

6 Multivariate models

- 6.1 Basics of multivariate modelling
- 6.2 Normal mixture distributions
- 6.3 Spherical and elliptical distributions
- 6.4 Dimension reduction techniques

6.1 Basics of multivariate modelling

6.1.1 Random vectors and their distributions

Joint and marginal distributions

- Let $\mathbf{X} = (X_1, \dots, X_d) : \Omega \rightarrow \mathbb{R}^d$ be a d -dimensional *random vector* (representing risk-factor changes, risks, etc.).
- The *(joint) distribution function (df) F of \mathbf{X}* is

$$F(\mathbf{x}) = F_{\mathbf{X}}(\mathbf{x}) = \mathbb{P}(\mathbf{X} \leq \mathbf{x}) = \mathbb{P}(X_1 \leq x_1, \dots, X_d \leq x_d), \quad \mathbf{x} \in \mathbb{R}^d.$$

- The *j th margin or marginal df F_j of \mathbf{X}* is

$$\begin{aligned} F_j(x_j) &= \mathbb{P}(X_j \leq x_j) \\ &= \mathbb{P}(X_1 \leq \infty, \dots, X_{j-1} \leq \infty, X_j \leq x_j, X_{j+1} \leq \infty, \dots, X_d \leq \infty) \\ &= F(\infty, \dots, \infty, x_j, \infty, \dots, \infty), \quad x_j \in \mathbb{R}, \quad j \in \{1, \dots, d\}. \end{aligned}$$

(interpreted as a *limit*).

- Similarly for *k-dimensional margins*. Suppose we partition \mathbf{X} into $(\mathbf{X}'_1, \mathbf{X}'_2)'$, where $\mathbf{X}_1 = (X_1, \dots, X_k)'$ and $\mathbf{X}_2 = (X_{k+1}, \dots, X_d)'$, then the marginal distribution function of \mathbf{X}_1 is

$$F_{\mathbf{X}_1}(\mathbf{x}_1) = \mathbb{P}(\mathbf{X}_1 \leq \mathbf{x}_1) = F(x_1, \dots, x_k, \infty, \dots, \infty).$$

- *F is absolutely continuous* if

$$F(\mathbf{x}) \underset{(*)}{=} \int_{-\infty}^{x_d} \dots \int_{-\infty}^{x_1} f(z_1, \dots, z_d) dz_1 \dots dz_d = \int_{(-\infty, \mathbf{x}]} f(\mathbf{z}) d\mathbf{z}$$

for some $f \geq 0$ known as the *(joint) density of X (or F)*. Similarly, the *jth marginal df F_j is absolutely continuous* if $F_j(x) = \int_{-\infty}^x f_j(z) dz$ for some $f_j \geq 0$ known as the *density of X_j (or F_j)*.

- In case f exists, $F_j(x_j) \underset{(*)}{=} \int_{-\infty}^{x_j} \int_{(-\infty, \infty)} f(\mathbf{z}) d\mathbf{z}_{-j} dz_j = \int_{-\infty}^{x_j} f_j(z_j) dz_j$, so that *$f_j(x_j)$* can be recovered from f via

$$\underbrace{\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty}}_{d-1\text{-many}} f(z_1, \dots, z_{j-1}, x_j, z_{j+1}, \dots, z_d) dz_1 \dots dz_{j-1} dz_{j+1} \dots dz_d.$$

- Existence of a **joint density** \Rightarrow Existence of **marginal densities** for all k -dimensional marginals, $1 \leq k \leq d - 1$. The **converse is false in general** (counter-examples can be constructed with singular **copulas**; see Chapter 7).
- By **replacing integrals by sums**, one obtains similar formulas for the **discrete case**, in which the notion of densities is replaced by **probability mass functions**.
- We sometimes work with the **survival function \bar{F} of \mathbf{X}** ,

$$\bar{F}(\mathbf{x}) = \bar{F}_{\mathbf{X}}(\mathbf{x}) = \mathbb{P}(\mathbf{X} > \mathbf{x}) = \mathbb{P}(X_1 > x_1, \dots, X_d > x_d), \quad \mathbf{x} \in \mathbb{R}^d,$$

with corresponding **j th marginal survival function \bar{F}_j**

$$\begin{aligned} \bar{F}_j(x_j) &= \mathbb{P}(X_j > x_j) \\ &= \bar{F}(-\infty, \dots, -\infty, x_j, -\infty, \dots, -\infty), \quad x_j \in \mathbb{R}, \quad j \in \{1, \dots, d\}. \end{aligned}$$

- Note that $\bar{F}(\mathbf{x}) \neq 1 - F(\mathbf{x})$ in general (unless $d = 1$).

Conditional distributions and independence

- A **multivariate model** for risks in the form of a joint df, survival function or density, **implicitly describes** their **dependence structure**. We can then make statements about **conditional probabilities**.
- As before, consider $\mathbf{X} = (\mathbf{X}'_1, \mathbf{X}'_2) \sim F$. The **conditional df of \mathbf{X}_2 given $\mathbf{X}_1 = \mathbf{x}_1$** is $F_{\mathbf{X}_2|\mathbf{X}_1}(\mathbf{x}_2 | \mathbf{x}_1) = \mathbb{P}(\mathbf{X}_2 \leq \mathbf{x}_2 | \mathbf{X}_1 = \mathbf{x}_1) = \mathbb{E}(I_{\{\mathbf{X}_2 \leq \mathbf{x}_2\}} | \mathbf{X}_1 = \mathbf{x}_1)$, where $\mathbb{E}(\cdot | \cdot)$ denotes conditional expectation (**not discussed here**).
- A **useful identity** for conditional dfs is

$$F(\mathbf{x}) = \int_{(-\infty, \mathbf{x}_1]} F_{\mathbf{X}_2|\mathbf{X}_1}(\mathbf{x}_2 | \mathbf{z}) dF_{\mathbf{X}_1}(\mathbf{z}); \quad (17)$$

see the **appendix** for a proof.

- ▶ If $\mathbf{x}_1 \rightarrow \infty$, then $F_{\mathbf{X}_2}(\mathbf{x}_2) = \int_{\mathbb{R}^d} F_{\mathbf{X}_2|\mathbf{X}_1}(\mathbf{x}_2 | \mathbf{z}) dF_{\mathbf{X}_1}(\mathbf{z})$.
- ▶ If F has a density f , then $f_{\mathbf{X}_2}(\mathbf{x}_2) = \int_{\mathbb{R}^d} f_{\mathbf{X}_2|\mathbf{X}_1}(\mathbf{x}_2 | \mathbf{z}) dF_{\mathbf{X}_1}(\mathbf{z})$.

- If F has density f and f_{X_1} denotes the density of \mathbf{X}_1 , then

$$\begin{aligned} f(\mathbf{x}_1, \mathbf{x}_2) &= \frac{\partial^2}{\partial \mathbf{x}_2 \partial \mathbf{x}_1} F(\mathbf{x}_1, \mathbf{x}_2) \stackrel{(17)}{=} \frac{\partial}{\partial \mathbf{x}_2} F_{\mathbf{X}_2|\mathbf{X}_1}(\mathbf{x}_2 | \mathbf{x}_1) f_{\mathbf{X}_1}(\mathbf{x}_1) \\ &= f_{\mathbf{X}_2|\mathbf{X}_1}(\mathbf{x}_2 | \mathbf{x}_1) f_{\mathbf{X}_1}(\mathbf{x}_1). \end{aligned}$$

We call

$$f_{\mathbf{X}_2|\mathbf{X}_1}(\mathbf{x}_2 | \mathbf{x}_1) = \frac{f(\mathbf{x}_1, \mathbf{x}_2)}{f_{\mathbf{X}_1}(\mathbf{x}_1)}$$

the *conditional density of \mathbf{X}_2 given $\mathbf{X}_1 = \mathbf{x}_1$* . In this case, the conditional df $F_{\mathbf{X}_2|\mathbf{X}_1}(\mathbf{x}_2 | \mathbf{x}_1)$ is given by

$$F_{\mathbf{X}_2|\mathbf{X}_1}(\mathbf{x}_2 | \mathbf{x}_1) = \int_{-\infty}^{x_{k+1}} \cdots \int_{-\infty}^{x_d} f_{\mathbf{X}_2|\mathbf{X}_1}(z_{k+1}, \dots, z_d | \mathbf{x}_1) dz_{k+1} \cdots dz_d.$$

- $\mathbf{X}_1, \mathbf{X}_2$ are *independent* if $F(\mathbf{x}_1, \mathbf{x}_2) = F_{\mathbf{X}_1}(\mathbf{x}_1)F_{\mathbf{X}_2}(\mathbf{x}_2)$ for all $\mathbf{x}_1, \mathbf{x}_2$.
- If F has density f , then $\mathbf{X}_1, \mathbf{X}_2$ are independent if $f(\mathbf{x}_1, \mathbf{x}_2) = f_{\mathbf{X}_1}(\mathbf{x}_1)f_{\mathbf{X}_2}(\mathbf{x}_2)$ for all $\mathbf{x}_1, \mathbf{x}_2$. In this case, $f_{\mathbf{X}_2|\mathbf{X}_1}(\mathbf{x}_2 | \mathbf{x}_1) = f_{\mathbf{X}_2}(\mathbf{x}_2)$.

- The components X_1, \dots, X_d of \mathbf{X} are *(mutually) independent* if $F(\mathbf{x}) = \prod_{j=1}^d F_j(x_j)$ for all \mathbf{x} or, if F has density f , if $f(\mathbf{x}) = \prod_{j=1}^d f_j(x_j)$ for all \mathbf{x} .

Moments and characteristic function

- If $\mathbb{E}|X_j| < \infty$, $j \in \{1, \dots, d\}$, the *mean vector of \mathbf{X}* is defined by

$$\mathbb{E}\mathbf{X} = (\mathbb{E}X_1, \dots, \mathbb{E}X_d).$$

One can show: X_1, \dots, X_d independent $\Rightarrow \mathbb{E}(X_1 \cdots X_d) = \prod_{j=1}^d \mathbb{E}(X_j)$

- If $\mathbb{E}(X_j^2) < \infty$ for all j , the *covariance matrix of \mathbf{X}* is defined by

$$\text{cov}(\mathbf{X}) = \mathbb{E}((\mathbf{X} - \mathbb{E}\mathbf{X})(\mathbf{X} - \mathbb{E}\mathbf{X})').$$

If we write $\Sigma = \text{cov}(\mathbf{X})$, its (i, j) th element is

$$\begin{aligned}\sigma_{ij} &= \Sigma_{ij} = \text{cov}(X_i, X_j) = \mathbb{E}((X_i - \mathbb{E}X_i)(X_j - \mathbb{E}X_j)) \\ &= \mathbb{E}(X_i X_j) - \mathbb{E}(X_i)\mathbb{E}(X_j);\end{aligned}$$

the diagonal elements are $\sigma_{jj} = \text{var}(X_j)$, $j \in \{1, \dots, d\}$.

- X_1, X_2 independent $\not\Rightarrow \text{cov}(X_1, X_2) = 0$ (counter-examples can be constructed with **copulas**; see Chapter 7).
- The **cross covariance matrix between** two random vectors \mathbf{X}, \mathbf{Y} is defined by $\text{cov}(\mathbf{X}, \mathbf{Y}) = \mathbb{E}((\mathbf{X} - \mathbb{E}\mathbf{X})(\mathbf{Y} - \mathbb{E}\mathbf{Y})')$; note that $\text{cov}(\mathbf{X}, \mathbf{X}) = \text{cov}(\mathbf{X})$.
- If $\mathbb{E}(X_j^2) < \infty$, $j \in \{1, \dots, d\}$, the **correlation matrix of \mathbf{X}** is defined by the matrix **corr(\mathbf{X})** with (i, j) th element

$$\text{corr}(X_i, X_j) = \frac{\text{cov}(X_i, X_j)}{\sqrt{\text{var}(X_i) \text{var}(X_j)}}, \quad i, j \in \{1, \dots, d\},$$

which is in $[-1, 1]$ with $\text{corr}(X_i, X_j) = \pm 1$ if and only if $X_j \stackrel{\text{a.s.}}{=} aX_i + b$ for some $a \neq 0$ and $b \in \mathbb{R}$.

- **Some properties of $\mathbb{E}()$ and $\text{cov}()$:**

1) For all $A \in \mathbb{R}^{k \times d}$, $\mathbf{b} \in \mathbb{R}^k$:

$$\blacktriangleright \mathbb{E}(A\mathbf{X} + \mathbf{b}) = A\mathbb{E}\mathbf{X} + \mathbf{b} = A\boldsymbol{\mu} + \mathbf{b};$$

$$\begin{aligned} \blacktriangleright \text{cov}(A\mathbf{X} + \mathbf{b}) &= A \text{cov}(\mathbf{X}) A' = A \Sigma A'; \text{ if } k = 1 \ (A = \mathbf{a}'), \\ \mathbf{a}' \Sigma \mathbf{a} &= \text{cov}(\mathbf{a}' \mathbf{X}) = \text{var}(\mathbf{a}' \mathbf{X}) \geq 0, \quad \mathbf{a} \in \mathbb{R}^d, \end{aligned} \quad (18)$$

i.e. *covariance matrices are positive semidefinite*.

$$\blacktriangleright \text{cov}(\mathbf{X}_1 + \mathbf{X}_2) = \text{cov}(\mathbf{X}_1) + \text{cov}(\mathbf{X}_2) + 2 \text{cov}(\mathbf{X}_1, \mathbf{X}_2)$$

2) If Σ is a *positive definite matrix* (i.e. $\mathbf{a}' \Sigma \mathbf{a} > 0$ for all $\mathbf{a} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$), one can show that Σ is invertible.

3) A *symmetric, positive (semi)definite* Σ can be written as

$$\Sigma = A A' \quad \text{Cholesky decomposition} \quad (19)$$

for a lower triangular matrix A with $A_{jj} > 0$ ($A_{jj} \geq 0$) for all j . A is known as *Cholesky factor* (and is also denoted by $\Sigma^{1/2}$).

■ Properties of \mathbf{X} can often be shown with the *characteristic function (cf)*

$$\phi_{\mathbf{X}}(\mathbf{t}) = \mathbb{E}(\exp(i\mathbf{t}'\mathbf{X})), \quad \mathbf{t} \in \mathbb{R}^d.$$

X_1, \dots, X_d are independent $\Leftrightarrow \phi_{\mathbf{X}}(\mathbf{t}) = \prod_{j=1}^d \phi_{X_j}(t_j)$ for all \mathbf{t} .

Proposition 6.1 (Characterization of covariance matrices)

A symmetric matrix Σ is a covariance matrix if and only if it is positive semidefinite.

Proof.

“ \Rightarrow ” As we have seen in (18), a covariance matrix Σ is positive semidefinite.

“ \Leftarrow ” Let Σ be positive semidefinite with Cholesky factor A . Let \mathbf{X} be a random vector with $\text{cov } \mathbf{X} = I_d = \text{diag}(1, \dots, 1)$ (e.g. $X_j \stackrel{\text{ind.}}{\sim} N(0, 1)$). Then $\text{cov}(A\mathbf{X}) = A \text{cov}(\mathbf{X}) A' = AA' = \Sigma$, i.e. Σ is a covariance matrix (namely that of $A\mathbf{X}$). \square

6.1.2 Standard estimators of covariance and correlation

- Assume $\mathbf{X}_1, \dots, \mathbf{X}_n \sim F$ (daily/weekly/monthly/yearly risk-factor changes) are **serially uncorrelated** (i.e. multivariate white noise) with $\mu := \mathbb{E}\mathbf{X}_1$, $\Sigma := \text{cov } \mathbf{X}_1$ and $P = \text{corr}(\mathbf{X}_1)$.

- Standard estimators of $\boldsymbol{\mu}, \Sigma, P$ are

$$\bar{\mathbf{X}} = \frac{1}{n} \sum_{i=1}^n \mathbf{X}_i \quad (\text{sample mean})$$

$$S = \frac{1}{n} \sum_{i=1}^n (\mathbf{X}_i - \bar{\mathbf{X}})(\mathbf{X}_i - \bar{\mathbf{X}})' \quad (\text{sample covariance matrix})$$

$$R = (R_{ij}) \text{ for } R_{ij} = \frac{S_{ij}}{\sqrt{S_{ii}S_{jj}}} \quad (\text{sample correlation matrix})$$

- Under joint normality (F multivariate normal), $\bar{\mathbf{X}}, S$ and R are also **MLEs**. S is biased, but an unbiased version can be obtained by

$$S_n = \frac{n}{n-1} S.$$

- Clearly, $\bar{\mathbf{X}}$ is unbiased. Since the \mathbf{X}_i 's are uncorrelated,

$$\text{cov}(\bar{\mathbf{X}}) = \frac{1}{n^2} \sum_{i=1}^n \text{cov}(\mathbf{X}_i) = \frac{1}{n} \text{cov}(\mathbf{X}_1) = \frac{1}{n} \Sigma.$$

- S_n is unbiased since

$$\begin{aligned}
 \mathbb{E}S_n &= \frac{1}{n-1} \sum_{i=1}^n \mathbb{E}((\mathbf{X}_i - \bar{\mathbf{X}})(\mathbf{X}_i - \bar{\mathbf{X}})') \\
 &= \frac{1}{n-1} \sum_{i=1}^n \mathbb{E}(((\mathbf{X}_i - \boldsymbol{\mu}) - (\bar{\mathbf{X}} - \boldsymbol{\mu}))((\mathbf{X}_i - \boldsymbol{\mu}) - (\bar{\mathbf{X}} - \boldsymbol{\mu}))') \\
 &= \frac{1}{n-1} \sum_{i=1}^n \mathbb{E}((\mathbf{X}_i - \boldsymbol{\mu})(\mathbf{X}_i - \boldsymbol{\mu})' - (\bar{\mathbf{X}} - \boldsymbol{\mu})(\bar{\mathbf{X}} - \boldsymbol{\mu})') \\
 &= \frac{1}{n-1} \sum_{i=1}^n (\Sigma - \text{cov } \bar{\mathbf{X}}) \stackrel{\text{cov}(\bar{\mathbf{X}}) = \frac{\Sigma}{n}}{=} \frac{n}{n-1} \left(1 - \frac{1}{n}\right) \Sigma = \Sigma.
 \end{aligned}$$

- Further properties of $\bar{\mathbf{X}}, S, R$ depend on F .

6.1.3 The multivariate normal distribution

Definition 6.2 (Multivariate normal distribution)

$\mathbf{X} = (X_1, \dots, X_d)$ has a *multivariate normal* (or *Gaussian*) *distribution* if

$$\mathbf{X} \stackrel{d}{=} \boldsymbol{\mu} + A\mathbf{Z}, \quad (20)$$

where $\mathbf{Z} = (Z_1, \dots, Z_k)$, $Z_l \stackrel{\text{ind.}}{\sim} N(0, 1)$, $A \in \mathbb{R}^{d \times k}$, $\boldsymbol{\mu} \in \mathbb{R}^d$.

- $\mathbb{E}\mathbf{X} = \boldsymbol{\mu} + A\mathbb{E}\mathbf{Z} = \boldsymbol{\mu}$
- $\text{cov}(\mathbf{X}) = \text{cov}(\boldsymbol{\mu} + A\mathbf{Z}) = A \text{cov}(\mathbf{Z}) A' = AA' =: \Sigma$

Proposition 6.3 (Cf of the multivariate normal distribution)

Let \mathbf{X} be as in (20) and $\Sigma = AA'$. Then the cf of \mathbf{X} is

$$\phi_{\mathbf{X}}(\mathbf{t}) = \mathbb{E}(\exp(it'\mathbf{X})) = \exp\left(it'\boldsymbol{\mu} - \frac{1}{2}\mathbf{t}'\Sigma\mathbf{t}\right), \quad \mathbf{t} \in \mathbb{R}^d.$$

Idea of proof. Using the fact that $\phi_Z(t) = \exp(-t^2/2)$ for $Z \sim N(0, 1)$ (see the appendix for a proof), we obtain that

$$\begin{aligned}
 \phi_{\mathbf{X}}(\mathbf{t}) &= \mathbb{E}(\exp(i\mathbf{t}'(\boldsymbol{\mu} + A\mathbf{Z}))) \underset{\tilde{\mathbf{t}} = \mathbf{t}'A}{=} \exp(i\mathbf{t}'\boldsymbol{\mu})\mathbb{E}(\exp(i\tilde{\mathbf{t}}'\mathbf{Z})) \\
 &\stackrel{\text{ind.}}{=} \exp(i\mathbf{t}'\boldsymbol{\mu}) \prod_{j=1}^d \mathbb{E}(\exp(i\tilde{t}_j Z_j)) = \exp\left(i\mathbf{t}'\boldsymbol{\mu} - \frac{1}{2} \sum_{j=1}^d \tilde{t}_j^2\right) \\
 &= \exp\left(i\mathbf{t}'\boldsymbol{\mu} - \frac{1}{2} \tilde{\mathbf{t}}'\tilde{\mathbf{t}}\right) = \exp\left(i\mathbf{t}'\boldsymbol{\mu} - \frac{1}{2} \mathbf{t}'A A'\mathbf{t}\right) \\
 &= \exp\left(i\mathbf{t}'\boldsymbol{\mu} - \frac{1}{2} \mathbf{t}'\Sigma\mathbf{t}\right) \quad \square
 \end{aligned}$$

- We see that the multivariate normal distribution is **characterized by $\boldsymbol{\mu}$ and Σ** , hence the notation $\mathbf{X} \sim N_d(\boldsymbol{\mu}, \Sigma)$.
- $N_d(\boldsymbol{\mu}, \Sigma)$ can be characterized by univariate normal distributions.

Proposition 6.4 (Characterization of $N_d(\boldsymbol{\mu}, \Sigma)$)

$$\mathbf{X} \sim N_d(\boldsymbol{\mu}, \Sigma) \iff \mathbf{a}'\mathbf{X} \sim N(\mathbf{a}'\boldsymbol{\mu}, \mathbf{a}'\Sigma\mathbf{a}) \quad \forall \mathbf{a} \in \mathbb{R}^d.$$

Proof. “ \Rightarrow ” via uniqueness of cfs; “ \Leftarrow ” via Corollary A.10 □

Consequences:

- Margins: $\mathbf{X} \sim N_d(\boldsymbol{\mu}, \Sigma) \xRightarrow{\mathbf{a}=\mathbf{e}_j} X_j \sim N(\mu_j, \Sigma_{jj}), \quad j \in \{1, \dots, d\}.$
- Sums: $\mathbf{X} \sim N_d(\boldsymbol{\mu}, \Sigma) \xRightarrow{\mathbf{a}=\mathbf{1}} \sum_{j=1}^d X_j \sim N(\sum_{j=1}^d \mu_j, \sum_{i,j} \Sigma_{ij}).$

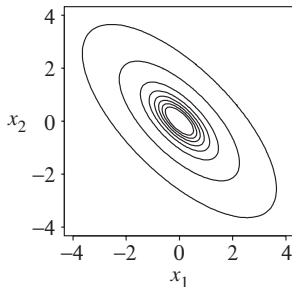
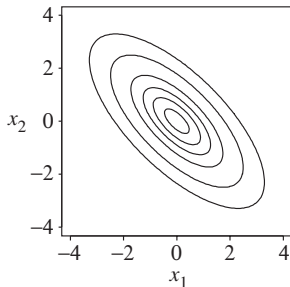
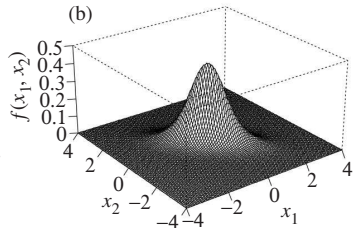
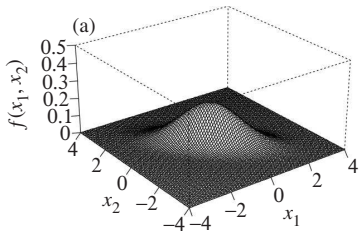
Proposition 6.5 (Density)

Let $\mathbf{X} \sim N_d(\boldsymbol{\mu}, \Sigma)$ with $\text{rank } \Sigma = d = k$ ($\Rightarrow \Sigma$ pos. definite, invertible). Via the Density Transformation Theorem, it is an exercise to show that \mathbf{X} has density

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{d/2} \sqrt{\det \Sigma}} \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})' \Sigma^{-1}(\mathbf{x} - \boldsymbol{\mu})\right), \quad \mathbf{x} \in \mathbb{R}^d.$$

Consequences:

- Sets of the form $S_c = \{\mathbf{x} \in \mathbb{R}^d : (\mathbf{x} - \boldsymbol{\mu})' \Sigma^{-1}(\mathbf{x} - \boldsymbol{\mu}) = c\}$, $c > 0$, describe points of equal density. Contours of equal density are thus ellipsoids. Whenever a multivariate density $f_{\mathbf{X}}(\mathbf{x})$ depends on \mathbf{x} only through the quadratic form $(\mathbf{x} - \boldsymbol{\mu})' \Sigma^{-1}(\mathbf{x} - \boldsymbol{\mu})$, it is the density of an elliptical distribution (see later).
- The components of $\mathbf{X} \sim N_d(\boldsymbol{\mu}, \Sigma)$ are mutually independent if and only if Σ is diagonal, i.e. if and only if the components of \mathbf{X} are uncorrelated.



Left: $N_d(\boldsymbol{\mu}, \Sigma)$ for $\boldsymbol{\mu} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\Sigma = \begin{pmatrix} 1 & -0.7 \\ -0.7 & 1 \end{pmatrix}$; Right: $t_\nu(\boldsymbol{\mu}, \frac{\nu-2}{\nu}\Sigma)$, $\nu = 4$,
 (same mean and covariance matrix as on the left-hand side)

The definition of $N_d(\boldsymbol{\mu}, \Sigma)$ in terms of a **stochastic representation** ($\mathbf{X} \stackrel{d}{=} \boldsymbol{\mu} + A\mathbf{Z}$) **directly justifies the following sampling algorithm**.

Algorithm 6.6 (Sampling $N_d(\boldsymbol{\mu}, \Sigma)$)

Let $\mathbf{X} \sim N_d(\boldsymbol{\mu}, \Sigma)$ with Σ symmetric and positive definite.

- 1) Compute the Cholesky factor A of Σ ; see, e.g. Press et al. (1992).
- 2) Generate $Z_j \stackrel{\text{ind.}}{\sim} N(0, 1)$, $j \in \{1, \dots, d\}$.
- 3) Return $\mathbf{X} = \boldsymbol{\mu} + A\mathbf{Z}$, where $\mathbf{Z} = (Z_1, \dots, Z_d)$.

Further useful properties of multivariate normal distributions

■ **Linear combinations**

If $\mathbf{X} \sim N_d(\boldsymbol{\mu}, \Sigma)$ and $B \in \mathbb{R}^{k \times d}$, $\mathbf{b} \in \mathbb{R}^k$, then

$$\begin{aligned} B\mathbf{X} + \mathbf{b} &= B(\boldsymbol{\mu} + A\mathbf{Z}) + \mathbf{b} = (B\boldsymbol{\mu} + \mathbf{b}) + BA\mathbf{Z} \\ &\sim N_k(B\boldsymbol{\mu} + \mathbf{b}, BA(BA)') = N_k(B\boldsymbol{\mu} + \mathbf{b}, B\Sigma B'). \end{aligned}$$

Special case (see variance-covariance method; or Proposition 6.4):
 $b'X \sim N(b'\mu, b'\Sigma b)$

■ Marginal dfs

Let $X \sim N_d(\mu, \Sigma)$ and write $X = (X_1', X_2')$, where $X_1 \in \mathbb{R}^k$, $X_2 \in \mathbb{R}^{d-k}$, and $\mu = (\mu_1', \mu_2')$, $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$. Then

$$X_1 \sim N_k(\mu_1, \Sigma_{11}) \quad \text{and} \quad X_2 \sim N_{d-k}(\mu_2, \Sigma_{22}).$$

Proof. Choose $B = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & I_{d-k} \end{pmatrix}$, respectively, in the above.

■ Conditional distributions

Let X be as before and Σ be positive definite. One can show that

$$X_2 | X_1 = x_1 \sim N_{d-k}(\mu_{2.1}, \Sigma_{22.1}),$$

where $\mu_{2.1} = \mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(x_1 - \mu_1)$ and $\Sigma_{22.1} = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$.

■ Quadratic forms

Let $X \sim N_d(\mu, \Sigma)$ and Σ be positive definite with Cholesky factor A .

Furthermore, let $\mathbf{Z} = \mathbf{A}^{-1}(\mathbf{X} - \boldsymbol{\mu})$. Then $\mathbf{Z} \sim \mathcal{N}_d(\mathbf{0}, \mathbf{I}_d)$. Moreover,

$$(\mathbf{X} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\mathbf{X} - \boldsymbol{\mu}) = \mathbf{Z}' \mathbf{Z} \sim \chi_d^2, \quad (21)$$

which is useful for (goodness-of-fit) testing of $\mathcal{N}_d(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

■ Convolutions

Let $\mathbf{X} \sim \mathcal{N}_d(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $\mathbf{Y} \sim \mathcal{N}_d(\tilde{\boldsymbol{\mu}}, \tilde{\boldsymbol{\Sigma}})$ be independent. Via cfs it is then an exercise to show that

$$\mathbf{X} + \mathbf{Y} \sim \mathcal{N}_d(\boldsymbol{\mu} + \tilde{\boldsymbol{\mu}}, \boldsymbol{\Sigma} + \tilde{\boldsymbol{\Sigma}}).$$

6.1.4 Testing multivariate normality

- For testing univariate normality, all tests of Section 3.1.2 can be applied.
- Now consider multivariate normality. By Proposition 6.4,

$$\mathbf{X}_1, \dots, \mathbf{X}_n \stackrel{\text{ind.}}{\sim} \mathcal{N}_d(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \Rightarrow \mathbf{a}' \mathbf{X}_1, \dots, \mathbf{a}' \mathbf{X}_n \stackrel{\text{ind.}}{\sim} \mathcal{N}(\mathbf{a}' \boldsymbol{\mu}, \mathbf{a}' \boldsymbol{\Sigma} \mathbf{a}).$$

This can be tested statistically (for some \mathbf{a}) with various goodness-of-fit tests (e.g. Q-Q plots) used for univariate normality (however, for $\mathbf{a} = \mathbf{e}_j$,

$j \in \{1, \dots, d\}$, we would **only test normality of the margins, not joint normality**). Alternatively, (21) can be used to test joint normality.

- Multivariate Shapiro–Wilk

- Mardia's test

- ▶ According to (21), if $\mathbf{X} \sim N_d(\boldsymbol{\mu}, \Sigma)$ with Σ positive definite, then $(\mathbf{X} - \boldsymbol{\mu})' \Sigma^{-1} (\mathbf{X} - \boldsymbol{\mu}) \sim \chi_d^2$.
- ▶ Let $D_i^2 = (\mathbf{X}_i - \bar{\mathbf{X}})' S^{-1} (\mathbf{X}_i - \bar{\mathbf{X}})$ denote the *squared Mahalanobis distances* and $D_{ij} = (\mathbf{X}_i - \bar{\mathbf{X}})' S^{-1} (\mathbf{X}_j - \bar{\mathbf{X}})$ the *Mahalanobis angles*.
- ▶ Let $b_d = \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n D_{ij}^3$ and $k_d = \frac{1}{n} \sum_{i=1}^n D_i^4$. Under the null hypothesis one can show that asymptotically for $n \rightarrow \infty$,

$$\frac{n}{6} b_d \sim \chi_{d(d+1)(d+2)/6}^2, \quad \frac{k_d - d(d+2)}{\sqrt{8d(d+2)/n}} \sim N(0, 1),$$

which can be used for testing; see Joensuu and Vogel (2014).

Example 6.7 (Multivariate (non-)normality of 10 Dow Jones stocks)

- We apply Mardia's test (of multivariate skewness and kurtosis) to daily/weekly/monthly/quarterly log-returns of 10 (of the 30) Dow Jones stocks from 1993–2000.

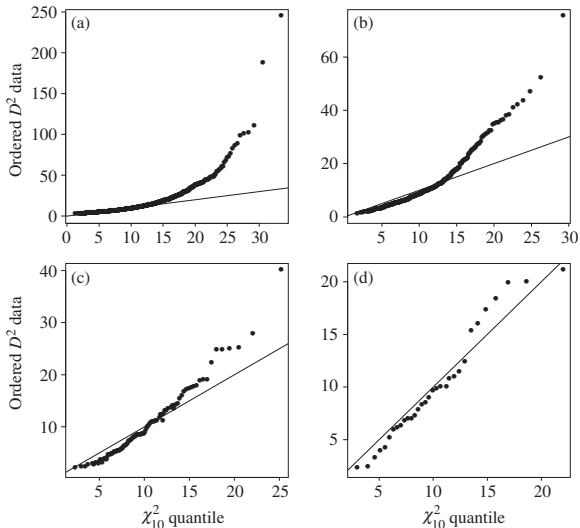
n	Daily 2020	Weekly 416	Monthly 96	Quarterly 32
b_{10}	9.31	9.91	21.10	50.10
p -value	0.00	0.00	0.00	0.02
k_{10}	242.45	177.04	142.65	120.83
p -value	0.00	0.00	0.00	0.44

- We can also compare D_i^2 data to a χ_{10}^2 graphically using a Q-Q plot.

Conclusion: Daily/weekly/monthly data: Evidence against joint normality; Quarterly data: CLT effect seems to take place (but too little data to say more); still evidence against joint normality.

Q-Q plot of D_i^2 data against a χ_{10}^2 distribution:

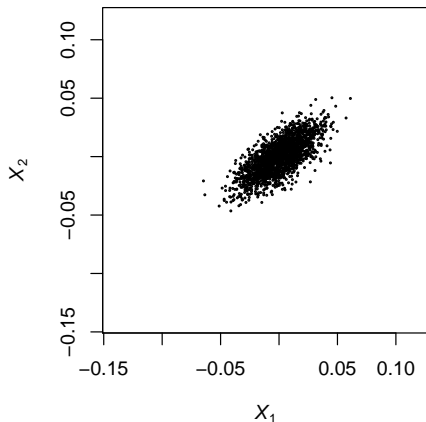
(a) daily data; (b) weekly data; (c) monthly data; and (d) quarterly data



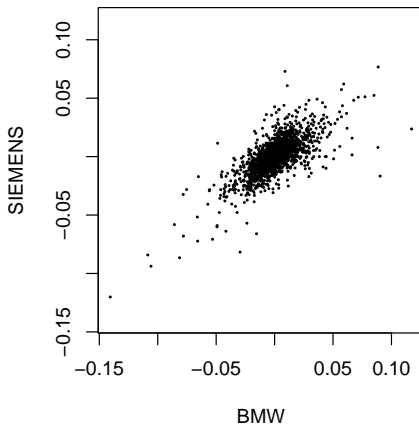
Example 6.8 (Simulated data vs BMW–Siemens)

Is the **BMW–Siemens data** (see Section 3.2.2) **jointly normal**?

Simulated data (fitted multivariate normal)

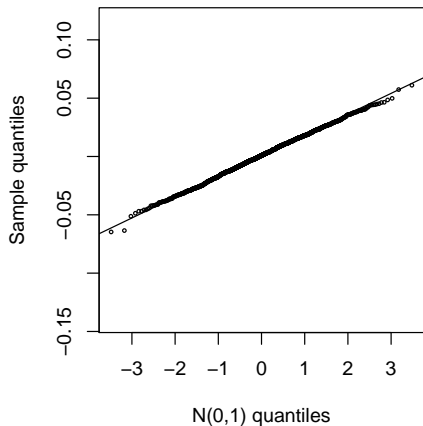


Real risk-factor changes

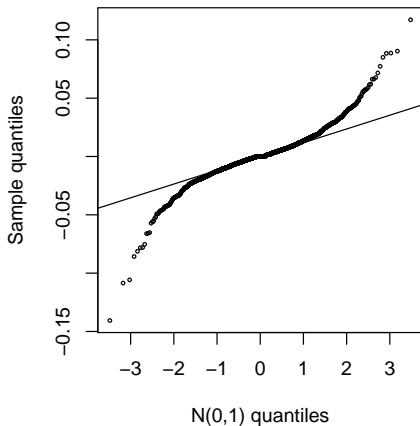


Considering the **first margin** only:

Q-Q plot for margin 1 (simulated data)

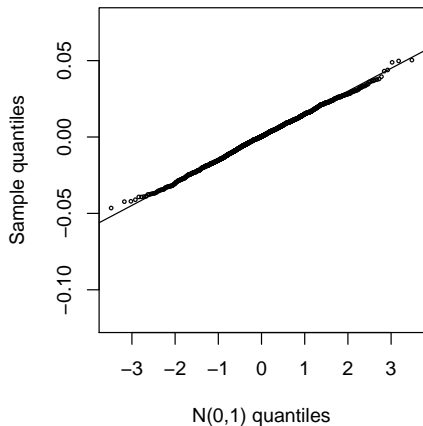


Q-Q plot for margin 1 (real data)

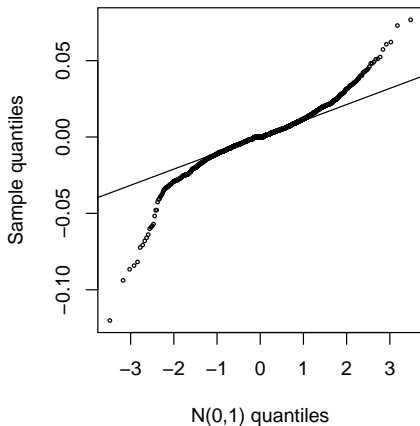


Considering the **second margin** only:

Q-Q plot for margin 2 (simulated data)

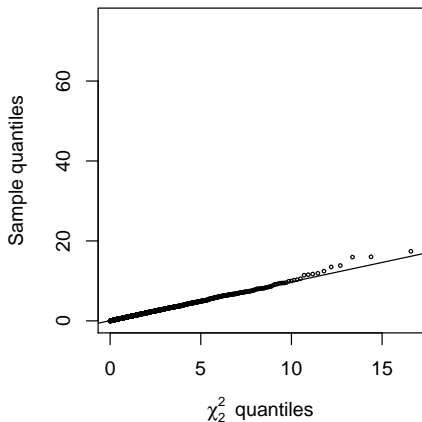


Q-Q plot for margin 2 (real data)

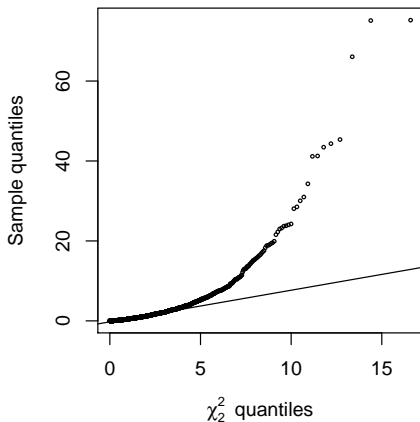


Q-Q plot of the simulated (left) or real (right) D_i^2 's against a χ_2^2 :

Q-Q plot of D_i^2 (simulated data)



Q-Q plot of D_i^2 (real data)



Advantages of $N_d(\mu, \Sigma)$

- Inference “easy”.
- Distribution is determined by μ and Σ .
- Linear combinations are normal (\Rightarrow VaR_α and ES_α calculations for portfolios are easy).
- Marginal distributions are normal.
- Conditional distributions are normal.
- Quadratic forms are known.
- Convolutions are normal.
- Sampling is straightforward.
- Independence and uncorrelatedness are equivalent.

Drawbacks of $N_d(\boldsymbol{\mu}, \Sigma)$ for modelling risk-factor changes

- 1) Tails of univariate (normal) margins are too thin (generate too few extreme events).
- 2) Joint tails are too thin (generate too few joint extreme events).
 $N_d(\boldsymbol{\mu}, \Sigma)$ cannot capture the notion of tail dependence (see Chapter 7).
- 3) Very strong symmetry known as radial symmetry: \mathbf{X} is called *radially symmetric about $\boldsymbol{\mu}$* if $\mathbf{X} - \boldsymbol{\mu} \stackrel{d}{=} \boldsymbol{\mu} - \mathbf{X}$. This is true for $N_d(\boldsymbol{\mu}, \Sigma)$.

Short outlook:

- Normal variance mixtures (or, more general, elliptical distributions can address 1) and 2) while sharing many of the desirable properties of $N_d(\boldsymbol{\mu}, \Sigma)$.
- Normal mean-variance mixtures can also address 3) (but at the expense of tractability in comparison to $N_d(\boldsymbol{\mu}, \Sigma)$).

6.2 Normal mixture distributions

Idea: Randomize Σ (and μ) with a non-negative rv W .

6.2.1 Normal variance mixtures

Definition 6.9 (Multivariate normal variance mixtures)

The random vector \mathbf{X} has a (multivariate) *normal variance mixture distribution* if

$$\mathbf{X} \stackrel{d}{=} \boldsymbol{\mu} + \sqrt{W} \mathbf{A} \mathbf{Z}, \quad (22)$$

where $\mathbf{Z} \sim N_k(\mathbf{0}, I_k)$, $W \geq 0$ is a rv independent of \mathbf{Z} , $\mathbf{A} \in \mathbb{R}^{d \times k}$, and $\boldsymbol{\mu} \in \mathbb{R}^d$. $\boldsymbol{\mu}$ is called *location vector* and $\Sigma = \mathbf{A} \mathbf{A}'$ *scale* (or *dispersion matrix*).

Observe that $(\mathbf{X} | W = w) \stackrel{d}{=} \boldsymbol{\mu} + \sqrt{w} \mathbf{A} \mathbf{Z} = N_d(\boldsymbol{\mu}, w \mathbf{A} \mathbf{A}') = N_d(\boldsymbol{\mu}, w \Sigma)$; or $(\mathbf{X} | W) \stackrel{d}{=} N_d(\boldsymbol{\mu}, W \Sigma)$. W can be interpreted as a *shock* affecting the variances of all risk factors.

Properties of multivariate normal variance mixtures

Let $\mathbf{X} = \boldsymbol{\mu} + \sqrt{W}\mathbf{A}\mathbf{Z}$ and $\mathbf{Y} = \boldsymbol{\mu} + \mathbf{A}\mathbf{Z}$. Assume that $\text{rank}(\mathbf{A}) = d \leq k$ and that Σ is positive definite.

■ If $\mathbb{E}\sqrt{W} < \infty$, then $\mathbb{E}(\mathbf{X}) \stackrel{\text{ind.}}{=} \boldsymbol{\mu} + \mathbb{E}(\sqrt{W})\mathbf{A}\mathbb{E}(\mathbf{Z}) = \boldsymbol{\mu} + \mathbf{0} = \boldsymbol{\mu} = \mathbb{E}\mathbf{Y}$

■ If $\mathbb{E}W < \infty$, then

$$\text{cov}(\mathbf{X}) = \text{cov}(\sqrt{W}\mathbf{A}\mathbf{Z}) = \mathbb{E}((\sqrt{W}\mathbf{A}\mathbf{Z})(\sqrt{W}\mathbf{A}\mathbf{Z})')$$

$$\stackrel{\text{ind.}}{=} \mathbb{E}(W) \cdot \mathbb{E}(\mathbf{A}\mathbf{Z}\mathbf{Z}'\mathbf{A}') = \mathbb{E}(W) \cdot \mathbf{A}\mathbb{E}(\mathbf{Z}\mathbf{Z}')\mathbf{A}'$$

$$= \mathbb{E}(W)\mathbf{A}\mathbf{I}_k\mathbf{A}' = \mathbb{E}(W)\Sigma \neq \Sigma \quad (\text{in general} = \text{cov}(\mathbf{Y}))$$

■ However, if they exist (i.e. if $\mathbb{E}W < \infty$) $\text{corr}(\mathbf{X}) = \text{corr}(\mathbf{Y})$ since

$$\begin{aligned}\text{corr}(X_i, X_j) &= \frac{\text{cov}(X_i, X_j)}{\sqrt{\text{var}(X_i)\text{var}(X_j)}} = \frac{\mathbb{E}(W)\Sigma_{ij}}{\sqrt{\mathbb{E}(W)\Sigma_{ii}\mathbb{E}(W)\Sigma_{jj}}} \\ &= \frac{\Sigma_{ij}}{\sqrt{\Sigma_{ii}\Sigma_{jj}}} = \text{corr}(Y_i, Y_j), \quad i, j \in \{1, \dots, d\}.\end{aligned}$$

Lemma 6.10 (Independence in normal variance mixtures)

Let $\mathbf{X} = \boldsymbol{\mu} + \sqrt{W}\mathbf{I}_d\mathbf{Z}$ with $\mathbb{E}W < \infty$ (uncorrelated normal variance mixture). Then

X_i and X_j are independent $\iff W$ is a.s. constant (i.e. $\mathbf{X} \sim \mathcal{N}_d$).

See the appendix for a proof. Intuitively, W affects all components of \mathbf{X} and thus creates dependence (unless it is constant).

Recall: If $\mathbf{X} \sim \mathcal{N}_d(\boldsymbol{\mu}, \Sigma)$, then $\phi_{\mathbf{X}}(\mathbf{t}) = \exp(i\mathbf{t}'\boldsymbol{\mu} - \frac{1}{2}\mathbf{t}'\Sigma\mathbf{t})$.

Furthermore, $\mathbf{X} \mid W = w \sim \mathcal{N}_d(\boldsymbol{\mu}, w\Sigma)$

- **Characteristic function:** The cf of a multivariate normal variance mixtures is

$$\begin{aligned}\phi_{\mathbf{X}}(\mathbf{t}) &= \mathbb{E}(\exp(i\mathbf{t}'\mathbf{X})) = \mathbb{E}(\mathbb{E}(\exp(i\mathbf{t}'\mathbf{X}) \mid W)) \\ &= \mathbb{E}(\exp(i\mathbf{t}'\boldsymbol{\mu} - \frac{1}{2}W\mathbf{t}'\Sigma\mathbf{t})) = \exp(i\mathbf{t}'\boldsymbol{\mu})\mathbb{E}(\exp(-W\frac{1}{2}\mathbf{t}'\Sigma\mathbf{t})).\end{aligned}$$

- **LS transform:** The *Laplace-Stieltjes transform* of F_W is

$$\hat{F}_W(\theta) := \mathbb{E}(\exp(-\theta W)) = \int_0^\infty e^{-\theta w} dF_W(w).$$

Therefore, $\phi_{\mathbf{X}}(\mathbf{t}) = \exp(it'\boldsymbol{\mu})\hat{F}_W(\frac{1}{2}\mathbf{t}'\Sigma\mathbf{t})$. We thus introduce the notation $\mathbf{X} \sim M_d(\boldsymbol{\mu}, \Sigma, \hat{F}_W)$ for a d -dimensional multivariate normal variance mixture.

- **Density:** If Σ is positive definite, $\mathbb{P}(W = 0) = 0$, the density of \mathbf{X} is

$$\begin{aligned} f_{\mathbf{X}}(\mathbf{x}) &= \int_0^\infty f_{\mathbf{X}|W}(\mathbf{x} | w) dF_W(w) \\ &= \int_0^\infty \frac{1}{(2\pi)^{d/2} w^{d/2} |\Sigma|^{1/2}} \exp\left(-\frac{(\mathbf{x} - \boldsymbol{\mu})'\Sigma^{-1}(\mathbf{x} - \boldsymbol{\mu})}{2w}\right) dF_W(w). \end{aligned}$$

\Rightarrow Only depends on \mathbf{x} through $(\mathbf{x} - \boldsymbol{\mu})'\Sigma^{-1}(\mathbf{x} - \boldsymbol{\mu})$.

\Rightarrow Multivariate normal variance mixtures are **elliptical distributions**.

If Σ is diagonal and $\mathbb{E}W < \infty$, \mathbf{X} is **uncorrelated** (as $\text{cov}(\mathbf{X}) = \mathbb{E}(W)\Sigma$)
but **not independent** unless W is constant a.s.

- **Linear combinations:** For $\mathbf{X} \sim M_d(\boldsymbol{\mu}, \Sigma, \hat{F}_W)$ and $\mathbf{Y} = \mathbf{B}\mathbf{X} + \mathbf{b}$, where $\mathbf{B} \in \mathbb{R}^{k \times d}$ and $\mathbf{b} \in \mathbb{R}^k$, we have $\mathbf{Y} \sim M_k(\mathbf{B}\boldsymbol{\mu} + \mathbf{b}, \mathbf{B}\Sigma\mathbf{B}', \hat{F}_W)$; this can be shown via cfs. If $\mathbf{a} \in \mathbb{R}^d$ ($\mathbf{b} = \mathbf{0}$, $\mathbf{B} = \mathbf{a}' \in \mathbb{R}^{1 \times d}$), $\mathbf{a}'\mathbf{X} \sim M_1(\mathbf{a}'\boldsymbol{\mu}, \mathbf{a}'\Sigma\mathbf{a}, \hat{F}_W)$.
- **Sampling:**

Algorithm 6.11 (Simulation of $\mathbf{X} = \boldsymbol{\mu} + \sqrt{W}\mathbf{A}\mathbf{Z} \sim M_d(\boldsymbol{\mu}, \Sigma, \hat{F}_W)$)

- 1) Generate $\mathbf{Z} \sim N_d(\mathbf{0}, I_d)$.
- 2) Generate $W \sim F_W$ (with LS transform \hat{F}_W), independent of \mathbf{Z} .
- 3) Compute the Cholesky factor \mathbf{A} (such that $\mathbf{A}\mathbf{A}' = \Sigma$).
- 4) Return $\mathbf{X} = \boldsymbol{\mu} + \sqrt{W}\mathbf{A}\mathbf{Z}$.

Example 6.12 ($t_d(\nu, \boldsymbol{\mu}, \Sigma)$ distribution)

For Step 2), generate $V \sim \chi_\nu^2$ and set $W = \frac{\nu}{V} \sim \text{Ig}(\nu/2, \nu/2)$; or $W = \frac{1}{V}$ with $V \sim \Gamma(\frac{\nu}{2}, \frac{\nu}{2})$ ($\Gamma(\alpha, \beta)$ density: $f(x) = \beta^\alpha x^{\alpha-1} e^{-\beta x} / \Gamma(\alpha)$).

Examples of multivariate normal variance mixtures

- **Multivariate normal distribution**

$W = 1$ a.s. (degenerate case)

- **Two point mixture**

$$W = \begin{cases} w_1 & \text{with probability } p, \\ w_2 & \text{with probability } 1 - p \end{cases} \quad w_1, w_2 > 0, w_1 \neq w_2.$$

Can be used to model **ordinary and stress regimes**; extends to k regimes.

- **Symmetric generalised hyperbolic distribution**

W has a generalised inverse Gaussian distribution (GIG); see McNeil et al. (2015, p. 187)

- **Multivariate t distribution**

W has an inverse gamma distribution $W = 1/V$ for $V \sim \Gamma(\nu/2, \nu/2)$.

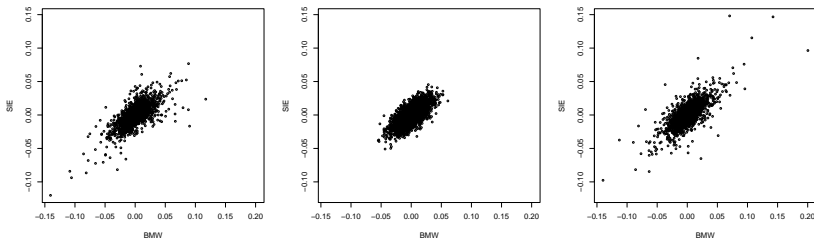
- ▶ $\mathbb{E}(W) = \frac{\nu}{\nu-2} \Rightarrow \text{cov}(\mathbf{X}) = \frac{\nu}{\nu-2} \Sigma$. For finite variances/correlations, $\nu > 2$ is required. For finite mean, $\nu > 1$ is required.

- The density of the multivariate t distribution is given by

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{\Gamma((\nu + d)/2)}{\Gamma(\nu/2)(\nu\pi)^{d/2}|\Sigma|^{1/2}} \left(1 + \frac{(\mathbf{x} - \boldsymbol{\mu})'\Sigma^{-1}(\mathbf{x} - \boldsymbol{\mu})}{\nu} \right)^{-\frac{\nu+d}{2}},$$

where $\boldsymbol{\mu} \in \mathbb{R}^d$, $\Sigma \in \mathbb{R}^{d \times d}$ is a positive definite matrix, and ν is the degrees of freedom. Notation: $\mathbf{X} \sim t_d(\nu, \boldsymbol{\mu}, \Sigma)$.

- $t_d(\nu, \boldsymbol{\mu}, \Sigma)$ has heavier marginal and joint tails than $N_d(\boldsymbol{\mu}, \Sigma)$.
- BMW–Siemens data; simulations from fitted $N_d(\boldsymbol{\mu}, \Sigma)$ and $t_d(3, \boldsymbol{\mu}, \Sigma)$:



6.2.2 Normal mean-variance mixtures

- Radial symmetry implies that **all one-dimensional margins of normal variance mixtures are symmetric**.
- Often visible in data: **joint losses have heavier tails** than joint gains.

Idea: Introduce **asymmetry by mixing** normal distributions **with different means and variances**.

\mathbf{X} has a (multivariate) *normal mean-variance mixture distribution* if

$$\mathbf{X} \stackrel{d}{=} \mathbf{m}(W) + \sqrt{W} \mathbf{A} \mathbf{Z}, \quad (23)$$

where

- $\mathbf{Z} \sim N_k(\mathbf{0}, I_k)$;
- $W \geq 0$ is a scalar random variable which is independent of \mathbf{Z} ;
- $\mathbf{A} \in \mathbb{R}^{d \times k}$ is a matrix of constants;
- $\mathbf{m} : [0, \infty) \rightarrow \mathbb{R}^d$ is a measurable function.

- Normal mean-variance mixtures add **skewness**: Let $\Sigma = AA'$ and observe that $\mathbf{X} | W = w \sim N_d(\mathbf{m}(w), w\Sigma)$. In general, **they are no longer elliptical** (see later).

Example 6.13

- Suppose we have $\mathbf{m}(W) = \boldsymbol{\mu} + W\boldsymbol{\gamma}$. Since

$$\begin{aligned}\mathbb{E}(\mathbf{X} | W) &= \boldsymbol{\mu} + W\boldsymbol{\gamma}, \\ \text{cov}(\mathbf{X} | W) &= W\Sigma\end{aligned}$$

we have

$$\begin{aligned}\mathbb{E}\mathbf{X} &= \mathbb{E}(\mathbb{E}(\mathbf{X} | W)) = \boldsymbol{\mu} + \mathbb{E}(W)\boldsymbol{\gamma} \quad \text{if } \mathbb{E}W < \infty, \\ \text{cov}(\mathbf{X}) &= \mathbb{E}(\text{cov}(\mathbf{X} | W)) + \text{cov}(\mathbb{E}(\mathbf{X} | W)) \\ &= \mathbb{E}(W)\Sigma + \text{var}(W)\boldsymbol{\gamma}\boldsymbol{\gamma}' \quad \text{if } \mathbb{E}(W^2) < \infty.\end{aligned}$$

- If W has a GIG distribution, then \mathbf{X} follows a *generalised hyperbolic distribution*. $\boldsymbol{\gamma} = \mathbf{0}$ leads to (elliptical) normal variance mixtures; see McNeil et al. (2015, Sections 6.2.3) for details.

6.3 Spherical and elliptical distributions

Empirical examples (see McNeil et al. (2015, Sections 6.2.4)) show that

- 1) $M_d(\boldsymbol{\mu}, \Sigma, \hat{F}_W)$ (e.g. multivariate t , NIG) provide superior models to $N_d(\boldsymbol{\mu}, \Sigma)$ for daily/weekly US stock-return data;
- 2) the more general skewed normal mean-variance mixture distributions offer only a modest improvement.

We study elliptical distributions, a generalization of $M_d(\boldsymbol{\mu}, \Sigma, \hat{F}_W)$.

6.3.1 Spherical distributions

Definition 6.14 (Spherical distribution)

A random vector $\mathbf{Y} = (Y_1, \dots, Y_d)$ has a spherical distribution if for every orthogonal $U \in \mathbb{R}^{d \times d}$ (i.e. $U \in \mathbb{R}^{d \times d}$ with $UU' = U'U = I_d$)

$\mathbf{Y} \stackrel{d}{=} U\mathbf{Y}$ (distributionally invariant under rotations and reflections)

Theorem 6.15 (Characterization of spherical distributions)

Let $\|\mathbf{t}\| = (t_1^2 + \dots + t_d^2)^{1/2}$, $\mathbf{t} \in \mathbb{R}^d$. The following are equivalent:

- 1) \mathbf{Y} is spherical (notation: $\mathbf{Y} \sim S_d(\psi)$ for ψ as below).
- 2) \exists a characteristic generator $\psi : [0, \infty) \rightarrow \mathbb{R}$, such that $\phi_{\mathbf{Y}}(\mathbf{t}) = \mathbb{E}(e^{i\mathbf{t}'\mathbf{Y}}) = \psi(\|\mathbf{t}\|^2)$, $\forall \mathbf{t} \in \mathbb{R}^d$.
- 3) For every $\mathbf{a} \in \mathbb{R}^d$, $\mathbf{a}'\mathbf{Y} \stackrel{d}{=} \|\mathbf{a}\|Y_1$ (lin. comb. are of the same type).
 \Rightarrow Subadditivity of VaR_α for jointly elliptical losses

Theorem 6.16 (Stochastic representation)

$\mathbf{Y} \sim S_d(\psi)$ if and only if $\mathbf{Y} \stackrel{d}{=} R\mathbf{S}$ for an independent radial part $R \geq 0$ and $\mathbf{S} \sim \text{U}(\{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| = 1\})$.

- See the appendix for proofs for Theorems 6.15 and 6.16.
- If \mathbf{Y} has a density $f_{\mathbf{Y}}$, it satisfies $f_{\mathbf{Y}}(\mathbf{y}) = g(\|\mathbf{y}\|^2)$ for a function $g : [0, \infty) \rightarrow [0, \infty)$ referred to as density generator (i.e. $f_{\mathbf{Y}}$ is constant on spheres); see the appendix for a proof.

Corollary 6.17

If $\mathbf{Y} \sim S_d(\psi)$ and $\mathbb{P}(\mathbf{Y} = \mathbf{0}) = 0$, then $(\|\mathbf{Y}\|, \frac{\mathbf{Y}}{\|\mathbf{Y}\|}) \stackrel{d}{=} (R, \mathbf{S})$ since

$$(\|\mathbf{Y}\|, \frac{\mathbf{Y}}{\|\mathbf{Y}\|}) \stackrel{d}{=} (\|R\mathbf{S}\|, \frac{R\mathbf{S}}{\|R\mathbf{S}\|}) = (|R|\|\mathbf{S}\|, \frac{R\mathbf{S}}{|R|\|\mathbf{S}\|}) = (R, \mathbf{S}).$$

In particular, $\|\mathbf{Y}\|$ and $\mathbf{Y}/\|\mathbf{Y}\|$ are independent (\Rightarrow goodness-of-fit).

Example 6.18 (Standardized normal variance mixtures)

- $\mathbf{Y} \sim M_d(\mathbf{0}, I_d, \hat{F}_W)$ is spherical (recall: $\mathbf{Y} \stackrel{d}{=} \mathbf{0} + \sqrt{W}I_d\mathbf{Z}$) since

$$\begin{aligned}\phi_{\mathbf{Y}}(\mathbf{t}) &= \mathbb{E}(\exp(i\mathbf{t}'\sqrt{W}\mathbf{Z})) = \mathbb{E}_W(\mathbb{E}(\exp(i(\mathbf{t}\sqrt{W})'\mathbf{Z}) \mid W)) \\ &= \mathbb{E}(\exp(-\tfrac{1}{2}W\mathbf{t}'\mathbf{t})) = \hat{F}_W(\tfrac{1}{2}\mathbf{t}'\mathbf{t}) = \hat{F}_W(\tfrac{1}{2}\|\mathbf{t}\|^2),\end{aligned}$$

so $\mathbf{Y} \sim S_d(\psi)$ by Theorem 6.15 Part 2). We thus have $\psi(t) = \hat{F}_W(t/2)$.

- For $\mathbf{Y} \sim N_d(\mathbf{0}, I_d)$, $\psi(t) = \exp(-t/2)$. By Corollary 6.17, simulating $\mathbf{S} \sim U(\{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| = 1\})$ can thus be done via $\mathbf{S} \stackrel{d}{=} \mathbf{Y}/\|\mathbf{Y}\|$. Fang et al. (1990, pp. 48) show that ψ generates $S_d(\psi)$ for all $d \in \mathbb{N}$ if and only if it is the characteristic generator of a normal mixture.

Example 6.19 ($R, S, \text{cov}, \text{corr}$)

- It follows from $\mathbf{Y} \sim N_d(\mathbf{0}, I_d)$ and $R^2 = \mathbf{Y}'\mathbf{Y} \sim \chi_d^2$ that

$$\mathbf{0} = \mathbb{E}\mathbf{Y} \stackrel{\text{Th. 6.16}}{=} \mathbb{E}R\mathbb{E}S \Rightarrow \mathbb{E}S = \mathbf{0},$$

$$I_d = \underset{\text{Th. 6.16}}{\text{cov}} \mathbf{Y} = \text{cov}(RS) = \mathbb{E}(R^2) \text{cov} S = d \text{cov} S \Rightarrow \text{cov} S = I_d/d. \quad (24)$$

- For $\mathbf{Y} \sim S_d(\psi)$ with $\mathbb{E}(R^2) < \infty$, it follows that

$$\underset{\text{Th. 6.16}}{\text{cov}} \mathbf{Y} = \text{cov}(RS) = \mathbb{E}(R^2) \text{cov} S = \frac{\mathbb{E}(R^2)}{d} I_d$$

$$\text{and thus } \text{corr} \mathbf{Y} = \frac{(\mathbb{E}(R^2)/d)I_d}{\sqrt{(\mathbb{E}(R^2)/d)(\mathbb{E}(R^2)/d)}} = I_d.$$

- For $\mathbf{X} = \boldsymbol{\mu} + A\mathbf{Y}$ with $\mathbb{E}(R^2) < \infty$ and Cholesky factor A of a covariance matrix Σ , we have $\text{cov} \mathbf{X} = \frac{\mathbb{E}(R^2)}{d} \Sigma$ and $\text{corr} \mathbf{X} = P$ (the correlation matrix corresponding to Σ).

Example 6.20 (t distribution)

For $\mathbf{Y} \sim t_d(\nu, \mathbf{0}, I_d)$, $R^2 = \mathbf{Y}'\mathbf{Y} = W\mathbf{Z}'\mathbf{Z}$ for $\mathbf{Z} \sim N_d(\mathbf{0}, I_d)$. Therefore,

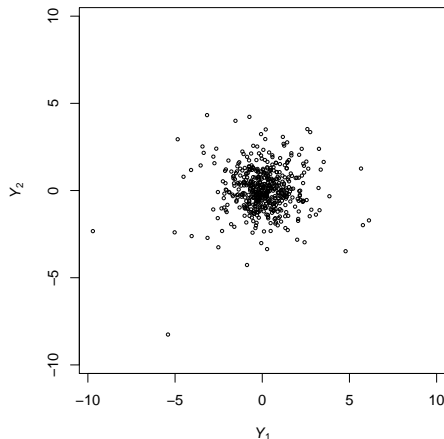
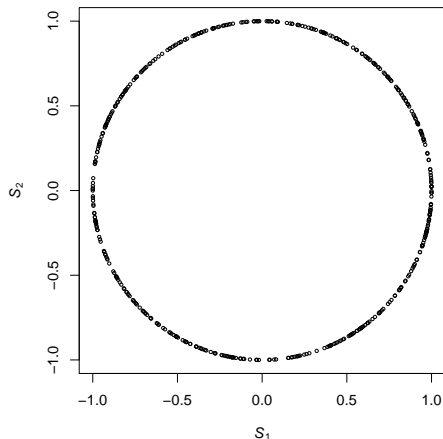
$$\frac{R^2}{d} = \frac{\mathbf{Z}'\mathbf{Z}/d}{(\nu/W)/\nu} = \frac{\chi_d^2/d}{\chi_\nu^2/\nu} \sim F(d, \nu)$$

and thus $\mathbb{E}(R^2/d) = \frac{\nu}{\nu-2}$.

- This, together with Example 6.19, implies that $\mathbf{X} \sim t_d(\nu, \boldsymbol{\mu}, \Sigma)$ has $\text{cov } \mathbf{X} = \frac{\nu}{\nu-2}\Sigma$ and $\text{corr } \mathbf{X} = P$ (which we already know from Section 6.2.1); note that in the univariate case $X \sim t(\nu, \mu, \sigma^2)$ and $\text{var}(X) = \frac{\nu}{\nu-2}\sigma^2$.
- We also see that we can use a Q-Q plot of the order statistics of $R^2/d = \|\mathbf{Y}\|^2/d$ versus the theoretical quantiles of a (hypothesized) $F(d, \nu)$ distribution to check the goodness-of-fit of the hypothesized t distribution (in any dimensions).
- See the appendix for the form of the density generator g .

Example 6.21 (Understanding spherical distributions)

$n = 500$ realizations of \mathbf{S} (left) and $\mathbf{Y} = R\mathbf{S}$ (right) for $R \sim \sqrt{dF(d, \nu)}$, $d = 2$, $\nu = 4$ (as for the multivariate t distribution with $\nu = 4$).



6.3.2 Elliptical distributions

Definition 6.22 (Elliptical distribution)

A random vector $\mathbf{X} = (X_1, \dots, X_d)$ has an *elliptical distribution* if

$$\mathbf{X} \stackrel{d}{=} \boldsymbol{\mu} + A\mathbf{Y}, \quad (\text{multivariate affine transformation})$$

where $\mathbf{Y} \sim S_k(\psi)$, $A \in \mathbb{R}^{d \times k}$ (*scale matrix* $\Sigma = AA'$), and (*location vector*) $\boldsymbol{\mu} \in \mathbb{R}^d$.

- By Theorem 6.16, an elliptical random vector **admits the stochastic representation** $\mathbf{X} \stackrel{d}{=} \boldsymbol{\mu} + RAS$, with R and S as before.
- The **cf** of an elliptical random vector \mathbf{X} is $\phi_{\mathbf{X}}(\mathbf{t}) = \mathbb{E}(e^{it'\mathbf{X}}) = \mathbb{E}(e^{it'(\boldsymbol{\mu} + A\mathbf{Y})}) = e^{it'\boldsymbol{\mu}} \mathbb{E}(e^{i(A'\mathbf{t})'\mathbf{Y}}) = e^{it'\boldsymbol{\mu}} \psi(\mathbf{t}'\Sigma\mathbf{t})$. Notation: $\mathbf{X} \sim E_d(\boldsymbol{\mu}, \Sigma, \psi)$ ($= E_d(\boldsymbol{\mu}, c\Sigma, \psi(\cdot/c))$, $c > 0$).
- If Σ is positive definite with Cholesky factor A , then $\mathbf{X} \sim E_d(\boldsymbol{\mu}, \Sigma, \psi)$ if and only if $\mathbf{Y} = A^{-1}(\mathbf{X} - \boldsymbol{\mu}) \sim S_d(\psi)$.

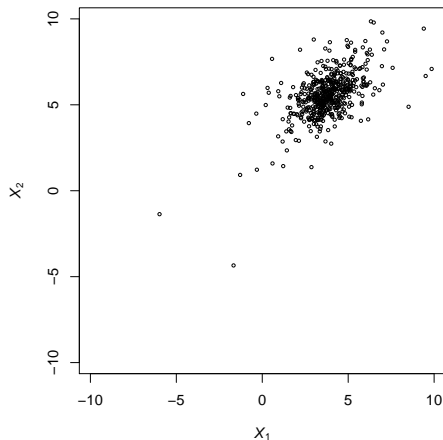
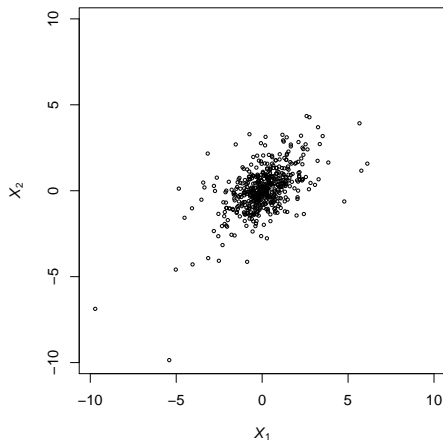
- Normal variance mixture distributions are elliptical (most useful examples) since $\mathbf{X} \stackrel{d}{=} \boldsymbol{\mu} + \sqrt{W}A\mathbf{Z} = \boldsymbol{\mu} + \sqrt{W}\|\mathbf{Z}\|A\mathbf{Z}/\|\mathbf{Z}\| = \boldsymbol{\mu} + RAS$ with $R = \sqrt{W}\|\mathbf{Z}\|$ and $S = \mathbf{Z}/\|\mathbf{Z}\|$. By Corollary 6.17, R and S are indeed independent.
- If $\mathbf{X} \sim E_d(\boldsymbol{\mu}, \Sigma, \psi)$ with $\mathbb{P}(\mathbf{X} = \boldsymbol{\mu}) = 0$, then $\mathbf{Y} = A^{-1}(\mathbf{X} - \boldsymbol{\mu}) \sim S_d(\psi)$. Corollary 6.17 implies that

$$\left(\sqrt{(\mathbf{X} - \boldsymbol{\mu})' \Sigma^{-1} (\mathbf{X} - \boldsymbol{\mu})}, \frac{A^{-1}(\mathbf{X} - \boldsymbol{\mu})}{\sqrt{(\mathbf{X} - \boldsymbol{\mu})' \Sigma^{-1} (\mathbf{X} - \boldsymbol{\mu})}} \right) \stackrel{d}{=} (R, S), \quad (25)$$

which can be used for testing elliptical symmetry.

Example 6.23 (Understanding elliptical distributions)

$n = 500$ realizations of $\mathbf{X} = \mathbf{RAS}$ (left) and $\mathbf{X} = \boldsymbol{\mu} + \mathbf{RAS}$ (right) for $R \sim \sqrt{dF(d, \nu)}$, $d = 2$, $\nu = 4$; based on the same samples as in Example 6.21.



6.3.3 Properties of elliptical distributions

- **Density:** Let Σ be positive definite and $\mathbf{Y} \sim S_d(\psi)$ have density generator g . The **Density Transformation Theorem** implies that $\mathbf{X} = \boldsymbol{\mu} + A\mathbf{Y}$ has density

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{\sqrt{\det \Sigma}} g((\mathbf{x} - \boldsymbol{\mu})' \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu})),$$

which depends on \mathbf{x} only through $(\mathbf{x} - \boldsymbol{\mu})' \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu})$, i.e. is constant on ellipsoids (hence the name “elliptical”).

- **Linear combinations:** For $\mathbf{X} \sim E_d(\boldsymbol{\mu}, \Sigma, \psi)$, $B \in \mathbb{R}^{k \times d}$ and $\mathbf{b} \in \mathbb{R}^k$,

$$B\mathbf{X} + \mathbf{b} \sim E_k(B\boldsymbol{\mu} + \mathbf{b}, B\Sigma B', \psi) \quad (\text{via cfs}).$$

If $\mathbf{a} \in \mathbb{R}^d$ (take $\mathbf{b} = \mathbf{0}$ and $B = \mathbf{a}' \in \mathbb{R}^{1 \times d}$),

$$\mathbf{a}'\mathbf{X} \sim E_1(\mathbf{a}'\boldsymbol{\mu}, \mathbf{a}'\Sigma\mathbf{a}, \psi) \quad (\text{as for } N(\boldsymbol{\mu}, \Sigma)). \quad (26)$$

From $\mathbf{a} = \mathbf{e}_j = (0, \dots, 0, 1, 0, \dots, 0)$ we see that all marginal distributions are of the same type.

- **Marginal dfs:** As for $N_d(\boldsymbol{\mu}, \Sigma)$, it immediately follows that $\mathbf{X} = (\mathbf{X}'_1, \mathbf{X}'_2)' \sim E_d(\boldsymbol{\mu}, \Sigma, \psi)$ satisfies $\mathbf{X}_1 \sim E_k(\boldsymbol{\mu}_1, \Sigma_{11}, \psi)$ and that $\mathbf{X}_2 \sim E_{d-k}(\boldsymbol{\mu}_2, \Sigma_{22}, \psi)$; i.e. **margins of elliptical distributions are elliptical**.
- **Conditional distributions:** One can also show that **conditional distributions of elliptical distributions are elliptical**; see Embrechts et al. (2002). For $N_d(\boldsymbol{\mu}, \Sigma)$ the characteristic generator remains the same.
- **Quadratic forms:** (25) implies that $(\mathbf{X} - \boldsymbol{\mu})' \Sigma^{-1} (\mathbf{X} - \boldsymbol{\mu}) \stackrel{d}{=} R^2$. If $\mathbf{X} \sim N_d(\boldsymbol{\mu}, \Sigma)$, $R^2 \sim \chi_d^2$; and if $\mathbf{X} \sim t_d(\nu, \boldsymbol{\mu}, \Sigma)$, $R^2/d \sim F(d, \nu)$.
- **Convolutions:** Let $\mathbf{X} \sim E_d(\boldsymbol{\mu}, \Sigma, \psi)$ and $\mathbf{Y} \sim E_d(\tilde{\boldsymbol{\mu}}, c\Sigma, \tilde{\psi})$ be **independent**. Then **$a\mathbf{X} + b\mathbf{Y}$ is elliptically distributed** for $a, b \in \mathbb{R}$, $c > 0$.
- **Conditional correlations remain invariant** See Proposition A.11.

Many (but not all) **nice properties of $N_d(\boldsymbol{\mu}, \Sigma)$ are preserved**. For estimating $\boldsymbol{\mu}$, Σ , P , see the appendix. The following result shows why elliptical distributions are known as the “Garden of Eden” of QRM.

Proposition 6.24 (Subadditivity of VaR in elliptical models)

Let $L_i = \lambda_i' X$, $\lambda_i \in \mathbb{R}^d$, $i \in \{1, \dots, n\}$, with $X \sim E_d(\mu, \Sigma, \psi)$. Then $\text{VaR}_\alpha(\sum_{i=1}^n L_i) \leq \sum_{i=1}^n \text{VaR}_\alpha(L_i)$ for all $\alpha \in [1/2, 1]$.

Proof. Consider a generic $L = \lambda' X \stackrel{d}{=} \lambda' \mu + \lambda' A Y$ for $Y \sim S_k(\psi)$. By Theorem 6.15 Part 3), $\lambda' A Y \stackrel{d}{=} \|\lambda' A\| Y_1$, so $L \stackrel{d}{=} \lambda' \mu + \|\lambda' A\| Y_1$ (all L_i 's are of the same type). By translation invariance and positive homogeneity,

$$\text{VaR}_\alpha(L) = \lambda' \mu + \|\lambda' A\| \text{VaR}_\alpha(Y_1). \quad (27)$$

Applying (27) once to $L = \sum_{i=1}^n L_i = (\sum_{i=1}^n \lambda_i)' X$ and to each $L = L_i = \lambda_i' X$, $i \in \{1, \dots, n\}$, and using that $\text{VaR}_\alpha(Y_1) \geq 0$ for $\alpha \in [1/2, 1]$, we obtain $\text{VaR}_\alpha(\sum_{i=1}^n L_i) \stackrel{(27)}{=} \sum_{i=1}^n \lambda_i' \mu + \|\sum_{i=1}^n \lambda_i' A\| \text{VaR}_\alpha(Y_1)$
 $\leq \sum_{i=1}^n \lambda_i' \mu + (\sum_{i=1}^n \|\lambda_i' A\|) \text{VaR}_\alpha(Y_1) = \sum_{i=1}^n (\lambda_i' \mu + \|\lambda_i' A\| \text{VaR}_\alpha(Y_1))$
 $\stackrel{(27)}{=} \sum_{i=1}^n \text{VaR}_\alpha(L_i)$. For $\lambda_i = e_i$, $\text{VaR}_\alpha(\sum_{i=1}^n X_i) \leq \sum_{i=1}^n \text{VaR}_\alpha(X_i)$. \square

6.4 Dimension reduction techniques

6.4.1 Factor models

Explain the variability of \mathbf{X} in terms of common factors.

Definition 6.25 (p -factor model)

\mathbf{X} follows a *p -factor model* if

$$\mathbf{X} = \mathbf{a} + B\mathbf{F} + \boldsymbol{\varepsilon}, \quad (28)$$

where

- 1) $B \in \mathbb{R}^{d \times p}$ is a *matrix of factor loadings* and $\mathbf{a} \in \mathbb{R}^d$;
- 2) $\mathbf{F} = (F_1, \dots, F_p)$ is the random vector of *(common) factors* with $p < d$ and $\boldsymbol{\Omega} := \text{cov}(\mathbf{F})$, (*systematic risk*);
- 3) $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_d)$ is the random vector of *idiosyncratic error terms* with $\mathbb{E}(\boldsymbol{\varepsilon}) = \mathbf{0}$, $\boldsymbol{\Upsilon} := \text{cov}(\boldsymbol{\varepsilon})$ diag., $\text{cov}(\mathbf{F}, \boldsymbol{\varepsilon}) = (\mathbf{0})$ (*idiosync. risk*).

- **Goals:** Identify or estimate \mathbf{F}_t , $t \in \{1, \dots, n\}$, then model the distribution/dynamics of the (lower-dimensional) factors (instead of \mathbf{X}_t , $t \in \{1, \dots, n\}$).
- Factor models imply that $\Sigma := \text{cov}(\mathbf{X}) = B\Omega B' + \Upsilon$.
- With $B^* = B\Omega^{1/2}$ and $\mathbf{F}^* = \Omega^{-1/2}(\mathbf{F} - \mathbb{E}(\mathbf{F}))$, we have

$$\mathbf{X} = \boldsymbol{\mu} + B^* \mathbf{F}^* + \boldsymbol{\varepsilon},$$

where $\boldsymbol{\mu} = \mathbb{E}(\mathbf{X})$. We have $\Sigma = B^*(B^*)' + \Upsilon$. Conversely, if $\text{cov}(\mathbf{X}) = BB' + \Upsilon$ for some $B \in \mathbb{R}^{d \times p}$ with $\text{rank}(B) = p < d$ and diagonal matrix Υ , then \mathbf{X} has a factor-model representation for a p -dimensional \mathbf{F} and d -dimensional $\boldsymbol{\varepsilon}$.

- For a one-factor/equicorrelation example, see the appendix.

6.4.2 Statistical estimation strategies

Consider $\mathbf{X}_t = \mathbf{a} + B\mathbf{F}_t + \varepsilon_t$, $t \in \{1, \dots, n\}$. Three types of factor model are commonly used:

- 1) *Macroeconomic factor models*: Here we assume that \mathbf{F}_t is observable, $t \in \{1, \dots, n\}$. Estimation of B, \mathbf{a} is accomplished by time series regression.
- 2) *Fundamental factor models*: Here we assume that the matrix of factor loadings B is known but the factors \mathbf{F}_t are unobserved (and have to be estimated from \mathbf{X}_t , $t \in \{1, \dots, n\}$, using cross-sectional regression at each t).
- 3) *Fundamental factor models*: Here we assume that neither the factors \mathbf{F}_t nor the factor loadings B are observed (both have to be estimated from \mathbf{X}_t , $t \in \{1, \dots, n\}$). The factors can be found with principal component analysis.

6.4.3 Estimating macroeconomic factor models

This is achieved by [time series regression](#).

Univariate regression

- Consider the (univariate) [time series regression](#) model

$$X_{t,j} = a_j + \mathbf{b}'_j \mathbf{F}_t + \varepsilon_{t,j}, \quad t \in \{1, \dots, n\}.$$

- To justify the use of the [ordinary least-squares \(OLS\)](#) method to derive statistical properties of the method it is usually [assumed that](#), conditional on the factors, the errors $\varepsilon_{1,j}, \dots, \varepsilon_{n,j}$ [form a white noise process](#) (i.e. are identically distributed and serially uncorrelated).
- \hat{a}_j estimates a_j , $\hat{\mathbf{b}}_j$ estimates the j th row of B .

Models can also be estimated simultaneously using [multivariate regression](#); see McNeil et al. (2015).

6.4.4 Estimating fundamental factor models

- Consider the cross-sectional regression model $\mathbf{X}_t = B\mathbf{F}_t + \boldsymbol{\varepsilon}_t$ (B known; \mathbf{F}_t to be estimated; $\text{cov}(\boldsymbol{\varepsilon}) = \Upsilon$); note that \mathbf{a} can be absorbed into \mathbf{F}_t . To obtain precision in estimating \mathbf{F}_t , we need $d \gg p$.
- First estimate \mathbf{F}_t via OLS by $\hat{\mathbf{F}}_t^{\text{OLS}} = (B'B)^{-1}B'\mathbf{X}_t$. This is the best linear unbiased estimator if the $\boldsymbol{\varepsilon}$ is homoskedastic. However, it is possible to obtain linear unbiased estimates with a smaller covariance matrix via generalized least squares (GLS).
- To this end, estimate Υ by $\hat{\Upsilon}$ via the diagonal of the sample covariance matrix of the residuals $\hat{\boldsymbol{\varepsilon}}_t = \mathbf{X}_t - B\hat{\mathbf{F}}_t^{\text{OLS}}$, $t \in \{1, \dots, n\}$.
- Then estimate \mathbf{F}_t via $\hat{\mathbf{F}}_t = (B'\hat{\Upsilon}^{-1}B)^{-1}B'\hat{\Upsilon}^{-1}\mathbf{X}_t$.

6.4.5 Principal component analysis

- **Goal:** Reduce the dimensionality of highly correlated data by finding a small number of uncorrelated linear combinations which account for most of the variance in the data; this can be used for finding factors.
- **Key:** Any symmetric A admits a *spectral decomposition*

$$A = \Gamma \Lambda \Gamma',$$

where

- 1) $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_d)$ is the diagonal matrix of eigenvalues of A which, w.l.o.g., are ordered so that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_d$; and
 - 2) Γ is an orthogonal matrix whose columns are eigenvectors of A standardized to have length 1.
- Let $\Sigma = \Gamma \Lambda \Gamma'$ with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_d \geq 0$ (positive semidefiniteness \Rightarrow all eigenvalues ≥ 0) and $\mathbf{Y} = \Gamma'(\mathbf{X} - \boldsymbol{\mu})$ (the so-called *principal component transform*). The j th component $Y_j = \boldsymbol{\gamma}_j'(\mathbf{X} - \boldsymbol{\mu})$ is the j th *principal component of \mathbf{X}* (where $\boldsymbol{\gamma}_j$ is the j th column of Γ).

- We have $\mathbb{E}\mathbf{Y} = \mathbf{0}$ and $\text{cov}(\mathbf{Y}) = \Gamma'\Sigma\Gamma = \Gamma'\Gamma\Lambda\Gamma'\Gamma = \Lambda$, so the principal components are uncorrelated and $\text{var}(Y_j) = \lambda_j$, $j \in \{1, \dots, d\}$. The principal components are thus ordered by decreasing variance.
- One can show:
 - ▶ The first principal component is that standardized linear combination of \mathbf{X} which has maximal variance among all such combinations, i.e. $\text{var}(\gamma_1'\mathbf{X}) = \max\{\text{var}(\mathbf{a}'\mathbf{X}) : \mathbf{a}'\mathbf{a} = 1\}$.
 - ▶ For $j \in \{2, \dots, d\}$, the j th principal component is that standardized linear combination of \mathbf{X} which has maximal variance among all such linear combinations which are orthogonal to (and hence uncorrelated with) the first $j - 1$ -many linear combinations.
- $\sum_{j=1}^d \text{var}(Y_j) = \sum_{j=1}^d \lambda_j = \text{trace}(\Sigma) = \sum_{j=1}^d \text{var}(X_j)$, so we can interpret $\sum_{j=1}^k \lambda_j / \sum_{j=1}^d \lambda_j$ as the fraction of total variance explained by the first k principal components.

Principal components as factors

- Inverting the principal component transform $\mathbf{Y} = \Gamma'(\mathbf{X} - \boldsymbol{\mu})$, we have

$$\mathbf{X} = \boldsymbol{\mu} + \Gamma \mathbf{Y} = \boldsymbol{\mu} + \Gamma_1 \mathbf{Y}_1 + \Gamma_2 \mathbf{Y}_2 =: \boldsymbol{\mu} + \Gamma_1 \mathbf{Y}_1 + \boldsymbol{\varepsilon}$$

where $\mathbf{Y}_1 \in \mathbb{R}^k$ contains the first k principal components. This is reminiscent of the basic factor model.

- Although $\varepsilon_1, \dots, \varepsilon_d$ will tend to have small variances, the assumptions of the factor model are generally violated (since they need not have a diagonal covariance matrix and need not be uncorrelated with \mathbf{Y}_1). Nevertheless, principal components are often interpreted as factors.
- In principle, the same can be applied to the sample covariance matrix to obtain the sample principal components; see the appendix.