5 Extreme value theory

5.1 Maxima

5.2 Threshold exceedances

5.1 Maxima

Consider a series of financial losses $(X_k)_{k\in\mathbb{N}}$.

5.1.1 Generalized extreme value distribution

Convergence of sums

Let $(X_k)_{k\in\mathbb{N}}$ be iid with $\mathbb{E}(X_1^2)<\infty$ (mean μ, variance σ^2) and $S_n=\sum_{k=1}^n X_k$. As $n\to\infty$, $\bar{X}_n \overset{\text{a.s.}}{\to} \mu$ by the Strong Law of Large Numbers (SLLN), so $(\bar{X}_n-\mu)/\sigma \overset{\text{a.s.}}{\to} 0$. By the CLT,

$$\sqrt{n}\frac{\bar{X}_n-\mu}{\sigma}=\frac{S_n-n\mu}{\sqrt{n}\sigma}\underset{n\uparrow\infty}{\overset{\text{d}}{\to}}\mathrm{N}(0,1)\quad\text{or}\quad\lim_{n\to\infty}\mathbb{P}\Big(\frac{S_n-d_n}{c_n}\le x\Big)=\Phi(x),$$

where the sequences $c_n=\sqrt{n}\sigma$ and $d_n=n\mu$ give normalization and where $\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-z^2/2}dz$. More generally ($\sigma^2=\infty$), the limiting distributions for appropriately normalized sums are the class of α-stable distributions ($\alpha\in(0,2]$; $\alpha=2$: normal distribution).

Convergence of maxima

QRM is concerned with maximal losses (orst-case losses). Let $(X_i)_{i\in\mathbb{N}} \overset{\text{ind.}}{\sim} F$ (can be relaxed to a strictly stationary time series) and F continuous. Then the *block maximum* is given by

$$M_n = \max\{X_1, \dots, X_n\}.$$

One can show that, for $n \to \infty$, $M_n \overset{\text{a.s.}}{\to} x_F$ (similar as in the SLLN) where $x_F := \sup\{x \in \mathbb{R} : F(x) < 1\} = F^{\leftarrow}(1) \leq \infty$ denotes the *right endpoint of F* (similar to the SLLN).

Question: Is there a "CLT" for block maxima?

Idea CLT: What about linear transformations (the simplest possible)?

Definition 5.1 (Maximum domain of attraction)

Suppose we find *normalizing sequences* of real numbers $(c_n) > 0$ and (d_n) such that $(M_n - d_n)/c_n$ converges in distribution, i.e.

$$\mathbb{P}((M_n - d_n)/c_n \le x) = \mathbb{P}(M_n \le c_n x + d_n) = F^n(c_n x + d_n) \underset{n\uparrow\infty}{\to} H(x),$$

for some *non-degenerate* df H (not a unit jump). Then F is in the *maximum domain of attraction* of H ($F \in \mathrm{MDA}(H)$).

One can show that H is determined up to location/scale, i.e. H specifies a unique *type* of distribution. This is guaranteed by the convergence to types theorem; see the appendix.

Question: What does H look like?

Definition 5.2 (Generalized extreme value (GEV) distribution)

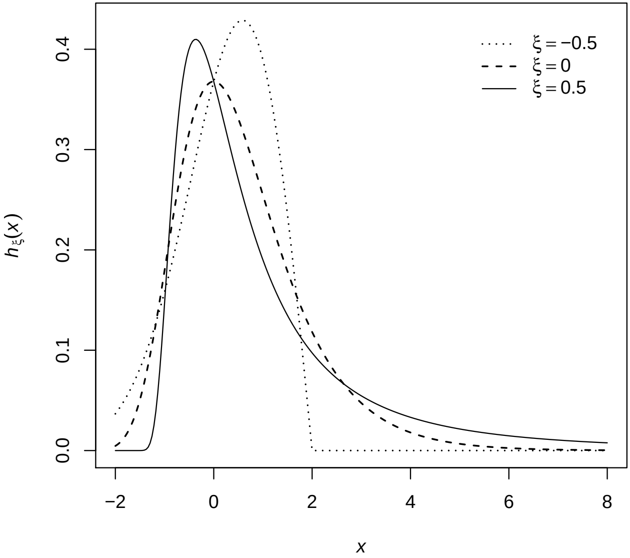
The (standard) *generalized extreme value (GEV) distribution* is given by

$$H_\xi(x) = \begin{cases} \exp(-(1+\xi x)^{-1/\xi}), & \text{if } \xi \neq 0, \\ \exp(-e^{-x}), & \text{if } \xi = 0, \end{cases}$$

where $1 + \xi x > 0$ (MLE!). A three-parameter family is obtained by a location-scale transform $H_{\xi,\mu,\sigma}(x) = H_\xi((x-\mu)/\sigma)$, $\mu \in \mathbb{R}$, $\sigma > 0$.

- The parameterization is continuous in ξ (simplifies statistical modelling).
- The larger ξ, the heavier tailed H_ξ (if $\xi > 0$, $\mathbb{E}(X^k) = \infty$ iff $k \geq \frac{1}{\xi}$).
- ξ is the *shape* (determines moments, tail). Special cases:
  1) $\xi < 0$: the Weibull df, short-tailed, $x_{H_\xi} < \infty$;
  2) $\xi = 0$: the Gumbel df, $x_{H_0} = \infty$, decays exponentially;
  3) $\xi > 0$: the Fréchet df, $x_{H_\xi} = \infty$, heavy-tailed ($\bar{H}_\xi(x) \approx (\xi x)^{-1/\xi}$), most important case for practice

Density h_ξ for $\xi \in \{-0.5, 0, 0.5\}$ (dotted, dashed, solid)



Theorem 5.3 (Fisher–Tippett–Gnedenko)
If $F \in \mathrm{MDA}(H)$ for some non-degenerate H, then H must be of GEV type, i.e. $H = H_\xi$ for some $\xi \in \mathbb{R}$.

Proof. Non-trivial. For a sketch, see Embrechts et al. (1997, p. 122). □

- **Interpretation:** If location-scale transformed maxima converge in distribution to a non-degenerate limit, the limiting distribution must be a GEV distribution.
- We can always choose normalizing sequences $(c_n) > 0$, (d_n) such that H_ξ appears in standard form.
- All commonly encountered continuous distributions are in the MDA of a GEV distribution.

Example 5.4 (Exponential distribution)

For $(X_i)_{i\in\mathbb{N}} \overset{\text{ind.}}{\sim} \mathrm{Exp}(\lambda)$, choosing $c_n = 1/\lambda$, $d_n = \log(n)/\lambda$, one obtains

$$\begin{aligned} F^n(c_n x + d_n) &= \big(1 - \exp(-\lambda((1/\lambda)x + \log(n)/\lambda))\big)^n \\ &= (1 - \exp(-x)/n)^n \underset{n\uparrow\infty}{\to} \exp(-e^{-x}) = H_0(x) \text{ (Gumbel)} \end{aligned}$$

Example 5.5 (Pareto distribution)

For $(X_i)_{i\in\mathbb{N}} \overset{\text{ind.}}{\sim} \mathrm{Par}(\theta,\kappa)$ with $F(x) = 1 - (\frac{\kappa}{\kappa+x})^\theta$, $x \ge 0$, $\theta,\kappa > 0$, choosing $c_n = \kappa n^{1/\theta}/\theta$, $d_n = \kappa(n^{1/\theta} - 1)$, one obtains

$$\begin{aligned} F^n(c_n x + d_n) &= \Big(1 - \Big(\frac{\kappa}{\kappa + x\kappa n^{1/\theta}/\theta + \kappa(n^{1/\theta} - 1)}\Big)^\theta\Big)^n \\ &= \Big(1 - \Big(\frac{1}{1 + x n^{1/\theta}/\theta + n^{1/\theta} - 1}\Big)^\theta\Big)^n = \Big(1 - \frac{(1/(x/\theta))^\theta}{n}\Big)^n \\ &= \Big(1 - \frac{(\theta/x)^\theta}{n}\Big)^n \underset{n\uparrow\infty}{\to} \exp(-(\theta/x)^\theta) = H_{1/\theta,\theta,1}(x) \text{ (Fréchet)} \end{aligned}$$

Therefore, $F \in \mathrm{MDA}(H_{1/\theta})$.

5.1.2 Maximum domains of attraction

All commonly applied continuous F belong to $\mathrm{MDA}(H_\xi)$ for some $\xi \in \mathbb{R}$. μ, σ can be estimated, but how can we characterize/determine ξ? All $F \in \mathrm{MDA}(H_\xi)$ for $\xi > 0$ have an elegant characterization involving the following notions.

Definition 5.6 (Slowly/regularly varying functions)

1) A positive, Lebesgue-measurable function L on $(0, \infty)$ is *slowly varying at* ∞ if $\lim_{x\to\infty} \frac{L(tx)}{L(x)} = 1$, $t > 0$. The class of all such functions is denoted by $\mathcal{R}_0$; e.g. $c, \log \in \mathcal{R}_0$.
2) A positive, Lebesgue-measurable function h on $(0, \infty)$ is *regularly varying at* ∞ *with index* $\alpha \in \mathbb{R}$ if $\lim_{x\to\infty} \frac{h(tx)}{h(x)} = t^\alpha$, $t > 0$. The class of all such functions is denoted by $\mathcal{R}_\alpha$; e.g. $x^\alpha L(x) \in \mathcal{R}_\alpha$.

If $\bar{F} \in \mathcal{R}_{-\alpha}$, $\alpha > 0$, the tail of F decays like a power function (Pareto like).

The Fréchet case

Theorem 5.7 (Fréchet MDA, Gnedenko (1943))
For $\xi > 0$, $F \in \mathrm{MDA}(H_\xi)$ if and only if $\bar{F}(x) = x^{-1/\xi}L(x)$ for some $L \in \mathcal{R}_0$. If $F \in \mathrm{MDA}(H_\xi)$, $\xi > 0$, the normalizing sequences can be chosen as $c_n = F^{\leftarrow}(1 - 1/n)$ and $d_n = 0$, $n \in \mathbb{N}$.

Proof. Non-trivial. For a sketch, see Embrechts et al. (1997, p. 131). □

- **Interpretation:** Distributions in $\mathrm{MDA}(H_\xi)$, $\xi > 0$, are those whose tails decay like power functions; $\alpha = 1/\xi$ is known as *tail index*.
- If $X \sim F \in \mathrm{MDA}(H_\xi)$, $\xi > 0$, $X \geq 0$, then $\mathbb{E}(X^k) < \infty$ if $k < \alpha = 1/\xi$, $\mathbb{E}(X^k) = \infty$ if $k > \alpha = 1/\xi$; see Embrechts et al. (1997, p. 568).
- **Examples in $\mathrm{MDA}(H_\xi)$, $\xi > 0$:** Inverse gamma, Student t, log-gamma, F, Cauchy, α-stable with $0 < \alpha < 2$, Burr and Pareto

Example 5.8 (Pareto distribution)

For $F = \mathrm{Par}(\theta, \kappa)$, $\bar{F}(x) = (\kappa/(\kappa + x))^\theta = (1 + x/\kappa)^{-\theta} = x^{-\theta}L(x)$, $x \geq 0$, $\theta, \kappa > 0$, where $L(x) = (\kappa^{-1} + x^{-1})^{-\theta} \in \mathcal{R}_0$. We (again) see that $F \in \mathrm{MDA}(H_\xi)$, $\xi > 0$.

The Gumbel case

- The characterization of this class is more complicated; see the appendix and Embrechts et al. (1997, p. 142).
- Essentially $\mathrm{MDA}(H_0)$ contains dfs whose tails decay roughly exponentially *(light-tailed)*, but the tails can be quite different (up to moderately heavy). All moments exist for distributions in the Gumbel class, but both $x_F < \infty$ and $x_F = \infty$ are possible.
- **Examples in** $\mathrm{MDA}(H_0)$**:** Normal, log-normal, exponential, gamma (exponential, Erlang, χ^2), standard Weibull, Benktander type I and II, generalized hyperbolic (except Student t).

The Weibull case

Theorem 5.9 (Weibull MDA)
For $\xi < 0$, $F \in \mathrm{MDA}(H_\xi)$ if and only if $x_F < \infty$ and $\bar{F}(x_F - 1/x) = x^{1/\xi} L(x)$ for some $L \in \mathcal{R}_0$; the normalizing sequences can be chosen as $c_n = x_F - F^{\leftarrow}(1 - 1/n)$ and $d_n = x_F$, $n \in \mathbb{N}$.

Proof. Non-trivial. For a sketch, see Embrechts et al. (1997, p. 135). □

Examples in $\mathrm{MDA}(H_\xi)$, $\xi < 0$: beta (uniform). All $F \in \mathrm{MDA}(H_\xi)$, $\xi < 0$, share $x_F < \infty$.

5.1.3 Maxima of strictly stationary time series

What about maxima of strictly stationary time series?

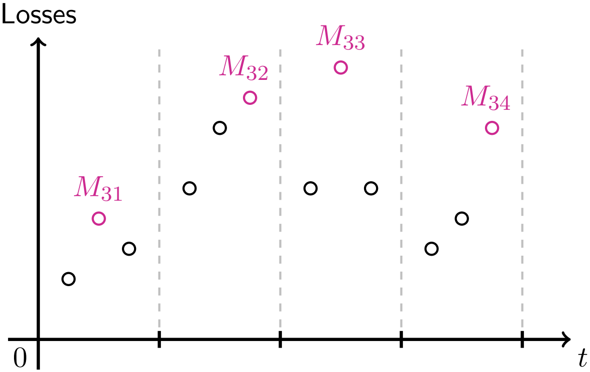
- Let $(X_k)_{k \in \mathbb{Z}}$ denote a strictly stationary time series with stationary distribution $X_k \sim F$, $k \in \mathbb{Z}$.

- Let $\tilde{X}_k \overset{\text{ind.}}{\sim} F$, $k \in \mathbb{Z}$, and $\tilde{M}_n = \max\{\tilde{X}_1, \dots, \tilde{X}_n\}$. For many processes one can show that there exists a real number $\theta \in (0, 1]$ such that $\lim_{n\uparrow\infty} \mathbb{P}((M_n - d_n)/c_n \le x) = H^\theta(x)$ if and only if $\lim_{n\uparrow\infty} \mathbb{P}((\tilde{M}_n - d_n)/c_n \le x) = H(x)$ (non-degenerate); θ is known as the *extremal index*.
- If $F \in \mathrm{MDA}(H_\xi)$ for some $\xi \Rightarrow M_n$ converges in distribution to H_ξ^θ. Since H_ξ^θ is of the same type as H_ξ, the limiting distribution of the block maxima of the dependent series is the same as in the iid case (only location/scale may change).
- For large n, $\mathbb{P}((M_n - d_n)/c_n \le x) \approx H^\theta(x) \approx F^{n\theta}(c_n x + d_n)$, so the distribution of M_n from a time series with extremal index θ can be approximated by the distribution $\tilde{M}_{n\theta}$ of the maximum of $n\theta < n$ observations from the associated iid series. $\Rightarrow n\theta$ counts the number of roughly independent clusters in n observations (θ is often interpreted as "1/mean cluster size").

- If $\theta = 1$, large sample maxima behave as in the iid case; if $\theta \in (0, 1)$, large sample maxima tend to cluster.
- **Examples** (see Embrechts et al. (1997, pp. 216, pp. 415, pp. 476))
  - Strict white noise (iid rvs): $\theta = 1$;
  - ARMA processes with (ε_t) strict white noise: $\theta = 1$ (Gaussian); $\theta \in (0, 1)$ (if df of ε_t is in $\mathrm{MDA}(H_\xi)$, $\xi > 0$);
  - GARCH processes: $\theta \in (0, 1)$.

5.1.4 The block maxima method (BMM)

The basic idea in a picture based on losses $X_1, \dots, X_{12}$:



Consider the maximal loss from each block and fit $H_{\xi,\mu,\sigma}$ to them.

Fitting the GEV distribution

- Suppose $(x_i)_{i\in\mathbb{N}}$ are realizations of $(X_i)_{i\in\mathbb{N}} \overset{\text{ind.}}{\sim} F \in \mathrm{MDA}(H_\xi)$, $\xi \in \mathbb{R}$, or of a process with an extremal index such as GARCH. The Fisher–Tippett–Gnedenko Theorem implies that

$$\mathbb{P}(M_n \le x) = \mathbb{P}((M_n - d_n)/c_n \le (x - d_n)/c_n) \underset{n \text{ large}}{\approx} H_{\xi,\mu=d_n,\sigma=c_n}(x).$$

- For fitting $\boldsymbol{\theta} = (\xi, \mu, \sigma)$, divide the realizations into m blocks of size n denoted by $M_{n1}, \dots, M_{nm}$ (e.g. daily log-returns $\Rightarrow$ monthly maxima)
- Assume the block size n to be sufficiently large so that (regardless of whether the underlying data are dependent or not), the block maxima can be considered independent.
- The density h_ξ of H_ξ is

$$h_\xi(x) = \begin{cases} (1+\xi x)^{-1/\xi-1} H_\xi(x) I_{\{1+\xi x>0\}}, & \text{if } \xi \ne 0, \\ e^{-x} H_0(x), & \text{if } \xi = 0. \end{cases}$$

The log-likelihood is thus

$$\ell(\boldsymbol{\theta}; M_{n1}, \ldots, M_{nm}) = \sum_{i=1}^{m} \log\Big(\frac{1}{\sigma} h_\xi\Big(\frac{M_{ni}-\mu}{\sigma}\Big) I_{\{1+\xi(M_{ni}-\mu)/\sigma>0\}}\Big).$$

Maximize w.r.t. $\boldsymbol{\theta} = (\xi, \mu, \sigma)$ to get $\hat{\boldsymbol{\theta}} = (\hat{\xi}, \hat{\mu}, \hat{\sigma})$.

Remark 5.10

1) Sufficiently many/large blocks require large amounts of data.
2) Bias and variance must be traded off (*bias-variance tradeoff*):
   - Block size $n \uparrow \Rightarrow$ GEV approximation more accurate $\Rightarrow$ bias $\downarrow$
   - Number of blocks $m \uparrow \Rightarrow$ more data for MLE $\Rightarrow$ variance $\downarrow$
3) There is no general best strategy known to find the optimal block size.
4) The support of the density depends on the parameters $\Rightarrow$ not differentiable; classical MLE regularity conditions for consistency and asymptotic efficiency do not applied. For $\xi > -1/2$ (fine for practice), Smith (1985) showed that the MLE is regular.

Return levels and stress losses (exceedances)

The fitted GEV model can be used to estimate the...

1) ... size of an event with prescribed frequency (*return-level problem*)

2) ... frequency of an event with prescribed size (*return-period problem*)

Definition 5.11 (Return level, return period)

Let $M_n \sim H$ (exact or estimated).

- The k *n-block return level* is $r_{n,k} = H^{\leftarrow}(1 - 1/k)$.
- The *return period* of the event $\{M_n > u\}$ is $k_{n,u} = 1/\bar{H}(u)$.

- $r_{n,k}$ is the level which is expected to be exceeded in one out of every k n-blocks, so $r_{n,k}$ solves $\mathbb{P}(M_n > r_{n,k}) = 1/k$ (e.g. 10-year return level $r_{260,10}$ = level exceeded in one out of every 10 years; 260d $\approx$ 1 year).
- $k_{n,u}$ is the number of n-blocks for which we expect to see a single n-block exceeding u, so $k_{n,u}$ solves $r_{n,k_{n,u}} = H^{\leftarrow}(1 - 1/k_{n,u}) = u$.

- Parametric estimators are given by

$$\hat{r}_{n,k} = H^{\leftarrow}_{\hat{\xi},\hat{\mu},\hat{\sigma}}(1-1/k) = \hat{\mu} + \frac{\hat{\sigma}}{\hat{\xi}}((-\log(1-1/k))^{-\hat{\xi}} - 1),$$
$$\hat{k}_{n,u} = 1/\bar{H}_{\hat{\xi},\hat{\mu},\hat{\sigma}}(u).$$

  Confidence intervals for $r_{n,k}$, $k_{n,u}$ can be constructed via profile-likelihoods; see Davison (2003, pp. 126) and McNeil et al. (2005, p. 274).

Example 5.12 (Block maxima analysis of S&P500)

Suppose it is Friday 1987-10-16; the Friday before Black Monday (1987-10-19). The S&P 500 index fell by 10.0% this week. On that Friday alone the index is down 5.4%. We fit a GEV distribution to (bi)annual maxima of daily negative log-returns $X_t = \log(S_t/S_{t-1})$ since 1960-01-01.

Analysis 1: Annual maxima ($m = 28$; including the latest from the incomplete year 1987): $\hat{\boldsymbol{\theta}} = (0.30, 0.02, 0.007) \Rightarrow$ heavy-tailed Fréchet distribution (infinite fourth moment). The corresponding standard errors are $(0.21, 0.002, 0.001) \Rightarrow$ High uncertainty (m small) for estimating ξ.

Analysis 2: Biannual maxima ($m = 56$): $\hat{\boldsymbol{\theta}} = (0.34, 0.02, 0.006)$ with standard errors $(0.14, 0.0009, 0.0008) \Rightarrow$ Even heavier tails. In what follows we work with the annual maxima.

- What is the probability that next year's maximal risk-factor change exceeds all previous ones? $1 - H_{\hat{\xi},\hat{\mu},\hat{\sigma}}(\max_i\{X_i\})$

- Was a risk-factor change of the size/level as of Black Monday foreseeable?
  - Based on data up to and including Friday 1987-10-16, the 10-year return level $r_{260,10}$ is estimated as $\hat{r}_{260,10} = 4.42\%$.
  - Index drop Black Monday: 25.7% $\Rightarrow X_{t+1} = 22.9\% \gg \hat{r}_{260,10}$.
  - One can show that 22.9% is in the 95% confidence interval of $r_{260,50}$ (estimated as $\hat{r}_{260,50} = 7.49\%$), but the 28 maxima are too few to get a reliable estimate of a once-in-50-years event.
- Based on the available data, what is the (estimated) return period of a risk-factor change at least as large as on Black Monday?
  - The estimated return period $k_{260,0.229}$ is $\hat{k}_{260,0.229} = 1877$ years.
  - One can show that the 95% confidence interval encompasses everything from 45 years to essentially never! $\Rightarrow$ Very high uncertainty involved in estimating $k_{260,0.229}$.

In summary, on 1987-10-16 we simply did not have enough data to say anything meaningful about an event of this magnitude. This illustrates the difficulties of quantifying events beyond our empirical experience.

5.2 Threshold exceedances

The BMM is wasteful of data (only the maxima of large blocks are used). It has been largely superseded in practice by methods based on threshold exceedances (*peaks-over-threshold (POT)* approach), where all data above a designated high threshold u are used.

5.2.1 Generalized Pareto distribution

Definition 5.13 (Generalized Pareto distribution (GPD))

The *generalized Pareto distribution (GPD)* is given by

$$G_{\xi,\beta}(x) = \begin{cases} 1-(1+\xi x/\beta)^{-1/\xi}, & \text{if } \xi \neq 0, \\ 1-\exp(-x/\beta), & \text{if } \xi = 0, \end{cases}$$

where $\beta > 0$, and the support is $x \ge 0$ when $\xi \ge 0$ and $x \in [0, -\beta/\xi]$ when $\xi < 0$.

- The parameterization is continuous in ξ.
- The larger ξ, the heavier tailed $G_{\xi,\beta}$ (if $\xi > 0$, $\mathbb{E}(X^k) = \infty$ iff $k \geq \frac{1}{\xi}$; if $\xi < 1$, then $\mathbb{E}X = \beta/(1-\xi)$).
- ξ is known as *shape*; β as *scale*. Special cases:

  1) $\xi > 0$: $\mathrm{Par}(1/\xi, \beta/\xi)$

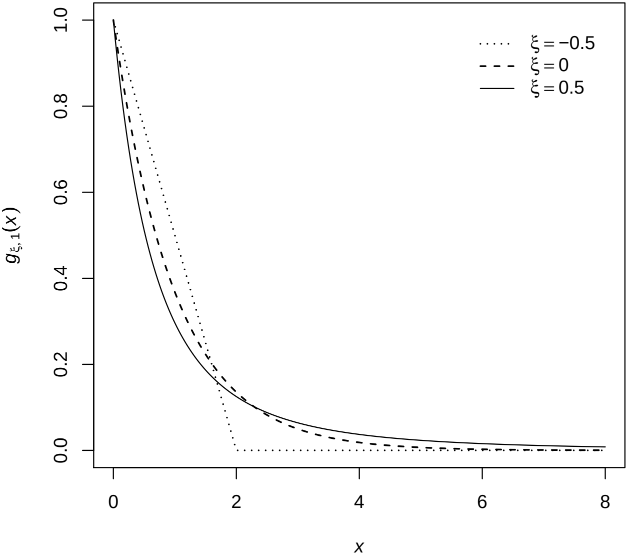
  2) $\xi = 0$: $\mathrm{Exp}(1/\beta)$

  3) $\xi < 0$: short-tailed Pareto type II distribution
- The density $g_{\xi,\beta}$ of $G_{\xi,\beta}$ is given by

$$g_{\xi,\beta}(x) = \begin{cases} \frac{1}{\beta}(1+\xi x/\beta)^{-1/\xi-1}, & \text{if } \xi \neq 0, \\ \frac{1}{\beta}\exp(-x/\beta), & \text{if } \xi = 0, \end{cases}$$

  where $x \geq 0$ when $\xi \geq 0$ and $x \in [0, -\beta/\xi)$ when $\xi < 0$ (MLE!).
- $G_{\xi,\beta} \in \mathrm{MDA}(H_\xi)$, $\xi \in \mathbb{R}$.

Density $g_{\xi,1}$ for $\xi \in \{-0.5, 0, 0.5\}$ (dotted, dashed, solid)



Definition 5.14 (Excess distribution over u, mean excess function)

Let $X \sim F$. The *excess distribution over the threshold u* is defined by

$$F_u(x) = \mathbb{P}(X - u \le x \,|\, X > u) = \frac{F(x+u) - F(u)}{1 - F(u)}, \quad x \in [0, x_F - u).$$

If $\mathbb{E}|X| < \infty$, the *mean excess function* is defined by

$$e(u) = \mathbb{E}(X - u \,|\, X > u) \quad \text{(i.e. the mean w.r.t. } F_u\text{)}$$

Interpretation

F_u describes the distribution of the loss over u (excess), given that u is exceeded. $e(u)$ is the mean of F_u as a function in u.

- One can show the useful formula $e(u) = \frac{1}{\bar{F}(u)} \int_u^{x_F} \bar{F}(x)\,dx$.
- For continuous $X \sim F$ with $\mathbb{E}|X| < \infty$, the following formula holds:

$$\mathrm{ES}_\alpha(X) = e(\mathrm{VaR}_\alpha(X)) + \mathrm{VaR}_\alpha(X), \quad \alpha \in (0,1) \tag{10}$$

Example 5.15 (F_u, $e(u)$ for $\mathrm{Exp}(\lambda)$, $G_{\xi,\beta}$)

1) If F is $\mathrm{Exp}(\lambda)$, then $F_u(x) = 1 - e^{-\lambda x}$, $x \geq 0$ (so again $\mathrm{Exp}(\lambda)$; lack-of-memory property). The mean excess function is $e(u) = 1/\lambda = \mathbb{E}X$.

2) If F is $G_{\xi,\beta}$, then $F_u(x) = G_{\xi,\beta+\xi u}(x)$, $x \geq 0$ (so again GPD, with the same shape, only the scale grows linearly in u). The mean excess function of $G_{\xi,\beta}$ is

$$e(u) = \frac{\beta + \xi u}{1 - \xi}, \quad \text{for all } u : \beta + \xi u > 0,$$

which is linear in u (this is a characterizing property of the GPD and used to determine u). Note that ξ determines the slope of $e(u)$.

Theorem 5.16 (Pickands–Balkema–de Haan (1974/75))
There exists a positive, measurable function $\beta(u)$, such that

$$\lim_{u \uparrow x_F} \sup_{0 \le x < x_F - u} |F_u(x) - G_{\xi,\beta(u)}(x)| = 0.$$

if and only if $F \in \mathrm{MDA}(H_\xi)$, $\xi \in \mathbb{R}$.

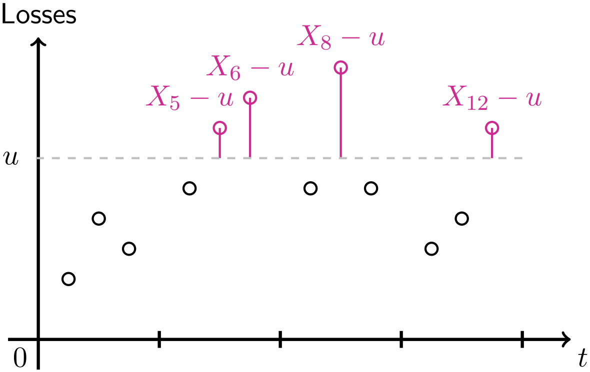
Proof. Non-trivial; see, e.g. Pickands (1975) and Balkema and de Haan (1974). □

Interpretation

- GPD = Canonical df for modelling excess losses over high u.
- The result is also a characterization of $\mathrm{MDA}(H_\xi)$, $\xi \in \mathbb{R}$. All $F \in \mathrm{MDA}(H_\xi)$ form a set of df for which the excess distribution converges to the GPD $G_{\xi,\beta}$ with the same ξ as in H_ξ as the threshold u is raised.

5.2.2 Modelling excess losses

The basic idea in a picture based on losses $X_1, \dots, X_{12}$.



Consider all excesses over u and fit $G_{\xi,\beta}$ to them.

The method

- Given losses $X_1, \dots, X_n \sim F \in \mathrm{MDA}(H_\xi)$, $\xi \in \mathbb{R}$, let
  - $N_u = |\{i \in \{1, \dots, n\} : X_i > u\}|$ denote the *number of exceedances* over the (given; see later) threshold u;
  - $\tilde{X}_1, \dots, \tilde{X}_{N_u}$ denote the *exceedances*; and
  - $Y_k = \tilde{X}_k - u$, $k \in \{1, \dots, N_u\}$, the corresponding *excesses*.
- If $Y_1, \dots, Y_{N_u}$ are iid and (roughly) distributed as $G_{\xi,\beta}$, the log-likelihood is given by

$$\begin{aligned}\ell(\xi, \beta; Y_1, \dots, Y_{N_u}) &= \sum_{k=1}^{N_u} \log g_{\xi,\beta}(Y_k) \\ &= -N_u \log(\beta) - (1 + 1/\xi) \sum_{k=1}^{N_u} \log(1 + \xi Y_k/\beta)\end{aligned}$$

$\Rightarrow$ Maximize w.r.t. $\beta > 0$ and $1 + \xi Y_k/\beta > 0$ for all $k \in \{1, \dots, N_u\}$.

Excesses over higher thresholds

Once a model is fitted to F_u, we can infer a model for F_v, $v \geq u$.

Lemma 5.17

Assume, for some u, $F_u(x) = G_{\xi,\beta}(x)$ for $0 \leq x < x_F - u$. Then $F_v(x) = G_{\xi,\beta+\xi(v-u)}(x)$ for all $v \geq u$.

Proof. Recall that $F_u(x) = \mathbb{P}(X - u \leq x \,|\, X > u) = \frac{F(u+x)-F(u)}{\bar{F}(u)}$, so $\bar{F}_u(x) = \bar{F}(u+x)/\bar{F}(u)$. For $v \geq u$, we have

$$\bar{F}_v(x) = \frac{\bar{F}(v+x)}{\bar{F}(v)} = \frac{\bar{F}(u+(v+x-u))}{\bar{F}(u)} \frac{\bar{F}(u)}{\bar{F}(u+(v-u))}$$
$$= \frac{\bar{F}_u(v+x-u)}{\bar{F}_u(v-u)} = \frac{\bar{G}_{\xi,\beta}(x+v-u)}{\bar{G}_{\xi,\beta}(v-u)} \underset{\text{check}}{=} \bar{G}_{\xi,\beta+\xi(v-u)}(x) \qquad \square$$

$\Rightarrow$ The excess distribution over $v \geq u$ remains GPD with the same ξ (and β growing linearly in v); makes sense for a limiting distribution for $u \uparrow$.

If $\xi < 1$ (so if it exists), the mean excess function is given by

$$e(v) = \frac{\xi}{1-\xi} v + \frac{\beta - \xi u}{1-\xi}, \quad v \in [u, \infty) \text{ if } \xi \in [0,1), \tag{11}$$

and $v \in [u, u - \beta/\xi]$ if $\xi < 0$. This forms the basis for a graphical method for choosing u.

Sample mean excess plot and choice of the threshold

Definition 5.18 (Sample mean excess function, mean excess plot)
Based on positive loss data $X_1, \dots, X_n$, the *sample mean excess function* is defined by

$$e_n(v) = \frac{\sum_{i=1}^n (X_i - v) I_{\{X_i > v\}}}{\sum_{i=1}^n I_{\{X_i > v\}}}, \quad X_{(n)} > v.$$

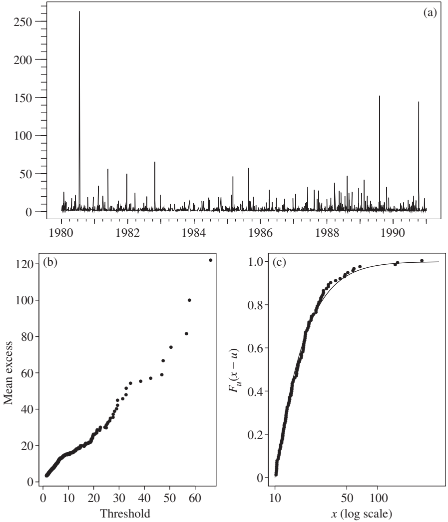
The *mean excess plot* is the plot of $\{(X_{(i)}, e_n(X_{(i)})) : 1 \le i \le n-1\}$, where $X_{(i)}$ denotes the ith order statistic.

- If the data supports the GPD model over u, $e_n(v)$ should become increasingly "linear" for higher values of $v \geq u$. An upward/zero/downward trend indicates $\xi > 0/\xi = 0/\xi < 0$.
- The sample mean excess plot is rarely perfectly linear (particularly for large u where one averages over a small number of excesses).
- The choice of a good threshold u is as difficult as finding an adequate block size for the Block Maxima method. There are data-driven tools (e.g. sample mean excess plot), but there is no general method to determine an optimal threshold (without second-order assumptions on $L \in \mathcal{R}_0$).
- Typically, select u as the smallest point where $e_n(v)$, $v \geq u$, becomes linear. Rule-of-thumb: One needs a couple of thousand data points and can often take u around the 0.9-quantile.
- One should always analyze the data for several u and check the sensitivity of the choice of u.

Example 5.19 (Danish fire loss data)

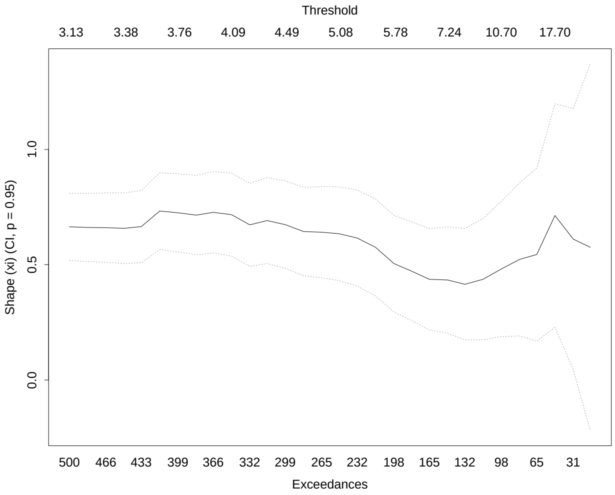
- 2156 fire insurance losses over 1M Danish kroner from 1980-01-03 to 1990-12-31; combined loss for a building and its contents, in some cases also a loss of business earnings. The losses are inflation adjusted to reflect values as of 1985.
- The mean excess function shows a "kink" below 10; "straightening out" above 10 $\Rightarrow$ Our choice is $u = 10$ (so 10M Danish kroner).
- MLE $(\hat{\xi}, \hat{\beta}) = (0.50, 7.0)$ (with standard errors $(0.14, 1.1)$)
  $\Rightarrow$ very heavy-tailed, infinite-variance model
- We can then estimate the expected loss given exceedance of 10M kroner or any higher threshold (via $e(v)$ in (11) based on $\hat{\xi}, \hat{\beta}$ and the chosen u), even beyond the data.
  $\Rightarrow$ EVT allows us to estimate "in the data" and then "scale up".

(a): Losses (> 1M; in M); (b): $e_n(u)$ (↑); (c) empirical $F_u(x-u)$, $G_{\hat{\xi},\hat{\beta}}$



⇒ Choose the threshold $u = 10$.

Sensitivity of the estimated shape parameter $\hat{\xi}$ to changes in u:



$\Rightarrow$ The higher u, the wider the confidence intervals (also support $u = 10$).

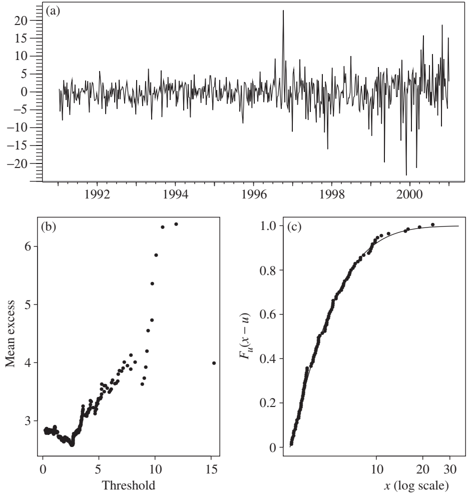
Example 5.20 (AT&T weekly loss data)

- Let (X_t) denote weekly log-returns and consider the percentage one-week loss as a fraction of S_t, given by

$$100L_{t+1}/S_t \underset{(1)}{=} 100(-S_t(\exp(X_{t+1})-1))/S_t = 100(1-\exp(X_{t+1})).$$

- We have 521 such losses (period 1991–2000).
- The estimated GPD parameters are $\hat{\xi} = 0.22$ and $\hat{\beta} = 2.1$ (MLEs) with standard errors 0.13 and 0.34, respectively. The fitted model is thus close to having an infinite fourth moment.
- Note that we ignored here that monthly data over 1993–2000 is not consistent with the iid assumption (absolute values of log-returns reject the hypothesis of serial uncorrelatedness via the Ljung–Box test).

(a): % losses (1991–2000); (b): $e_n(u)$; (c): empirical $F_u(x-u)$, $G_{\hat{\xi},\hat{\beta}}$.



⇒ Choose the threshold $u = 2.75\%$ (102 exceedances)

5.2.3 Modelling tails and measures of tail risk

- How can the fitted GPD model be used to estimate the tail of the loss distribution F and associated risk measures?
- Assume $F_u(x) = G_{\xi,\beta}(x)$ for $0 \le x < x_F - u$, $\xi \neq 0$ and some u.
- We obtain the following GPD-based formula for tail probabilities:

$$\begin{aligned}\bar{F}(x) &= \mathbb{P}(X > u)\mathbb{P}(X > x \mid X > u) \\ &= \bar{F}(u)\mathbb{P}(X - u > x - u \mid X > u) = \bar{F}(u)\bar{F}_u(x-u) \\ &= \bar{F}(u)\Big(1 + \xi\frac{x-u}{\beta}\Big)^{-1/\xi}, \quad x \ge u.\end{aligned}$$

- Assuming we know $\bar{F}(u)$, inverting this formula for $\alpha \ge F(u)$ leads to

$$\mathrm{VaR}_\alpha = F^{\leftarrow}(\alpha) = u + \frac{\beta}{\xi}\Big(\Big(\frac{1-\alpha}{\bar{F}(u)}\Big)^{-\xi} - 1\Big), \tag{12}$$

$$\mathrm{ES}_\alpha = \frac{\mathrm{VaR}_\alpha}{1-\xi} + \frac{\beta - \xi u}{1-\xi}, \quad \xi < 1. \tag{13}$$

The formula for ES_α can also be obtained from $e(\cdot)$ via (10) and (11).

- $\bar{F}(x)$, VaR_α and ES_α are all of the form $g(\xi, \beta, \bar{F}(u))$. If we have sufficient samples above u, we obtain semi-parametric plug-in estimators via $g(\hat{\xi}, \hat{\beta}, N_u/n)$.
- We hope to gain over empirical estimators by using a kind of extrapolation based on the GPD for more extreme tail probabilities and risk measures.
- In this spirit, Smith (1987) proposed the *tail estimator*

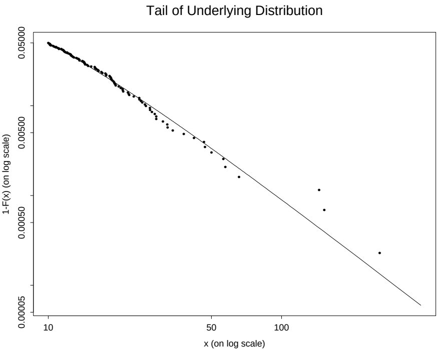
$$\hat{\bar{F}}(x) = \frac{N_u}{n}\left(1 + \hat{\xi}\,\frac{x-u}{\hat{\beta}}\right)^{-1/\hat{\xi}}, \quad x \ge u;$$

  also known as the *Smith estimator* (note that it is only valid for $x \ge u$). It faces a bias-variance tradeoff: If u is increased, the bias of parametrically estimating $\bar{F}_u(x-u)$ decreases, but the variance of it and the nonparametrically estimated $\bar{F}(u)$ increases.
- GPD-based $\widehat{\mathrm{VaR}}_\alpha$, $\widehat{\mathrm{ES}}_\alpha$ for $\alpha \ge 1 - N_u/n$ can be obtained similarly from (12), (13).

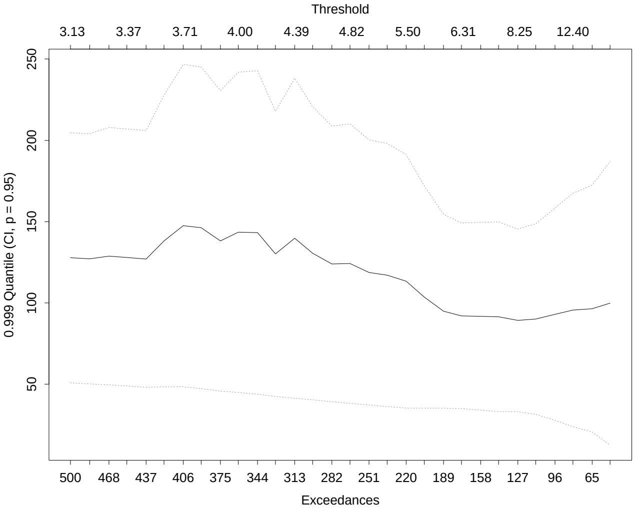
- Confidence intervals for $\bar{F}(x)$, $x \geq u$, VaR_α, ES_α can be obtained likelihood-based (neglecting the uncertainty in N_u/n): Reparametrize the GPD model in terms of $\phi = g(\xi, \beta, N_u/n)$ and construct a confidence interval for ϕ based on the likelihood ratio test.

Example 5.21 (Danish fire loss data (continued))

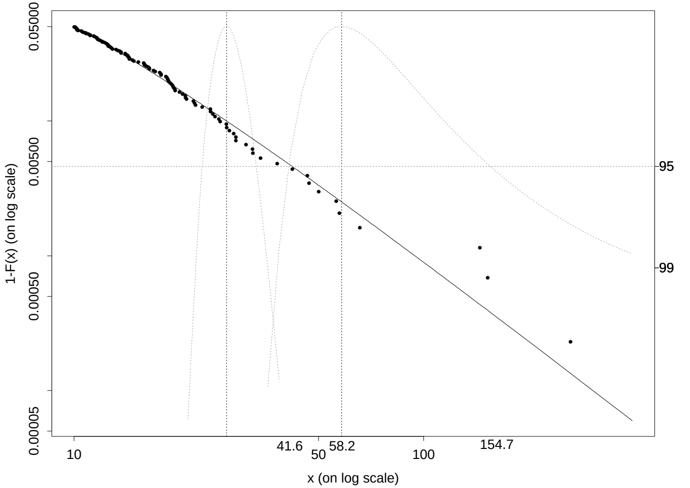
The semi-parametric Smith/tail estimator $\hat{\hat{\bar{F}}}(x)$, $x \geq u$ is given by:



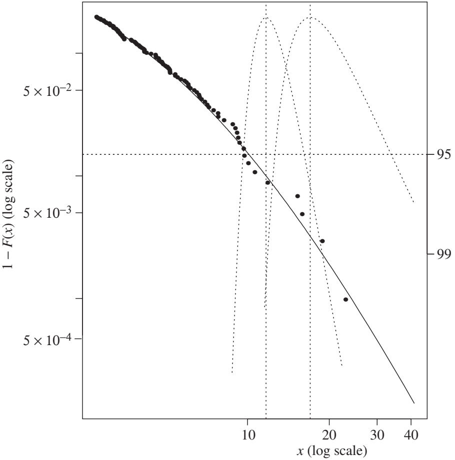
It is important to check the sensitivity of $\hat{\bar{F}}$ (or $\widehat{\mathrm{VaR}}_\alpha$, $\widehat{\mathrm{ES}}_\alpha$) w.r.t. u.



Here are $\hat{\bar{F}}(x)$, $x \geq u$, $\widehat{\mathrm{VaR}}_{0.99}$, $\widehat{\mathrm{ES}}_{0.99}$ including confidence intervals.



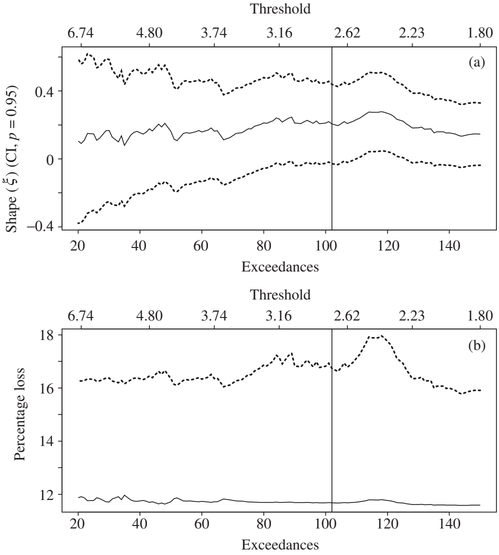
Example 5.22 (AT&T weekly loss data (continued))



- Fitted GPD model as in Example 5.20.
- Plot of $\hat{\bar{F}}(x)$.
- Vertical lines: $\widehat{\mathrm{VaR}}_{0.99}$, $\widehat{\mathrm{ES}}_{0.99}$
- log-log scale often good: $\bar{F}(x) = x^{-\alpha}L(x)$ and therefore

  $$\log \bar{F}(x) = -\alpha \log(x) + \log L(x)$$

  $\approx$ linear in $\log(x)$



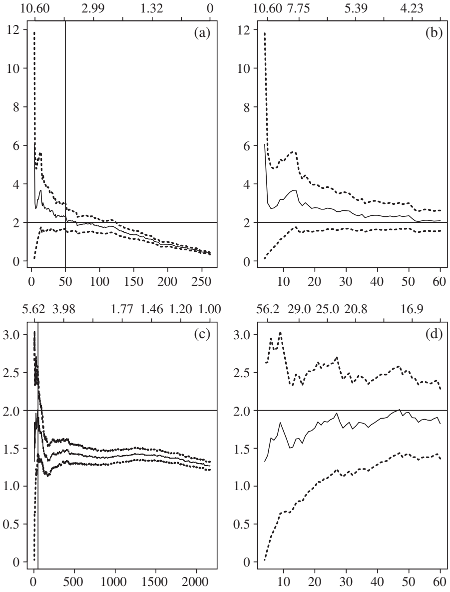
- Sensitivity w.r.t. u
- **Top:** $\hat{\xi}$ for different u or N_u, including a 95% CI based on standard error
- **Bottom:** Corresponding $\widehat{\mathrm{VaR}}_{0.99}$ (solid line), $\widehat{\mathrm{ES}}_{0.99}$ (dotted line)

5.2.4 The Hill estimator

- Assume $F \in \mathrm{MDA}(H_\xi)$, $\xi > 0$, so that $\bar{F}(x) = x^{-\alpha} L(x)$, $\alpha > 0$.
- The standard form of the *Hill estimator* of the tail index α is

$$\hat{\alpha}^{(\mathsf{H})}_{k,n} = \left(\frac{1}{k} \sum_{i=1}^{k} \log X_{i,n} - \log X_{k,n} \right)^{-1}, \quad 2 \le k \le n,\ k \text{ sufficiently small.}$$

  This can be derived by noting that the mean excess function $e(\log u)$ of $\log X$ at $\log u$ is roughly $1/\alpha$ for large u (by Karamata's Theorem), then using $e_n(\log X_{k,n})$ as an estimator for $e(\log u)$ and solving for α; see the appendix.
- Choosing k: Find a small k where the *Hill plot* $\{(k, \hat{\alpha}^{(\mathsf{H})}_{k,n}) : 2 \le k \le n\}$ stabilizes (typically, $k = \lceil \beta n \rceil$, $\beta \in [0.01, 0.05]$).
- Interpreting Hill plots can be difficult. If F does not have a regularly varying tail (or if it has serial dependence), Hill plots can be very misleading.



- Hill plots showing estimates of $\alpha = 1/\xi$ for (a), (b) the AT&T data and (c),(d) the Danish fire loss data (rhs = zoomed-in version of the lhs).
- (a),(b) suggest estimates of $\alpha \in [1.5, 2]$ ($\xi \in [1/2, 2/3]$; close to the estimated $\hat{\xi} = 0.50$, see Example 5.19); (c),(d) suggest estimates of $\alpha \in [2, 4]$ ($\xi \in [1/4, 1/2]$; larger than the estimated $\hat{\xi} = 0.22$, see Example 5.20)

Hill-based tail and risk measure estimates

- Assume $\bar{F}(x) = cx^{-\alpha}$, $x \ge u > 0$ (replacing L by a constant). Estimate α by $\hat{\alpha}^{(\mathrm{H})}_{k,n}$ and u by $X_{k,n}$ (for k sufficiently small).
- Note that $c = u^{\alpha}\bar{F}(u)$ so $\hat{c} = X_{k,n}^{\hat{\alpha}^{(\mathrm{H})}_{k,n}} \hat{\bar{F}}_n(X_{k,n}) \approx X_{k,n}^{\hat{\alpha}^{(\mathrm{H})}_{k,n}} \frac{k}{n}$. We thus obtain the semi-parametric *Hill tail estimator*

$$\hat{\bar{F}}(x) = \frac{k}{n}\left(\frac{x}{X_{k,n}}\right)^{-\hat{\alpha}^{(\mathrm{H})}_{k,n}}, \quad x \ge X_{k,n}.$$

- From this result we obtain the semi-parametric *Hill VaR estimator*

$$\widehat{\mathrm{VaR}}_{\alpha}(X) = \left(\frac{n}{k}(1-\alpha)\right)^{-\frac{1}{\hat{\alpha}^{(\mathrm{H})}_{k,n}}} X_{k,n}, \quad \alpha \ge F(u) \approx 1 - \frac{k}{n},$$

and, for $\hat{\alpha}^{(\mathrm{H})}_{k,n} > 1$, $\alpha \ge F(u) \approx 1 - \frac{k}{n}$, the semi-param. *Hill ES estimator*

$$\widehat{\mathrm{ES}}_{\alpha}(X) = \frac{\left(\frac{n}{k}\right)^{-\frac{1}{\hat{\alpha}^{(\mathrm{H})}_{k,n}}} X_{k,n}}{1-\alpha} \int_{\alpha}^{1} (1-z)^{-\frac{1}{\hat{\alpha}^{(\mathrm{H})}_{k,n}}}\, dz = \frac{\hat{\alpha}^{(\mathrm{H})}_{k,n}}{\hat{\alpha}^{(\mathrm{H})}_{k,n} - 1} \widehat{\mathrm{VaR}}_{\alpha}(X).$$

5.2.5 Simulation study of EVT quantile estimators

We compare estimators for ξ (Study 1) and $\mathrm{VaR}_{0.99}$ (Study 2) based on

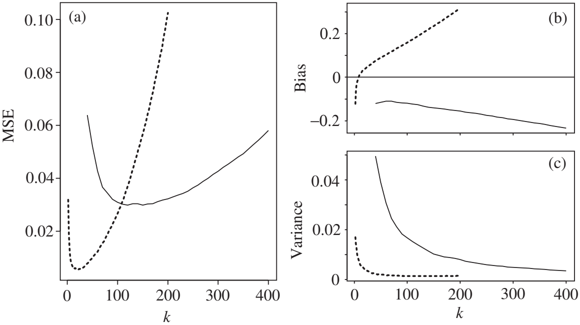
$$\begin{aligned}\mathrm{MSE}(\hat{\theta}) &= \mathbb{E}((\hat{\theta}-\theta)^2) = \mathbb{E}((\hat{\theta}-\mathbb{E}[\hat{\theta})+\mathbb{E}(\hat{\theta})-\theta)^2] \\ &= \mathbb{E}((\hat{\theta}-\mathbb{E}[\hat{\theta}))^2] + \mathbb{E}(2(\hat{\theta}-\mathbb{E}[\hat{\theta}))(\mathbb{E}(\hat{\theta})-\theta)] + \mathbb{E}((\mathbb{E}[\hat{\theta})-\theta)^2] \\ &= (\mathbb{E}(\hat{\theta})-\theta)^2 + \mathrm{var}(\hat{\theta}) = \mathrm{bias}(\hat{\theta})^2 + \mathrm{var}(\hat{\theta})\end{aligned}$$

with a Monte Carlo study (Sample size $N = 1000$; from a t_4 distribution with corresponding true $\xi = 1/4$); analytical evaluation of bias and variance is not possible.

Study 1: Estimating ξ

We estimate ξ with a fitted GPD (via MLE; $k \in \{30, 40, \dots, 400\}$) and with the Hill estimator ($\hat{\xi} = 1/\hat{\alpha}^{(\mathsf{H})}_{k,n}$; $k \in \{2, 3, \dots, 200\}$). Note that the t_4 distribution has a well-behaved regularly varying tail.

(a): $\widehat{\text{MSE}}(\hat{\xi})$; (b): $\widehat{\text{bias}}(\hat{\xi})$; (c): $\widehat{\text{var}}(\hat{\xi})$ (solid: GPD; dotted: Hill)

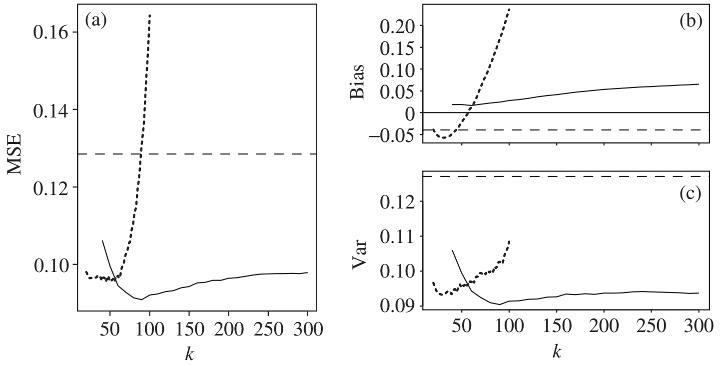


- The Hill estimator outperforms the GPD estimator (optimal k around 20–30) according to the variance for small k (number of order statistics)
- The biases are closer; the Hill (GPD) estimator tends to overestimate (underestimate) ξ.
- For the GPD method, the optimal u is around 100–150 exceedances.

Study 2: Estimating $\mathrm{VaR}_{0.99}$

Estimate $\mathrm{VaR}_{0.99}$ based on a fitted GPD, with the Hill VaR estimator and with the empirical quantile estimator. Here the situation changes.

(a): $\widehat{\mathrm{MSE}}(\widehat{\mathrm{VaR}}_{0.99})$; (b): $\widehat{\mathrm{bias}}(\widehat{\mathrm{VaR}}_{0.99})$; (c): $\widehat{\mathrm{var}}(\widehat{\mathrm{VaR}}_{0.99})$ (solid: GPD; dotted: Hill; dashed: empirical quantile estimator)



- The empirical $\mathrm{VaR}_{0.99}$ estimator has a negative bias.
- The Hill $\mathrm{VaR}_{0.99}$ estimator has a negative bias for small k but a rapidly growing positive bias for larger k.
- The GPD $\mathrm{VaR}_{0.99}$ estimator has a positive bias which grows much more slowly.
- The GPD $\mathrm{VaR}_{0.99}$ estimator attains lowest MSE for a value of k around 100, but the MSE is very robust to the choice of k (because of the slow growth of the bias) $\Rightarrow$ Choice of u less critical
- The Hill $\mathrm{VaR}_{0.99}$ estimator performs well for $20 \leq k \leq 75$ (we only use k values that lead to a quantile estimate beyond the effective threshold $X_{k,n}$) but then deteriorates rapidly.
- Both EVT methods outperform the empirical quantile estimator.

5.2.6 Conditional EVT for financial time series

- The GPD method is an unconditional approach for estimating $\bar{F}$ and associated risk measures. A conditional (time-dependent) risk-measurement approach may be more appropriate.
- We now consider a simple adaptation of the GPD method to obtain conditional risk-measure estimates in a GARCH context.
- Assume $X_{t-n+1}, \dots, X_t$ are negative log-returns generated by a strictly stationary time series process (X_t) of the form

$$X_t = \mu_t + \sigma_t Z_t,$$

where μ_t and σ_t are $\mathcal{F}_{t-1}$-measurable and $Z_t \overset{\text{ind.}}{\sim} F_Z$; e.g. ARMA model with GARCH errors. Furthermore, let $Z \sim F_Z$.

- VaR_α^t and ES_α^t based on $F_{X_{t+1}|\mathcal{F}_t}$ are given by

$$\mathrm{VaR}_\alpha^t(X_{t+1}) = \mu_{t+1} + \sigma_{t+1}\,\mathrm{VaR}_\alpha(Z),$$
$$\mathrm{ES}_\alpha^t(X_{t+1}) = \mu_{t+1} + \sigma_{t+1}\,\mathrm{ES}_\alpha(Z).$$