamma(n-1) \\ \vdots & \ddots & \vdots & \vdots \\ \cdots & \cdots & \gamma(0) & \vdots \\ \cdots & \cdots & \cdots & \gamma(0) \end{pmatrix} \begin{pmatrix} \phi_{n,1} \\ \vdots \\ \phi_{n,n-1} \\ \phi_{n,n} \end{pmatrix} = \begin{pmatrix} \gamma(1) \\ \vdots \\ \gamma(n-1) \\ \gamma(n) \end{pmatrix}$$

$$\Leftrightarrow \Gamma_{n-1} \begin{pmatrix} \phi_{n,1} \\ \vdots \\ \phi_{n,n-1} \end{pmatrix} + \underbrace{\phi_{n,n} \begin{pmatrix} \gamma(n-1) \\ \vdots \\ \gamma(1) \end{pmatrix}}_{\stackrel{=}{\underset{\text{YW}}{\Gamma_{n-1} \begin{pmatrix} \phi_{n-1,n-1} \\ \vdots \\ \phi_{n-1,1} \end{pmatrix}}}} = \underbrace{\begin{pmatrix} \gamma(1) \\ \vdots \\ \gamma(n-1) \end{pmatrix}}_{\stackrel{=}{\underset{\text{YW}}{\Gamma_{n-1} \begin{pmatrix} \phi_{n-1,1} \\ \vdots \\ \phi_{n-1,n-1} \end{pmatrix}}}},$$

and $\sum_{k=1}^{n-1} \gamma(n-k)\phi_{n,k} + \phi_{n,n}\gamma(0) \stackrel{(***)}{=} \gamma(n)$. Multiplying with Γ_{n-1}^{-1} leads (**). For (*), use the k th row $\phi_{n,k} = \phi_{n-1,k} - \phi_{n,n}\phi_{n-1,n-k}$ in (***) and solve w.r.t. $\phi_{n,n}$. \square

Definition A.5 (PACF)

The *partial autocorrelation function (PACF)* of a stationary time series $(X_t)_{t \in \mathbb{Z}}$ with $\mu(t) = 0$, $t \in \mathbb{Z}$, $\gamma(0) > 0$, $\gamma(h) \rightarrow 0$ ($h \rightarrow \infty$) is

$$\begin{aligned}\phi(h) &= \text{corr}(X_0 - P_{\mathcal{H}_{h-1}}X_0, X_h - P_{\mathcal{H}_{h-1}}X_h) \\ &= \frac{\mathbb{E}(X_0(X_h - P_{\mathcal{H}_{h-1}}X_h))}{\mathbb{E}((X_h - P_{\mathcal{H}_{h-1}}X_h)(X_h - P_{\mathcal{H}_{h-1}}X_h))} \\ &= \frac{\mathbb{E}(X_0X_h) - \sum_{k=1}^{h-1} \phi_{h-1,k} \mathbb{E}(X_0X_{h-k})}{\mathbb{E}(X_h(X_h - P_{\mathcal{H}_{h-1}}X_h))} \\ &= \frac{\gamma(h) - \sum_{k=1}^{h-1} \gamma(h-k) \phi_{h-1,k}}{\text{station. } \gamma(0) - \sum_{k=1}^{h-1} \gamma(k) \phi_{h-1,k}} \stackrel{\text{DL algorithm}}{=} \phi_{h,h}, \quad h \in \mathbb{Z}.\end{aligned}$$

- PACF for MA(1): Let $\theta = \theta_1$.

$$\rho(h) = \begin{cases} 1, & \text{if } h = 0, \\ \frac{\theta}{1+\theta^2}, & \text{if } |h| = 1, \\ 0, & \text{if } |h| > 1. \end{cases}$$

Yule–Walker equations $\Leftrightarrow P_h \phi_h = \rho_h$. One can show by induction (or the Durbin–Levinson algorithm) that

$$\phi_{h,h} = -\frac{(-\theta)^h(1-\theta^2)}{1-\theta^{2(h+1)}}, \quad h \in \mathbb{N},$$

$$\phi_{h,h-k} = (-\theta)^{-k} \left(\frac{1-\theta^{2k}}{1-\theta^2} \right) \phi_{h,h}, \quad k \in \{1, \dots, h-1\}.$$

In particular, $\phi(h) = \phi_{h,h} \searrow 0$ exponentially.

- PACF for AR(p): For $h > p$, let $Y \in \mathcal{H}_{h-1} = \text{span}\{X_1, \dots, X_{h-1}\}$. Since $(X_t)_{t \in \mathbb{Z}}$ is causal, $Y \in \text{span}\{\varepsilon_s : s \leq h-1\}$. Thus,

$$\left\langle X_h - \sum_{k=1}^p \phi_k X_{h-k}, Y \right\rangle = \langle \varepsilon_t, Y \rangle = 0.$$

Prediction equations $\Rightarrow \sum_{k=1}^p \phi_k X_{h-k}$ is the best linear approximation in the L^2 -sense to X_h from X_1, \dots, X_{h-1} , so $\sum_{k=1}^p \phi_k X_{h-k} = P_{\mathcal{H}_{h-1}} X_h$. Hence,

$$\phi(h) = \text{corr}\left(\underbrace{X_0 - P_{\mathcal{H}_{h-1}} X_0}_{\in \text{span}\{X_0, \dots, X_{h-1}\}}, \underbrace{X_h - P_{\mathcal{H}_{h-1}} X_h}_{=\varepsilon_h} \right) \underset{\text{causality}}{=} 0.$$

Proof idea of Theorem 4.10.

“ \Leftarrow ” $\phi(z) \neq 0$, $|z| \leq 1 \Rightarrow 1/\phi(z)$ holomorphic on $|z| < 1 + \varepsilon$ for some $\varepsilon > 0 \Rightarrow 1/\phi(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k(1 + \varepsilon/2)^k \rightarrow 0$ ($k \rightarrow \infty$) $\Rightarrow \exists c > 0 : |a_k| < c(1 + \varepsilon/2)^{-k}$, $k \in \mathbb{N}_0 \Rightarrow \sum_{k=0}^{\infty} |a_k| < \infty$. Proposition 4.9 $\Rightarrow \varepsilon_t/\phi(B)$ is stationary. $\phi(B)X_t = \theta(B)\varepsilon_t \Rightarrow X_t = \frac{1}{\phi(B)}\phi(B)X_t = \theta(B)\varepsilon_t/\phi(B)$ is stationary (and causal).

“ \Rightarrow ” $X_t = \sum_{k=0}^{\infty} \psi_k \varepsilon_{t-k} = \psi(B)\varepsilon_t$, $\sum_{k=0}^{\infty} |\psi_k| < \infty \Rightarrow \theta(B)\varepsilon_t = \phi(B)X_t = \eta(B)\varepsilon_t$ for $\eta(B) = \phi(B)\psi(B)$. Let $\eta(z) = \phi(z)\psi(z) = \sum_{k=0}^{\infty} \eta_k z^k$, $|z| \leq 1$. With $\theta_0 = 1$, it follows that $\sum_{k=0}^q \theta_k \varepsilon_{t-k} = \sum_{k=0}^{\infty} \eta_k \varepsilon_{t-k}$. Applying $\mathbb{E}(\cdot \varepsilon_{t-j})$ ($\langle \cdot, \varepsilon_{t-j} \rangle$) and using that $(\varepsilon_t) \sim \text{WN}(0, \sigma^2)$, we obtain $\eta_k = \theta_k$, $k \in \{0, \dots, q\}$, and $\eta_k = 0$, $k > q$. This implies that $\theta(z) = \eta(z) = \phi(z)\psi(z)$ for all $|z| \leq 1$. Assume $\phi(z_0) = 0$ for some $|z_0| \leq 1$. Then $0 \neq \theta(z_0) = 0 \cdot \psi(z_0)$. Since $|\psi(z)| \leq \sum_{k=0}^{\infty} |\psi_k| < \infty$ for all $|z| \leq 1$, we obtain a contradiction. Thus $\phi(z) \neq 0$ for all $|z| \leq 1$. \square

Properties of (Q)MLEs

- We consider **two situations**: The model which has been fitted...
 - 1) ... has been **correctly specified**;
 - 2) ... has the correct dynamics but the **innovation distribution is erroneously assumed to be Gaussian** (in this case the MLE is known as **quasi-maximum likelihood estimator (QMLE)**).
- The **asymptotic results** for GARCH models are **similar to the results in the iid case**; they have been derived in a series of papers. We only treat pure GARCH models, the form of the results will apply more generally (e.g. to ARMA models with GARCH errors).
- Under 1), one can show that for a **GARCH(p, q)** model with Gaussian innovations,

$$\sqrt{n}(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}) \xrightarrow[(n \rightarrow \infty)]{d} N_{p+q+1}(\mathbf{0}, I(\boldsymbol{\theta})^{-1}),$$

where

$$I(\boldsymbol{\theta}) := \mathbb{E}\left(\frac{\partial \ell_t(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \left(\frac{\partial \ell_t(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right)'\right) = -\mathbb{E}\left(\frac{\partial^2 \ell_t(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}^2}\right) =: J(\boldsymbol{\theta})$$

is the *Fisher (or: expected) information* matrix. Thus we have a *consistent and asymptotically normal estimator*.

- In practice, the $I(\boldsymbol{\theta})$ is often approximated by an *observed information matrix*. Two candidates are

$$\bar{I}(\boldsymbol{\theta}) = \frac{1}{n} \sum_{t=1}^n \left(\frac{\partial \ell_t(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \left(\frac{\partial \ell_t(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right)'\right) \quad \text{and} \quad \bar{J}(\boldsymbol{\theta}) = -\frac{1}{n} \sum_{t=1}^n \frac{\partial^2 \ell_t(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}^2},$$

where the former has *outer-product* and the latter has *Hessian* form. Evaluating them at the MLEs leads to $\bar{I}(\hat{\boldsymbol{\theta}}_n)$ or $\bar{J}(\hat{\boldsymbol{\theta}}_n)$; in practice, the derivatives are often approximated using first and second-order differences. Under 1), $\bar{I}(\hat{\boldsymbol{\theta}}_n) \approx \bar{J}(\hat{\boldsymbol{\theta}}_n)$. One could also take the *sandwich estimator* $\bar{J}(\hat{\boldsymbol{\theta}}_n) \bar{I}(\hat{\boldsymbol{\theta}}_n)^{-1} \bar{J}(\hat{\boldsymbol{\theta}}_n)$.

- Under 2), one *still obtains a consistent estimator*. If the true innovation

distribution has finite fourth moment, then

$$\sqrt{n}(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}) \underset{(n \rightarrow \infty)}{\overset{d}{\rightarrow}} N_{p+q+1}(\mathbf{0}, J(\boldsymbol{\theta})^{-1} I(\boldsymbol{\theta}) J(\boldsymbol{\theta})^{-1}),$$

Note that $I(\boldsymbol{\theta})$ and $J(\boldsymbol{\theta})$ typically differ in this case. $J(\boldsymbol{\theta})^{-1} I(\boldsymbol{\theta}) J(\boldsymbol{\theta})^{-1}$ can be **estimated by the sandwich estimator**.

- If model checking suggests that the **dynamics** have been **adequately described** by the GARCH model, but **the Gaussian assumption seems doubtful**, then **standard errors for parameter estimates should be computed based on this covariance matrix estimate**.

The SARIMA model

$(X_t)_{t \in \mathbb{Z}}$ is a SARIMA(p, d, q) \times ($\tilde{p}, \tilde{d}, \tilde{q}$)_s (Seasonal; Integrated) process if

$$\underbrace{\phi(B)}_{\text{order } p} \underbrace{\tilde{\phi}(B^s)}_{\text{order } s\tilde{p}} \overbrace{(1-B)^d (1-B^s)^{\tilde{d}}}^{\text{integrated part}} X_t = \underbrace{\theta(B)}_{\text{order } q} \underbrace{\tilde{\theta}(B^s)}_{\text{order } s\tilde{q}} \varepsilon_t, \quad t \in \mathbb{Z}.$$

We see that this is also an ARMA($d + p + s(\tilde{d} + \tilde{p}), q + s\tilde{q}$) process. (Seasonal) “differences” are taken to get data from a **stationary** model.

A.5 Extreme value theory

The convergence to types theorem

Theorem A.6 (Convergence to types)

Suppose $(M_n)_n$ is a sequence of rvs such that $\frac{M_n - d_n}{c_n} \xrightarrow{d} Y$ for a rv Y and $d_n \in \mathbb{R}$, $c_n > 0$. Then

$$\frac{M_n - \delta_n}{\gamma_n} \xrightarrow{d} Z$$

for a rv Z and $\delta_n \in \mathbb{R}$, $\gamma_n > 0$ if and only if

$$(c_n/\gamma_n) \rightarrow c \in [0, \infty), \quad (d_n - \delta_n)/\gamma_n \rightarrow d \in \mathbb{R},$$

in which case $Z \stackrel{d}{=} cY + d$ (i.e. Y and Z are of the same type) and c, d are the unique such constants.

Proof. See Embrechts et al. (1997, p. 554). □

The Gumbel MDA

Theorem A.7 (Gumbel MDA)

$F \in \text{MDA}(H_0)$ if and only if there exists $z < x_F \leq \infty$ such that

$$\bar{F}(x) = c(x) \exp\left(-\int_z^x \frac{g(t)}{a(t)} dt\right), \quad x \in (z, x_F),$$

where c and g are measurable functions satisfying $c(x) \rightarrow c > 0$, $g(x) \rightarrow 1$ for $x \uparrow x_F$ and $a(x) > 0$ with density a' satisfying $\lim_{x \uparrow x_F} a'(x) = 0$.

If $F \in \text{MDA}(H_0)$, the normalizing sequences can be chosen as $c_n = a(d_n)$ for $a(x) = \int_x^{x_F} \bar{F}(t) dt / \bar{F}(x)$, $x < x_F$, (the mean excess function), and $d_n = F^{\leftarrow}(1 - 1/n)$, $n \in \mathbb{N}$.

Derivation of the Hill estimator

Let e be the mean excess function for $\log X$. Using partial integration ($\int H dG = [HG] - \int G dH$), we obtain

$$\begin{aligned} e(\log u) &= \mathbb{E}(\log X - \log u \mid \log X > \log u) \\ &= \frac{1}{\bar{F}(u)} \int_u^\infty (\log x - \log u) dF(x) = -\frac{1}{\bar{F}(u)} \int_u^\infty \log\left(\frac{x}{u}\right) d\bar{F}(x) \\ &= -\frac{1}{\bar{F}(u)} \left(\underbrace{\left[\log\left(\frac{x}{u}\right) \bar{F}(x) \right]_u^\infty}_{=0} - \int_u^\infty \bar{F}(x) \frac{1}{x} dx \right) \\ &= \frac{1}{\bar{F}(u)} \int_u^\infty \frac{\bar{F}(x)}{x} dx = \frac{1}{\bar{F}(u)} \int_u^\infty x^{-\alpha-1} L(x) dx. \end{aligned}$$

For u sufficiently large, $L(x) \approx L(u)$, $x \geq u$ (by **Karamata's Theorem**), so

$$e(\log u) \underset{u \text{ large}}{\approx} \frac{L(u)u^{-\alpha}/\alpha}{\bar{F}(u)} = \frac{1}{\alpha}.$$

For n large and k sufficiently small, replace $e(\cdot)$ by $e_n(\cdot)$ and use $u = X_{k,n}$.

We obtain that

$$\begin{aligned}\frac{1}{\alpha} &\approx e_n(\log X_{k,n}) = \frac{\sum_{i=1}^n (\log X_i - \log X_{k,n}) I_{\{\log X_i > \log X_{k,n}\}}}{\sum_{i=1}^n I_{\{\log X_i > \log X_{k,n}\}}} \\ &= \frac{\sum_{i=1}^{k-1} (\log X_{i,n} - \log X_{k,n})}{k-1} = \frac{1}{k-1} \sum_{i=1}^{k-1} \log X_{i,n} - \log X_{k,n}\end{aligned}$$

The standard form of the estimator is typically written with the average taken over the largest k (instead of $k-1$) terms.

Non-iid data

- If X_1, \dots, X_n are serially dependent and show no tendency of clusters of extreme values (extremal index $\theta = 1$), asymptotic theory of point processes suggests a limiting model for high-level threshold exceedances, in which exceedances occur according to a Poisson process and the excess losses are iid generalized Pareto distributed.
- If extremal clustering is present ($\theta < 1$; e.g. GARCH processes), the assumption of independent excess losses is less satisfactory. Easiest approach: neglect the problem, simply apply MLE which is then a quasi-MLE (QMLE) (likelihood misspecified); point estimates should still be reasonable, standard errors may be too small.
- See the following section for more details on threshold exceedances.

Point process models

So far: **loss size distribution**. Now: **loss frequency distribution**

Threshold exceedances for strict white noise

- Consider a **strict white noise** $(X_i)_{i \in \mathbb{N}}$ (iid from $F \in \text{MDA}(H_\xi)$; can be extended to dependent processes with extremal index $\theta = 1$).
- Let $u_n(x) = c_n x + d_n$ (x fixed). We know $F^n(u_n(x)) \xrightarrow{n \uparrow \infty} H_\xi(x)$. Taking $-\log(\cdot)$ and using $-\log y \approx 1 - y$ for $y \rightarrow 1$, we obtain $n\bar{F}(u_n(x)) \approx -n \log F(u_n(x)) = -\log(F^n(u_n(x))) \xrightarrow{n \uparrow \infty} -\log H_\xi(x)$.
- $N_{u_n(x)}$ (exceedances among X_1, \dots, X_n) fulfills $N_{u_n(x)} \sim \text{B}(n, \bar{F}(u_n(x)))$
- The **Poisson Limit Theorem** ($n \rightarrow \infty$, $p = \bar{F}(u_n(x)) \rightarrow 0$, $np = n\bar{F}(u_n(x)) \rightarrow \lambda = -\log H_\xi(x)$) **implies** $N_{u_n(x)} \xrightarrow{n \uparrow \infty} \text{Poi}(-\log H_\xi(x))$.
- One can show: **Not only is** $N_{u_n(x)}$ **asymptotically Poisson**, but the **exceedances occur according to a Poisson process**.

1) On point processes

- Suppose Y_1, \dots, Y_n take values in some *state space* \mathcal{X} (e.g. \mathbb{R} , \mathbb{R}^2). Define for any $A \subseteq \mathcal{X}$, the counting rv

$$N(A) = \sum_{i=1}^n I_{\{Y_i \in A\}}.$$

Under technical conditions, see Embrechts et al. (1997, pp. 220), $N(\cdot)$ defines a point process.

- $N(\cdot)$ is a *Poisson point process* on \mathcal{X} with *intensity measure* Λ if:

1) For $A \subseteq \mathcal{X}$ and $k \geq 0$,

$$\mathbb{P}(N(A) = k) = \begin{cases} e^{-\Lambda(A)} \frac{\Lambda(A)^k}{k!}, & \text{if } \Lambda(A) < \infty, \\ 0, & \text{if } \Lambda(A) = \infty. \end{cases}$$

2) $N(A_1), \dots, N(A_m)$ are independent for any mutually disjoint subsets A_1, \dots, A_m of \mathcal{X} .

- Note that $\mathbb{E}N(A) = \Lambda(A)$. Also, the *intensity (function)* is the function $\lambda(x)$ which satisfies $\Lambda(A) = \int_A \lambda(x) dx$.

2) Asymptotic behaviour of the point process of exceedances

- For $n \in \mathbb{N}$ and $i \in \{1, \dots, n\}$ let $Y_{i,n} = \frac{i}{n} I_{\{X_i > u_n(x)\}}$. The *point process of exceedances over u_n* is the process $N_n(\cdot)$ with state space $\mathcal{X} = (0, 1]$ given by

$$N_n(A) = \sum_{i=1}^n I_{\{Y_{i,n} \in A\}}, \quad A \subseteq \mathcal{X}.$$

- N_n is an element of the sequence of point processes (N_n) . N_n counts the *exceedances with time of occurrence in A* and we are interested in the behaviour of N_n as $n \rightarrow \infty$.
- Embrechts et al. (1997, Theorem 5.3.2) show that $N_n(\cdot)$ converges in distribution on \mathcal{X} to a Poisson process $N(\cdot)$ with intensity $\Lambda(\cdot)$ satisfying $\Lambda(A) = (t_2 - t_1)\lambda(x)$ for $A = (t_1, t_2) \subseteq \mathcal{X}$, $\lambda(x) = -\log H_\xi(x)$.

- In particular, $\mathbb{E}N_n(A) \xrightarrow{n \uparrow \infty} \mathbb{E}N(A) = \Lambda(A) = (t_2 - t_1)\lambda(x)$. λ does not depend on time and takes the constant value $\lambda = \lambda(x)$.
- We refer to the limiting process as a *homogeneous Poisson process with intensity* (or rate) λ .

3) Application of the result in practice

- Fix a large n and $u = c_n x + d_n$ for some x .
- Approximate N_u by a Poisson rv and the point process of exceedances of u by a homogeneous Poisson process with rate $\lambda = -\log H_\xi(x) = -\log H_\xi((u - d_n)/c_n) = -\log H_{\xi, \mu=d_n, \sigma=c_n}(u)$.
 \Rightarrow Relationship between the GEV model and a Poisson model for the occurrence in time of exceedances of u .
- We see that exceedances of iid data over u are separated by iid exponential waiting times.

The POT model

- Putting the pieces together, we obtain an asymptotic model for threshold exceedances in regularly spaced iid data (or data with $\theta = 1$).
- This so-called *peaks-over-threshold (POT) model* makes the following assumptions:
 - 1) Exceedance times occur according to a homogeneous Poisson process.
 - 2) Excesses above u are iid and independent of exceedance times.
 - 3) The excess distribution is generalized Pareto.
- This model can also be viewed as a *marked Poisson point process* (exceedance times = points; GPD-distributed excesses = marks) or a (non-homogeneous) *two-dimensional Poisson* point process (point (t, x) = (time, magnitude of exceedance))

1) Two-dimensional Poisson formulation of POT model

- Assume that, on the state space $\mathcal{X} = (0, 1] \times (u, \infty)$, the point process defined by $N(A) = \sum_{i=1}^n I_{\{(i/n, X_i) \in A\}}$ is a Poisson process with intensity at (t, x) given by

$$\lambda(x) = \lambda(t, x) = \begin{cases} \frac{1}{\sigma} (1 + \xi \frac{x-\mu}{\sigma})^{-1/\xi-1}, & \text{if } (1 + \xi(x - \mu)/\sigma) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

- For $A = (t_1, t_2) \times (x, \infty) \subseteq \mathcal{X}$, the intensity measure is

$$\Lambda(A) = \int_{t_1}^{t_2} \int_x^{\infty} \lambda(y) dy dt = -(t_2 - t_1) \log H_{\xi, \mu, \sigma}(x)$$

Thus, for any $x \geq u$, the one-dimensional process of exceedances of x is a homogeneous Poisson process with intensity $\tau(x) = -\log H_{\xi, \mu, \sigma}(x)$.

- $\bar{F}_u(x)$ can be calculated as the ratio of the rates of exceeding $u + x$ and u via

$$\bar{F}_u(x) = \frac{\tau(u+x)}{\tau(u)} = \left(1 + \frac{\xi x}{\sigma + \xi(u-\mu)}\right)^{-1/\xi} = \bar{G}_{\xi, \sigma + \xi(u-\mu)}(x)$$

This is precisely the **POT model**.

- The model also implies the **GEV model**. Consider $\{M_n \leq x\}$ for some $x \geq u$, i.e. the event that there are no points in $A = (0, 1] \times (x, \infty)$. Thus, $\mathbb{P}(M_n \leq x) = \mathbb{P}(N(A) = 0) = \exp(-\Lambda(A)) = H_{\xi, \mu, \sigma}(x)$, $x \geq u$, which is precisely the GEV model.

2) Statistical estimation of the POT model

- Given the **exceedances** $\tilde{X}_1 < \dots < \tilde{X}_{N_u}$, $A = (0, 1] \times (u, \infty)$ and $\Lambda(A) = \tau(u) =: \tau_u$, **the likelihood** $L(\xi, \sigma, \mu; \tilde{X}_1, \dots, \tilde{X}_{N_u})$ is

$$\underbrace{N_u!}_{\text{ordered sample}} \underbrace{e^{-\Lambda(A)} \frac{\Lambda(A)^{N_u}}{N_u!}}_{\text{prob. of } N_u \text{ samples}} \underbrace{\prod_{i=1}^{N_u} \frac{\lambda(\tilde{X}_i)}{\Lambda(A)}}_{\text{density of } \tilde{X}_i} = e^{-\Lambda(A)} \prod_{i=1}^{N_u} \lambda(\tilde{X}_i) = e^{-\tau_u} \prod_{i=1}^{N_u} \lambda(\tilde{X}_i).$$

- Reparametrizing λ by $\tau_u = -\log H_{\xi, \mu, \sigma}(u) = (1 + \xi \frac{u - \mu}{\sigma})^{-1/\xi}$ and

$\beta = \sigma + \xi(u - \mu)$, we obtain

$$\begin{aligned}
\lambda(x) &= \frac{1}{\sigma} \left(1 + \xi \frac{x - \mu}{\sigma}\right)^{-\frac{1}{\xi}-1} = \frac{1}{\sigma} \left(\left(1 + \xi \frac{u - \mu}{\sigma}\right) \left(1 + \frac{\xi \frac{x-u}{\sigma}}{1 + \xi \frac{u-\mu}{\sigma}}\right) \right)^{-\frac{1}{\xi}-1} \\
&= \frac{\tau_u}{\sigma(1 + \xi \frac{u-\mu}{\sigma})} \left(1 + \frac{\xi \frac{x-u}{\sigma}}{1 + \xi \frac{u-\mu}{\sigma}}\right)^{-\frac{1}{\xi}-1} = \frac{\tau_u}{\beta} \left(1 + \frac{\xi(x-u)}{\sigma + \xi(u-\mu)}\right)^{-\frac{1}{\xi}-1} \\
&= \frac{\tau_u}{\beta} \left(1 + \frac{\xi(x-u)}{\beta}\right)^{-\frac{1}{\xi}-1} = \tau_u g_{\xi, \beta}(x-u),
\end{aligned}$$

where $\xi \in \mathbb{R}$ and $\tau_u, \beta > 0$. Therefore, $\ell(\xi, \sigma, \mu; \tilde{X}_1, \dots, \tilde{X}_{N_u})$ equals

$$\begin{aligned}
&= -\tau_u + \sum_{i=1}^{N_u} \log \lambda(\tilde{X}_i) = -\tau_u + N_u \log \tau_u + \sum_{i=1}^{N_u} \overbrace{(\log \lambda(\tilde{X}_i) - \log \tau_u)}^{= \log g_{\xi, \beta}(\tilde{X}_i - u)} \\
&= \ell_{\text{Poi}}(\tau_u; N_u) - N_u \log(T) + \ell_{\text{GPD}}(\xi, \beta; \tilde{X}_1 - u, \dots, \tilde{X}_{N_u} - u),
\end{aligned} \tag{137}$$

where ℓ_{Poi} is the log-likelihood for a one-dimensional homogeneous Poisson process with rate τ_u and ℓ_{GPD} is the log-likelihood for fitting a GPD to the excesses $\tilde{X}_i - u$, $i \in \{1, \dots, N_u\}$.

- We can thus separate inferences about (ξ, β) and τ_u . Estimate (ξ, β) in a GPD analysis and then τ_u by its MLE N_u . Use these estimates to infer estimates of $\mu = u - \beta(1 - \tau_u^\xi)/\xi$ and $\sigma = \tau_u^\xi \beta$.

3) Advantages of the POT model formulation

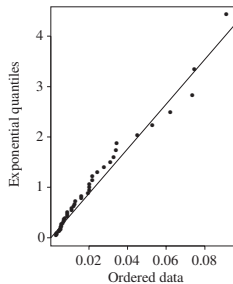
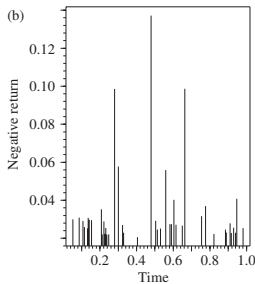
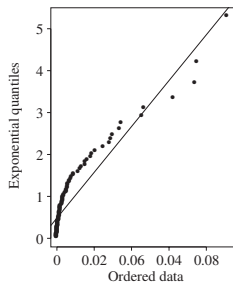
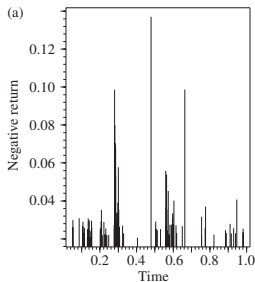
- One advantage of the two-dimensional Poisson point process model is that ξ , μ and σ do not depend on u (unlike β in the GPD model).
 \Rightarrow In practice, we would expect the estimated parameters of the Poisson model to be roughly stable over a range of high thresholds.
- The intensity λ is thus often used to introduce covariates to obtain Poisson processes which are non-homogeneous in time, e.g. by replacing μ and σ by parameters that vary over time as functions of covariates; see, e.g. Chavez-Demoulin et al. (2014).

4) Applicability of the POT model to return series data

- Returns do not really form genuine point events in time (in contrast to, e.g. water levels). They are discrete-time measurements that describe short-term changes (a day or a week). Nonetheless, assume that under a longer-term perspective, such data can be approximated by point events in time.
- Exceedances of u for daily financial return series do not necessarily occur according to a homogeneous Poisson process. They tend to cluster. Thus the standard POT model is not directly applicable.
- For stochastic processes with extremal index $\theta < 1$, e.g. GARCH processes, the extremal clusters themselves should occur according to a homogeneous Poisson process in time \Rightarrow Individual exceedances occur according to a *Poisson cluster process*; see Leadbetter (1991). Thus a suitable model for the occurrence and magnitude of exceedances in a

financial return series might be some form of marked Poisson cluster process.

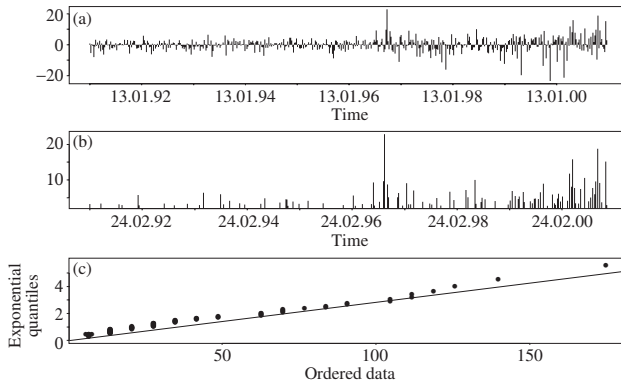
- *Declustering* may circumvent the problem. One identifies clusters (ad hoc; not easy) of exceedances and then applies the POT model to cluster maxima only.
- A possible declustering algorithm is the *runs method*. A run size r is fixed and two successive exceedances are said to belong to two different clusters if they are separated by a run of at least r values below u ; see Embrechts et al. (1997, pp. 422).
- In the following figure the DAX daily negative returns have been declustered with $r = 10$ trading days; this reduces the 100 exceedances to 42 cluster maxima.



- (a): DAX daily negative returns and a Q-Q plot of their spacings
- (b): Declustered data (runs method with $r = 10$ trading days \Rightarrow spacings are more consistent with a Poisson model)
- However, by neglecting the modelling of cluster formation, we cannot make more dynamic statements about the intensity of occurrence of exceedances.

Example A.8 (POT analysis of AT&T weekly losses (continued))

Consider the 102 weekly percentage losses exceeding $u = 2.75\%$:



- Inter-exceedance times seem to follow an exponential distribution.
- But exceedances become more frequent over time (which contradicts a homogeneous Poisson process)

- Using the log-likelihood (137), we fit a two-dimensional Poisson model to the 102 exceedances of $u = 2.75\%$. The parameter estimates are $\hat{\xi} = 0.22$, $\hat{\mu} = 19.9$ and $\hat{\sigma} = 5.95$.
- The implied GPD scale parameter is $\hat{\beta} = \hat{\sigma} + \hat{\xi}(u - \hat{\mu}) = 2.1 \Rightarrow$ The same $\hat{\xi}$ and $\hat{\beta}$ as in Example 5.19.
- The estimated exceedance rate over $u = 2.75$ is $\hat{\tau}(u) = -\log H_{\hat{\xi}, \hat{\mu}, \hat{\sigma}}(u) = 102$ (= number of exceedances; as theory suggests).
- Higher thresholds, e.g. 15%: Since $\hat{\tau}(15) = 2.50$, losses exceeding 15% occur as a Poisson process with rate 2.5 losses per 10-year period (\approx a four-year event). \Rightarrow The Poisson model provides an alternative method of defining the return period of an event.
- Similarly, estimate return levels: If the 10-year return level is the level which is exceeded according to a Poisson process with rate one loss per 10 years, estimate the level by solving $\hat{\tau}(u) = 1$ w.r.t. u , so

$u = H_{\hat{\xi}, \hat{\mu}, \hat{\sigma}}^{-1}(\exp(-1)) = 19.9$ so the 10-year event is a weekly loss of roughly 20%.

- Confidence intervals for such quantities can be constructed via profile likelihoods.

A.6 Multivariate models

Conditional distributions and independence

Proof of (17). We have

$$\begin{aligned}& \int_{(-\infty, x_1]} F_{X_2|X_1}(\mathbf{x}_2 | z) dF_{X_1}(z) \\&= \int_{\mathbb{R}^d} I_{\{z \leq x_1\}} \mathbb{E}(I_{\{X_2 \leq x_2\}} | \mathbf{X}_1 = z) dF_{X_1}(z) \\&= \mathbb{E}(I_{\{\mathbf{X}_1 \leq x_1\}} \mathbb{E}(I_{\{X_2 \leq x_2\}} | \mathbf{X}_1)) = \mathbb{E}(\mathbb{E}(I_{\{\mathbf{X}_1 \leq x_1, X_2 \leq x_2\}} | \mathbf{X}_1)) \\&\stackrel[\text{property}]{\text{tower}}{=} \mathbb{E}(I_{\{\mathbf{X}_1 \leq x_1, X_2 \leq x_2\}}) = F(\mathbf{x}),\end{aligned}$$

where the second-last equality holds by the **tower property** of conditional expectations. □

The multivariate normal distribution

Proof of the form of the cf of $N(0, 1)$; see Proposition 6.3. The rv $Z \sim N(0, 1)$ has density $\varphi(x) = \exp(-x^2/2)/\sqrt{2\pi}$ which satisfies

- i) $\varphi(x) = \varphi(-x)$;
- ii) $\varphi'(x) = -x\varphi(x)$.

By Euler's Formula, the characteristic function $\phi_Z(t)$ of Z is given by

$$\phi_Z(t) = \int_{-\infty}^{\infty} (\cos(tx) + i \sin(tx)) \varphi(x) dx \stackrel{\text{i)}}{=} \int_{-\infty}^{\infty} \cos(tx) \varphi(x) dx.$$

Hence,

$$\phi'_Z(t) = \int_{-\infty}^{\infty} \sin(tx)(-x)\varphi(x) dx \stackrel{\text{ii)}}{=} \int_{-\infty}^{\infty} \sin(tx)\varphi'(x) dx \stackrel{\text{by parts}}{=} -t\phi_Z(t).$$

We also know that $\phi_Z(0) = 1$. This initial value problem has the unique solution $\phi_Z(t) = \exp(-t^2/2)$. □

Theorem A.9 (Cramér–Wold)

Let \mathbf{X}, \mathbf{X}_n , $n \in \mathbb{N}$, be random vectors. Then

$$\mathbf{X}_n \xrightarrow[n \uparrow \infty]{d} \mathbf{X} \iff \mathbf{a}'\mathbf{X}_n \xrightarrow[n \uparrow \infty]{d} \mathbf{a}'\mathbf{X} \quad \forall \mathbf{a} \in \mathbb{R}^d$$

Proof.

“ \Rightarrow ” This follows from the Continuous Mapping Theorem with the map $g(\mathbf{x}) = \mathbf{a}'\mathbf{x}$.

“ \Leftarrow ” Note that $\phi_{\mathbf{X}_n}(\mathbf{t}) = \mathbb{E}(\exp(i \cdot \mathbf{1} \cdot \mathbf{t}'\mathbf{X}_n)) = \phi_{\mathbf{t}'\mathbf{X}_n}(1) \xrightarrow[n \uparrow \infty]{} \phi_{\mathbf{t}'\mathbf{X}}(1) = \phi_{\mathbf{X}}(\mathbf{t})$ for all \mathbf{t} and apply the Lévy Continuity Theorem. \square

Corollary A.10

Let \mathbf{X}, \mathbf{Y} be two random vectors. Then

$$\mathbf{X} \stackrel{d}{=} \mathbf{Y} \iff \mathbf{a}'\mathbf{X} \stackrel{d}{=} \mathbf{a}'\mathbf{Y} \quad \forall \mathbf{a} \in \mathbb{R}^d.$$

Properties of multivariate normal variance mixtures

Proof of Lemma 6.10. W.l.o.g. assume $\mu = \mathbf{0}$.

$$\begin{aligned} \Rightarrow \mathbb{E}|X_i| \mathbb{E}|X_j| &\stackrel{\text{ind.}}{=} \mathbb{E}(|X_i||X_j|) = \mathbb{E}(W|Z_i||Z_j|) \stackrel{\text{ind.}}{=} \mathbb{E}(W) \mathbb{E}|Z_i| \mathbb{E}|Z_j| \\ &\stackrel{\text{Jensen}}{\geq} \mathbb{E}(\sqrt{W})^2 \mathbb{E}|Z_i| \mathbb{E}|Z_j| \stackrel{\text{ind.}}{=} \mathbb{E}|\sqrt{W}Z_i| \mathbb{E}|\sqrt{W}Z_j| = \mathbb{E}|X_i| \mathbb{E}|X_j| \end{aligned}$$

\Rightarrow We must have “=” in Jensen’s inequality. This holds if and only if W is constant a.s.; so $\mathbf{X} \sim N_d(\mathbf{0}, WI_d)$ in this case.

\Leftarrow W a.s. constant $\Rightarrow \mathbf{X} \sim N_d(\mathbf{0}, WI_d) \Rightarrow X_i, X_j$ independent. \square

Spherical distributions

Proof of Theorem 6.15.

1) \Rightarrow 2): $\phi_{\mathbf{Y}}(\mathbf{t}) = \phi_{U\mathbf{Y}}(\mathbf{t}) = \phi_{\mathbf{Y}}(U'\mathbf{t})$ for all $U \in \mathbb{R}^{d \times d}$ orthogonal. Since U can only change the direction of \mathbf{t} but not its length, $\phi_{\mathbf{Y}}(\mathbf{t})$ only depends on $\|\mathbf{t}\|$, i.e. the length of $\mathbf{t} \Rightarrow$ we can define $\psi(\|\mathbf{t}\|^2) = \phi_{\mathbf{Y}}(\mathbf{t})$.

$$\begin{aligned}
2) \Rightarrow 3): \phi_{Y_1}(t) &= \phi_Y(te_1) \stackrel{2)}{=} \psi(t^2) (*). \text{ Now } \phi_{a'Y}(t) = \phi_Y(ta) \stackrel{2)}{=} \\
&\psi(t^2\|a\|^2) = \psi((t\|a\|)^2) \stackrel{(*)}{=} \phi_{Y_1}(t\|a\|) = \phi_{\|a\|Y_1}(t) \\
3) \Rightarrow 1): \phi_{UY}(t) &= \mathbb{E}(\exp(i(U't)'Y)) \stackrel{U't=:a}{=} \mathbb{E}(\exp(ia'Y)) \stackrel{3)}{=} \mathbb{E}(\exp(i\|a\|Y_1)) \\
&= \mathbb{E}(\exp(i\|t\|Y_1)) \stackrel{3)}{=} \mathbb{E}(\exp(it'Y)) = \phi_Y(t) \quad \square
\end{aligned}$$

Proof of Theorem 6.16. Let Ω_d be the characteristic generator of S .

$$\begin{aligned}
\text{"}\Rightarrow\text{" } Y \sim S_d(\psi) &\Rightarrow \phi_Y(\|t\|u) \stackrel{2)}{=} \psi(\|t\|^2 u'u) = \psi(\|t\|^2) \text{ for all } u \in \mathbb{R}^d : \\
&\|u\| = 1. \text{ Replacing } u \text{ by } S \text{ and integrating leads to } \psi(\|t\|^2) = \\
&\mathbb{E}_S(\phi_Y(\|t\|S)) = \mathbb{E}_S(\mathbb{E}_Y(e^{i\|t\|S'Y})) \stackrel{\text{Fubini}}{=} \mathbb{E}_Y(\mathbb{E}_S(e^{i\|t\|S'Y})) = \\
&\mathbb{E}_Y(\phi_S(\|t\|Y)) \stackrel{2)}{=} \mathbb{E}_Y(\Omega_d(\|t\|^2 Y'Y)). \text{ We thus obtain that} \\
\phi_Y(t) &\stackrel{2)}{=} \psi(\|t\|^2) \stackrel{R:=\|Y\|}{=} \mathbb{E}_R(\Omega_d(\|t\|^2 R^2)) = \int_0^\infty \Omega_d(\|t\|^2 r^2) dF_R(r) \\
&\stackrel{2)}{=} \int_0^\infty \phi_S(rt) dF_R(r) = \phi_{RS}(t) \text{ for all } t \in \mathbb{R}^d.
\end{aligned}$$

$$\text{"}\Leftarrow\text{" Let } Z \sim N_d(0, I_d). \text{ Since } Z \text{ is spherical and } \|Z/\|Z\|\| = \|Z\|/\|Z\| =$$

1, $\mathbf{S} \stackrel{d}{=} \mathbf{Z}/\|\mathbf{Z}\|$. As such, \mathbf{S} itself is spherical, since $U\mathbf{S} \stackrel{d}{=} U\mathbf{Z}/\|\mathbf{Z}\| \stackrel{d}{=} \mathbf{Z}/\|\mathbf{Z}\| \stackrel{d}{=} \mathbf{S}$ for any orthogonal $U \in \mathbb{R}^{d \times d}$. Theorem 6.15 Part 2) implies that $\phi_{\mathbf{S}}(\mathbf{t}) = \Omega_d(\|\mathbf{t}\|^2)$, so $\phi_{R\mathbf{S}}(\mathbf{t}) = \mathbb{E}(\exp(i\mathbf{t}'R\mathbf{S})) = \mathbb{E}_R(\mathbb{E}(\exp(i\mathbf{t}'R\mathbf{S}) \mid R)) = \mathbb{E}_R(\phi_{\mathbf{S}}(R\mathbf{t})) = \mathbb{E}_R(\Omega_d(R^2\|\mathbf{t}\|^2))$, which is a function of $\|\mathbf{t}\|^2$ and thus, by 2), $R\mathbf{S}$ is spherical. \square

Density of $\mathbf{Y} \sim S_d(\psi)$ constant on spheres

If \mathbf{Y} admits a density $f_{\mathbf{Y}}$, then $f_{\mathbf{Y}}(\mathbf{y})$ is constant on hyperspheres in \mathbb{R}^d . The *inversion formula* $f_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\mathbf{t}'\mathbf{y}} \phi_{\mathbf{Y}}(\mathbf{t}) d\mathbf{t}$ and Theorem 6.15 Part 2) show that for any orthogonal U ,

$$\begin{aligned} f_{\mathbf{Y}}(U\mathbf{y}) &\stackrel{\text{inv.}}{=} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i(U'\mathbf{t})'\mathbf{y}} \phi_{\mathbf{Y}}(\mathbf{t}) d\mathbf{t} \\ &\stackrel{\text{subs.}}{=} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\mathbf{s}'\mathbf{y}} \phi_{\mathbf{Y}}(U\mathbf{s}) d\mathbf{s} \\ &\stackrel{2)}{=} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\mathbf{s}'\mathbf{y}} \psi((U\mathbf{s})'U\mathbf{s}) d\mathbf{s} \end{aligned}$$

$$= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-is'y} \psi(s's) ds \underset{\text{backwards}}{=} \cdots = f_Y(y).$$

This implies that $f_Y(y) = g(\|y\|^2)$ for a function $g : [0, \infty) \rightarrow [0, \infty)$, the density generator. For $Y \sim t_d(\nu, \mathbf{0}, I_d)$, one can show that $g(x) = \frac{\Gamma((\nu+d)/2)}{\Gamma(\nu/2)(\pi\nu)^{d/2}} (1 + \frac{x}{\nu})^{-(\nu+d)/2}$.

Elliptical distributions

Proposition A.11

Let $X \sim E_d(\mu, \Sigma, \psi)$ for positive definite Σ and $\mathbb{E}(R^2) < \infty$ (i.e. $\text{cov}(X)$ finite). For any $c \geq 0$ such that $\mathbb{P}((X - \mu)' \Sigma^{-1} (X - \mu) \geq c) > 0$,

$$\text{corr}(X \mid (X - \mu)' \Sigma^{-1} (X - \mu) \geq c) = \text{corr}(X).$$

Proof. $X \mid ((X - \mu)' \Sigma^{-1} (X - \mu) \geq c) \stackrel{d}{=} \mu + RAS \mid (R^2 \geq c) \stackrel{\text{ind.}}{=} \mu + \tilde{R}AS$
 where $\tilde{R} \stackrel{d}{=} (R \mid R^2 \geq c)$. A (and thus Σ) remains the same. \square

Estimating scale and correlation

- Suppose $\mathbf{X}_1, \dots, \mathbf{X}_n \sim E_d(\boldsymbol{\mu}, \Sigma, \psi)$. How can we estimate $\boldsymbol{\mu}, \Sigma$ and P ? (P is the correlation matrix corresponding to Σ ; this always exists)
- $\bar{\mathbf{X}}, S, R$ may not be the best options for heavy-tailed data (e.g. concerning robustness against contamination).

M-estimators for $\boldsymbol{\mu}, \Sigma$ (see Maronna (1976))

- **Goal:** Improve given estimators $\hat{\boldsymbol{\mu}}, \hat{\Sigma}$.
- **Idea:** Compute improved estimates by downweighting observations with large $D_i = \sqrt{(\mathbf{X}_i - \hat{\boldsymbol{\mu}})' \hat{\Sigma}^{-1} (\mathbf{X}_i - \hat{\boldsymbol{\mu}})}$ (these are the ones which tend to distort $\hat{\boldsymbol{\mu}}, \hat{\Sigma}$ most).
- This can be turned into an iterative procedure that converges to so-called *M-estimates* of location and scale ($\hat{\Sigma}$ is in general biased).

Algorithm A.12 (M-estimators of location and scale)

1) Set $k = 1$, $\hat{\mu}^{[1]} = \bar{X}$ and $\hat{\Sigma}^{[1]} = S$.

2) Repeat until convergence:

2.1) For $i \in \{1, \dots, n\}$ set $D_i = \sqrt{(\mathbf{X}_i - \hat{\mu}^{[k]})' \hat{\Sigma}^{[k]-1} (\mathbf{X}_i - \hat{\mu}^{[k]})}$.

2.2) Update:

$$\hat{\mu}^{[k+1]} = \frac{\sum_{i=1}^n w_1(D_i) \mathbf{X}_i}{\sum_{i=1}^n w_1(D_i)},$$

where w_1 is a weight function, e.g. $w_1(x) = (d + \nu)/(x^2 + \nu)$ (or $I_{x \leq a} + (a/x)I_{x > a}$ for some value a).

2.3) Update:

$$\hat{\Sigma}^{[k+1]} = \frac{1}{n} \sum_{i=1}^n w_2(D_i^2) (\mathbf{X}_i - \hat{\mu}^{[k]})(\mathbf{X}_i - \hat{\mu}^{[k]})',$$

where w_2 is a weight function, e.g. $w_2(x) = w_1(\sqrt{x})$ (or $(w_1(\sqrt{x}))^2$).

2.4) Set k to $k + 1$.

Factor models

Example A.13 (One-factor/equicorrelation model)

Let $\mathbb{E}(\mathbf{X}) = \mathbf{0}$, $\Sigma = \text{cov}(\mathbf{X}) = \rho J_d + (1 - \rho)I_d$ ($J_d = (1) \in \mathbb{R}^{d \times d}$).

- Then $\Sigma = BB' + \Upsilon$ for $B = \sqrt{\rho}\mathbf{1}$ and $\Upsilon = (1 - \rho)I_d$.
- Any Y with $\mathbb{E}Y = 0$, $\text{var } Y = 1$ independent of \mathbf{X} leads to the *factor decomposition* of \mathbf{X}

$$F = \frac{\sqrt{\rho}}{1 + \rho(d-1)} \sum_{j=1}^d X_j + \sqrt{\frac{1 - \rho}{1 + \rho(d-1)}} Y, \quad \varepsilon_j = X_j - \sqrt{\rho}F.$$

We have $\mathbb{E}(F) = 0$, $\text{var}(F) = 1$, so $\mathbf{X} = \mathbf{0} + BF + \varepsilon = \sqrt{\rho}\mathbf{1}F + \varepsilon$.

- The requirements of Definition 6.25 are fulfilled since $\text{cov}(F, \varepsilon_j) = 0$, $\text{cov}(\varepsilon_j, \varepsilon_k) = 0$ for all $j \neq k$.
- $\text{var}(\bar{X}_n) = \text{var}(\sqrt{\rho}F + \bar{\varepsilon}_d) = \rho + \frac{1-\rho}{d} \xrightarrow{(d \rightarrow \infty)} \rho$ (systematic factor matters!)

- If $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$, take $Y \sim \mathcal{N}(0, 1)$ (then F is also normal). One typically writes this (one-factor) equicorrelation model as $\mathbf{X} = \sqrt{\rho}F + \sqrt{1 - \rho}\mathbf{Z}$, where $F, Z_1, \dots, Z_d \stackrel{\text{ind.}}{\sim} \mathcal{N}(0, 1)$.

Multivariate regression

- Here, construct large matrices:

$$X = \underbrace{\begin{pmatrix} \mathbf{X}'_1 \\ \vdots \\ \mathbf{X}'_n \end{pmatrix}}_{n \times d}, \quad F = \underbrace{\begin{pmatrix} 1 & \mathbf{F}'_1 \\ \vdots & \vdots \\ 1 & \mathbf{F}'_n \end{pmatrix}}_{n \times (p+1)}, \quad \tilde{B} = \underbrace{\begin{pmatrix} \mathbf{a}' \\ B' \end{pmatrix}}_{(p+1) \times d}, \quad E = \underbrace{\begin{pmatrix} \boldsymbol{\varepsilon}'_1 \\ \vdots \\ \boldsymbol{\varepsilon}'_n \end{pmatrix}}_{n \times d}.$$

This model can be expressed by $X = F\tilde{B} + E$ (estimate \tilde{B}).

- Assume the unobserved $\boldsymbol{\varepsilon}_1, \dots, \boldsymbol{\varepsilon}_n$ form a white noise process. Then, conditional on $\mathbf{F}_1, \dots, \mathbf{F}_n$, we have a multivariate linear regression, see, e.g. Mardia et al. (1979), with estimator $\hat{\tilde{B}} = (F'F)^{-1}F'X$.

- Now examine the conditions of Definition 6.25: Do the errors vectors ε_t come from a distribution with diagonal covariance matrix, and are they uncorrelated with the factors?
- Consider the sample correlation matrix of $\hat{E} = X - F\hat{B}$ (model residual matrix; hopefully shows that there is little correlation in the errors) and take the diagonal elements as an estimator $\hat{\Upsilon}$ of Υ .

Sample principal components

- Assume $\mathbf{X}_1, \dots, \mathbf{X}_n$ with identical distribution, **unknown mean vector μ and covariance matrix Σ** with the **spectral decomposition $\Sigma = \Gamma\Lambda\Gamma'$** as before.
- **Estimate μ by $\bar{\mathbf{X}}$ and Σ by $S_x = \frac{1}{n} \sum_{t=1}^n (\mathbf{X}_t - \bar{\mathbf{X}})(\mathbf{X}_t - \bar{\mathbf{X}})'$.**
- Apply the **spectral decomposition** to S_x to get $S_x = GLG'$, where G is the eigenvector matrix and $L = \text{diag}(l_1, \dots, l_d)$ is the diagonal matrix consisting of ordered eigenvalues.

- Define the “sample principle component transforms” $\mathbf{Y}_t = G'(\mathbf{X}_t - \bar{\mathbf{X}})$, $t \in \{1, \dots, n\}$. The j th component $Y_{t,j} = \mathbf{g}'_j(\mathbf{X}_t - \bar{\mathbf{X}})$ is the j th sample principal component at time t (\mathbf{g}_j is the j th column of G).
- The rotated vectors $\mathbf{Y}_1, \dots, \mathbf{Y}_n$ have sample covariance matrix L :

$$\begin{aligned} S_y &= \frac{1}{n} \sum_{t=1}^n (\mathbf{Y}_t - \bar{\mathbf{Y}})(\mathbf{Y}_t - \bar{\mathbf{Y}})' = \frac{1}{n} \sum_{t=1}^n \mathbf{Y}_t \mathbf{Y}_t' \\ &= \frac{1}{n} \sum_{t=1}^n G'(\mathbf{X}_t - \bar{\mathbf{X}})(\mathbf{X}_t - \bar{\mathbf{X}})'G = G' S_x G = L. \end{aligned}$$

Thus the rotated vectors show no correlation between components and the components are ordered by their sample variances, from largest to smallest.

- Now use G and \mathbf{Y}_t to calibrate an approximate factor model. We assume our data are realizations from the model

$$\mathbf{X}_t = \bar{\mathbf{X}} + G_1 \mathbf{F}_t + \boldsymbol{\varepsilon}_t, \quad t \in \{1, \dots, n\},$$

where G_1 consists of the first k columns of G and $F_t = (Y_{t,1}, \dots, Y_{t,k})$, $t \in \{1, \dots, n\}$.

- In practice, the errors ϵ_t do not have a diagonal covariance matrix and are not uncorrelated with F_t . Nevertheless the method is a popular approach to constructing time series of statistically explanatory factors from multivariate time series of risk-factor changes.

A.7 Copulas and dependence

An example

Let $C(\mathbf{u}) = \lambda C_1(\mathbf{u}) + (1 - \lambda)C_2(\mathbf{u})$ for copulas C_1, C_2 and $\lambda \in [0, 1]$ (convex combination). Then C is again a copula since:

1) Analytical proof:

- Let $\mathbf{u}_j = (u_1, \dots, u_{j-1}, 0, u_{j+1}, \dots, u_d)$. Then

$$C(\mathbf{u}_j) = \lambda C_1(\mathbf{u}_j) + (1 - \lambda)C_2(\mathbf{u}_j) = \lambda \cdot 0 + (1 - \lambda) \cdot 0 = 0$$

since C_1, C_2 are grounded. Hence, C is grounded.

- Let $\mathbf{u}_j = (1, \dots, 1, u_j, 1, \dots, 1)$. Then

$$C(\mathbf{u}_j) = \lambda C_1(\mathbf{u}_j) + (1 - \lambda)C_2(\mathbf{u}_j) = \lambda u_j + (1 - \lambda)u_j = u_j$$

since C_1, C_2 have $U[0, 1]$ margins. Hence, C has $U[0, 1]$ margins.

- $\Delta_{(a,b]}C = \lambda \Delta_{(a,b]}C_1 + (1 - \lambda)\Delta_{(a,b]}C_2 \geq 0$, so C is d -increasing.

2) Stochastic proof:

Let $U_k \sim C_k$, $k \in \{1, 2\}$ and let $X \sim B(1, \lambda)$, independent of each other. Furthermore, let

$$U = \begin{cases} U_1, & \text{if } X = 1, \\ U_2, & \text{if } X = 0. \end{cases}$$

The Law of Total Probability implies that

$$\begin{aligned} \mathbb{P}(U \leq \mathbf{u}) &= \mathbb{P}(U \leq \mathbf{u}, X = 1) + \mathbb{P}(U \leq \mathbf{u}, X = 0) \\ &= \mathbb{P}(U_1 \leq \mathbf{u}, X = 1) + \mathbb{P}(U_2 \leq \mathbf{u}, X = 0) \\ &= \mathbb{P}(U_1 \leq \mathbf{u}) \mathbb{P}(X = 1) + \mathbb{P}(U_2 \leq \mathbf{u}) \mathbb{P}(X = 0) \\ &= C_1(\mathbf{u}) \lambda + C_2(\mathbf{u}) (1 - \lambda) = C(\mathbf{u}). \end{aligned}$$

So $U \sim C$ and hence C is a df. From the same calculation it follows that U has uniform margins, hence C is a copula.

Basic properties

Lemma A.14

For any copula C , $|C(\mathbf{b}) - C(\mathbf{a})| \leq \sum_{j=1}^d |b_j - a_j|$ for all $\mathbf{a}, \mathbf{b} \in [0, 1]^d$.

Proof. Using a telescoping sum expansion and the triangle inequality, we obtain

$$|C(\mathbf{b}) - C(\mathbf{a})| \leq \sum_{j=1}^d |C(b_1, \dots, b_{j-1}, b_j, a_{j+1}, \dots, a_d) - C(b_1, \dots, b_{j-1}, a_j, a_{j+1}, \dots, a_d)|.$$

W.l.o.g. let $\mathbf{a} \leq \mathbf{b}$. By d -increasingness, $C \nearrow$ component-wise, so omit $|\cdot|$. Since, by d -increasingness, the j th summand is \nearrow in each component $\neq j$, let $b_1, \dots, b_{j-1}, a_{j+1}, \dots, a_d \nearrow 1$ to obtain the upper bound $\sum_{j=1}^d C(1, \dots, 1, b_j, 1, \dots, 1) - C(1, \dots, 1, a_j, 1, \dots, 1) = b_j - a_j$ for summand j . \square

Generalized inverses

$T \nearrow$ means that T is *increasing*; $T \uparrow$ means that T is *strictly increasing*;
 $\text{ran } T = \{T(x) : x \in \mathbb{R}\}$ denotes the *range* of T .

Proposition A.15 (Working with generalized inverses)

Let $T : \mathbb{R} \rightarrow \mathbb{R} \nearrow$ with $T(-\infty) = \lim_{x \downarrow -\infty} T(x)$ and $T(\infty) = \lim_{x \uparrow \infty} T(x)$
and let $x, y \in \mathbb{R}$. Then,

- (GI1) $T^{\leftarrow}(y) = -\infty$ if and only if $T(x) \geq y$ for all $x \in \mathbb{R}$. Similarly,
 $T^{\leftarrow}(y) = \infty$ if and only if $T(x) < y$ for all $x \in \mathbb{R}$.
- (GI2) $T^{\leftarrow} \nearrow$. If $T^{\leftarrow}(y) \in (-\infty, \infty)$, T^{\leftarrow} is left-continuous at y and
admits a limit from the right at y .
- (GI3) $T^{\leftarrow}(T(x)) \leq x$. If $T \uparrow$, then $T^{\leftarrow}(T(x)) = x$.
- (GI4) Let T be right-continuous and $\text{ran } T$ denote the *range of T* , i.e.
 $\text{ran } T = \{T(x) : x \in \mathbb{R}\}$. $T^{\leftarrow}(y) < \infty$ implies $T(T^{\leftarrow}(y)) \geq y$.
Furthermore, $y \in \text{ran } T \cup \{\inf T, \sup T\}$ implies $T(T^{\leftarrow}(y)) = y$.

Moreover, if $y < \inf T$ then $T(T^{\leftarrow}(y)) > y$ and if $y > \sup T$ then $T(T^{\leftarrow}(y)) < y$.

- (GI5) $T(x) \geq y$ implies $x \geq T^{\leftarrow}(y)$. The other implication holds if T is right-continuous. Furthermore, $T(x) < y$ implies $x \leq T^{\leftarrow}(y)$.
- (GI6) $(T^{\leftarrow}(y-), T^{\leftarrow}(y+)) \subseteq \{x \in \mathbb{R} : T(x) = y\} \subseteq [T^{\leftarrow}(y-), T^{\leftarrow}(y+)]$, where $T^{\leftarrow}(y-) = \lim_{z \uparrow y} T^{\leftarrow}(z)$ and $T^{\leftarrow}(y+) = \lim_{z \downarrow y} T^{\leftarrow}(z)$.
- (GI7) T is continuous if and only if $T^{\leftarrow} \uparrow$ on $[\inf T, \sup T]$.
 $T \uparrow$ if and only if T^{\leftarrow} is continuous on $\text{ran } T$.
- (GI8) If T_1 and T_2 are right-continuous transformations with properties as T , then $(T_1 \circ T_2)^{\leftarrow} = T_2^{\leftarrow} \circ T_1^{\leftarrow}$.

Proof. See Embrechts and Hofert (2013a). □

Proof of Lemma 7.2. Note that the *range of a rv X* is defined by

$$\text{ran } X = \{x \in \mathbb{R} : \mathbb{P}(X \in (x - h, x]) > 0 \text{ for all } h > 0\}.$$

Since F is continuous on \mathbb{R} , (GI7) implies that $F^{\leftarrow} \uparrow$ on $[\inf F, \sup F] = [0, 1]$. Thus

$$\begin{aligned} \mathbb{P}(F(X) \leq u) &\stackrel{\text{(GI7)}}{=} \mathbb{P}(F^{\leftarrow}(F(X)) \leq F^{\leftarrow}(u)) \stackrel{\text{(GI3)}}{=} \mathbb{P}(X \leq F^{\leftarrow}(u)) \\ &= F(F^{\leftarrow}(u)) \stackrel{\text{(GI4)}}{=} u, \quad u \in [0, 1], \end{aligned}$$

where (GI3) applies since $F \uparrow$ on $\text{ran } X$. □

Proof of Lemma 7.6.

$$\begin{aligned} \text{"}\Rightarrow\text{" } \mathbb{P}(F_j(X_j) \leq u_j \forall j) &\stackrel{\text{cont.}}{=} \mathbb{P}(F_j(X_j) < u_j \forall j) \stackrel{\text{(GI5)}}{=} \mathbb{P}(X_j < F_j^{\leftarrow}(u_j) \forall j) \\ &\stackrel{\text{cont.}}{=} \mathbb{P}(X_j \leq F_j^{\leftarrow}(u_j) \forall j) = F(F_1^{\leftarrow}(u_1), \dots, F_d^{\leftarrow}(u_d)) \stackrel{\text{Sklar}}{=} C(\mathbf{u}). \end{aligned}$$

$\text{"}\Leftarrow\text{"}$ Since $F_j \uparrow$ on $\text{ran } X_j$, $j \in \{1, \dots, d\}$,

$$\begin{aligned} F(\mathbf{x}) &\stackrel{\text{(GI3)}}{=} \mathbb{P}(F_j^{\leftarrow}(F_j(X_j)) \leq x_j \forall j) \stackrel{\text{(GI5)}}{=} \mathbb{P}(F_j(X_j) \leq F_j(x_j) \forall j) \\ &\stackrel{\text{ass.}}{=} C(F_1(x_1), \dots, F_d(x_d)) \stackrel{\text{Sklar}}{\Rightarrow} \mathbf{X} \text{ has copula } C \quad \square \end{aligned}$$

Proof of Theorem 7.8.

- 1) ■ By Lemma A.14, $1 - C(\mathbf{u}) = C(\mathbf{1}) - C(\mathbf{u}) \leq \sum_{j=1}^d (1 - u_j) = d - \sum_{j=1}^d u_j$, so $C(\mathbf{u}) \geq \sum_{j=1}^d u_j - d + 1$. Also, $C(\mathbf{u}) \geq 0$. So $C(\mathbf{u}) \geq W(\mathbf{u})$.
 - Since copulas are componentwise increasing, $C(\mathbf{u}) \leq C(1, \dots, 1, u_j, 1, \dots, 1) = u_j$ for all j . Hence, $C(\mathbf{u}) \leq \min_{1 \leq j \leq d} \{u_j\} = M(\mathbf{u})$.
- 2) W is a copula for $d = 2$ since $(U, 1 - U) \sim W$ for $U \sim U(0, 1)$. W is not a copula for $d \geq 3$ since

$$\begin{aligned}
 & \Delta_{(\frac{1}{2}, 1]} W \\
 &= \sum_{\mathbf{i} \in \{0, 1\}^d} (-1)^{\sum_{j=1}^d i_j} W\left(\frac{1}{2}^{i_1}, \dots, \frac{1}{2}^{i_d}\right) \\
 &= \max\{1 + 1 + 1 + \dots + 1 - d + 1, 0\} \quad (i_j = 0 \ \forall j) \\
 & \quad - d \max\{\frac{1}{2} + 1 + 1 + \dots + 1 - d + 1, 0\} \quad (\exists! j : i_j = 1)
 \end{aligned}$$

$$\begin{aligned}
& + \binom{d}{2} \max\{\tfrac{1}{2} + \tfrac{1}{2} + 1 + \cdots + 1 - d + 1, 0\} \quad (\exists! \text{ two } j : i_j = 1) \\
& - \cdots + (-1)^d \max\{\tfrac{1}{2} + \cdots + \tfrac{1}{2} - d + 1, 0\} \quad (i_j = 1 \ \forall j) \\
& = 1 - \frac{d}{2} < 0 \quad \text{for } d \geq 3.
\end{aligned}$$

3) M is a copula for all $d \geq 2$ since $(U, \dots, U) \sim M$ for $U \sim \text{U}(0, 1)$. \square

Extreme value and Marshall–Olkin copulas

- *Extreme value copulas* are the *copulas* C of limiting distributions of properly location-scale transformed componentwise maxima of a sequence of random vectors.
- They are given by

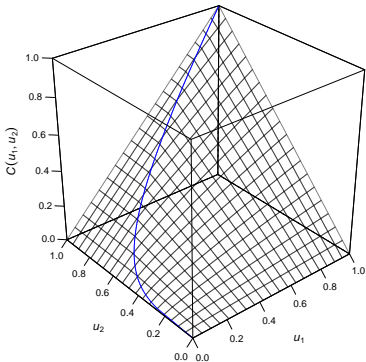
$$C(\mathbf{u}) = \left(\prod_{j=1}^d u_j \right)^{A\left(\frac{\log u_1}{\log \Pi(\mathbf{u})}, \dots, \frac{\log u_d}{\log \Pi(\mathbf{u})}\right)}$$

for a *Pickands dependence function* A ; see Ressel (2013) for a characterization of A .

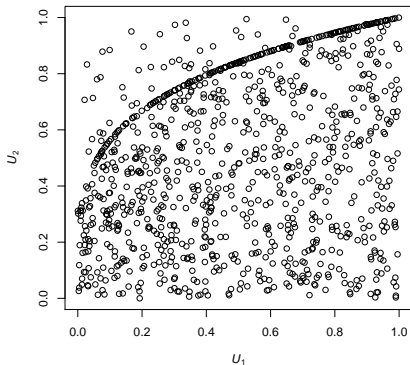
- **Examples:** Gumbel copula, Marshall–Olkin copulas.
- For more details, see Jaworski et al. (2010, Chapter 6).

Another class of copulas is given by $C(u_1, u_2) = \min\{u_1 u_2^{1-\alpha_2}, u_1^{1-\alpha_1} u_2\}$, $\alpha_1, \alpha_2 \in [0, 1]$. Such copulas are called *Marshall–Olkin copulas* and one of their characteristics is a *singular component* (set of Lebesgue measure 0 in which (U_1, U_2) take values with a positive probability).

MO copula with singular component ($\alpha_1 = 0.2, \alpha_2 = 0.8, \tau = 0.19$)



Scatter plot MO copula ($n = 1000, \alpha_1 = 0.2, \alpha_2 = 0.8, \tau = 0.19$)



Perfect dependence

Proof of Proposition 7.14. Consider Part 1); Part 2) works similarly.

“ \Rightarrow ” By assumption, $\mathbb{P}(X_2 \leq x)$ equals $\mathbb{P}(F_2^{\leftarrow}(1 - F_1(X_1)) \leq x) \stackrel{(\text{GI5})}{=}$

$\mathbb{P}(1 - F_1(X_1) \leq F_2(x)) = F_2(x)$. If (X_1, X_2) has copula C , then

$$C(\mathbf{u}) \stackrel{\text{L.7.6}}{\underset{\text{“only if”}}{=}} \mathbb{P}(F_1(X_1) \leq u_1, F_2(F_2^{\leftarrow}(1 - F_1(X_1))) \leq u_2)$$

$$\stackrel{(\text{GI4})}{=} \mathbb{P}(F_1(X_1) \leq u_1, 1 - F_1(X_1) \leq u_2)$$

$$= \mathbb{P}(1 - u_2 < U \leq u_1) = W(u_1, u_2) \quad \text{for } U \sim U(0, 1).$$

“ \Leftarrow ” $W(u_1, u_2) = 0$ for all $u_1, u_2 \in [0, 1]$ such that $u_1 + u_2 - 1 < 0$, so W puts no mass below the secondary diagonal. Similarly one shows that W puts no mass above the diagonal. This implies that W puts mass 1 on the secondary diagonal. Since $F_2 \uparrow \text{ran } X_2$, we thus obtain $\mathbb{P}(X_2 = F_2^{\leftarrow}(1 - F_1(X_1))) = \mathbb{P}(F_2(X_2) = F_2(F_2^{\leftarrow}(1 - F_1(X_1)))) \stackrel{(\text{GI4})}{=} \mathbb{P}(F_2(X_2) = 1 - F_1(X_1)) = \mathbb{P}(U_2 = 1 - U_1) = 1. \quad \square$

Proof of Proposition 7.15. Consider $T(u) = F_1^{\leftarrow}(u) + \cdots + F_d^{\leftarrow}(u) \nearrow$, left-continuous and let $U \sim U(0, 1)$. We first show that $F_{T(U)}^{\leftarrow}(u) = T(u)$, for all $u \in [0, 1]$.

$$1) \ T \text{ left-continuous} \Rightarrow T(u) \leq x \Leftrightarrow u \leq u_x := \sup\{u : T(u) \leq x\}$$

$$2) \ 1) \Rightarrow \{T(U) \leq x\} = \{U \leq u_x\} \Rightarrow F_{T(U)}^{\leftarrow}(x) = F_U(u_x) = u_x.$$

$$\Rightarrow F_{T(U)}^{\leftarrow}(u) \leq x \underset{(G15)}{\Leftrightarrow} F_{T(U)}^{\leftarrow}(x) \geq u \underset{2)}{\Leftrightarrow} u_x \geq u \underset{1)}{\Leftrightarrow} T(u) \leq x$$

Choosing $x = T(u)$ and $x = F_{T(U)}^{\leftarrow}(u)$ in the last line, we see that $F_{T(U)}^{\leftarrow}(u) = T(u)$. Now Proposition 7.14 2) implies that

$$(X_1, \dots, X_d) \stackrel{d}{=} (F_1^{\leftarrow}(U), \dots, F_d^{\leftarrow}(U)),$$

so that

$$F_{\sum_{j=1}^d X_j}(x) = \mathbb{P}\left(\sum_{j=1}^d X_j \leq x\right) = \mathbb{P}\left(\sum_{j=1}^d F_j^{\leftarrow}(U) \leq x\right) = \mathbb{P}(T(U) \leq x)$$

$$\text{and thus } F_{\sum_{j=1}^d X_j}^{\leftarrow}(\alpha) = F_{T(U)}^{\leftarrow}(\alpha) = T(\alpha) = \sum_{j=1}^d F_j^{\leftarrow}(\alpha).$$

□

Linear correlation

Proof of Proposition 7.16. Let (X'_1, X'_2) be an iid-copy of (X_1, X_2) . Consider

$$\begin{aligned} & 2 \operatorname{cov}(X_1, X_2) \\ &= \mathbb{E}((X_1 - \mathbb{E}X_1)(X_2 - \mathbb{E}X_2)) + \mathbb{E}((X'_1 - \mathbb{E}X'_1)(X'_2 - \mathbb{E}X'_2)) \\ &= \mathbb{E}(((X_1 - \mathbb{E}X_1) - (X'_1 - \mathbb{E}X'_1)) \cdot ((X_2 - \mathbb{E}X_2) - (X'_2 - \mathbb{E}X'_2))) \\ &\stackrel{\text{check}}{=} \mathbb{E}((X_1 - X'_1)(X_2 - X'_2)). \end{aligned}$$

With $b - a = \int_{-\infty}^{\infty} (I_{\{a \leq x\}} - I_{\{b \leq x\}}) dx$ for all $a, b \in \mathbb{R}$, we obtain that

$$\begin{aligned} & 2 \operatorname{cov}(X_1, X_2) \\ &= \mathbb{E} \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (I_{\{X'_1 \leq x_1\}} - I_{\{X_1 \leq x_1\}})(I_{\{X'_2 \leq x_2\}} - I_{\{X_2 \leq x_2\}}) dx_1 dx_2 \right] \\ &\stackrel{\text{Fubini}}{=} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbb{E}(\dots) dx_1 dx_2 \stackrel{\text{multiply ind.}}{=} 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (F(x_1, x_2) - F_1(x_1)F_2(x_2)) dx_1 dx_2. \end{aligned}$$

□

Rank correlation

To overcome (some) of the deficiencies of ρ , Scarsini (1984) introduced:

Definition A.16 (Rank correlation coefficient)

A measure of association $\kappa = \kappa(X_1, X_2) = \kappa(C)$ between two continuously distributed random variables X_1 and X_2 with copula C is a *rank correlation coefficient (or measure of concordance)* if

- 1) κ exists for every pair (X_1, X_2) of cont. distributed random variables;
- 2) $-1 \leq \kappa \leq 1$, $\kappa(W) = -1$, and $\kappa(M) = 1$;
- 3) $\kappa(X_1, X_2) = \kappa(X_2, X_1)$;
- 4) X_1 and X_2 being independent implies $\kappa(X_1, X_2) = \kappa(\Pi) = 0$;
- 5) $\kappa(-X_1, X_2) = -\kappa(X_1, X_2)$;
- 6) $C_1(\mathbf{u}) \leq C_2(\mathbf{u})$ for all $\mathbf{u} \in [0, 1]^2$ implies $\kappa(C_1) \leq \kappa(C_2)$;
- 7) $C_n \rightarrow C$ ($n \rightarrow \infty$) pointwise implies $\lim_{n \rightarrow \infty} \kappa(C_n) = \kappa(C)$.

Proposition A.17 (Basic properties of κ)

Let κ be a **rank correlation coefficient** for two continuously distributed random variables $X_1 \sim F_1$ and $X_2 \sim F_2$. Then

- 1) $\kappa(X_1, X_2) = \kappa(C)$ (κ only depends on C).
- 2) if T_j is a **strictly increasing function** on $\text{ran } X_j$, $j \in \{1, 2\}$, then $\kappa(T_1(X_1), T_2(X_2)) = \kappa(X_1, X_2)$.

Proof.

- 1) Set $(U_1, U_2) = (F_1(X_1), F_2(X_2))$. By the **invariance principle**, (X_1, X_2) and (U_1, U_2) have the **same copula C** . Thus, by 6), $\kappa(U_1, U_2) \leq \kappa(X_1, X_2)$, but also $\kappa(X_1, X_2) \leq \kappa(U_1, U_2)$, so $\kappa(X_1, X_2) = \kappa(U_1, U_2)$ (\Rightarrow only depends on C).
- 2) **Invariance principle** \Rightarrow The copula C of (X_1, X_2) equals the copula of $(T_1(X_1), T_2(X_2))$. Hence $\kappa(T_1(X_1), T_2(X_2)) = \kappa(C) = \kappa(X_1, X_2)$.

□

Kendall's tau and Spearman's rho

Proof of Proposition 7.20. Let (X'_1, X'_2) be an independent copy of (X_1, X_2) . Then

$$\begin{aligned}\rho_\tau &= \mathbb{P}((X_1 - X'_1)(X_2 - X'_2) > 0) - \mathbb{P}((X_1 - X'_1)(X_2 - X'_2) < 0) \\ &= 2 \underbrace{\mathbb{P}((X_1 - X'_1)(X_2 - X'_2) > 0)}_{=2\mathbb{P}(X_1 < X'_1, X_2 < X'_2)} - 1 = 4\mathbb{P}(U_1 \leq U'_1, U_2 \leq U'_2) - 1 \\ &= 4 \int_0^1 \int_0^1 \mathbb{P}(U_1 \leq u_1, U_2 \leq u_2) dC(u_1, u_2) - 1\end{aligned}\quad \square$$

For computing ρ_τ , $\int_{[0,1]^2} C(\mathbf{u}) d\tilde{C}(\mathbf{u}) = \frac{1}{2} - \int_{[0,1]^2} D_1 C(\mathbf{u}) D_2 \tilde{C}(\mathbf{u}) d\mathbf{u}$ is often helpful; see Li et al. (2002). One can also show that for any bivariate copulas C, \tilde{C} , $\int_{[0,1]^2} C(\mathbf{u}) d\tilde{C}(\mathbf{u}) = \int_{[0,1]^2} \tilde{C}(\mathbf{u}) dC(\mathbf{u})$.

Rank correlation for elliptical copulas

Lemma A.18

Let $\mathbf{X} \sim E_2(\mathbf{0}, \Sigma, \psi)$ with $\mathbb{P}(\mathbf{X} = \mathbf{0}) = 0$ and $\rho = P_{12} = \text{corr}(\Sigma)_{12}$. Then

$$\mathbb{P}(X_1 > 0, X_2 > 0) = \frac{1}{4} + \frac{\arcsin \rho}{2\pi}.$$

Proof.

- Note that $\mathbf{Y} = \begin{pmatrix} 1/\sqrt{\sigma_{11}} & 0 \\ 0 & 1/\sqrt{\sigma_{22}} \end{pmatrix} \mathbf{X} \sim E_2(\mathbf{0}, P, \psi)$ with $P = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$.
- Let $A = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix}$ so that $AA' = P$. Then $\mathbf{Y} \stackrel{d}{=} R\mathbf{A}\mathbf{U} \stackrel{d}{=} R\mathbf{A} \begin{pmatrix} \cos \Theta \\ \sin \Theta \end{pmatrix}$, $\Theta \sim U(-\pi, \pi)$ independent of R .
- With $\varphi = \arcsin \rho$, we have $\mathbf{Y} \stackrel{d}{=} R \begin{pmatrix} \cos \Theta \\ \sin \varphi \cos \Theta + \cos \varphi \sin \Theta \end{pmatrix} = \begin{pmatrix} \cos \Theta \\ \sin(\varphi + \Theta) \end{pmatrix}$.
- Thus $\mathbb{P}(X_1 > 0, X_2 > 0) = \mathbb{P}(Y_1 > 0, Y_2 > 0) = \mathbb{P}(\cos \Theta > 0, \sin(\varphi + \Theta) > 0) = \mathbb{P}(\Theta \in (-\frac{\pi}{2}, \frac{\pi}{2}), \varphi + \Theta \in (0, \pi)) = \mathbb{P}(\Theta \in (-\frac{\pi}{2}, \frac{\pi}{2}), \Theta \in (-\varphi, \pi - \varphi)) = \mathbb{P}(\Theta \in (-\varphi, \frac{\pi}{2}]) = (\frac{\pi}{2} - (-\varphi))/(2\pi)$. \square

Lemma A.19 (Representation of Spearman's rho)

Let $(U_1, U_2) \sim C$ and $\tilde{U}_1, \bar{U}_2 \stackrel{\text{ind.}}{\sim} U(0, 1)$ be independent. Then $\rho_S = \rho_S(U_1, U_2) = 12\mathbb{P}(U_1 \leq \tilde{U}_1, U_2 \leq \bar{U}_2) - 3$.

Proof. $12\mathbb{P}(U_1 \leq \tilde{U}_1, U_2 \leq \bar{U}_2) - 3 = 12\mathbb{E}(\mathbb{P}(\tilde{U}_1 > U_1, \bar{U}_2 > U_2 \mid U_1, U_2)) - 3 = 12\mathbb{E}((1 - U_1)(1 - U_2)) - 3 = 12\mathbb{E}(U_1 U_2) - 3 = \rho_S(U_1, U_2)$. \square
ind.

Proof of Proposition 7.26. $\mathbf{X} \stackrel{d}{=} \sqrt{W}\mathbf{Z}$ for $\mathbf{Z} \sim N_2(\mathbf{0}, P)$. Let $\tilde{Z}, \bar{Z} \sim N(0, 1)$ and assume $\mathbf{Z}, \tilde{Z}, \bar{Z}, W, \tilde{W}$ and \bar{W} are all independent. Let

$$\tilde{X} = \sqrt{\tilde{W}}\tilde{Z}, \quad \bar{X} = \sqrt{\bar{W}}\bar{Z},$$

$$Y_1 = X_1 - \tilde{X} = \sqrt{W}Z_1 - \sqrt{\tilde{W}}\tilde{Z},$$

$$Y_2 = X_2 - \bar{X} = \sqrt{W}Z_2 - \sqrt{\bar{W}}\bar{Z}.$$

$$\rho_S(X_1, X_2) \stackrel{\text{L. A.19}}{\underset{\Phi^{-1}}{=}} 12\mathbb{P}(X_1 \leq \tilde{X}_1, X_2 \leq \bar{X}_2) - 3$$

$$\begin{aligned}
&= 6\mathbb{P}((X_1 - \tilde{X}_1)(X_2 - \bar{X}_2) > 0) - 3 \\
&= 3(2\mathbb{E}(\mathbb{P}(Y_1 Y_2 > 0 \mid W, \tilde{W}, \bar{W})) - 1) \\
&= 3(4\mathbb{E}(\mathbb{P}(Y_1 > 0, Y_2 > 0 \mid W, \tilde{W}, \bar{W})) - 1).
\end{aligned}$$

Now note that $\mathbf{Y} \mid W, \tilde{W}, \bar{W} \sim \mathbf{N}_2(\mathbf{0}, \begin{pmatrix} W+\tilde{W} & W\rho \\ W\rho & W+\bar{W} \end{pmatrix})$ with $\rho(Y_1, Y_2) = \frac{W\rho}{\sqrt{(W+\tilde{W})(W+\bar{W})}}$. Apply Lemma A.18 to see that

$$\rho_S(X_1, X_2) = 3\left(4\mathbb{E}\left(\frac{1}{4} + \frac{\arcsin \rho}{2\pi}\right) - 1\right) = \frac{12}{2\pi}\mathbb{E}(\arcsin \rho(Y_1, Y_2)).$$

For Gauss copulas, $F_W(x) = I_{\{x \geq 1\}}$, thus $W = \tilde{W} = \bar{W} = 1$ a.s. and the result follows. \square

Proof of Proposition 7.27.

- Let (X'_1, X'_2) be an independent copy of (X_1, X_2) . We have already seen that $\rho_\tau = 2\mathbb{P}((X_1 - X'_1)(X_2 - X'_2) > 0) - 1$.
- With $\mathbf{X} \stackrel{d}{=} R\mathbf{A}\mathbf{U}$ and $\mathbf{X}' \stackrel{d}{=} R'\mathbf{A}\mathbf{U}' (\stackrel{d}{=} -\mathbf{X}')$ we have $\mathbf{Y} = \mathbf{X} - \mathbf{X}' \stackrel{d}{=} \mathbf{0} + \mathbf{A}(R\mathbf{U} - R'\mathbf{U}')$. Note that the characteristic function of $-\mathbf{X}'$ is

$\phi_{-X'}(\mathbf{t}) = \phi_{X'}(\mathbf{t}) = \phi_X(\mathbf{t})$ so that $\phi_Y(\mathbf{t}) \stackrel{\text{ind.}}{=} \phi_X(\mathbf{t})\phi_{-X'}(\mathbf{t}) = \phi_X(\mathbf{t})^2$,
hence $\mathbf{Y} \sim E_2(\mathbf{0}, P, \psi^2)$.

- We thus obtain that

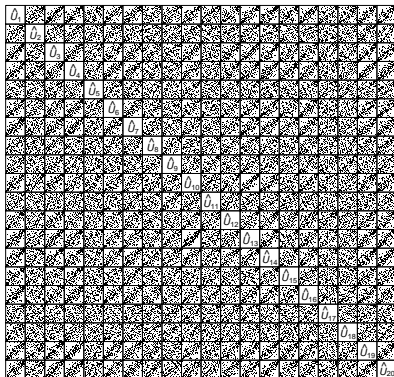
$$\begin{aligned}\rho_\tau &= 2\mathbb{P}(\mathbf{Y}_1\mathbf{Y}_2 > \mathbf{0}) - 1 = 2(\mathbb{P}(Y_1 > 0, Y_2 > 0) + \mathbb{P}(Y_1 < 0, Y_2 < 0)) - 1 \\ &= 4\mathbb{P}(\mathbf{Y} > \mathbf{0}) - 1 \stackrel{\text{cont.}}{\stackrel{\text{L.A.18}}{=}} \frac{2}{\pi} \arcsin \rho.\end{aligned}\quad \square$$

For a generalization to componentwise n.d. \mathbf{X} , see Lindskog et al. (2003).

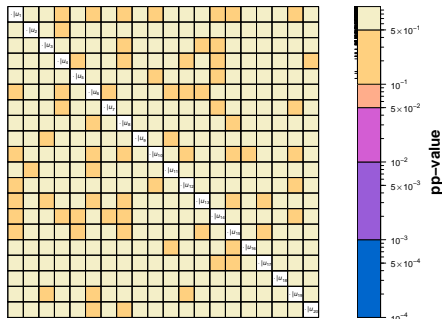
Goodness-of-fit

A graphical goodness-of-fit approach by Hofert and Mächler (2014) based on daily log-returns of the SMI from 2011-09-09 to 2012-03-28.

Pseudo-observations of the log-returns of the SMI



Pairwise Rosenblatt transformed pseudo-observations
to test $H_0: C$ is $t_{11.96}$



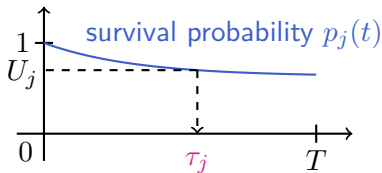
pp-values: minimum: 0.12; global (Bonferroni/Holm): 1

How intensity-/copula-based default models work

Intensity-based **default model**:

$$p_j(t) = \exp\left(-\int_0^t \lambda_j(s) ds\right)$$

$$\tau_j = \inf\{t \geq 0 : p_j(t) \leq U_j\}$$



Note: $\lambda_U = 0$ (as for the Gauss copula!)

\Rightarrow (Almost) **no joint defaults!** (p_j typically **very flat**)

Copulas for the **triggers** U :

- 1) Li (2000): **Gauss** (Sibuya (1960): $\lambda_U = 0$)
- 2) Schönbucher and Schubert (2001): **Archimedean** ($\lambda_U > 0$)
- 3) Hofert and Scherer (2011): **nested Archimedean** ($\lambda_U > 0$, **hierarchies**)

Typical application: **CDO pricing models** based on iTraxx data.

A.8 Aggregate risk

Proof of Proposition 8.11.

1) Recall that $L \stackrel{d}{=} F_L^{\leftarrow}(U)$, $U \sim U(0, 1)$. Therefore,

$$\begin{aligned}\frac{\mathbb{E}((L - F_L^{\leftarrow}(\alpha))_+)}{1 - \alpha} &= \frac{1}{1 - \alpha} \int_0^1 (F_L^{\leftarrow}(u) - F_L^{\leftarrow}(\alpha))_+ du \\ &= \frac{1}{1 - \alpha} \int_{\alpha}^1 (F_L^{\leftarrow}(u) - F_L^{\leftarrow}(\alpha)) du \\ &= \text{ES}_{\alpha}(L) - F_L^{\leftarrow}(\alpha).\end{aligned}$$

2) First note that

$$\begin{aligned}\mathbb{E}((L - F_L^{\leftarrow}(\alpha))_+) &= \mathbb{E}((L - F_L^{\leftarrow}(\alpha))I_{\{L > F_L^{\leftarrow}(\alpha)\}}) \\ &= \mathbb{E}(LI_{\{L > F_L^{\leftarrow}(\alpha)\}}) - F_L^{\leftarrow}(\alpha)\mathbb{E}(I_{\{L > F_L^{\leftarrow}(\alpha)\}}) \\ &= \mathbb{E}(LI_{\{L > F_L^{\leftarrow}(\alpha)\}}) - F_L^{\leftarrow}(\alpha)\bar{F}_L(F_L^{\leftarrow}(\alpha)).\end{aligned}$$

Now apply 1), divide by $1 - \alpha$ and add $F_L^{\leftarrow}(\alpha)$.

Proof of Corollary 8.12.

1) Since $\bar{F}_L(F_L^{\leftarrow}(\alpha)) = 1 - F_L(F_L^{\leftarrow}(\alpha)) = 1 - \alpha$ for all $\alpha \in \text{ran } F_L \cup \{\inf F_L, \sup F_L\} \supseteq (0, 1)$, the claim follows from Proposition 8.11 2).

2) First note that

$$\begin{aligned} F_{L|L>F_L^{\leftarrow}(\alpha)}(x) &= \mathbb{P}(L \leq x \mid L > F_L^{\leftarrow}(\alpha)) = \frac{\mathbb{P}(F_L^{\leftarrow}(\alpha) < L \leq x)}{\mathbb{P}(L > F_L^{\leftarrow}(\alpha))} \\ &= \frac{F_L(x) - F_L(F_L^{\leftarrow}(\alpha))}{1 - F_L(F_L^{\leftarrow}(\alpha))} I_{\{x > F_L^{\leftarrow}(\alpha)\}} = \frac{F_L(x) - \alpha}{1 - \alpha} I_{\{x > F_L^{\leftarrow}(\alpha)\}}, \end{aligned}$$

where the latter equality holds since $\alpha \in \text{ran } F_L$. This implies

$$\begin{aligned} \mathbb{E}(L \mid L > F_L^{\leftarrow}(\alpha)) &= \int_{\mathbb{R}} x dF_{L|L>F_L^{\leftarrow}(\alpha)}(x) = \int_{F_L^{\leftarrow}(\alpha)}^{\infty} x \frac{dF_L(x)}{1 - \alpha} \\ &= \frac{\mathbb{E}(L I_{\{L > F_L^{\leftarrow}(\alpha)\}})}{1 - \alpha} \stackrel{1)}{=} \text{ES}_{\alpha}(L). \end{aligned} \quad \square$$

We now present an elementary proof for subadditivity of ES. We start with some auxiliary results.

Lemma A.20

$\mathbb{P}(L = F_L^{\leftarrow}(\alpha)) = 0$ implies $F_L(F_L^{\leftarrow}(\alpha)) = \alpha$.

Proof. $F_L(F_L^{\leftarrow}(\alpha)) - F_L(F_L^{\leftarrow}(\alpha)-) = \mathbb{P}(L = F_L^{\leftarrow}(\alpha)) = 0$, so F_L does not jump in $F_L^{\leftarrow}(\alpha)$. By definition of F_L^{\leftarrow} , $F_L(F_L^{\leftarrow}(\alpha)) \geq \alpha$ and $F_L(F_L^{\leftarrow}(\alpha)-) < \alpha$, which implies $F_L(F_L^{\leftarrow}(\alpha)) = \alpha$. \square

For the following result let

$$I_{\{L>q\}}^{(\alpha)} = \begin{cases} I_{\{L>q\}}, & \text{if } \mathbb{P}(L = q) = 0, \\ I_{\{L>q\}} + \frac{1-\alpha-\bar{F}_L(q)}{\mathbb{P}(L=q)} I_{\{L=q\}}, & \text{if } \mathbb{P}(L = q) > 0. \end{cases}$$

Lemma A.21 (Properties of $I_{\{L > F_L^{\leftarrow}(\alpha)\}}^{(\alpha)}$)

- 1) $I_{\{L > F_L^{\leftarrow}(\alpha)\}}^{(\alpha)} \in [0, 1]$
- 2) $\mathbb{E}(I_{\{L > F_L^{\leftarrow}(\alpha)\}}^{(\alpha)}) = 1 - \alpha$

Proof.

- 1) If $\mathbb{P}(L = F_L^{\leftarrow}(\alpha)) = 0$ we are done, so consider $\mathbb{P}(L = F_L^{\leftarrow}(\alpha)) > 0$.
 On the set of all $\omega \in \Omega$ such that $L(\omega) > F_L^{\leftarrow}(\alpha)$, we are again done.
 Now consider all $\omega \in \Omega$ such that $L(\omega) = F_L^{\leftarrow}(\alpha)$. Then $I_{\{L > F_L^{\leftarrow}(\alpha)\}}^{(\alpha)} = \frac{1 - \alpha - \bar{F}_L(F_L^{\leftarrow}(\alpha))}{\mathbb{P}(L = F_L^{\leftarrow}(\alpha))}$. By definition, $F_L(F_L^{\leftarrow}(\alpha)) \geq \alpha$, so $\bar{F}_L(F_L^{\leftarrow}(\alpha)) \leq 1 - \alpha$, thus $I_{\{L > F_L^{\leftarrow}(\alpha)\}}^{(\alpha)} \geq 0$. Also, $F_L(F_L^{\leftarrow}(\alpha) -) < \alpha$, so $I_{\{L > F_L^{\leftarrow}(\alpha)\}}^{(\alpha)} = \frac{1 - \alpha - (1 - F_L(F_L^{\leftarrow}(\alpha)))}{\mathbb{P}(L = F_L^{\leftarrow}(\alpha))} = \frac{F_L(F_L^{\leftarrow}(\alpha)) - \alpha}{F_L(F_L^{\leftarrow}(\alpha)) - F_L(F_L^{\leftarrow}(\alpha) -)} < 1$.

2) We have

$$\mathbb{E}(I_{\{L>q\}}^{(\alpha)}) = \begin{cases} \bar{F}_L(q), & \text{if } \mathbb{P}(L = q) = 0, \\ \bar{F}_L(q) + \frac{1-\alpha-\bar{F}_L(q)}{\mathbb{P}(L=q)}\mathbb{P}(L = q) = 1 - \alpha, & \text{if } \mathbb{P}(L = q) > 0. \end{cases}$$

Consider $\mathbb{P}(L = q) = 0$. Since $q = F_L^{\leftarrow}(\alpha)$, Lemma A.20 implies that $\bar{F}_L(q) = 1 - F_L(F_L^{\leftarrow}(\alpha)) = 1 - \alpha$. Thus $\mathbb{E}(I_{\{L>q\}}^{(\alpha)}) = 1 - \alpha$. \square

Lemma A.22 (Representation of ES_α in terms of $I_{\{L>F_L^{\leftarrow}(\alpha)\}}^{(\alpha)}$)

$$\text{ES}_\alpha(L) = \frac{\mathbb{E}(LI_{\{L>F_L^{\leftarrow}(\alpha)\}}^{(\alpha)})}{1 - \alpha}$$

Proof.

- If $\mathbb{P}(L = F_L^{\leftarrow}(\alpha)) = 0$, Lemma A.20 implies that $\bar{F}_L(F_L^{\leftarrow}(\alpha)) = 1 - \alpha$. By Proposition 8.11 2) and since $\mathbb{P}(L = F_L^{\leftarrow}(\alpha)) = 0$,

$$\text{ES}_\alpha(L) = \frac{\mathbb{E}(LI_{\{L>F_L^{\leftarrow}(\alpha)\}}^{(\alpha)}) + F_L^{\leftarrow}(\alpha)(1 - \alpha - (1 - \alpha))}{1 - \alpha}$$

$$= \frac{\mathbb{E}(LI_{\{L > F_L^{\leftarrow}(\alpha)\}})}{1 - \alpha} = \frac{\mathbb{E}(LI_{\{L > F_L^{\leftarrow}(\alpha)\}}^{(\alpha)})}{1 - \alpha}.$$

- If $\mathbb{P}(L = F_L^{\leftarrow}(\alpha)) > 0$, $\mathbb{E}(LI_{\{L > F_L^{\leftarrow}(\alpha)\}}^{(\alpha)})$ equals

$$\begin{aligned} \mathbb{E}(LI_{\{L > F_L^{\leftarrow}(\alpha)\}}) + \frac{1 - \alpha - \bar{F}_L(F_L^{\leftarrow}(\alpha))}{\mathbb{P}(L = F_L^{\leftarrow}(\alpha))} \underbrace{\mathbb{E}(LI_{\{L = F_L^{\leftarrow}(\alpha)\}})}_{= \mathbb{E}(F_L^{\leftarrow}(\alpha) I_{\{L = F_L^{\leftarrow}(\alpha)\}}) = F_L^{\leftarrow}(\alpha) \mathbb{P}(L = F_L^{\leftarrow}(\alpha))} \\ = \mathbb{E}(F_L^{\leftarrow}(\alpha) I_{\{L = F_L^{\leftarrow}(\alpha)\}}) = F_L^{\leftarrow}(\alpha) \mathbb{P}(L = F_L^{\leftarrow}(\alpha)) \end{aligned}$$

So, $\mathbb{E}(LI_{\{L > F_L^{\leftarrow}(\alpha)\}}^{(\alpha)}) = \mathbb{E}(LI_{\{L > F_L^{\leftarrow}(\alpha)\}}) + F_L^{\leftarrow}(\alpha)(1 - \alpha - \bar{F}_L(F_L^{\leftarrow}(\alpha)))$,
which, by Proposition 8.11 2), equals $(1 - \alpha) \text{ES}_\alpha(L)$. \square

Proposition A.23 (Subadditivity of ES)

ES_α is **subadditive** for all $\alpha \in (0, 1)$.

Proof. It suffices to show that

$$(1 - \alpha)(\text{ES}_\alpha(L_1) + \text{ES}_\alpha(L_2) - \text{ES}_\alpha(L_1 + L_2)) \geq 0.$$

By Lemma A.22, this equals

$$\begin{aligned}
 & \left(\sum_{j=1}^2 \mathbb{E}(L_j I_{\{L_j > F_{L_j}^{\leftarrow}(\alpha)\}}^{(\alpha)}) \right) - \mathbb{E}((L_1 + L_2) I_{\{L_1 + L_2 > F_{L_1 + L_2}^{\leftarrow}(\alpha)\}}^{(\alpha)}) \\
 & \stackrel{\text{Linearity}}{=} \sum_{j=1}^2 \mathbb{E}(L_j (I_{\{L_j > F_{L_j}^{\leftarrow}(\alpha)\}}^{(\alpha)} - I_{\{L_1 + L_2 > F_{L_1 + L_2}^{\leftarrow}(\alpha)\}}^{(\alpha)})). \tag{138}
 \end{aligned}$$

- $L_j > F_{L_j}^{\leftarrow}(\alpha) \Rightarrow I_{\{L_j > F_{L_j}^{\leftarrow}(\alpha)\}}^{(\alpha)} - I_{\{L_1 + L_2 > F_{L_1 + L_2}^{\leftarrow}(\alpha)\}}^{(\alpha)} = 1 - \dots \geq 0$
- $L_j < F_{L_j}^{\leftarrow}(\alpha) \Rightarrow I_{\{L_j > F_{L_j}^{\leftarrow}(\alpha)\}}^{(\alpha)} - I_{\{L_1 + L_2 > F_{L_1 + L_2}^{\leftarrow}(\alpha)\}}^{(\alpha)} = 0 - \dots \leq 0$

In both cases, we make the expectations in (138) **smaller** by replacing L_j by $F_{L_j}^{\leftarrow}(\alpha)$. Hence

$$\begin{aligned}
 (138) & \geq \sum_{j=1}^2 F_{L_j}^{\leftarrow}(\alpha) \underbrace{\mathbb{E}(I_{\{L_j > F_{L_j}^{\leftarrow}(\alpha)\}}^{(\alpha)} - I_{\{L_1 + L_2 > F_{L_1 + L_2}^{\leftarrow}(\alpha)\}}^{(\alpha)})}_{\stackrel{\text{Lem. A.21 2)}}{=} (1-\alpha) - (1-\alpha) = 0} \geq 0. \quad \square
 \end{aligned}$$

Proof of Proposition 8.16. $\nu([0, t]) := \phi(t)$ defines a measure on $[0, 1]$. For every function $f : [0, 1] \rightarrow \mathbb{R}$ we have

$$\int_0^1 f(\alpha) d\nu(\alpha) = f(0)\phi(0) + \int_0^1 f(\alpha) d\phi(\alpha).$$

Define a further measure μ on $[0, 1]$ by $\frac{d\mu}{d\nu}(\alpha) = 1 - \alpha$, that is, $\int_0^1 f(\alpha) d\mu(\alpha) = f(0)\phi(0) + \int_0^1 f(\alpha)(1 - \alpha) d\phi(\alpha)$. Using this for $f \equiv 1$,

$$\begin{aligned} \mu([0, 1]) &= 1 \cdot \phi(0) + \int_0^1 1 \cdot (1 - \alpha) d\phi(\alpha) \stackrel{\text{by parts}}{=} \phi(0) + \int_0^1 \phi(\alpha) d\alpha - \phi(0) \\ &= \int_0^1 \phi(\alpha) d\alpha = D(1) - D(0) = 1, \end{aligned}$$

so μ is a probability measure. By Fubini's Theorem we have that

$$\begin{aligned} \varrho(L) &= \int_0^1 q_u(L) \phi(u) du = \int_0^1 q_u(L) \int_0^u 1 d\nu(\alpha) du \\ &= \int_0^1 \int_0^1 q_u(L) 1_{\{\alpha \leq u\}} d\nu(\alpha) du \stackrel{\text{Fubini}}{=} \int_0^1 \int_{\alpha}^1 q_u(L) du d\nu(\alpha) \\ &= \int_0^1 \text{ES}_{\alpha}(L)(1 - \alpha) d\nu(\alpha) = \int_0^1 \text{ES}_{\alpha}(L) d\mu(\alpha). \end{aligned}$$

□

Proof of Proposition 8.19. The expectile y solves $\alpha\mathbb{E}(|L - y|) = \alpha\mathbb{E}((L - y)^+) + \alpha\mathbb{E}((L - y)^-)$ $\stackrel{(44)}{=} \mathbb{E}((L - y)^-) = \int_{-\infty}^y (y - x) dF_L(x) = yF_L(y) - \mu(y)$, where

$$\begin{aligned}\mathbb{E}(|L - y|) &= \int_{-\infty}^y (y - x) dF_L(x) + \int_y^{\infty} (x - y) dF_L(x) \\ &= 2 \int_{-\infty}^y (y - x) dF_L(x) + \int_{-\infty}^{\infty} (x - y) dF_L(x) \\ &= 2(yF_L(y) - \mu(y)) + \mu - y\end{aligned}$$

and hence $\alpha = \tilde{F}_L(y)$ with \tilde{F}_L as defined in (47). We have left to show that \tilde{F}_L is a df. The derivative of \tilde{F}_L can be easily computed to be

$$\tilde{f}_L(y) = \frac{\mu F_L(y) - \mu(y)}{(2(yF_L(y) - \mu(y)) + \mu - y)^2} = \frac{F_L(y)(\mu - \mathbb{E}(L | L \leq y))}{(2(yF_L(y) - \mu(y)) + \mu - y)^2},$$

which is nonnegative for all y and strictly positive on $D = \{y : 0 < F_L(y) < 1\}$; hence \tilde{F}_L is increasing for all y and strictly increasing on D . It is easy to check that for $y_0 = \inf D$ and $y_1 = \sup D$, $[y_0, y_1]$ is the support of \tilde{F}_L and $\lim_{y \rightarrow y_0} \tilde{F}_L(y) = 0$ and $\lim_{y \rightarrow y_1} \tilde{F}_L(y) = 1$. □

Proof of Proposition 8.21. For $L \in \mathcal{M}$ and $y \in \mathbb{R}$ define

$$\begin{aligned} g(L, y, \alpha) &= \alpha \mathbb{E}((L - y)^+) - (1 - \alpha) \mathbb{E}((L - y)^-) \\ &= (2\alpha - 1) \mathbb{E}((L - y)^+) + (1 - \alpha) \mathbb{E}(L - y). \end{aligned}$$

For fixed L , g is a decreasing function of y and, for fixed y , g is monotonic in L so that $L_1 \leq L_2 \Rightarrow g(L_1, y, \alpha) \leq g(L_2, y, \alpha)$.

Translation invariance and positive homogeneity follow easily from the fact that if $g(L, y, \alpha) = 0$ (i.e. if $e_\alpha(L) = y$) then $g(L + m, y + m, \alpha) = 0$ for $m \in \mathbb{R}$ and $g(\lambda L, \lambda y, \alpha) = 0$ for $\lambda > 0$.

For monotonicity, fix α and let $y_1 = e_\alpha(L_1)$, $y_2 = e_\alpha(L_2)$. If $L_2 \geq L_1$ then $g(L_2, y_1, \alpha) \geq g(L_1, y_1, \alpha) = 0 = g(L_2, y_2, \alpha)$. Since g is decreasing in y , it must be the case that $y_2 \geq y_1$.

For subadditivity, again let $y_1 = e_\alpha(L_1)$, $y_2 = e_\alpha(L_2)$. We have that

$$\begin{aligned} & g(L_1 + L_2, y_1 + y_2, \alpha) \\ &= (2\alpha - 1)\mathbb{E}((L_1 + L_2 - y_1 - y_2)^+) + (1 - \alpha)\mathbb{E}(L_1 + L_2 - y_1 - y_2) \\ &= (2\alpha - 1)\mathbb{E}((L_1 + L_2 - y_1 - y_2)^+) + (1 - \alpha)\mathbb{E}(L_1 - y_1) + (1 - \alpha)\mathbb{E}(L_2 - y_2) \end{aligned}$$

and, since $(2\alpha - 1)\mathbb{E}((L_i - y_i)^+) + (1 - \alpha)\mathbb{E}(L_i - y_i) = 0$ for $i = 1, 2$, we get

$$\begin{aligned} & g(L_1 + L_2, y_1 + y_2, \alpha) \\ &= (2\alpha - 1) \left(\mathbb{E}((L_1 + L_2 - y_1 - y_2)^+) - \mathbb{E}((L_1 - y_1)^+) - \mathbb{E}((L_2 - y_2)^+) \right) \end{aligned}$$

where we have used the fact that $(x_1 + x_2)^+ \leq x_1^+ + x_2^+$. So $g(L_1 + L_2, y_1 + y_2, \alpha) \leq 0$ and thus, as g is decreasing in y , $g(L_1 + L_2, y, \alpha) = 0$ for some $y \leq y_1 + y_2 = e_\alpha(L_1) + e_\alpha(L_2)$ with $y = e_\alpha(L_1 + L_2)$. \square

Proof of Euler's rule. Let $f : \mathbb{R}^d \setminus \{\mathbf{0}\} \rightarrow \mathbb{R}$ be differentiable and positive homogeneous of order $r > 0$, so that

$$f(t\mathbf{x}) = t^r f(\mathbf{x}), \quad \text{for all } t > 0, \mathbf{x} \in \mathbb{R}^d \setminus \{\mathbf{0}\}.$$

Differentiating w.r.t. t (use the chain rule on the left-hand side) leads to $rt^{r-1}f(\mathbf{x}) = \nabla f(t\mathbf{x})\mathbf{x} = \sum_{j=1}^d \frac{\partial f}{\partial x_j}(t\mathbf{x}) \cdot x_j$ and thus and thus (for $t = 1$)

$$rf(\mathbf{x}) = \nabla f(\mathbf{x})\mathbf{x} = \sum_{j=1}^d \frac{\partial f}{\partial x_j}(\mathbf{x}) \cdot x_j. \quad (139)$$

On the other hand, if $f : \mathbb{R}^d \setminus \{\mathbf{0}\} \rightarrow \mathbb{R}$ fulfills (139) and is differentiable, consider $g(t) = f(t\mathbf{x})$, $t > 0$. Then g satisfies

$$g'(t) = \nabla f(t\mathbf{x})\mathbf{x} = \frac{1}{t} \sum_{j=1}^d \frac{\partial f}{\partial x_j}(t\mathbf{x}) \cdot tx_j \stackrel{(139)}{=} \frac{1}{t} rf(t\mathbf{x}) = \frac{r}{t} g(t)$$

with the initial value $g(1) = f(\mathbf{x})$. The only solution is $g(t) = t^r f(\mathbf{x})$, so $f(t\mathbf{x}) = t^r f(\mathbf{x})$ and thus f is homogeneous of order $r > 0$. \square