2 Basics concepts in risk management

2.1 Risk management for a financial firm

2.1.1 Assets, liabilities and the balance sheet

The risks of a financial firm can be understood from its *balance sheet* (financial statement showing *assets* (investments) and *liabilities* (how funds have been raised; obligations)). A stylized balance sheet for a bank is:

Assets Investments of the firm	Liabilities Obligations from fundraising
	Debt capital
- Cash	- Customer deposits
- Securities	- Bonds issued
- Loans, mortgages	- Reserves for losses on loans from banks
- Property	
	Equity

A stylized balance sheet for an insurer (sells contracts, collects premiums, raises funds by issuing bonds $\Rightarrow$ Liabilities are thus obligations to policy holders (reserve against future claims; obligations to bondholders)) is:

Assets Investments of the firm	Liabilities Obligations to policy holders
- Investments (e.g., bonds, stocks) - Investements for unit-linked contracts - Property	Debt capital - Reserves for policies written - Bonds issued
	Equity

- Balance sheet equation: Assets $=$ Liabilities $=$ Debt $+$ Equity.
  If equity > 0, the company is *solvent*, otherwise *insolvent*. Distinction to default (not able to pay): Note that a solvent company can default because of liquidity problems.

- Valuation of the items on the balance sheet is a non-trivial task.
  - *Amortized cost accounting* values a position a *book value* at its inception and this is carried forward/progressively reduced over time.
  - (Similar to market consistent valuation (a variant of)) *fair-value accounting* values assets at prices they are sold and liabilities at prices that would have to be paid in the market. This can be challenging for non-traded or illiquid assets or liabilities.

  There is a tendency in the financial industry to move towards fair-value accounting.

2.1.2 Risks faced by a financial firm

- Decrease in the value of the investments on the asset side of the balance sheet (e.g., losses from securities trading or credit risk)
- *Maturity mismatch* (large parts of the assets are relatively illiquid (long-term) whereas large parts of the liabilities are rather short-term obligations. This can lead to a default of a solvent bank and even a bank run).
- The prime risk of an insurer is *insolvency* (risk that claims of policy holders cannot be met). On the asset side, risks are similar to those of a bank. On the liability side, the main risk is that reserves are insufficient to cover future claim payments. Note that the liabilities of a life insurer are of a long-term nature and subject to multiple categories of risk (e.g., interest rate risk, inflation risk and longevity risk).
- So risk is found on both sides of the balance sheet and thus RM should not focus on the asset side alone.

2.1.3 Capital

- There are different notions of capital. One distinguishes:

  Equity capital
  - Value of assets − debt;
  - Measures the firm's value to its shareholders;
  - Can be split into *shareholder capital* (initial capital invested in the firm) and *retained earnings* (accumulated earnings not paid out to shareholders).

  Regulatory capital
  - Capital required according to regulatory rules;
  - For European insurance companies: MCR + SCR (see Solvency II);
  - A regulatory framework also specifies the capital quality. Here one distinguishes *Tier 1 capital* (i.e., shareholder capital + retained earnings;

can act in full as buffer) and *Tier 2 capital* (includes other positions on the balance sheet, e.g., subordinated debt).

Economic capital
- Capital required to control the probability of becoming insolvent (typically over a one-year horizon);
- Internal assessment or risk capital;
- Aims at a holistic view (assets and liabilities) and works with fair values of balance sheet items.

- All of these notions refer to items on the liability side that entail no (or very limited) obligations to outside creditors and that can thus serve as a buffer against losses.

2.2 Modeling value and value change

2.2.1 Mapping of risks

Prob. space $(\Omega, \mathcal{F}, \mathbb{P})$
X random variable (rv)
$x = X(\omega)$ a realization
(ω = state of nature)

We now set up a general mathematical model for value and changes in value caused by financial risks. For this we assume to work on a *probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ and consider a risk or loss as a *random variable* $X : \Omega \to \mathbb{R}$ (or: L, a random vector $\boldsymbol{X}$, ...).

- Consider a portfolio of assets and possibly liabilities. The *value* of the portfolio at time t (*today*) is denoted by V_t (a random variable; assumed to be known at t; its df is typically not trivial to determine!).
- We consider a given *time horizon* Δt (e.g., 1 d or 10 d for market risk; 1 y for credit risk; 20 y for pension funds) and assume:
  1) the portfolio composition remains fixed over Δt;
  2) there are no intermediate payments during Δt

  $\Rightarrow$ Fine for $\Delta t \in \{1\,\text{d}, 10\,\text{d}\}$ but unlikely to hold for $\Delta t \in \{1\,\text{y}, 20\,\text{y}\}$.

- The *change* in value of our portfolio is then given by

$$\Delta V_{t+1} = V_{t+1} - V_t$$

and we define the (random) *loss* as the sign-adjusted value change

$$L_{t+1} = -\Delta V_{t+1}$$

(as QRM is mainly concerned with losses).

Remark 2.1

1) The distribution of L_{t+1} is called *loss distribution* (df F_L or simply F).
2) Practitioners often consider the *profit-and-loss (P&L) distribution* which is the distribution of $-L_{t+1} = \Delta V_{t+1}$.
3) For longer time intervals, $\Delta V_{t+1} = V_{t+1}/(1 + r) - V_t$ (r = risk-free interest rate) would be more adequate, but we will mostly neglect this issue.

- V_t is typically modeled as a function f of time t and a d-dimensional random vector $\boldsymbol{Z} = (Z_{t,1}, \ldots, Z_{t,d})$ of *risk factors* (d typically large), that is,

$$V_t = f(t, \boldsymbol{Z}_t) \quad (\textit{mapping} \text{ of risks})$$

  for some measurable $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$. The choice of f and $\boldsymbol{Z}_t$ is problem-specific (but typically known to a bank).

- It is often convenient to work with the *risk-factor changes*

$$\boldsymbol{X}_t = \boldsymbol{Z}_t - \boldsymbol{Z}_{t-1}.$$

  We can rewrite L_{t+1} in terms of $\boldsymbol{X}_t$ via

$$\begin{aligned} L_{t+1} &= -(V_{t+1} - V_t) = -(f(t+1, \boldsymbol{Z}_{t+1}) - f(t, \boldsymbol{Z}_t)) \\ &= -(f(t+1, \boldsymbol{Z}_t + \boldsymbol{X}_{t+1}) - f(t, \boldsymbol{Z}_t)) =: L(\boldsymbol{X}_{t+1}); \end{aligned}$$

  $L(\cdot)$ is known as *loss operator*. We see that the loss df is determined by the loss df of $\boldsymbol{X}_{t+1}$.

- If f is differentiable, its first-order (Taylor) approximation is

$$f(t+1, \boldsymbol{Z}_t + \boldsymbol{X}_{t+1}) \approx f(t, \boldsymbol{Z}_t) + f_t(t, \boldsymbol{Z}_t) \cdot 1 + \sum_{j=1}^{d} f_{z_j}(t, \boldsymbol{Z}_t) \cdot X_{t+1,j}$$

  We can thus approximate L_{t+1} by the *linearized loss*

$$L_{t+1}^{\Delta} = -\left(f_t(t, \boldsymbol{Z}_t) + \sum_{j=1}^{d} f_{z_j}(t, \boldsymbol{Z}_t) X_{t+1,j} \right) = -(c_t + \boldsymbol{b}_t^{\top} \boldsymbol{X}_{t+1}),$$

  a linear function of $X_{t+1,1}, \dots, X_{t+1,d}$ (indices denote partial derivatives). The approximation is best if the $|X_{t+1,j}|$'s are small (typically if Δt is small; questionable for extreme market changes) and if V_{t+1} is almost linear in $\boldsymbol{Z}_t$ (i.e., if mixed partial derivatives $|f_{z_i z_j}|$ are small in absolut value).

Example 2.2 (Stock portfolio)

Consider a portfolio $\mathcal{P}$ of d stocks $S_{t,1}, \dots, S_{t,d}$ ($S_{t,j}$ = value of stock j at time t) and denote by λ_j the number of shares of stock j in $\mathcal{P}$. In finance and risk management, one typically uses logarithmic prices as risk factors, i.e., $Z_{t,j} = \log S_{t,j}$, $j \in \{1, \dots, d\}$. Then

$$V_t = f(t, \boldsymbol{Z}_t) = \sum_{j=1}^{d} \lambda_j S_{t,j} = \sum_{j=1}^{d} \lambda_j e^{Z_{t,j}}.$$

- The one-period ahead loss is then given by

$$\begin{aligned} L_{t+1} &= -(V_{t+1} - V_t) = -\sum_{j=1}^{d} \lambda_j (e^{Z_{t,j}+X_{t+1,j}} - e^{Z_{t,j}}) \\ &= -\sum_{j=1}^{d} \lambda_j e^{Z_{t,j}} (e^{X_{t+1,j}} - 1) = -\sum_{j=1}^{d} \lambda_j S_{t,j} (e^{X_{t+1,j}} - 1). \end{aligned} \quad (1)$$

- With $f_{z_j}(t, \boldsymbol{Z}_t) = \lambda_j e^{Z_{t,j}} = \lambda_j S_{t,j}$, the linearized loss is

$$L^{\Delta}_{t+1} = -\Big(0 + \sum_{j=1}^{d} f_{z_j}(t, \boldsymbol{Z}_t) X_{t+1,j}\Big) = -\sum_{j=1}^{d} \lambda_j S_{t,j} X_{t+1,j}$$
$$= -\sum_{j=1}^{d} \tilde{w}_{t,j} X_{t+1,j} = -V_t \sum_{j=1}^{d} w_{t,j} X_{t+1,j},$$

  where $\tilde{w}_{t,j} = \lambda_j S_{t,j}$ and $w_{t,j} = \lambda_j S_{t,j}/V_t$ (proportion of V_t invested in stock j). Note that $c_t = 0$ and $\boldsymbol{b}_t = \tilde{\boldsymbol{w}}_t$ here.

- If $\boldsymbol{\mu} = \mathbb{E}\boldsymbol{X}_{t+1}$ and $\Sigma = \operatorname{Cov} \boldsymbol{X}_{t+1}$ are known, then expectation and variance of the (linearized) one-period ahead loss are

$$\mathbb{E} L^{\Delta}_{t+1} = -\tilde{\boldsymbol{w}}_t^{\top} \boldsymbol{\mu} = -V_t \boldsymbol{w}_t^{\top} \boldsymbol{\mu},$$
$$\operatorname{Var} L^{\Delta}_{t+1} = \tilde{\boldsymbol{w}}_t^{\top} \Sigma \tilde{\boldsymbol{w}}_t = V_t^2 \boldsymbol{w}_t^{\top} \Sigma \boldsymbol{w}_t.$$

Example 2.3 (European call option)

Consider a portfolio consisting of a European call option on a non-dividend-paying[3] stock S_t with maturity T and strike (exercise price) K. The Black–Scholes formula says that

$$V_t = C^{\mathrm{BS}}(t, S_t; r, \sigma, K, T) = S_t \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2), \qquad (2)$$

where

- t is the time in years;
- Φ is the df of $\mathrm{N}(0,1)$;
- r is the continuously compounded risk-free interest rate;
- $d_1 = \frac{\log(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$ and $d_2 = d_1 - \sigma\sqrt{T-t}$; and
- σ is the annualized volatility (standard deviation) of S_t.

[3] No distribution of profits to the shareholders (dividends).

While (2) assumes r, σ to be constant, this is often not true in real markets. Hence, besides $\log S_t$, we consider r_t, σ_t as risk factors, so

$$\boldsymbol{Z}_t = (\log S_t, r_t, \sigma_t) \;\Rightarrow\; \boldsymbol{X}_{t+1} = (\log(S_{t+1}/S_t), r_{t+1} - r_t, \sigma_{t+1} - \sigma_t).$$

This implies that the mapping f is given by

$$V_t = C^{\mathsf{BS}}(t, e^{Z_{t,1}}; Z_{t,2}, Z_{t,3}, K, T) =: f(t, \boldsymbol{Z}_t)$$

and the linearized one-day ahead loss (omitting the arguments of C^{BS}) is

$$\begin{aligned} L^{\Delta}_{t+1} &= -\Big(f_t(t, \boldsymbol{Z}_t) + \sum_{j=1}^{3} f_{z_j}(t, \boldsymbol{Z}_t) X_{t+1,j}\Big) \\ &= -(C^{\mathsf{BS}}_t \Delta t + C^{\mathsf{BS}}_{S_t} S_t X_{t+1,1} + C^{\mathsf{BS}}_{r_t} X_{t+1,2} + C^{\mathsf{BS}}_{\sigma_t} X_{t+1,3}). \end{aligned}$$

Here $\Delta t = 1/250$ (as our risk management horizon is 1 d here) and the *"Greeks"* enter (C^{BS}_t is the *theta* of the option; $C^{\mathsf{BS}}_{S_t}$ the *delta*; $C^{\mathsf{BS}}_{r_t}$ the *rho*; $C^{\mathsf{BS}}_{\sigma_t}$ the *vega*).

For portfolios of derivatives, L^{Δ}_{t+1} can be a rather poor approximation to L_{t+1} $\Rightarrow$ higher-order (Taylor) approximations such as the *delta-gamma-*

approximation (second-order) have been used, but one loses the tractability/ellipticality.

2.2.2 Valuation methods

Fair value accounting

The *fair value* of an asset (liability) is an estimate of the price which would be received (or paid) on an active market. This valuation principle only applies to a minority of balance sheet positions. US/worldwide accounting rules thus distinguish the following levels (of determining a fair value):

Level 1 *Mark-to-market*. The fair value of an investment is determined from quoted prices in an active market for the same instrument (e.g., the stock portfolio in Example 2.2 above).

Level 2 *Mark-to-model with objective inputs*. The fair value of an instrument is determined using quoted prices in active markets for similar instruments or by using valuation techniques/models with inputs based on observable market data (e.g., the European call option in Example 2.3 above)

Level 3 *Mark-to-model with subjective inputs*. The fair value of an instrument is determined using valuation techniques/models for which some inputs are not observable in the market (e.g., determining default risk of portfolios of loans to companies for which no CDS spreads[4] are available).

Risk-neutral valuation

- . . . is widely used for pricing financial products, e.g., derivatives
- *value* of a financial instrument today = expected discounted values of future cash flows; the expectation is taken w.r.t. to the *risk-neutral pricing measure* Q (also called *equivalent martingale measure (EMM)*

[4] Annual amount the protection buyer must pay the protection seller over $[0, T]$, expressed as a fraction (often in 1 basis point = 0.01%) of the notional amount.

as it turns discounted prices into martingales, i.e., fair bets) as opposed to the real world/physical measure $\mathbb{P}$.

Example 2.4 ($\mathbb{P}$ vs Q; one-period default model)

- Consider a defaultable bond with principal 1 and maturity $T = 1\,y$. In case of a default (real world probability $p = 0.01$), the recovery rate is $R = 60\%$. The risk-free interest rate is $r = 0.05$. Moreover, assume the bond's current price to be $V_0 = 0.941$ $(t = 0)$.
- The expected discounted value of the bond is
$$\frac{1}{1+r}(1\cdot(1-p)+R\cdot p)=\frac{1}{1.05}(0.99+0.6p)=0.949$$
which is $> V_0$ since investors demand a premium for bearing the bond's default risk.
- An risk-neutral pricing measure is a probability measure Q such that the expectation of the discounted payoff w.r.t. Q equals V_0 (investing

becomes a fair bet). Here, Q is determined by specifying q such that

$$\frac{1}{1+r}(1 \cdot (1-q) + R \cdot q) = V_0 \quad \Rightarrow \quad q = 0.03 > 0.01 = p$$

(the larger q reflects the risk premium).

- $\mathbb{P}$ is estimated from historical data whereas Q is calibrated to current market prices.
- Risk-neutral valuation at t of a claim H at T is done via the *risk-neutral pricing rule*

$$V_0^H = \mathbb{E}_{Q,t}[e^{-r(T-t)}H], \quad t < T,$$

where $\mathbb{E}_{Q,t}[\cdot]$ denotes expectation w.r.t. Q given the information up to and including time t.

- Risk-neutral pricing applied to non-traded financial products is a typical example of level 2 valuation: Prices of traded securities are used to calibrate model parameters under the risk-neutral measure Q; this measure is then used to price the non-traded products.

- There are two theoretical justifications for risk-neutral pricing:
  - *(First) Fundamental Theorem of Asset Pricing*: A model for security prices is arbitrage free if and only if it admits at least one EMM Q.
  - In financial models it is often possible to replicate the pay-off of a product by trading in the assets, a practice known as *(dynamic) hedging*, and it is well-known that in a frictionless market the cost of heding is given by the risk-neutral pricing rule.

Example 2.5 (European call option continued)

- Suppose that options with our desired strike K and/or maturity time T are not traded, but that other options on the same stock are traded.
- Under P the stock price (S_t) is assumed to follow a geometric Brownian motion (GBM) (the so-called *Black–Scholes model*) with dynamics

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$$

for constants $\mu \in \mathbb{R}$ (the drift) and $\sigma > 0$ (the volatility), and a standard Brownian motion (W_t).

- It is well known that there is an EMM Q under which $(e^{-rt}S_t)$ is a martingale; under Q, S_t follows a GBM with drift r and volatility σ. The European call option payoff is $H = \max\{S_T - K, 0\}$ and the risk-neutral valuation formula may be shown to be

$$V_t = E_t^Q(e^{-r(T-t)}(S_T - K)^+) = C^{\mathsf{BS}}(t, S_t; r, \sigma, K, T), \quad t < T; \quad (3)$$

  where t, S_t, r, K, T are known.

- We would typically use quoted prices $C^{\mathsf{BS}}(t, S_t; r, \sigma, K^*, T^*)$ for options on the stock with different K^*, T^* to infer the unknown σ and then plug this so-called *implied volatility* into (3).

2.2.3 Loss distributions

From Example 2.2 we can identify the following key tasks of QRM:

1) Find a statistical model for $\boldsymbol{X}_{t+1}$ (typically an estimated *projection model* used to forecast $\boldsymbol{X}_{t+1}$; can also be a *valuation model*, see, e.g., Black–Scholes formula);
2) Compute/Derive the df $F_{L_{t+1}}$ (requires the df of $f(t+1, \boldsymbol{Z}_t + \boldsymbol{X}_{t+1})$);
3) Compute a risk measure from $F_{L_{t+1}}$.

There are three general methods to approach the challenges 1) and 2).

1) Analytical method

Idea: Choose $F_{\boldsymbol{X}_{t+1}}$ and f such that $F_{L_{t+1}}$ can be determined explicitly.

The prime example is the *variance-covariance method*; see RiskMetrics (1996):

Assumption 1 $\boldsymbol{X}_{t+1} \sim \mathrm{N}(\boldsymbol{\mu}, \Sigma)$ (e.g., if $(\boldsymbol{Z}_t)$ is a Brownian motion, $(\boldsymbol{S}_t)$ a geometric Brownian motion)

Assumption 2 $F_{L^{\Delta}_{t+1}}$ is a good approximation to $F_{L_{t+1}}$.

$L^{\Delta}_{t+1} = -(c_t + \boldsymbol{b}_t^{\top} \boldsymbol{X}_{t+1}) \underset{\text{Ass. 1}}{\Rightarrow} L^{\Delta}_{t+1} \sim \mathrm{N}(-c_t - \boldsymbol{b}_t^{\top} \boldsymbol{\mu}, \boldsymbol{b}_t^{\top} \Sigma \boldsymbol{b}_t)$.

Advantages:
- $F_{L_{t+1}}$ explicit ($\Rightarrow$ typically explicit risk measures)
- (Typically) easy to implement

Drawbacks: Assumptions. Especially Assumption 1 is unlikely to be realistic for daily (probably also weekly/monthly) data. Stylized facts about risk-factor changes (see later)) suggest that $F_{\boldsymbol{X}_{t+1}}$ is *leptokurtic*, i.e., thinner body and heavier tail than $\mathrm{N}(\boldsymbol{\mu}, \Sigma)$. $\Rightarrow$ $\boldsymbol{X}_{t+1} \sim \mathrm{N}(\boldsymbol{\mu}, \Sigma)$ underestimates the tail of $F_{L_{t+1}}$ and thus risk measures such as VaR.

Remark 2.6

- We have not talked about how to estimate $\boldsymbol{\mu}, \Sigma$ yet.

- When dynamic models for $\boldsymbol{X}_{t+1}$ are considered (e.g., time series models), different estimation methods are possible depending on whether we focus on conditional distributions $F_{\boldsymbol{X}_{t+1}|(\boldsymbol{X}_s)_{s\le t}}$ or the equilibrium distribution $F_{\boldsymbol{X}}$ in a stationary model.

2) Historical simulation

Idea: Estimate $F_{L_{t+1}}$ by its *empirical distribution function* based on the past risk-factor changes $\boldsymbol{X}_{t-n+1},\dots,\boldsymbol{X}_t$, so

$$F_{L_{t+1}}(x)=\frac{1}{n}\sum_{i=1}^{n}\mathbb{1}_{\{\tilde{L}_{t-i+1}\le x\}},\quad x\in\mathbb{R},$$

where $\tilde{L}_k=L(\boldsymbol{X}_k)=-(f(t+1,\boldsymbol{Z}_t+\boldsymbol{X}_k)-f(t,\boldsymbol{Z}_t))$.

The values $\tilde{L}_{t-n+1},\dots,\tilde{L}_t$ show what would happen to the current portfolio if the risk-factor changes in periods $k\in\{t-n+1,\dots,t\}$ were to recur.

Advantages:
- Easy to implement

- No estimation of the unknown distribution of $\boldsymbol{X}_{t+1}$ required

Drawbacks:
- Sufficient (synchronized) data for all risk-factor changes required
- Only considers past losses ("driving a car by looking in the back mirror")

3) Monte Carlo method

Idea: Take any (adequate) model for $\boldsymbol{X}_{t+1}$, simulate from it, compute the corresponding simulated losses and $F_{L_{t+1}}$ via the empirical df.

Advantages:
- Quite general (applicable for any model of $\boldsymbol{X}_{t+1}$ which is easy to sample)

Drawbacks:
- Unclear how to find an appropriate model for $\boldsymbol{X}_{t+1}$ (any result is only as good as the chosen $F_{\boldsymbol{X}_{t+1}}$)

- Computational cost (every simulation requires to evaluate the portfolio; expensive, e.g., if the latter contains derivatives which are priced via Monte Carlo themselves $\Rightarrow$ Nested Monte Carlo simulations)

Remark 2.7

- So-called *economic scenario generators* (i.e., economically motivated dynamic models for the evolution and interaction of different risk factors) used in insurance also fall under the heading of Monte Carlo methods.
- Furthermore, there are methods from extreme value theory based on approximations of the tail of the loss df $F_{L_{t+1}}$ (see later).

2.3 Risk measurement

Definition 2.8 (Risk measure)
A *risk measure* for a financial position with (random) loss L is a real number which measures the "riskiness of L". It can be interepreted as the amount of capital required (today) to account for future actual losses (realizations of L) in that position.

- Alternatively, ... the amount of capital required to make a position with loss L acceptable to an (internal/external) regulator (>0 if and only if not acceptable; equivalent to the amount of money to put aside now).
- Some reasons for using risk measures in practice:
  - To determine the amount of capital to hold as a buffer against unexpected future losses on a portfolio (in order to satisfy a regulator/manager concerned with the institution's solvency).

- By management, as a tool for limiting the amount of risk of a business unit (e.g., by requiring that the daily 95% Value-at-Risk (i.e., the 95%-quantile) of a trader's position should not exceed a given bound).
- To determine the riskiness (and thus fair premium) of an insurance contract.

2.3.1 Approaches to risk measurement

Existing approaches to measuring risk can be grouped into three categories:

1) **Notional-amount approach**

   - oldest approach
   - "standardized approaches" of Basel II (e.g., OpRisk) still use it
   - *risk of a portfolio* $\mathcal{P}$: $\sum_{\text{securities in } \mathcal{P}}$ "notional value of the security" $\cdot$ "riskiness factor of the corresponding asset class"

- Advantages:
  - simplicity

  Drawbacks:
  - No differentiation between long and short positions and no netting: the risk of a long position in corporate bonds hedged by an offsetting position in credit default swaps is counted as twice the risk of the unhedged bond position.
  - No diversification benefits: risk of a portfolio of loans to many companies = risk of a portfolio where the whole amount is lent to a single company.
  - Problems for portfolios of derivatives: notional amount of the underlying can widely differ from the economic value of the derivative position.

2) **Risk measures based on loss distributions**

- Most modern risk measures are characteristics of the underlying (conditional or unconditional) loss distribution over some predetermined time horizon Δt.
- Examples: variance, Value-at-Risk, expected shortfall (see later)
- Advantages:
  - The concept of a loss distribution makes sense on all levels of aggregation (from single portfolios to the overall position of a financial institution).
  - If estimated properly, loss distributions reflect netting and diversification effects.

  Drawbacks:
  - Estimates of loss distributions are typically based on past data.
  - It is difficult to estimate loss distributions accurately (especially for large portfolios).

$\Rightarrow$ Risk measures should be complemented by information from scenarios (forward-looking).

3) **Scenario-based risk measures**

- This approach to risk measurement is typically considered in stress testing.
- One considers possible future risk-factor changes (*scenarios*; e.g., a 20% drop in a market index).
- *Risk of a portfolio* = maximum (weighted) loss of the portfolio under all scenarios.
- If $\mathcal{X} = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_n\}$ denote the risk-factor changes (scenarios) with corresponding weights $\boldsymbol{w} = (w_1, \dots, w_n)$, the risk is

$$\psi_{\mathcal{X},\boldsymbol{w}} = \max_{1 \le i \le n} \{w_i L(\boldsymbol{x}_i)\}, \tag{4}$$

where $L(\cdot)$ is the loss operator. Many risk measures used in practice are of the form (4); see, e.g., *CME SPAN: Standard Portfolio Analysis of Risk* (2010).

- Mathematical interpretation of (4):
  - Assume $L(\mathbf{0}) = 0$ ($\checkmark$ if Δt small) and $w_i \in [0, 1]$, $i \in \{1, \dots, n\}$.
  - $w_i L(\boldsymbol{x}_i) = \mathbb{E}_{\mathbb{P}_i}[L(\boldsymbol{X}_i)]$ where $\boldsymbol{X}_i \sim \mathbb{P}_i = w_i \delta_{\boldsymbol{x}_i} + (1 - w_i)\delta_{\mathbf{0}}$ ($\delta_{\boldsymbol{x}}$ the Dirac measure at $\boldsymbol{x}$) is a probability measure on $\mathbb{R}^d$.

  Therefore, $\psi_{\mathcal{X},\boldsymbol{w}} = \max\{\mathbb{E}_{\mathbb{P}}[L(\boldsymbol{X})] : \boldsymbol{X} \sim \mathbb{P} \in \{\mathbb{P}_1, \dots, \mathbb{P}_n\}\}$. Such a risk measure is known as *generalized scenario*; they play an important role in the theory of coherent risk measures.

- Advantages:
  - Useful for portfolios with few risk factors.
  - Useful complementary information to risk measures based on loss distributions (past data).

  Drawbacks:
  - Determining scenarios and weights.

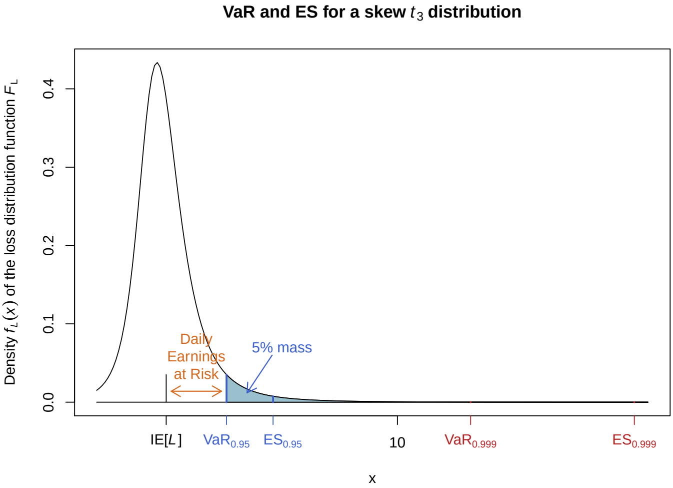
2.3.2 Value-at-Risk

One possible risk measure is the maximum loss $\inf\{x \in \mathbb{R} : F_L(x) = 1\}$. However, this is ∞ for most distributions of interest and neglects any probabilistic information. Idea of Value-at-Risk: replace "maximum loss" by "maximum loss not exceeded with a given high probability".

Definition 2.9 (Value-at-Risk)
For a loss $L \sim F_L$, *Value-at-Risk* (VaR) at confidence level $\alpha \in (0,1)$ is defined by $\mathrm{VaR}_\alpha = \mathrm{VaR}_\alpha(L) = F_L^-(\alpha) = \inf\{x \in \mathbb{R} : F_L(x) \geq \alpha\}$.

- VaR_α is simply the α-quantile of F_L. As such, $F_L(x) < \alpha$ for all $x < \mathrm{VaR}_\alpha(L)$ and $F_L(\mathrm{VaR}_\alpha(L)) = F_L(F_L^-(\alpha)) \geq \alpha$.
- Known since 1994: Weatherstone 4^{15} report (J.P. Morgan; RiskMetrics)
- VaR is the most widely used risk measure (suggested by Basel II)
- $\mathrm{VaR}_\alpha(L)$ also depends on the estimator of F_L and the time horizon.

- VaR is not a what if risk measure: $\mathrm{VaR}_\alpha(L)$ does not provide information about the severity of losses which occur with probability $\leq 1-\alpha$ (only about the loss frequency).



Example 2.10 (VaR for $\mathrm{N}(\mu,\sigma^2)$, $t_\nu(\mu,\sigma^2)$, $\mathrm{Par}(\theta)$)

1) Let $L \sim \mathrm{N}(\mu,\sigma^2)$. Then $F_L(x) = \mathbb{P}(L \le x) = \mathbb{P}((L-\mu)/\sigma \le (x-\mu)/\sigma) = \Phi((x-\mu)/\sigma)$. This implies that

$$\mathrm{VaR}_\alpha(L) = F_L^-(\alpha) = F_L^{-1}(\alpha) = \mu + \sigma\Phi^{-1}(\alpha).$$

2) Let $L \sim t_\nu(\mu,\sigma^2)$, so $(L-\mu)/\sigma \sim t_\nu$ and thus, as above,

$$\mathrm{VaR}_\alpha(L) = \mu + \sigma t_\nu^{-1}(\alpha).$$

Note that $X \sim t_\nu = t_\nu(0,1)$ has density $f_X(x) = \frac{\Gamma((\nu+1)/2)}{\sqrt{\nu\pi}\Gamma(\nu/2)}(1 + x^2/\nu)^{-\frac{\nu+1}{2}}$, $\mathbb{E}X = 0$ (if $\nu > 1$) and $\mathrm{Var}\,X = \frac{\nu}{\nu-2}$ (if $\nu > 2$).

3) Let $L \sim \mathrm{Par}(\theta)$, $\theta > 0$, so $L \sim F_L(x) = 1 - x^{-\theta}$, $x \ge 1$. Then

$$\mathrm{VaR}_\alpha(L) = (1-\alpha)^{-1/\theta}.$$

Choices of parameters $\Delta t, \alpha$:

- Δt should reflect the time period over which the portfolio is held (unchanged) (e.g., insurance companies: $\Delta t = 1\,$y)
- Δt should be relatively small (more risk-factor change data is available).
- Typical choices:
  - For limiting traders: $\alpha = 0.95$, $\Delta t = 1\,$d
  - According to Basel II:
    - Market risk: $\alpha = 0.99$, $\Delta t = 10\,$d (2 trading weeks)
    - Credit risk and operational risk: $\alpha = 0.999$, $\Delta t = 1\,$y
    - Economic capital: $\alpha = 0.9997$, $\Delta t = 1\,$y
  - According to Solvency II: $\alpha = 0.995$, $\Delta t = 1\,$y
- Backtesting often needs to be carried out at lower confidence levels in order to have sufficient statistical power to detect poor models.

- Be cautious with strict interpretations of $\mathrm{VaR}_\alpha(L)$ and other risk measures, there is typically considerable model/liquidity risk behind.

Interlude: Generalized inverses

$T \nearrow$ means that T is *increasing*, i.e., $T(x) \le T(y)$ for all $x < y$.

Definition 2.11 (Generalized inverse)

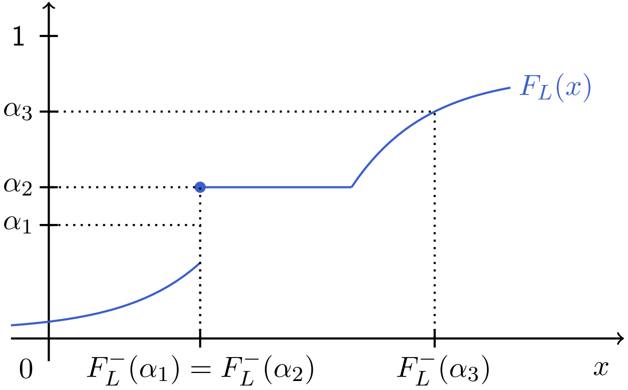
For any increasing function $T : \mathbb{R} \to \mathbb{R}$, with $T(-\infty) = \lim_{x\downarrow -\infty} T(x)$ and $T(\infty) = \lim_{x\uparrow\infty} T(x)$, the *generalized inverse* $T^- : \mathbb{R} \to \bar{\mathbb{R}} = [-\infty, \infty]$ of T is defined by

$$T^-(y) = \inf\{x \in \mathbb{R} : T(x) \ge y\}, \quad y \in \mathbb{R},$$

with the convention that $\inf \emptyset = \infty$. If T is a df, $T^- : [0,1] \to \bar{\mathbb{R}}$ is the *quantile function* of T.

- If T is continuous and $\uparrow$, then $T^- \equiv T^{-1}$ (ordinary inverse).

- There are rules for working with T^- (similar to T^{-1}), see Embrechts and Hofert (2013a).
- F_L^- visualized (here: for a df F_L):



2.3.3 VaR in risk capital calculations

1) VaR **in regulatory capital calculations for the trading book**

For banks using the *internal model (IM)* approach for market risk in Basel II, the daily risk capital formula is

$$\mathrm{RC}^t = \max\Big\{\mathrm{VaR}_{0.99}^{t,10},\ \frac{k}{60}\sum_{i=1}^{60}\mathrm{VaR}_{0.99}^{t-i+1,10}\Big\} + c.$$

- $\mathrm{VaR}_{\alpha}^{s,10}$ denotes the 10-day VaR_α calculated at day s ($t =$ today).
- $k \in [3,4]$ is a multiplier (or *stress factor*).
- $c =$ stressed VaR charge (calculated from data from a volatile market period) $+$ incremental risk charge (IRC; $\mathrm{VaR}_{0.999}$-estimate of the annual distribution of losses due to defaults and downgrades) $+$ charges for specific risks.

The averaging tends to lead to smooth changes in the capital charge over time unless $\mathrm{VaR}_{0.99}^{t,10}$ is large.

2) **The Solvency Capital Requirement in Solvency II**

The *Solvency Capital Requirement (SCR)* is the amount of capital that enables the insurer to meet its obligations over $\Delta t = 1\,\mathrm{y}$ with $\alpha = 0.995$. Let $V_t = A_t - B_t$ (assets − liabilities; aka *own funds*) denote the equity capital. The insurer wants to determine the minimum amount of extra capital x_0 to put aside to be solvent in Δt with probability $(\geq)\alpha$. So

$$\begin{aligned} x_0 &= \inf\{x \in \mathbb{R} : \mathbb{P}(V_{t+1} + x(1+r) \geq 0) \geq \alpha\} \\ &= \inf\Big\{x \in \mathbb{R} : \mathbb{P}\Big(-\Big(\frac{V_{t+1}}{1+r} - V_t\Big) \leq x + V_t\Big) \geq \alpha\Big\} \\ &= \inf\{x \in \mathbb{R} : \mathbb{P}(L_{t+1} \leq x + V_t) \geq \alpha\} \\ &= \inf\{x \in \mathbb{R} : F_{L_{t+1}}(x + V_t) \geq \alpha\} \\ &= \inf\{z - V_t \in \mathbb{R} : F_{L_{t+1}}(z) \geq \alpha\} = \mathrm{VaR}_\alpha(L_{t+1}) - V_t \end{aligned}$$

and thus $\text{SCR} = V_t + x_0 = \mathrm{VaR}_\alpha(L_{t+1})$ (available capital now + capital required to be solvent in Δt with probability $(\geq)\alpha$). For a well-capitalized company $(x_0 \leq 0)$, $-x_0$ (= own funds − SCR $\mathrm{VaR}_\alpha(L_{t+1})$)

is called the *excess capital*.

3) **Median shortfall**

The more robust alternative to expected shortfall (see later) *median shortfall* ($\mathrm{MS}_\alpha(L) = F^{-}_{L,\alpha}(1/2)$ where $F_{L,\alpha}(x) = \frac{F_L(x)-\alpha}{1-\alpha}\mathbb{1}_{\{x \ge F_L^{-}(\alpha)\}}$) is just $\mathrm{VaR}_{\frac{1+\alpha}{2}}$.

Watch out for (badly defined) VaR

The "bible" on VaR is Jorion (2007). The following "definition" is very common:

> "VaR is the *maximum* expected loss of a portfolio over a given time horizon with a certain confidence level."

It is however mathematically meaningless and potentially misleading. In no sense is VaR a maximum loss! We can lose more, sometimes much more, depending on the heaviness of the tail of the loss distribution.

2.3.4 Other risk measures based on loss distributions

1) **Variance**

- $\mathrm{Var}[L]$ is historically the dominating risk measure in finance (due to Markowitz)
- Drawbacks:
  - $\mathbb{E}[L^2] < \infty$ required (not justifiable for non-life insurance or operational risk)
  - no distinction between positive/negative deviations from the mean (Var is only a good risk measure for F_L (approx.) symmetric around $\mathbb{E}L$, but F_L is typically skewed in credit and operational risk)

2) **Upper partial moments**

- Risk management is mainly concerned with the upper tail of F_L.

- Given an exponent $k \geq 0$ and a reference point q, the *upper partial moment* is defined by

$$\mathrm{UPM}(k, q) = \int_q^\infty (x - q)^k \, dF_L(x).$$

- The larger k, the more conservative is this risk measure as more weight is put on large deviations from q.

3) **Expected shortfall**

Definition 2.12 (Expected shortfall)
For a loss $L \sim F_L$ with $\mathbb{E}|L| < \infty$, *expected shortfall* (ES) at confidence level $\alpha \in (0, 1)$ is defined by

$$\mathrm{ES}_\alpha = \mathrm{ES}_\alpha(L) = \frac{1}{1-\alpha} \int_\alpha^1 \mathrm{VaR}_u(L) \, du. \qquad (5)$$

- Besides VaR, ES is the most important risk measure in practice.

- ES_α is the average over VaR_u for all $u \geq \alpha$ (if F_L is continuous, ES_α is the average loss beyond VaR_α) $\Rightarrow$ $\mathrm{ES}_\alpha \geq \mathrm{VaR}_\alpha$
- ES_α looks further into the tail of F_L, it is a what if risk measure (VaR_α is frequency-based; ES_α is severity-based).
- Due to considering the tail of F_L, ES_α is more difficult to estimate and backtest than VaR_α (larger sample size required).
- $\mathrm{ES}_\alpha(L) < \infty$ requires $\mathbb{E}|L| < \infty$ (can be violated for OpRisk).
- Subadditivity and elicitability
  - In contrast to VaR_α, ES_α is subadditive (see later)
  - In contrast to ES_α (see Gneiting (2011) or Kou and Peng (2014)), VaR_α is elicitable (and also exists if $\mathbb{E}|L| = \infty$)
  - Concerning going from VaR_α to ES_α, see BIS (2012, p. 41, Question 8).

- A risk measure ρ is *elicitable* w.r.t. a class of dfs $\mathcal{F}$ if there exists a forecasting objective function $S : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\rho(L) = \operatorname*{arginf}_x \int_{\mathbb{R}} S(x,y)\, dF_L(y), \quad \forall\, F_L \in \mathcal{F} \tag{6}$$

(e.g., $S(x,y) = (x-y)^2 \Rightarrow \rho(L) = \mathbb{E}L$; $S(x,y) = |x-y| \Rightarrow \rho(L) = \mathrm{med}(L) = F_L^-(1/2)$). Not being elicitable implies that it is difficult/impossible to correctly compare models or optimize/minimize error functionals of type (6).

Proposition 2.13 (ES formulas)

Let $(x)_+ = \max\{x, 0\}$. For $\alpha \in (0,1)$,

1) $\mathrm{ES}_\alpha(L) = \dfrac{\mathbb{E}[(L - F_L^-(\alpha))_+]}{1-\alpha} + F_L^-(\alpha)$;

2) $\mathrm{ES}_\alpha(L) = \dfrac{\mathbb{E}[L \mathbb{1}_{\{L > F_L^-(\alpha)\}}]}{1-\alpha} + \dfrac{F_L^-(\alpha)(1 - \alpha - \bar{F}_L(F_L^-(\alpha)))}{1-\alpha}$.

Proof.

1) $L \stackrel{\mathrm{d}}{=} F_L^-(U)$, $U \sim \mathrm{U}[0,1]$, since $\mathbb{P}(F_L^-(U) \le x) = \mathbb{P}(U \le F_L(x)) = F_L(x)$. Therefore,

$$\begin{aligned}\frac{\mathbb{E}[(L - F_L^-(\alpha))_+]}{1-\alpha} &= \frac{1}{1-\alpha}\int_0^1 (F_L^-(u) - F_L^-(\alpha))_+\,du\\ &= \frac{1}{1-\alpha}\int_\alpha^1 (F_L^-(u) - F_L^-(\alpha))\,du\\ &= \mathrm{ES}_\alpha(L) - F_L^-(\alpha).\end{aligned}$$

2) First note that

$$\begin{aligned}\mathbb{E}[(L - F_L^-(\alpha))_+] &= \mathbb{E}[(L - F_L^-(\alpha))\mathbb{1}_{\{L > F_L^-(\alpha)\}}]\\ &= \mathbb{E}[L\mathbb{1}_{\{L > F_L^-(\alpha)\}}] - F_L^-(\alpha)\mathbb{E}[\mathbb{1}_{\{L > F_L^-(\alpha)\}}]\\ &= \mathbb{E}[L\mathbb{1}_{\{L > F_L^-(\alpha)\}}] - F_L^-(\alpha)\bar{F}_L(F_L^-(\alpha)).\end{aligned}$$

Now apply 1), divide by $1-\alpha$ and add $F_L^-(\alpha)$. $\square$

Corollary 2.14 (ES formulas under continuous F_L)

Let F_L be continuous. Then

1) $\mathrm{ES}_\alpha(L) = \frac{\mathbb{E}[L\mathbb{1}_{\{L>F_L^-(\alpha)\}}]}{1-\alpha}$
2) $\mathrm{ES}_\alpha(L) = \mathbb{E}[L \,|\, L > F_L^-(\alpha)]$ (i.e., *conditional VaR (CVaR)*)

Proof.

1) Since $\bar{F}_L(F_L^-(\alpha)) = 1 - F_L(F_L^-(\alpha)) = 1 - \alpha$ for all $\alpha \in \operatorname{ran} F_L \cup \{\inf F_L,\ \sup F_L\} \supseteq (0, 1)$, the claim follows from Proposition 2.13 2).

2) First note that

$$F_{L|L>F_L^-(\alpha)}(x) = \mathbb{P}(L \le x \,|\, L > F_L^-(\alpha)) = \frac{\mathbb{P}(F_L^-(\alpha) < L \le x)}{\mathbb{P}(L > F_L^-(\alpha))}$$

$$= \frac{F_L(x) - F_L(F_L^-(\alpha))}{1 - F_L(F_L^-(\alpha))}\mathbb{1}_{\{x>F_L^-(\alpha)\}} = \frac{F_L(x) - \alpha}{1 - \alpha}\mathbb{1}_{\{x>F_L^-(\alpha)\}},$$

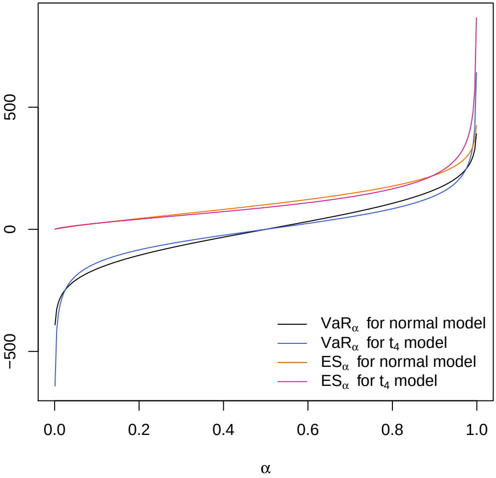
where the latter equality holds since $\alpha \in \operatorname{ran} F_L$. This implies

$$\mathbb{E}[L \mid L > F_L^-(\alpha)] = \int_{\mathbb{R}} x \, dF_{L|L>F_L^-(\alpha)}(x) = \int_{F_L^-(\alpha)}^{\infty} x \, \frac{dF_L(x)}{1-\alpha}$$
$$= \frac{\mathbb{E}[L \mathbb{1}_{\{L>F_L^-(\alpha)\}}]}{1-\alpha} \underset{1)}{=} \mathrm{ES}_\alpha(L). \qquad \square$$

Example 2.15 (VaR and ES for stock returns)

- Consider a portfolio consisting of a single stock $V_t = S_t = 10\,000$. Example 2.2 implies that $L_{t+1}^{\Delta} = -V_t X_{t+1}$, where $X_{t+1} = \log(S_{t+1}/S_t)$.
- Let $\sigma = 0.2/\sqrt{250}$ (annualized volatility of 20%) and assume
  1) $X_{t+1} \sim \mathrm{N}(0, \sigma^2) \Rightarrow L_{t+1}^{\Delta} \sim \mathrm{N}(0, V_t^2\sigma^2)$;
  2) $X_{t+1} \sim t_4(0, \sigma^2 \frac{\nu-2}{\nu})$ ($\operatorname{Var} X_{t+1} = \sigma^2$) or $X_{t+1} = \sqrt{\sigma^2 \frac{\nu-2}{\nu}} Y$ for $Y \sim t_4 \Rightarrow L_{t+1}^{\Delta} = -V_t \sqrt{\sigma^2 \frac{\nu-2}{\nu}} Y \sim t_4(0, V_t^2 \sigma^2 \frac{\nu-2}{\nu})$ ($\Rightarrow \operatorname{Var}[L_{t+1}^{\Delta}] = V_t^2 \sigma^2$).

- Note that $\mathrm{ES}_\alpha^{\text{normal}} \le \mathrm{ES}_\alpha^{t_4}$ for all α, but $\mathrm{VaR}_\alpha^{\text{normal}} \le \mathrm{VaR}_\alpha^{t_4}$; in particular, the t_4 model is not always "riskier" than the normal model when VaR_α is used as a risk measures.



Example 2.16 (Example 2.10 continued)

1) Let $\tilde{L} \sim \mathrm{N}(0,1)$. Then $\mathrm{VaR}_\alpha(\tilde{L}) = 0 + 1 \cdot \Phi^{-1}(\alpha)$ and thus

$$\mathrm{ES}_\alpha(\tilde{L}) = \frac{1}{1-\alpha}\int_\alpha^1 \Phi^{-1}(u)\,du \underset{x=\Phi^{-1}(u)}{=} \frac{1}{1-\alpha}\int_{\Phi^{-1}(\alpha)}^\infty x\varphi(x)\,dx,$$

where $\varphi(x) = \Phi'(x) = \exp(-x^2/2)/\sqrt{2\pi}$. Note that $x\varphi(x) = -\varphi'(x)$, so that

$$\mathrm{ES}_\alpha(\tilde{L}) = \frac{-[\varphi(x)]_{\Phi^{-1}(\alpha)}^{\infty}}{1-\alpha} = \frac{-(0-\varphi(\Phi^{-1}(\alpha)))}{1-\alpha} = \frac{\varphi(\Phi^{-1}(\alpha))}{1-\alpha}.$$

This implies that $L \sim \mathrm{N}(\mu, \sigma^2)$ has expected shortfall

$$\mathrm{ES}_\alpha(L) = \mu + \sigma\,\mathrm{ES}_\alpha(\tilde{L}) = \mu + \sigma\frac{\varphi(\Phi^{-1}(\alpha))}{1-\alpha}.$$

By l'Hôpital's Rule (case "0/0") and using $\varphi'(x) = -x\varphi(x)$, one can show that

$$1 \overset{\checkmark}{\leq} \lim_{\alpha\uparrow 1}\frac{\mathrm{ES}_\alpha(L)}{\mathrm{VaR}_\alpha(L)} = 1.$$

2) **Benchmark model in finance**

Let $L \sim t_\nu(\mu, \sigma^2)$, $\nu > 1$. Similarly as above, one obtains that

$$\mathrm{ES}_\alpha(L) = \mu + \sigma \frac{f_{t_\nu}(t_\nu^{-1}(\alpha))(\nu + t_\nu^{-1}(\alpha)^2)}{(1-\alpha)(\nu-1)},$$

where f_{t_ν} denotes the density of t_ν (see Example 2.10). Again by l'Hôpital's Rule (case "0/0"), one can show that

$$1 \overset{\checkmark}{\leq} \lim_{\alpha \uparrow 1} \frac{\mathrm{ES}_\alpha(L)}{\mathrm{VaR}_\alpha(L)} = \frac{\nu}{\nu - 1} > 1 \quad (\text{and } \uparrow \infty \text{ for } \nu \downarrow 1).$$

In finance, often $\nu \in (3, 5)$. With $\nu = 3$, $\mathrm{ES}_\alpha(L)$ is 50% larger than $\mathrm{VaR}_\alpha(L)$ (in the limit for large α).

3) If $L \sim \mathrm{Par}(\theta)$, $\theta > 1$, then $\mathrm{VaR}_\alpha(L) = (1-\alpha)^{-1/\theta}$, which implies

$$\mathrm{ES}_\alpha(L) = \frac{\theta}{\theta - 1} \mathrm{VaR}_\alpha(L)$$

and thus

$$1 \overset{\checkmark}{\leq} \lim_{\alpha \uparrow 1} \frac{\mathrm{ES}_\alpha(L)}{\mathrm{VaR}_\alpha(L)} = \frac{\theta}{\theta - 1} > 1 \quad (\text{and } \uparrow \infty \text{ for } \theta \downarrow 1).$$

Conclusion:

For losses with *heavy* (power-like) tails, the difference between using VaR and ES as risk measures for computing risk capital can be huge (for large α as required by Basel II).

2.3.5 Coherent and convex risk measures

- Artzner et al. (1999) (coherent risk measures) and Föllmer and Schied (2002) (convex risk measures) propose axioms a good risk measure should have.
- Here we assume that risk measures ρ are real-valued functions defined on a linear space of random variables $\mathcal{M}$ (including constants).
- There are two possible interpretations of elements of $\mathcal{M}$:
  1) Future net asset values of portfolios/positions
     Elements of $\mathcal{M}$ are V_{t+1}; a risk measure $\tilde{\rho}(V_{t+1})$ denotes the amount

of additional capital that needs to be added to a position with future net asset value V_{t+1} to make it acceptable to a regulator.

2) Losses L (related to 1) by $L = -(V_{t+1} - V_t)$) Elements of $\mathcal{M}$ are losses L; a risk measure $\rho(L)$ denotes the total amount of equity capital necessary to back a position with loss L.

1) and 2) are related via $\rho(L) = V_t + \tilde{\rho}(V_{t+1})$ (total capital = available capital + additional capital). In what follows, we focus on 2).

Axiom 1 (monotonicity) $L_1, L_2 \in \mathcal{M},\ L_1 \le L_2$ (a.s.) $\Rightarrow \rho(L_1) \le \rho(L_2)$

Interpr.: Positions which lead to a higher loss in every state of the world require more risk capital.

Criticism: none

Axiom 2 (translation invar.) $\rho(L + l) = \rho(L) + l$ for all $L \in \mathcal{M}$, $l \in \mathbb{R}$

Interpr.: By adding $l \in \mathbb{R}$ to a position with loss L, we alter the capital requirements accordingly. If $\rho(L) > 0$, and $l = -\rho(L)$, then $\rho(L - \rho(L)) = \rho(L + l) = \rho(L) + l = 0$ so that adding $\rho(L)$ to a position with loss L makes it acceptable.

Criticism: Most people believe this to be reasonable; exception: B. Rémillard (adding a constant value does not make a position riskier)

Axiom 3 (subadditivity) $\rho(L_1 + L_2) \leq \rho(L_1) + \rho(L_2)$ for all $L_1, L_2 \in \mathcal{M}$

Interpr.:

- Reflects the idea that risk can be reduced by diversification
- Using a non-subadditive ρ encourages institutions to legally break up into subsidiaries to reduce regulatory capital requirements.

- Subadditivity makes decentralization possible: if we want to bound the overall loss $L = L_1 + L_2$ of two positions by M, we can choose M_j such that $L_j \le M_j$, $j \in \{1, 2\}$, with $M_1 + M_2 \le M$ and require $\rho(L_j) \le M_j$, $j \in \{1, 2\}$. Then $\rho(L) \underset{\text{subadd.}}{\le} \rho(L_1) + \rho(L_2) \le M_1 + M_2 \le M$.

Criticism: VaR is ruled out in certain situations. Note that VaR is monotone ($L_1 \le L_2$ (a.s.) $\Rightarrow F_{L_1}(x) \ge F_{L_2}(x)$, $x \in \mathbb{R} \Rightarrow F^{-}_{L_1}(u) \le F^{-}_{L_2}(u)$, $u \in (0, 1)$), translation invariant ($F_{L+l}(x) = F_L(x - l) \Rightarrow F^{-}_{L+l}(u) = F^{-}_{L}(u) + l$, $u \in (0, 1)$) and positive homogeneous ($F_{\lambda L}(x) = F_L(x/\lambda) \Rightarrow F^{-}_{\lambda L}(u) = \lambda F^{-}_{L}(u)$), but in general not subadditive, especially not under one of the following scenarios (see below):

1) Independent, light-tailed L_1, L_2 and small α;

2) L_1, L_2 have skewed distributions;

3) L_1, L_2 have heavy tailed distributions;

4) L_1, L_2 have special dependence.

Note that $\mathcal{M}$ is important here. If it is sufficiently small (e.g., all multivariate elliptical distributions), VaR_α is subadditive (see later)!

Axiom 4 (positive homogeneity) $\rho(\lambda L) = \lambda\rho(L)$ for all $L \in \mathcal{M}$, $\lambda > 0$

Interpr.: $\lambda = n \in \mathbb{N}$, subadditivity $\Rightarrow \rho(nL) \leq n\rho(L)$. But n times the same loss L means no diversification, so equality should hold.

Criticism: If $\lambda > 0$ is large, liquidity risk plays a role and one should rather have $\rho(\lambda L) > \lambda\rho(L)$ (also to penalize concentration or risk), but this contradicts subadditivity. This has led to convex risk measures.

Definition 2.17 (Coherent risk measure)
A risk measure ρ is *coherent* if it satisfies Axioms 1–4 above.

Example 2.18 (Generalized scenario risk measures)
The generalized scenario risk measure $\psi_{\mathcal{X},\boldsymbol{w}}(L) = \max\{\mathbb{E}_{\mathbb{P}}[L(\boldsymbol{X})] : \boldsymbol{X} \sim \mathbb{P} \in \{\mathbb{P}_1, \ldots, \mathbb{P}_n\}\}$ is coherent. Monotonicity, translation invariance, positive homogeneity are clear; for subadditivity, note that

$$\psi_{\mathcal{X},\boldsymbol{w}}(L_1 + L_2) = \max\{\underbrace{\mathbb{E}_{\mathbb{P}}[L_1(\boldsymbol{X}) + L_2(\boldsymbol{X})]}_{=\mathbb{E}_{\mathbb{P}}[L_1(\boldsymbol{X})]+\mathbb{E}_{\mathbb{P}}[L_2(\boldsymbol{X})]} : \boldsymbol{X} \sim \mathbb{P} \in \{\mathbb{P}_1, \ldots, \mathbb{P}_n\}\}$$
$$\leq \psi_{\mathcal{X},\boldsymbol{w}}(L_1) + \psi_{\mathcal{X},\boldsymbol{w}}(L_2),$$

where $L_j(\boldsymbol{x})$ denotes the hypothetical loss of position j under scenario $\boldsymbol{x}$ (risk-factor change). Note that all coherent risk measures can be represented as generalized scenarios via $\rho(L) = \sup\{\mathbb{E}_{\mathbb{P}}[L] : \mathbb{P} \in \mathcal{P}\}$ where $\mathcal{P}$ is a set of probability measures; for a proof, see McNeil et al. (2005, Prop. 6.11 (ii)) for $|\Omega| < \infty$ and Delbaen (2000), Delbaen (2002) for the general case.

Example 2.19 (A coherent premium principle)

- Fischer (2003) proposed a class of coherent risk measures which are potentially useful for an insurance company that wants to compute premiums on a coherent basis without deviating too far from standard actuarial practice.
- Let $p > 1$, $\alpha \in [0, 1)$, $\mathcal{M} = L^p(\Omega, \mathcal{F}, \mathbb{P})$, $\|L\|_p = \mathbb{E}[|L|^p]^{1/p}$ and

$$\rho_{\alpha,p}(L) = \mathbb{E}[L] + \alpha\|\max\{L - \mathbb{E}[L], 0\}\|_p.$$

  Risk = pure actuarial premium + *risk loading* (α-fraction of $(\int_{\mathbb{E}[L]}^{\infty}(x - \mathbb{E}[L])^p \, dF_L(x))^{1/p}$). The higher α or p, the more conservative is $\rho_{\alpha,p}(L)$.
- For subaddivity use $\max\{L_1 + L_2, 0\} \leq \max\{L_1, 0\} + \max\{L_2, 0\}$ and thus

$$\begin{aligned} &\|\max\{L_1 - \mathbb{E}[L_1] + L_2 - \mathbb{E}[L_2], 0\}\|_p \\ &\leq \|\max\{L_1 - \mathbb{E}[L_1], 0\} + \max\{L_2 - \mathbb{E}[L_2], 0\}\|_p \\ &\underset{\text{Minkowski}}{\leq} \|\max\{L_1 - \mathbb{E}[L_1], 0\}\|_p + \|\max\{L_2 - \mathbb{E}[L_2], 0\}\|_p. \end{aligned}$$

For monotonicity, let $L_1 \le L_2$ a.s. and write $L = L_1 - L_2 (\le 0)$. Thus, $\max\{L - \mathbb{E}[L], 0\} \le \max\{0 - \mathbb{E}[L], 0\} = -\mathbb{E}[L]$ a.s., so $\|\max\{L - \mathbb{E}[L], 0\}\|_p \le -\mathbb{E}[L]$. Since $\alpha \in [0,1)$, $\rho_{\alpha,p}(L) \le \mathbb{E}[L](1-\alpha) \le 0$. Using subadditivity, we obtain $\rho_{\alpha,p}(L_1) \le \rho_{\alpha,p}(L) + \rho_{\alpha,p}(L_2) \le \rho_{\alpha,p}(L_2)$. Translation invariance and positive homogeneity are trivial.

Definition 2.20 (Convex risk measure)
A risk measure ρ which is monotone, translation invariant and convex is called a *convex risk measure*.

- Justification for their study is again diversification (but they don't have to be positive homogeneous).
- Let ρ be coherent. Then for all $\lambda \in [0,1]$, $L_1, L_2 \in \mathcal{M}$, $\rho(\lambda L_1 + (1-\lambda)L_2) \underset{\text{subadd.}}{\le} \rho(\lambda L_1) + \rho((1-\lambda)L_2) \underset{\text{pos.hom.}}{=} \lambda\rho(L_1) + (1-\lambda)\rho(L_2)$ so ρ is convex.
  The converse is not true in general, but for positive homogeneous risk measures, convexity and subadditivity are equivalent.

- Examples of convex but not positive homogeneous risk measures:
  1) Let $\rho'(L) = \rho(L) + 1$ for any coherent ρ.
  2) The *entropic risk measure* $\rho(L) = \mathbb{E}[e^{bL}]/b$, $b > 0$. To see that this is convex, use Young's inequality ($ab \le a^p/p + b^q/q$ for all $a, b \ge 0$, $p, q \ge 1$ such that $1/p + 1/q = 1$) with $p = 1/\lambda$, $q = 1/(1-\lambda)$, $a = e^{\lambda b L_1}$, $b = e^{(1-\lambda) b L_2}$.

Proposition 2.21 (Coherence of ES)
ES is a coherent risk measure.

Proof. Monotonicity, translation invariance and positive homogeneity follow from VaR. Subadditivity follows from Proposition 2.25 below. □

Proof of subadditivity of ES: A (mostly) analytic proof

We start with some auxiliary results.

Lemma 2.22
$\mathbb{P}(L = F_L^-(\alpha)) = 0$ implies $F_L(F_L^-(\alpha)) = \alpha$.

Proof. $F_L(F_L^-(\alpha)) - F_L(F_L^-(\alpha)-) = \mathbb{P}(L = F_L^-(\alpha)) = 0$, so F_L does not jump in $F_L^-(\alpha)$. By definition of F_L^-, $F_L(F_L^-(\alpha)) \geq \alpha$ and $F_L(F_L^-(\alpha)-) < \alpha$, which implies $F_L(F_L^-(\alpha)) = \alpha$. $\square$

For the following result let

$$\mathbb{1}^{(\alpha)}_{\{L>q\}} = \begin{cases} \mathbb{1}_{\{L>q\}}, & \text{if } \mathbb{P}(L=q)=0, \\ \mathbb{1}_{\{L>q\}} + \frac{1-\alpha-\bar{F}_L(q)}{\mathbb{P}(L=q)}\mathbb{1}_{\{L=q\}}, & \text{if } \mathbb{P}(L=q)>0. \end{cases}$$

Lemma 2.23 (Properties of $\mathbb{1}^{(\alpha)}_{\{L>F_L^-(\alpha)\}}$)
1) $\mathbb{1}^{(\alpha)}_{\{L>F_L^-(\alpha)\}} \in [0,1]$
2) $\mathbb{E}[\mathbb{1}^{(\alpha)}_{\{L>F_L^-(\alpha)\}}] = 1-\alpha$

Proof.

1) If $\mathbb{P}(L = F_L^-(\alpha)) = 0$ we are done, so consider $\mathbb{P}(L = F_L^-(\alpha)) > 0$. On the set of all $\omega \in \Omega$ such that $L(\omega) > F_L^-(\alpha)$, we are again done. Now consider all $\omega \in \Omega$ such that $L(\omega) = F_L^-(\alpha)$. Then $\mathbb{1}^{(\alpha)}_{\{L > F_L^-(\alpha)\}} = \frac{1-\alpha-\bar{F}_L(F_L^-(\alpha))}{\mathbb{P}(L = F_L^-(\alpha))}$. By definition, $F_L(F_L^-(\alpha)) \geq \alpha$, so $\bar{F}_L(F_L^-(\alpha)) \leq 1-\alpha$, thus $\mathbb{1}^{(\alpha)}_{\{L > F_L^-(\alpha)\}} \geq 0$. Also, $F_L(F_L^-(\alpha)-) < \alpha$, so $\mathbb{1}^{(\alpha)}_{\{L > F_L^-(\alpha)\}}$ equals
$$\frac{1-\alpha-(1-F_L(F_L^-(\alpha)))}{\mathbb{P}(L = F_L^-(\alpha))} = \frac{F_L(F_L^-(\alpha))-\alpha}{F_L(F_L^-(\alpha)) - F_L(F_L^-(\alpha)-)} < 1.$$

2) We have
$$\mathbb{E}[\mathbb{1}^{(\alpha)}_{\{L>q\}}] = \begin{cases} \bar{F}_L(q), & \text{if } \mathbb{P}(L=q)=0, \\ \bar{F}_L(q) + \frac{1-\alpha-\bar{F}_L(q)}{\mathbb{P}(L=q)}\mathbb{P}(L=q) = 1-\alpha, & \text{if } \mathbb{P}(L=q) > 0. \end{cases}$$
Consider $\mathbb{P}(L = q) = 0$. Since $q = F_L^-(\alpha)$, Lemma 2.22 implies that $\bar{F}_L(q) = 1 - F_L(F_L^-(\alpha)) = 1-\alpha$. Thus $\mathbb{E}[\mathbb{1}^{(\alpha)}_{\{L>q\}}] = 1-\alpha$. $\square$

Lemma 2.24 (Representation of ES_α in terms of $\mathbb{1}^{(\alpha)}_{\{L>F_L^-(\alpha)\}}$)

$$\mathrm{ES}_\alpha(L) = \frac{\mathbb{E}[L\mathbb{1}^{(\alpha)}_{\{L>F_L^-(\alpha)\}}]}{1-\alpha}$$

Proof.

- If $\mathbb{P}(L = F_L^-(\alpha)) = 0$, Lemma 2.22 implies that $\bar{F}_L(F_L^-(\alpha)) = 1-\alpha$. By Proposition 2.13 2) and since $\mathbb{P}(L = F_L^-(\alpha)) = 0$,

$$\begin{aligned}\mathrm{ES}_\alpha(L) &= \frac{\mathbb{E}[L\mathbb{1}_{\{L>F_L^-(\alpha)\}}]}{1-\alpha} + \frac{F_L^-(\alpha)\big(1-\alpha-(1-\alpha)\big)}{1-\alpha} \\ &= \frac{\mathbb{E}[L\mathbb{1}_{\{L>F_L^-(\alpha)\}}]}{1-\alpha} = \frac{\mathbb{E}[L\mathbb{1}^{(\alpha)}_{\{L>F_L^-(\alpha)\}}]}{1-\alpha}.\end{aligned}$$

- If $\mathbb{P}(L = F_L^-(\alpha)) > 0$, $\mathbb{E}[L\mathbb{1}^{(\alpha)}_{\{L>F_L^-(\alpha)\}}]$ equals

$$\mathbb{E}[L\mathbb{1}_{\{L>F_L^-(\alpha)\}}] + \frac{1-\alpha-\bar{F}_L(F_L^-(\alpha))}{\mathbb{P}(L = F_L^-(\alpha))}\underbrace{\mathbb{E}[L\mathbb{1}_{\{L=F_L^-(\alpha)\}}]}_{=\mathbb{E}[F_L^-(\alpha)\mathbb{1}_{\{L=F_L^-(\alpha)\}}]=F_L^-(\alpha)\mathbb{P}(L=F_L^-(\alpha))}$$

So, $\mathbb{E}[L\mathbb{1}^{(\alpha)}_{\{L>F_L^-(\alpha)\}}] = \mathbb{E}[L\mathbb{1}_{\{L>F_L^-(\alpha)\}}] + F_L^-(\alpha)\big(1-\alpha-\bar{F}_L(F_L^-(\alpha))\big)$, which, by Proposition 2.13 2), equals $(1-\alpha)\,\mathrm{ES}_\alpha(L)$. $\square$

Proposition 2.25 (Subadditivity of ES)
ES_α is subadditive for all $\alpha \in (0,1)$.

Proof. It suffices to show that

$$(1-\alpha)(\mathrm{ES}_\alpha(L_1) + \mathrm{ES}_\alpha(L_2) - \mathrm{ES}_\alpha(L_1+L_2)) \geq 0.$$

Lemma 2.24 implies that

$$\Big(\sum_{j=1}^{2}\mathbb{E}[L_j\mathbb{1}^{(\alpha)}_{\{L_j>F^-_{L_j}(\alpha)\}}]\Big)-\mathbb{E}[(L_1+L_2)\mathbb{1}^{(\alpha)}_{\{L_1+L_2>F^-_{L_1+L_2}(\alpha)\}}]$$

$$\underset{\text{Linearity}}{=}\sum_{j=1}^{2}\mathbb{E}[L_j(\mathbb{1}^{(\alpha)}_{\{L_j>F^-_{L_j}(\alpha)\}}-\mathbb{1}^{(\alpha)}_{\{L_1+L_2>F^-_{L_1+L_2}(\alpha)\}})]. \tag{7}$$

- $L_j>F^-_{L_j}(\alpha)\Rightarrow\mathbb{1}^{(\alpha)}_{\{L_j>F^-_{L_j}(\alpha)\}}-\mathbb{1}^{(\alpha)}_{\{L_1+L_2>F^-_{L_1+L_2}(\alpha)\}}=1-\dots\geq 0$
- $L_j<F^-_{L_j}(\alpha)\Rightarrow\mathbb{1}^{(\alpha)}_{\{L_j>F^-_{L_j}(\alpha)\}}-\mathbb{1}^{(\alpha)}_{\{L_1+L_2>F^-_{L_1+L_2}(\alpha)\}}=0-\dots\leq 0$

In both cases, we make the expectations in (7) smaller by replacing L_j by $F^-_{L_j}(\alpha)$. Hence

$$(7)\geq\sum_{j=1}^{2}F^-_{L_j}(\alpha)\underbrace{\mathbb{E}[\mathbb{1}^{(\alpha)}_{\{L_j>F^-_{L_j}(\alpha)\}}-\mathbb{1}^{(\alpha)}_{\{L_1+L_2>F^-_{L_1+L_2}(\alpha)\}}]}_{\underset{\text{Lem. 2.23 2)}}{=}(1-\alpha)-(1-\alpha)=0}\geq 0. \qquad\square$$

Proof of subadditivity of ES: A (mostly) stochastic approach

Proposition 2.26 (Subadditivity of ES)

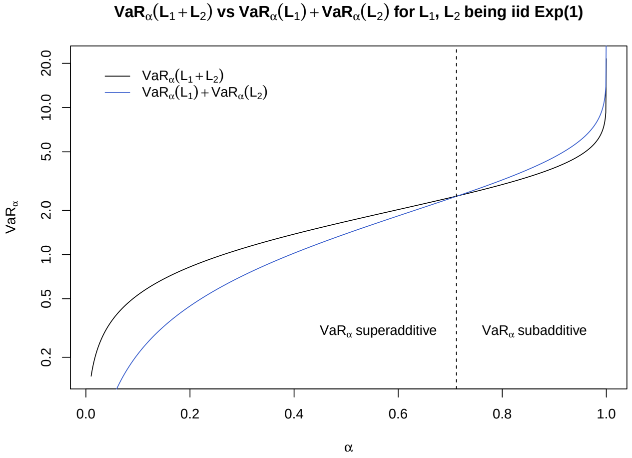
$\mathrm{ES}_\alpha(L) = \frac{\sup_{\{\tilde{L}\sim \mathrm{B}(1,1-\alpha)\}} \mathbb{E}[L\tilde{L}]}{1-\alpha}$ (which, trivially, is subadditive).

Proof (details become clear later). Let $L = F_L^-(U)$ for $U \sim \mathrm{U}[0,1]$ and $L' = \mathbb{1}_{\{U>\alpha\}} \sim \mathrm{B}(1, 1-\alpha)$. Then $\mathrm{ES}_\alpha(L) = \frac{1}{1-\alpha}\int_\alpha^1 F_L^-(u)\,du = \frac{1}{1-\alpha}\int_0^1 F_L^-(u)\mathbb{1}_{\{u>\alpha\}}\cdot 1\,du = \frac{1}{1-\alpha}\mathbb{E}[F_L^-(U)\mathbb{1}_{\{U>\alpha\}}] = \frac{1}{1-\alpha}\mathbb{E}[LL']$. Note that L and L' are comontone (see later), so that for any other $\tilde{L} \sim \mathrm{B}(1, 1-\alpha)$, Hoeffding's identity implies that $\mathbb{E}[L\tilde{L}] \le \mathbb{E}[LL']$. Hence $\mathrm{ES}_\alpha(L) = \sup_{\{\tilde{L}\sim \mathrm{B}(1,1-\alpha)\}} \mathbb{E}[L\tilde{L}]/(1-\alpha)$. From this representation, ES_α is easily seen to be subadditive. □

Superadditivity scenarios for VaR

Exercise 2.27 (Independent L_1, L_2 and small α)

If $L_1, L_2 \overset{\text{ind.}}{\sim} \mathrm{Exp}(1)$, VaR_α is superadditive $\iff \alpha < 0.71$.



Exercise 2.28 (Skewed loss distributions)

Consider a portfolio $\mathcal{P}$ of two independent defaultable zero-coupon bonds with maturity $T = 1$y, nominal/face value 100, equal default probability $p = 0.009$, no recovery and interest rate 5%. Hence, for $j \in \{1, 2\}$, the loss of bond j (investor's/lender's perspective) is

$$L_j = \begin{cases} -5, & \text{with prob. } 1 - p = 0.991, \\ 100, & \text{with prob. } p = 0.009, \end{cases}$$

Set $\alpha = 0.99$. Since $\mathbb{P}(L_j < -5) = 0 < \alpha$ and $\mathbb{P}(L_j \le -5) = 1 - p \ge \alpha$, $\mathrm{VaR}_\alpha(L_j) = -5$, $j \in \{1, 2\}$. Since L_1, L_2 are independent, the loss $L = L_1 + L_2$ of $\mathcal{P}$ is given by

$$L = \begin{cases} -10, & \text{with prob. } (1 - p)^2 = 0.982081, \\ 95, & \text{with prob. } 2p(1 - p) = 0.017838, \\ 200, & \text{with prob. } p^2 = 0.000081, \end{cases}$$

Since $\mathbb{P}(L < 95) < \alpha$ and $\mathbb{P}(L \le 95) = 0.999919 \ge \alpha$, $\mathrm{VaR}_\alpha(L) = 95 > -10 = \mathrm{VaR}_\alpha(L_1) + \mathrm{VaR}_\alpha(L_2)$. Hence VaR_α is superadditive. Note that

VaR_α punishes diversification since $\mathrm{VaR}_\alpha(0.5L_1+0.5L_2) = \mathrm{VaR}_\alpha(L)/2 = 47.5 > -5 = \mathrm{VaR}_\alpha(L_1)$.

Another example of this type is the following.

Exercise 2.29 (Skewed loss distributions; extended example)
Consider d independent defaultable bonds with maturity $T = 1$y, nominal/face value $b > 0$, yearly coupon of $a/b > 0$, default probability $p \in [0,1]$, and no recovery. Hence, for $j \in \{1,\dots,d\}$, the loss of bond j (investor's/lender's perspective) is

$$L_j = \begin{cases} -(b(1+a/b)-b) = -a, & \text{with prob. } 1-p, \\ b, & \text{with prob. } p. \end{cases}$$

Consider the two portfolios

$$\mathcal{P}_1 \text{ (“diversified”)}: \ L = \sum_{j=1}^d L_j, \quad \mathcal{P}_2 \text{ (“concentrated”)}: \ L = dL_1$$

and show that VaR_α is superadditive $\iff (1-p)^d < \alpha \le 1-p$.

Solution. Let $\tilde{L}_j = (L_j + a)/(b+a) \in \{0,1\}$. Then $\tilde{L}_j \sim \mathrm{B}(1,p)$, $j \in \{1,\dots,d\}$, with

$$F_{\mathrm{B}(1,p)}(x) = \begin{cases} 0 & \text{if } x \in (-\infty, 0), \\ 1-p & \text{if } x \in [0,1), \\ 1 & \text{if } x \in [1,\infty), \end{cases}$$

and

$$F^{-}_{\mathrm{B}(1,p)}(\alpha) = \begin{cases} -\infty, & \text{if } \alpha = 0, \\ 0, & \text{if } \alpha \in (0, 1-p], \\ 1, & \text{if } \alpha \in (1-p, 1]. \end{cases}$$

Furthermore, $\sum_{j=1}^d \tilde{L}_j \sim \mathrm{B}(d,p)$ with distribution function $F_{\mathrm{B}(d,p)}$.

- For $\mathcal{P}_1$, translation invariance and positive homogeneity imply

$$\mathrm{VaR}_\alpha\Big(\sum_{j=1}^d L_j\Big) = \mathrm{VaR}_\alpha\Big(\sum_{j=1}^d((b+a)\tilde{L}_j - a)\Big)$$
$$= (b+a)\,\mathrm{VaR}_\alpha\Big(\sum_{j=1}^d \tilde{L}_j\Big) - da = (b+a)F^-_{\mathrm{B}(d,p)}(\alpha) - da.$$

- For $\mathcal{P}_2$, $\mathrm{VaR}_\alpha(L) = \mathrm{VaR}_\alpha(dL_1) = d\,\mathrm{VaR}_\alpha(L_1) = d\,\mathrm{VaR}_\alpha((b+a)\tilde{L}_1 - a) = d(b+a)F^-_{\mathrm{B}(1,p)}(\alpha) - da$.

Since VaR_α is superadditive if and only if $\mathrm{VaR}_\alpha(\sum_{j=1}^d L_j) > \sum_{j=1}^d \mathrm{VaR}_\alpha(L_j) = d\,\mathrm{VaR}_\alpha(L_1)$, we obtain that

$$\begin{aligned}\mathrm{VaR}_\alpha \text{ superadd.} &\iff (b+a)F^-_{\mathrm{B}(d,p)}(\alpha) - da > d(b+a)F^-_{\mathrm{B}(1,p)}(\alpha) - da\\ &\iff F^-_{\mathrm{B}(d,p)}(\alpha) > dF^-_{\mathrm{B}(1,p)}(\alpha)\\ &\iff F_{\mathrm{B}(d,p)}(dF^-_{\mathrm{B}(1,p)}(\alpha)) < \alpha.\end{aligned}$$

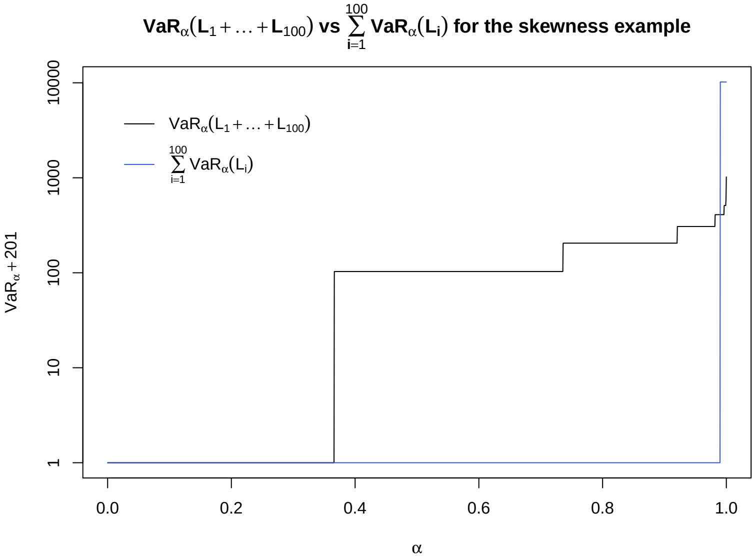
Since $dF^{-}_{\mathrm{B}(1,p)}(\alpha) \in \{-\infty, 0, d\}$ we have that $F_{\mathrm{B}(d,p)}(dF^{-}_{\mathrm{B}(1,p)}(\alpha))$ equals

$$F_{\mathrm{B}(d,p)}(dF^{-}_{\mathrm{B}(1,p)}(\alpha)) = \begin{cases} 0, & \text{if } \alpha = 0, \\ F_{\mathrm{B}(d,p)}(0) \underset{\mathbb{P}(\tilde{L}_j = 0\,\forall j)}{=} (1-p)^d, & \text{if } \alpha \in (0, 1-p], \\ F_{\mathrm{B}(d,p)}(d) = 1, & \text{if } \alpha \in (1-p, 1]. \end{cases}$$

For $\alpha = 0$ or $\alpha \in (1-p, 1]$, $F_{\mathrm{B}(d,p)}(dF^{-}_{\mathrm{B}(1,p)}(\alpha)) < \alpha$ is not possible $\Rightarrow$ VaR_α is superadditive if and only if $(1-p)^d < \alpha$ for $0 < \alpha \leq 1-p$. $\square$

- Note that the superadditivity range does not depend on a, b.
- For further generalizations of this result (e.g., to dependent bonds), see Hofert and McNeil (2014).

For $d = 100$, $T = 1$ y, nominal $b = 100$, coupon $a/b = 2\%$, $p = 1\%$:



Exercise 2.30 (Heavy tailed loss distributions)

$L_1, L_2 \overset{\text{ind.}}{\sim} \mathrm{Par}(1/2)$ with df $F(x) = 1 - x^{-1/2}$, $x \in [1, \infty)$. Show that VaR_α is superadditive for all $\alpha \in (0, 1)$.

Solution. F has density $f(x) = \frac{1}{2x^{3/2}}$, $x \in [1, \infty)$. Since L_1, L_2 are independent, we can compute the density of $L_1 + L_2$ as the convolution

$$f_{L_1+L_2}(x) = \int_{-\infty}^{\infty} f_{L_1}(t) f_{L_2}(x-t)\,dt = \frac{1}{4}\int_1^{x-1} \frac{1}{t^{3/2}} \frac{1}{(x-t)^{3/2}}\,dt$$
$$= \frac{1}{4}\int_1^{x-1} \frac{1}{(xt-t^2)^{3/2}}\,dt.$$

By completing the square and then substituting $s = t - x/2$, we obtain

$$f_{L_1+L_2}(x) = \frac{1}{4}\int_1^{x-1} \frac{1}{\left(\frac{x^2}{4} - \left(t - \frac{x}{2}\right)\right)^{3/2}}\,dt = \frac{1}{4}\int_{1-x/2}^{x/2-1} \frac{1}{\left(\frac{x^2}{4} - s^2\right)^{3/2}}\,ds.$$

Substituting $s = \frac{x}{2}\sin t$ ($t = \arcsin(2s/x)$; $ds = \frac{x}{2}\cos t\,dt$) leads to

$$f_{L_1+L_2}(x) = \frac{x}{8}\int_{\arcsin(2/x-1)}^{\arcsin(1-2/x)} \frac{\cos t}{\big(\frac{x^2}{4}(1-\sin^2 t)\big)^{3/2}}\,dt$$

$$= \frac{1}{x^2}\int_{\arcsin(2/x-1)}^{\arcsin(1-2/x)} \frac{1}{\cos^2 t}\,dt = \frac{1}{x^2}\Big[\tan t\Big]_{\arcsin(2/x-1)}^{\arcsin(1-2/x)}.$$

Note that $\tan\arcsin x = \frac{\sin\arcsin x}{\cos\arcsin x} = \frac{x}{\sqrt{1-x^2}}$. Hence,

$$f_{L_1+L_2}(x) = \frac{1}{x^2}\bigg(\frac{1-2/x}{\sqrt{1-(1-2/x)^2}} - \frac{2/x-1}{\sqrt{1-(2/x-1)^2}}\bigg)$$

$$= \frac{2}{x^2}\frac{x-2}{\sqrt{x^2-(x-2)^2}} = \frac{x-2}{x^2\sqrt{x-1}}, \quad x \in [2,\infty).$$

The corresponding df equals $F_{L_1+L_2}(x) = 1 - 2\sqrt{x-1}/x$, $x \in [2,\infty)$.

To determine $\mathrm{VaR}_\alpha(L_1+L_2) = F^{-}_{L_1+L_2}(\alpha)$ we have to solve $F_{L_1+L_2}(x) = \alpha$ with respect to x. We obtain

$$F_{L_1+L_2}(x) = \alpha \iff \frac{\sqrt{x-1}}{x} = \frac{1-\alpha}{2} \iff \Big(\frac{1-\alpha}{2}\Big)^2 x^2 - x + 1 = 0,$$

with solutions $x_{1,2} = \frac{1 \pm \sqrt{1-(1-\alpha)^2}}{(1-\alpha)^2/2}$. The solution has to satisfy $x_{1,2} \geq 2$, which happens if and only if $(1-\alpha)^2 \leq 1 \pm \sqrt{1-(1-\alpha)^2}$. Note that $x < \sqrt{x}$ for all $x \in (0,1)$, so this inequality is only valid for x_1. Thus

$$\begin{aligned}
\mathrm{VaR}_\alpha(L_1+L_2) &= \frac{1+\sqrt{1-(1-\alpha)^2}}{(1-\alpha)^2/2} = 2\frac{1+\sqrt{1-(1-\alpha)^2}}{(1-\alpha)^2} \\
&> 2\frac{1}{(1-\alpha)^2} = 2\,\mathrm{VaR}_\alpha(L_1) = \mathrm{VaR}_\alpha(L_1) + \mathrm{VaR}_\alpha(L_2)
\end{aligned}$$

for all $\alpha \in (0,1)$. □

Exercise 2.31 (Special dependence)

Let $\alpha \in (0,1)$, $L_1 \sim \mathrm{U}[0,1]$ and

$$L_2 \overset{\text{a.s.}}{=} \begin{cases} L_1, & \text{if } L_1 < \alpha, \\ 1+\alpha-L_1, & \text{if } L_1 \geq \alpha. \end{cases}$$

Let $\alpha \in (0,1)$. Show that $\mathrm{VaR}_{\alpha+\varepsilon}(L_1+L_2) > \mathrm{VaR}_{\alpha+\varepsilon}(L_1)+\mathrm{VaR}_{\alpha+\varepsilon}(L_2)$ for all $\varepsilon \in (0,(1-\alpha)/2)$.

Solution. We first show that $L_2 \sim \mathrm{U}[0,1]$. By the law of total probability, $\mathbb{P}(L_2 \leq x) = \mathbb{P}(L_2 \leq x, L_1 < \alpha) + \mathbb{P}(L_2 \leq x, L_1 \geq \alpha)$. By continuity, the first summand equals $\mathbb{P}(L_1 \leq x, L_1 < \alpha) = \mathbb{P}(L_1 \leq \min\{x,\alpha\}) = \min\{x,\alpha\}$. For the second summand, note that $1+\alpha-x \geq \alpha$ for all $x \in [0,1]$, so that it equals

$$\begin{aligned} &\mathbb{P}(1+\alpha-L_1 \leq x, L_1 \geq \alpha) = \mathbb{P}(L_1 \geq 1+\alpha-x, L_1 \geq \alpha) \\ &= \mathbb{P}(L_1 \geq \max\{1+\alpha-x,\alpha\}) = \mathbb{P}(L_1 \geq 1+\alpha-x) \\ &= \mathbb{P}(1+\alpha-x \leq L_1 \leq 1) = \max\{1-(1+\alpha-x),0\} = \max\{x-\alpha,0\}. \end{aligned}$$

Therefore, $\mathbb{P}(L_2 \le x) = \min\{x, \alpha\} + \max\{x - \alpha, 0\} = x$, $x \in [0, 1]$.

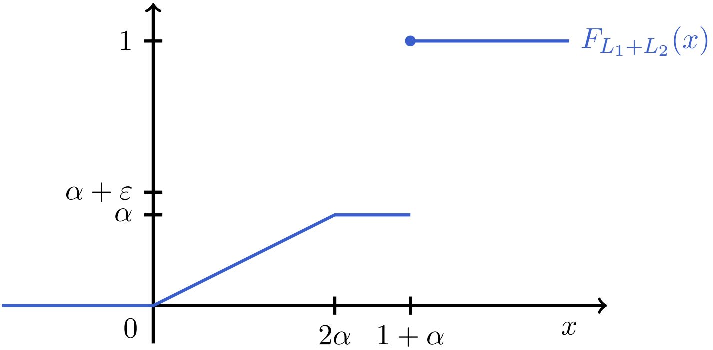
Note that

$$L_1 + L_2 = \begin{cases} 2L_1, & \text{if } L_1 < \alpha, \\ 1 + \alpha, & \text{if } L_1 \ge \alpha, \end{cases}$$

and $2\alpha = \alpha + \alpha < 1 + \alpha$ for all $\alpha \in (0, 1)$. Hence, $F_{L_1+L_2}(x)$ equals

$$\begin{aligned} &\mathbb{P}(L_1 + L_2 \le x) = \mathbb{P}(2L_1 \le x, L_1 < \alpha) + \mathbb{P}(1 + \alpha \le x, L_1 \ge \alpha) \\ &= \min\{x/2, \alpha\} + \mathbb{1}_{\{1+\alpha \le x\}}(1 - \alpha) \\ &= \begin{cases} 0, & \text{if } x < 0, \\ x/2, & \text{if } x \in [0, 2\alpha), \\ \alpha, & \text{if } x \in [2\alpha, 1 + \alpha), \\ 1, & \text{if } x \ge 1 + \alpha. \end{cases} \end{aligned}$$

A picture is worth a thousand words...



For all $\varepsilon \in (0,(1-\alpha)/2)$, we thus obtain

$$\mathrm{VaR}_{\alpha+\varepsilon}(L_1+L_2)=1+\alpha>2(\alpha+\varepsilon)=\mathrm{VaR}_{\alpha+\varepsilon}(L_1)+\mathrm{VaR}_{\alpha+\varepsilon}(L_2).$$

$\square$

Remark 2.32 (Special case of comonotone risks; elliptical risks)

- If $L_1 \overset{\text{a.s.}}{=} L_2$ (special case of comonotone risks (strongest positive dependence; see later)) then positive homogeneity of VaR_α implies that $\mathrm{VaR}_\alpha(L_1+L_2) = \mathrm{VaR}_\alpha(2L_1) = 2\,\mathrm{VaR}_\alpha(L_1) = \mathrm{VaR}_\alpha(L_1)+\mathrm{VaR}_\alpha(L_2)$ for all $\alpha \in (0,1)$, so VaR_α is *additive* (thus also subadditive). In comparison to Exercise 2.31, we see that the strongest positive dependence does not lead to the largest $\mathrm{VaR}_\alpha(L_1+L_2)$; if L_1 and L_2 have a special dependence structure, $\mathrm{VaR}_\alpha(L_1+L_2)$ can be larger than $\mathrm{VaR}_\alpha(L_1)+\mathrm{VaR}_\alpha(L_2)$.
- As we will see later, VaR_α is subadditive and thus coherent for all elliptical models (the "garden of eden of RM") if $\alpha \in [1/2,1]$. In the multivariate normal world, this can be seen as follows. Let $(L_1,L_2) \sim \mathrm{N}(\boldsymbol{\mu},\Sigma)$, $\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$. Then (see later)

$$L_1+L_2 \sim \mathrm{N}(\mu_1+\mu_2,\ \sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2).$$

Since $|\rho| \leq 1$,

$$\begin{aligned}\mathrm{VaR}_\alpha(L_1+L_2) &= \mu_1+\mu_2+\sqrt{\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2}\,\Phi^{-1}(\alpha)\\ &\leq \mu_1+\mu_2+\sqrt{(\sigma_1+\sigma_2)^2}\,\Phi^{-1}(\alpha) = \mathrm{VaR}_\alpha(L_1)+\mathrm{VaR}_\alpha(L_2),\end{aligned}$$

so VaR_α is subadditive for all $\alpha \in [1/2, 1)$ for $(L_1, L_2) \sim \mathrm{N}(\boldsymbol{\mu}, \Sigma)$.

- We also see that a statement like "VaR is coherent in the normal case" does not make sense unless we specify the joint distribution function of (L_1, L_2) (marginal dfs + dependence).
  - If the underlying copula is Gauss (see later), then (L_1, L_2) is multivariate normal and thus VaR_α, $\alpha \in [1/2, 1)$, is coherent.
  - If it is the copula underlying Exercise 2.31, then VaR_α is not coherent.

  Furthermore, $\mathrm{VaR}_\alpha(L_1+L_2)$ for $L_i \sim \mathrm{N}(\mu_i, \sigma_i^2)$, $i \in \{1, 2\}$, cannot even be computed unless we know the dependence between L_1, L_2.