Mathematics of Growth

Prediction is very difficult, especially if it's about the future.

—*Niels Bohr (1885–1962)*

1 Introduction

A sense of the future is an essential component in the mix of factors that should be influencing the environmental decisions we make today. In some circumstances, only a few factors may need to be considered, and the time horizon may be relatively short. For example, a wastewater treatment plant operator may need to predict the growth rate of bacteria in a digester over a period of hours or days. The designer of the plant, on the other hand, probably needs to estimate the local population growth rate over the next decade or two in order to size the facility. At the other extreme, to make even the crudest estimate of future carbon emissions and their effect on global warming, scientists need to forecast population and economic growth rates, anticipated improvements in energy efficiency, the global fossil fuel resource base and the consumption rate of individual fuels, the rate of deforestation or reforestation that could be anticipated, and so on. These estimates need to be made for periods of time that are often measured in hundreds of years.

As Niels Bohr suggests in the preceding quote, we cannot expect to make accurate predictions of the future, especially when the required time horizon may be extremely long. However, often simple estimates can be made that are robust enough that the insight they provide is most certainly valid. We can say, for example, with considerable certainty that world population growth at today's rates cannot continue for another hundred years. We can also use simple mathematical models to develop very useful "what if" scenarios: *If* population growth continues at a certain rate, and *if* energy demand is proportional to economic activity, and so forth, *then* the following would occur.

The purpose of this chapter is to develop some simple but powerful mathematical tools that can shed considerable light on the future of many environmental problems. We begin with what is probably the most useful and powerful mathematical function encountered in environmental studies: the *exponential function*. Other growth functions encountered often are also explored, including the *logistic* and the *Gaussian* functions. Applications that will be demonstrated using these functions include population growth, resource consumption, pollution accumulation, and radioactive decay.

2 Exponential Growth

Exponential growth occurs in any situation where the increase in some quantity is proportional to the amount currently present. This type of growth is quite common, and the mathematics required to represent it is relatively simple, yet extremely important. Exponential growth is a particular case of a first-order rate process. We will approach this sort of growth first as discrete, year-by-year increases and then in the more usual way as a continuous growth function.

Suppose something grows by a fixed percentage each year. For example, if we imagine our savings at a bank earning 5 percent interest each year, compounded once a year, then the amount of increase in savings over any given year is 5 percent of the amount available at the beginning of that year. If we start now with \$1,000, then at the end of one year we would have \$1,050 ($1{,}000 + 0.05 \times 1{,}000$); at the end of two years, we would have \$1,102.50 ($1{,}050 + 0.05 \times 1{,}050$); and so on. This can be represented mathematically as follows:

$$\begin{aligned} N_0 &= \text{initial amount} \\ N_t &= \text{amount after } t \text{ years} \\ r &= \text{growth rate (fraction per year)} \end{aligned}$$

Then,

$$N_{t+1} = N_t + rN_t = N_t(1 + r)$$

For example,

$$N_1 = N_0(1 + r);\ N_2 = N_1(1 + r) = N_0(1 + r)^2$$

and, in general,

$$N_t = N_0(1 + r)^t \tag{1}$$

EXAMPLE 1 U.S. Electricity Growth (Annual Compounding)

In 2005, the United States produced 4.0×10^{12} kWhr/yr of electricity. The average annual growth rate of U.S. electricity demand in the previous 15 years was about 1.8 percent. (For comparison, it had been about 7 percent per year for many decades before the 1973 oil embargo.) Estimate the electricity consumption in 2050 if the 1.8 percent per year growth rate were to remain constant over those 45 years.

Solution From (1), we have

$$\begin{aligned} N_{45} &= N_0(1 + r)^{45} \\ &= 4.0 \times 10^{12} \times (1 + 0.018)^{45} \\ &= 8.93 \times 10^{12} \text{ kWhr/yr} \end{aligned}$$

Continuous Compounding

For most events of interest in the environment, it is usually assumed that the growth curve is a smooth, continuous function without the annual jumps that (1) is based on. In financial calculations, this is referred to as continuous compounding. With the use of a little bit of calculus, the growth curve becomes the true exponential function that we will want to use most often.

One way to state the condition that leads to exponential growth is that the quantity grows in proportion to itself; that is, the rate of change of the quantity N is proportional to N. The proportionality constant r is called the rate of growth and has units of (time^{-1}).

$$\frac{dN}{dt} = rN \tag{2}$$

The solution is

$$N = N_0 e^{rt} \tag{3}$$

which is plotted in Figure 1.

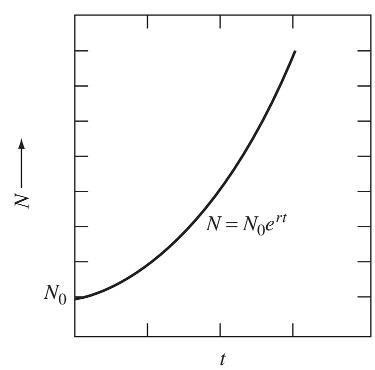

FIGURE 1 The exponential function.

EXAMPLE 2 **U.S. Electricity Growth (Continuous Compounding)**

Suppose we repeat Example 1, but now consider the 1.8 percent growth rate to be continuously compounded. Starting with the 2005 electricity consumption of 4.0×10^{12} kWhr/yr, what would consumption be in 2050 if the growth rate remains constant?

Solution Using (3),

$$\begin{aligned} N &= N_0 e^{rt} \\ &= 4.0 \times 10^{12} \times e^{0.018 \times 45} \\ &= 8.99 \times 10^{12} \text{ kWhr/yr} \end{aligned}$$

Example 1, in which increments were computed once each year, and Example 2, in which growth was continuously compounded, have produced nearly identical results. As either the period of time in question or the growth rate increases, the two approaches begin to diverge. At 12 percent growth, for example, those answers would differ by nearly 50 percent. In general, it is better to express growth rates as if they are continuously compounded so that (3) becomes the appropriate expression.

Doubling Time

Calculations involving exponential growth can sometimes be made without a calculator by taking advantage of the following special characteristic of the exponential function. A quantity that is growing exponentially requires a fixed amount of time to double in size, regardless of the starting point. That is, it takes the same amount of time to grow from N_0 to $2N_0$ as it does to grow from $2N_0$ to $4N_0$, and so on, as shown in Figure 2.

The *doubling time* (t_d) of a quantity that grows at a fixed exponential rate r is easily derived. From (3) the doubling time can be found by setting $N = 2N_0$ at $t = t_d$:

$$2N_0 = N_0 e^{rt_d}$$

Since N_0 appears on both sides of the equation, it can be canceled out, which is another way of saying the length of time required to double the quantity does not

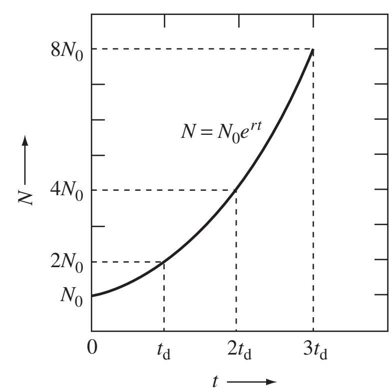

FIGURE 2 Illustrating the concept of doubling time.

depend on how much you start with. So canceling N_0 and taking the natural log of both sides gives

$$\ln 2 = rt_d$$

or

$$t_d = \frac{\ln 2}{r} \cong \frac{0.693}{r} \tag{4}$$

If the growth rate r is expressed as a percentage instead of as a fraction, we get the following important result:

$$t_d \cong \frac{69.3}{r(\%)} \cong \frac{70}{r(\%)} \tag{5}$$

Equation (5) is well worth memorizing; that is,

The length of time required to double a quantity growing at r percent is about equal to 70 divided by r percent.

If your savings earn 7 percent interest, it will take about 10 years to double the amount you have in the account. If the population of a country grows continuously at 2 percent, then it will double in size in 35 years, and so on.

EXAMPLE 3 Historical World Population Growth Rate

It took about 300 years for the world's population to increase from 0.5 billion to 4.0 billion. If we assume exponential growth at a constant rate over that period of time, what would that growth rate be? Do this example first with a calculator and (3), and then with the rule of thumb suggested in (5).

Solution Rearranging (3) and taking the natural log of both sides gives

$$\ln(e^{rt}) = rt = \ln\left(\frac{N}{N_0}\right)$$

$$r = \frac{1}{t}\ln\left(\frac{N}{N_0}\right) \tag{6}$$

$$r = \frac{1}{300}\ln\left(\frac{4.0}{0.5}\right) = 0.00693 = 0.693 \text{ percent}$$

Using the rule of thumb approach given by (5), three doublings would be required to have the population grow from 0.5 billion to 4.0 billion. Three doublings in 300 years means each doubling took 100 years.

$$t_d = 100 \text{ yrs} \cong \frac{70}{r\%}$$

so

$$r(\%) \cong \frac{70}{100} = 0.7 \text{ percent}$$

Our answers would have been exactly the same if the rule of thumb in (5) had not been rounded off.

TABLE 1

Exponential Growth Factors for Various Numbers of Doubling Times

Number of Doublings (n)	Growth Factor (2^n)
1	2
2	4
3	8
4	16
5	32
10	1024
20	$\approx 1.05 \times 10^6$
30	$\approx 1.07 \times 10^9$

Quantities that grow exponentially very quickly also increase in size at a deceptively fast rate. As Table 1 suggests, quantities growing at only a few percent per year increase in size with incredible speed after just a few doubling times. For example, at the 1995 rate of world population growth ($r = 1.5$ percent), the doubling time was about 46 years. If that rate were to continue for just 4 doubling times, or 184 years, world population would increase by a factor of 16, from 5.7 billion to 91 billion. In 20 doubling times, there would be 6 million billion (quadrillion) people, or more than one person for each square foot of surface area of the earth. The absurdity of these figures simply points out the impossibility of the underlying assumption that exponential growth, even at this relatively low-sounding rate, could continue for such periods of time.

Half-Life

When the rate of decrease of a quantity is proportional to the amount present, exponential growth becomes exponential decay. Exponential decay can be described using either a *reaction rate coefficient* (k) or a *half-life* ($t_{1/2}$), and the two are easily related to each other. A reaction rate coefficient for an exponential decay plays the same role that r played for exponential growth.

Exponential decay can be expressed as

$$N = N_0 e^{-kt} \tag{7}$$

where k is a reaction rate coefficient (time^{-1}), N_0 is an initial amount, and N is an amount at time t.

A plot of (7) is shown in Figure 3 along with an indication of half-life. To relate half-life to reaction rate, we set $N = N_0/2$ at $t = t_{1/2}$, giving

$$\frac{N_0}{2} = N_0 e^{-kt_{1/2}}$$

Canceling the N_0 term and taking the natural log of both sides gives us the desired expression

$$t_{1/2} = \frac{\ln 2}{k} \cong \frac{0.693}{k} \tag{8}$$

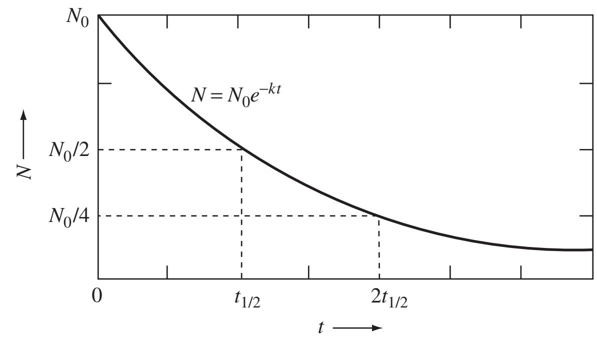

FIGURE 3 Exponential decay and half-life.

EXAMPLE 4 Radon Half-Life

If we start with a 1.0-Ci radon-222 source, what would its activity be after 5 days?

Solution Rearranging (8) to give a reaction rate coefficient results in

$$k = \frac{\ln 2}{t_{1/2}} = \frac{0.693}{3.8 \text{ day}} = 0.182/\text{day}$$

Using (7) gives

$$N = N_0 e^{-kt} = 1 \text{ Ci} \times e^{-0.182/\text{day} \times 5 \text{ days}} = 0.40 \text{ Ci}$$

Disaggregated Growth Rates

Often the quantity to be modeled can be considered to be the product of a number of individual factors. For example, to estimate future carbon emissions, we might *disaggregate* demand into the following three factors:

$$\text{Carbon emissions} = (\text{Population}) \times \left(\frac{\text{Energy}}{\text{Person}}\right) \times \left(\frac{\text{Carbon}}{\text{Energy}}\right) \tag{9}$$

By considering demand this way, we would hope to increase the accuracy of our forecast by estimating separately the rates of change of population, per capita energy consumption, and carbon emissions per unit of energy. Equation (9) is an example of the following simple conceptual model of factors that drive the environmental impacts of human activity:

$$\text{Impacts} = (\text{Population}) \times (\text{Affluence}) \times (\text{Technology}) \tag{10}$$

In (9), affluence is indicated by per capita energy demand, and technology is represented by the carbon emissions per unit of energy.

Expressing a quantity of interest as the product of individual factors, such as has been done in (9), leads to a very simple method of estimating growth. We begin with

$$P = P_1 P_2 \ldots P_n \tag{11}$$

If each factor is itself growing exponentially

$$P_i = p_i e^{r_i t}$$

then

$$\begin{aligned} P &= (p_1 e^{r_1 t})(p_2 e^{r_2 t}) \ldots (p_n e^{r_n t}) \\ &= (p_1 p_2 \ldots p_n) e^{(r_1 + r_2 + \ldots r_n)t} \end{aligned}$$

which can be written as

$$P = P_0 e^{rt}$$

where

$$P_0 = (p_1 p_2 \ldots p_n) \tag{12}$$

and

$$r = r_1 + r_2 + \cdots r_n \tag{13}$$

Equation (13) is a very simple, very useful result. That is,

> *If a quantity can be expressed as a product of factors, each growing exponentially, then the total rate of growth is just the sum of the individual growth rates.*

EXAMPLE 5 Future U.S. Energy Demand

One way to disaggregate energy consumption is with the following product:

$$\text{Energy demand} = (\text{Population}) \times \left(\frac{\text{GDP}}{\text{person}}\right) \times \left(\frac{\text{Energy}}{\text{GDP}}\right)$$

GDP is the gross domestic product. Suppose we project per capita GDP to grow at 2.3 percent and population at 0.6 percent. Assume that through energy-efficiency efforts, we expect energy required per dollar of GDP to *decrease* exponentially at the rate that it did between 1995 and 2005. In 1995, (energy/GDP) was 13.45 kBtu/$, and in 2005, it was 11.35 kBtu/$ (in constant 1987 dollars). Total U.S. energy demand in 2005 was 100.2 quads (quadrillion Btu). If the aforementioned rates of change continue, what would energy demand be in the year 2020?

Solution First, find the exponential rate of decrease of (energy/GDP) over the 10 years between 1995 and 2005. Using (6) gives

$$r_{\text{energy/GDP}} = \frac{1}{t} \ln\left(\frac{N}{N_0}\right) = \frac{1}{10} \ln\left(\frac{11.35}{13.45}\right) = -0.017 = -1.7 \text{ percent}$$

Notice that the equation automatically produces the negative sign. The overall energy growth rate is projected to be the sum of the three rates:

$$r = 0.6 + 2.3 - 1.7 = 1.2 \text{ percent}$$

(For comparison, energy growth rates in the United States before the 1973 oil embargo were typically 3 to 4 percent per year.) Projecting out 15 years to the year 2020, (3) gives

$$\begin{aligned} \text{Energy demand in 2020} &= 100.2 \times 10^{15} \text{ Btu/yr} \times e^{0.012 \times 15} \\ &= 120 \times 10^{15} \text{ Btu/yr} = 120 \text{ quads/yr} \end{aligned}$$

3 Resource Consumption

To maintain human life on Earth, we depend on a steady flow of energy and materials to meet our needs. Some of the energy used is *renewable* (e.g., hydroelectric power, windpower, solar heating systems, and even firewood if the trees are replanted), but most comes from *depletable* fossil fuels. The minerals extracted from the Earth's crust, such as copper, iron, and aluminum, are limited as well. The future of our way of life to a large extent depends on the availability of abundant supplies of inexpensive, depletable energy and materials.

How long will those resources last? The answer, of course, depends on how much there is and how quickly we use it. We will explore two different ways to model the rate of consumption of a resource: One is based on the simple exponential function that we have been working with, and the other is based on a more realistic bell-shaped curve.

Exponential Resource Production Rates

When a mineral is extracted from the Earth, geologists traditionally say the resource is being *produced* (rather than *consumed*). Plotting the rate of production of a resource versus time, as has been illustrated in Figure 4, the area under the curve between any two times will represent the total resource that has been produced during that time interval. That is, if P is the production rate (e.g., barrels of oil per day, tonnes of aluminum per year, etc.), and Q is the resource produced (total barrels, total tonnes) between times t_1 and t_2, we can write

$$Q = \int_{t_1}^{t_2} P\, dt \tag{14}$$

Assuming that the production rate of a resource grows exponentially, we can easily determine the total amount produced during any time interval; or, conversely, if the total amount to be produced is known, we can estimate the length of time that it will take to produce it. The basic assumption of exponential growth is probably not a good one if the time frame is very long, but nonetheless, it will give us some very useful insights. In the next section, we will work with a more reasonable production rate curve.

Figure 5 shows an exponential rate of growth in production. If the time interval of interest begins with $t = 0$, we can write

$$Q = \int_0^t P_0 e^{rt}\, dt = \frac{P_0}{r} e^{rt}\Big|_0^t$$

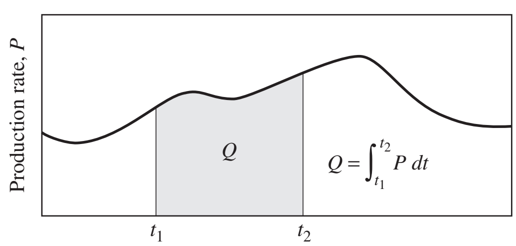

FIGURE 4 The total amount of a resource produced between times t_1 and t_2 is the shaded area under the curve.

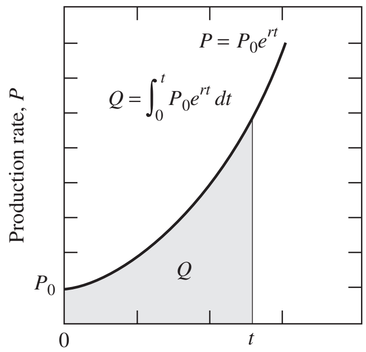

FIGURE 5 Total consumption of a resource experiencing exponential growth.

which has as a solution

$$Q = \frac{P_0}{r}(e^{rt} - 1) \tag{15}$$

where

Q = the total resource produced from time 0 to time t
P_0 = the initial production rate
r = the exponential rate of growth in production

Equation (15) tells us how much of the resource is produced in a given period of time if the production rate grows exponentially. If we want to know how long it will take to produce a given amount of the resource, rearrange (15) to give

$$t = \frac{1}{r}\ln\left(\frac{rQ}{P_0} + 1\right) \tag{16}$$

where t = the length of time required to produce an amount Q.

Applications of (16) often lead to startling results, as should be expected anytime we assume that exponential rates of growth will continue for a prolonged period of time.

EXAMPLE 6 World Coal Production

World coal production in 2005 was estimated to be 6.1 billion (short) tons per year, and the estimated total recoverable reserves of coal were estimated at 1.1 trillion tons. Growth in world coal production in the previous decade averaged 1.9 percent per year. How long would it take to use up those reserves at current production rates, and how long would it take if production continues to grow at 1.9 percent?

Solution At those rates, coal reserves would last

$$\frac{\text{Reserves}}{\text{Production}} = \frac{1.1 \times 10^{12}\ \text{tons}}{6.1 \times 10^{9}\ \text{tons/yr}} = 180\ \text{years}$$

If production grows exponentially, we need to use (16):

$$t = \frac{1}{r} \ln\left(\frac{rQ}{P_0} + 1\right)$$
$$= \frac{1}{0.019} \ln\left(\frac{0.019 \times 1.1 \times 10^{12}}{6.1 \times 10^{9}} + 1\right) = 78 \text{ years}$$

Even though the growth rate of 1.9 percent might seem modest, compared to a constant production rate, it cuts the length of time to deplete those reserves by more than half.

Example 6 makes an important point: By simply dividing the remaining amount of a resource by the current rate of production, we can get a misleading estimate of the remaining lifetime for that resource. If exponential growth is assumed, what would seem to be an abundant resource may actually be consumed surprisingly quickly. Continuing this example for world coal, let us perform a sensitivity analysis on the assumptions used. Table 2 presents the remaining lifetime of coal reserves, assuming an initial 2005 production rate of 6.1 billion tons per year, for various estimates of total available supply and for differing rates of production growth.

Notice how the lifetime becomes less and less sensitive to estimates of the total available resource as exponential growth rates increase. At a constant 3 percent growth rate, for example, a total supply of 500 billion tons of coal would last 41 years; if our supply is four times as large, 2,000 billion tons, the resource only lasts another 38 years. Having four times as much coal does not even double the projected lifetime.

This is a good time to mention the important distinction between the *reserves* of a mineral and the ultimately producible *resources*. As shown in Figure 6, *reserves* are quantities that can reasonably be assumed to exist and are producible with existing technology under present economic conditions. *Resources* include present reserves as well as deposits not yet discovered or deposits that have been identified but are not recoverable under present technological and economic conditions.

As existing reserves are consumed, further exploration, advances in extraction technology, and higher acceptable prices constantly shift mineral resources into the reserves category. Estimating the lifetime of a mineral based on the available reserves

TABLE 2

Years Required to Consume All of the World's Recoverable Coal Reserves, Assuming Various Reserve Estimates and Differing Production Growth Rates[a]

Growth rate (%)	500 Billion tons (yrs)	1,000 Billion tons (yrs)	2,000 Billion tons (yrs)
0	82	164	328
1	60	97	145
2	49	73	101
3	41	59	79
4	36	51	66
5	33	44	57

[a]Initial production rate 6.1 billion tons/year (2005); actual reserves estimated at 1,100 billion tons.

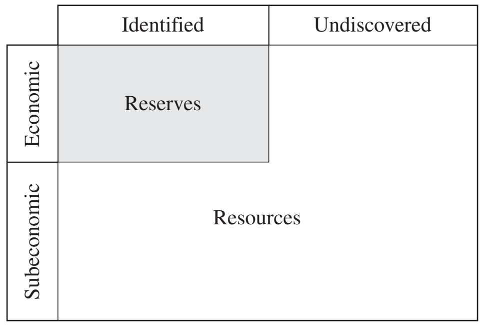

FIGURE 6 Classification of minerals. Reserves are a subcategory of resources.

rather than on the ultimately producible resources can therefore be misleading. World oil reserves in 1970, for example, were estimated at 550 billion barrels (1 barrel equals 42 gallons) while production was 17 billion barrels per year. The reserves-to-production ratio was about 32 years; that is, at 1970 production rates, those reserves would be depleted in 32 years. However, 35 years later (2005), instead of being out of reserves, they had grown to 1,300 billion barrels. Production in 2005 was 30.7 billion barrels per year, giving a reserves-to-production ratio of 42 years.

A Symmetrical Production Curve

We could imagine a complete production cycle to occur in many ways. The model of exponential growth until the resource is totally consumed that we just explored is just one example. It might seem an unlikely one since the day before the resource collapses, the industry is at full production, and the day after, it is totally finished. It is a useful model, however, to dispel any myths about the possibility of long-term exponential growth in the consumption rate of any finite resource.

A more reasonable approach to estimating resource cycles was suggested by M. King Hubbert (1969). Hubbert argued that consumption would more likely follow a course that might begin with exponential growth while the resource is abundant and relatively cheap. As new sources get harder to find, prices go up and substitutions would begin to take some of the market. Eventually, consumption rates would peak and then begin a downward trend as the combination of high prices and resource substitutions would prevail. The decline could be precipitous when the energy needed to extract and process the resource exceeds the energy derived from the resource itself. A graph of resource consumption rate versus time would therefore start at zero, rise, peak, and then decrease back to zero, with the area under the curve equaling the total resource consumed.

Hubbert suggested that a symmetrical production rate cycle that resembles a bell-shaped curve be used. One such curve is very common in probability theory, where it is called the *normal* or *Gaussian* function. Figure 7 shows a graph of the function and identifies some of the key parameters used to define it.

The equation for the complete production rate cycle of a resource corresponding to Figure 7 is

$$P = P_m \exp\left[-\frac{1}{2}\left(\frac{t - t_m}{\sigma}\right)^2\right] \tag{17}$$

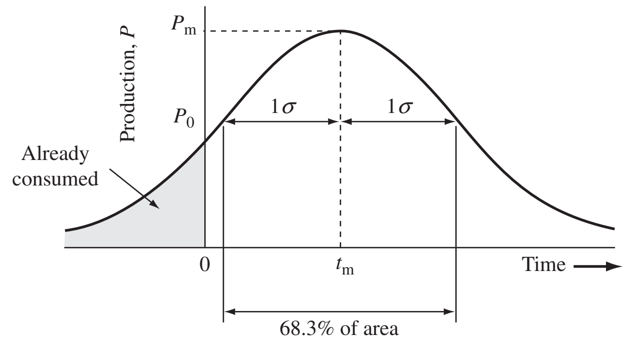

FIGURE 7 Resource production following a Gaussian distribution function.

where

P = the production rate of the resource
P_m = the maximum production rate
t_m = time at which the maximum production rate occurs
σ = standard deviation, a measure of the width of the bell-shaped curve
exp[ ] = the exponential function

The parameter σ is the standard deviation of a normal density function, and in this application, it has units of time. Within $\pm 1\ \sigma$ away from the time of maximum production, 68.3 percent of the production occurs; within $\pm 2\ \sigma$, 95 percent of the production occurs. Notice that with this bell-shaped curve, it is not possible to talk about the resource ever being totally exhausted. It is more appropriate to specify the length of time required for some major fraction of it to be used up. Hubbert uses 80 percent as his criterion, which corresponds to 1.3 σ after the year of maximum production.

We can find the total amount of the resource ever produced, Q_∞, by integrating (17)

$$Q_\infty = \int_{-\infty}^{\infty} P\,dt = \int_{-\infty}^{\infty} P_m \exp\left[-\frac{1}{2}\left(\frac{t - t_m}{\sigma}\right)^2\right]dt$$

which works out to

$$Q_\infty = \sigma P_m \sqrt{2\pi} \tag{18}$$

Equation (18) can be used to find σ if the ultimate amount of the resource ever to be produced Q_∞ and the maximum production rate P_m can be estimated.

It is also interesting to find the length of time required to reach the maximum production rate. If we set $t = 0$ in (17), we find the following expression for the initial production rate, P_0

$$P_0 = P_m \exp\left[-\frac{1}{2}\left(\frac{t_m}{\sigma}\right)^2\right] \tag{19}$$

which leads to the following expression for the time required to reach maximum production:

$$t_m = \sigma\sqrt{2 \ln \frac{P_m}{P_0}} \tag{20}$$

Let us demonstrate the use of these equations by fitting a Gaussian curve to data for U.S. coal resources.

EXAMPLE 7 **U.S. Coal Production**

Suppose ultimate total production of U.S. coal is double the 1995 recoverable reserves, which were estimated at 268×10^9 (short) tons. The U.S. coal production rate in 1995 was 1.0×10^9 tons/year. How long would it take to reach a peak production rate equal to four times the 1995 rate if a Gaussian production curve is followed?

Solution Equation (18) gives us the relationship we need to find an appropriate σ.

$$\sigma = \frac{Q_\infty}{P_m\sqrt{2\pi}} = \frac{2 \times 268 \times 10^9 \text{ ton}}{4 \times 1.0 \times 10^9 \text{ ton/yr}\sqrt{2\pi}} = 53.5 \text{ yr}$$

A standard deviation of 53.5 years says that in a period of 107 years ($2\,\sigma$), about 68 percent of the coal would be consumed.

To find the time required to reach peak production, use (20):

$$t_m = \sigma\sqrt{2\ln\frac{P_m}{P_0}} = 53.5\sqrt{2\ln 4} = 89 \text{ yr}$$

The complete production curve is given by (17)

$$P = P_m \exp\left[-\frac{1}{2}\left(\frac{t - t_m}{\sigma}\right)^2\right] = 4 \times 1.0 \times 10^9 \exp\left[-\frac{1}{2}\left(\frac{t - 89}{53.5}\right)^2\right] \text{tons/yr}$$

which is plotted in Figure 8 along with the production curves that result from the maximum production rate reaching two, four, and eight times the current production rates. The tradeoffs between achieving high production rates versus making the resource last beyond the next century are readily apparent from this figure.

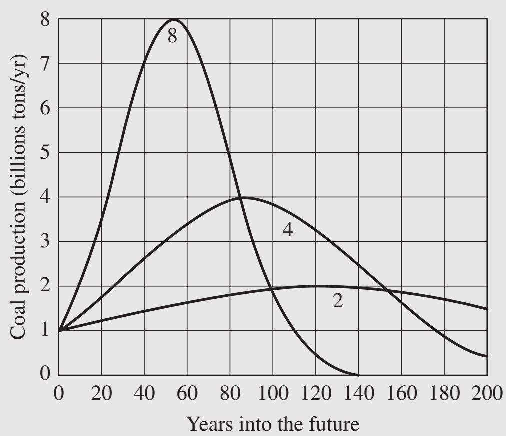

FIGURE 8 Gaussian curves fitted to U.S. production of coal. Ultimate production is set equal to twice the 1995 reserves. The plotted parameter is the ratio of peak production rate to initial (1995) production rate.

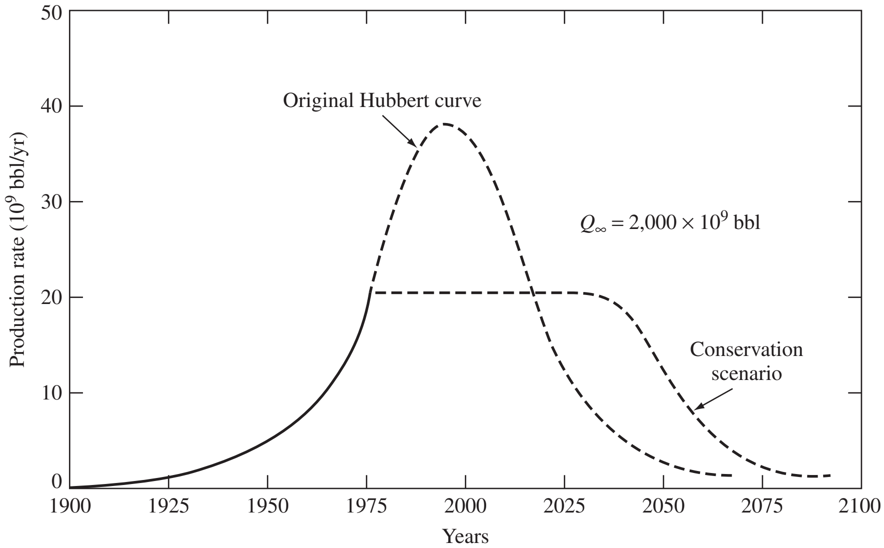

FIGURE 9 Two possible production rate curves for the world's ultimately recoverable crude oil resources, estimated by Hubbert in 1977 to be 2 trillion barrels. The original Hubbert curve indicates that peak production would have been reached before the year 2000, whereas the "conservation" scenario, which holds production to the 1975 level as long as possible, extends the resource for only a few decades more.
(*Source:* Hubbert, 1977.)

Hubbert's analysis of world oil production is presented in Figure 9 for two scenarios. One scenario is based on fitting a Gaussian curve to the historical production record before 1975; the other is based on a conservation scenario in which oil production follows the Gaussian curve until 1975 but then remains constant at the 1975 rate for as long as possible before a sudden decline takes place. With the bell-shaped curve, world oil consumption would have peaked just before the year 2000; with the conservation scenario, supplies would last until about 2040 before the decline would begin. The actual production curve between 1975 and 1995 fell between these two scenarios.

The curves drawn in Figure 9 were based on Hubbert's 1977 estimate of 2,000 billion barrels of oil ultimately produced. For perspective, by 2005, approximately 1,000 billion barrels had been produced (and consumed), and proved reserves stood at 1,300 billion barrels (two-thirds of which were in the Middle East). Amounts already produced plus proved reserves total 2,300 billion barrels, more than Hubbert's estimate for the total amount ever to be produced.

Will more oil ultimately be produced than the current 1.3 trillion barrels of proved reserves? Finding new sources, using enhanced oil recovery techniques to remove some of the oil that is now left behind, and eventually tapping into unconventional hydrocarbon resources such as extra heavy oil, bitumen, and shale oil could stretch petroleum supplies much farther into the future but at significantly higher environmental and economic costs.

4 Population Growth

The simple exponential growth model can serve us well for short time horizons, but obviously, even small growth rates cannot continue for long before environmental constraints limit further increases.

The typical growth curve for bacteria, shown in Figure 10, illustrates some of the complexities that natural biological systems often exhibit. The growth curve is divided into phases designated as lag, exponential, stationary, and death. The *lag phase,* characterized by little or no growth, corresponds to an initial period of time when bacteria are first inoculated into a fresh medium. After the bacteria have adjusted to their new environment, a period of rapid growth, the *exponential phase,* follows. During this time, conditions are optimal, and the population doubles with great regularity. (Notice that the vertical scale in Figure 10 is logarithmic so that exponential growth produces a straight line.) As the bacterial food supply begins to be depleted, or as toxic metabolic products accumulate, the population enters the no-growth, or *stationary phase*. Finally, as the environment becomes more and more hostile, the *death phase* is reached and the population declines.

Logistic Growth

Population projections are often mathematically modeled with a *logistic* or S-shaped (*sigmoidal*) growth curve such as the one shown in Figure 11. Such a curve has great intuitive appeal. It suggests an early exponential growth phase while conditions for growth are optimal, followed by slower and slower growth as the population nears the carrying capacity of its environment. Biologists have successfully used logistic curves to model populations of many organisms, including protozoa, yeast cells, water fleas, fruit flies, pond snails, worker ants, and sheep (Southwick, 1976).

Mathematically, the logistic curve is derived from the following differential equation:

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) \tag{21}$$

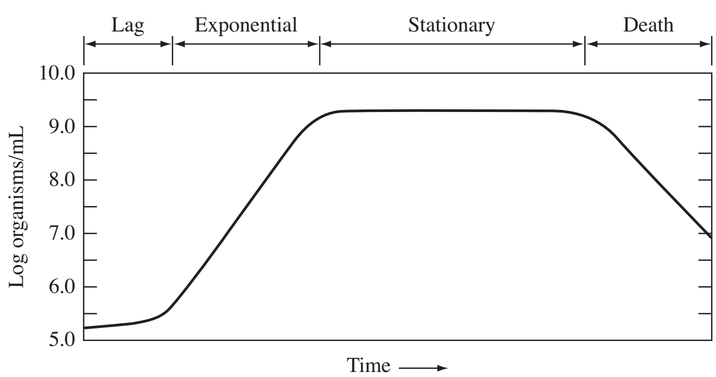

FIGURE 10 Typical growth curve for a bacterial population.
(*Source:* Brock, *Biology of Microorganisms,* 2nd ed., © 1974. Reprinted by permission of Prentice Hall, Inc., Englewood Cliffs, New Jersey.)

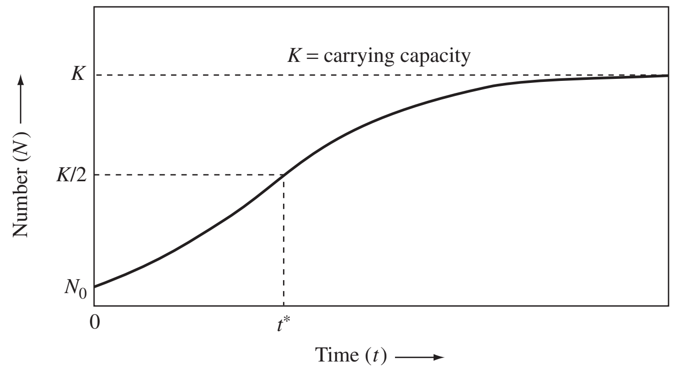

FIGURE 11 The logistic growth curve suggests a smooth transition from exponential growth to a steady-state population.

where N is population size, K is called the *carrying capacity* of the environment, and r is the exponential growth rate constant that would apply if the population size is far below the carrying capacity. Strictly speaking, r is the growth rate constant when $N = 0$; the population is zero. However, when population is much less than carrying capacity, population growth can be modeled with a rate constant r without significant error. As N increases, the rate of growth slows down, and eventually, as N approaches K, the growth stops altogether, and the population stabilizes at a level equal to the carrying capacity. The factor $(1 - N/K)$ is often called the *environmental resistance*. As population grows, the resistance to further population growth continuously increases.

The solution to (21) is

$$N = \frac{K}{1 + e^{-r(t-t^*)}} \tag{22}$$

Note that t^* corresponds to the time at which the population is half of the carrying capacity, $N = K/2$. Substituting $t = 0$ into (22) lets us solve for t^*. Applying the relationship into (22), we can derive the integrated expression of (21) as

$$N(t) = \frac{KN_0}{N_0 + (K - N_0)e^{-rt}} \tag{23}$$

where N_0 is the population at time $t = 0$. In the usual application of (23), the population growth rate is known at $t = 0$. However, the rate constant for this growth rate is not the same as the rate constant r. It cannot necessarily be assumed that the effect of environmental resistance on the growth rate is negligible. To find r from the growth rate measured at $t = 0$, the factor R_0 is introduced. Let $R_0 =$ instantaneous rate constant at $t = 0$ (but not necessarily when $N \ll K$). If we characterize growth at $t = 0$ as exponential, then

$$\left.\frac{dN}{dt}\right|_{t=0} = R_0 N_0 \tag{24}$$

But, from (21)

$$\left.\frac{dN}{dt}\right|_{t=0} = rN_0\left(1 - \frac{N_0}{K}\right) \tag{25}$$

so that equating (24) with (25) yields

$$r = \frac{R_0}{1 - N_0/K} \tag{26}$$

Equation (26) lets us use quantities that are known at $t = 0$, namely the population size N_0 and the population growth constant R_0, to find the appropriate growth constant r for (23). The relationship between the exponential rate constants r and R_0 can be confusing. Remember, there is a different R_0 corresponding to each value of N_0, but r has a single value corresponding to the exponential growth that occurs only when $N_0 = 0$. It may help to think of r as the intrinsic rate constant for the particular logistic growth curve, as shown in (21), and R_0 as a provisional rate constant for the instantaneous growth rate at any time, if the growth is assumed to be exponential (and not logistic).

The following example demonstrates the application of the logistic growth model.

EXAMPLE 8 Logistic Human Population Curve

Suppose the human population follows a logistic curve until it stabilizes at 15.0 billion. In 2006, the world's population was 6.6 billion, and its growth rate was 1.2 percent. When would the population reach 7.5 billion—one half of its assumed carrying capacity?

Solution Find r using (26)

$$r = \frac{R_0}{1 - N_0/K} = \frac{0.012}{(1 - 6.6 \times 10^9/15 \times 10^9)} = 0.0214$$

To solve for the time required to reach a population of 7.5 billion, rearrange (23) to solve for t:

$$t = \frac{1}{r} \ln \left(\frac{N(K - N_0)}{N_0(K - N)} \right) \tag{27}$$

Now substitute $N_0 = 6.6$ billion and $N(t) = 7.5$ billion into (27)

$$t = \frac{1}{0.0214} \ln \left(\frac{7.5 \times 10^9(15 \times 10^9 - 6.6 \times 10^9)}{6.6 \times 10^9(15 \times 10^9 - 7.5 \times 10^9)} \right) = 11 \text{ yrs}$$

The Earth would reach one-half its carrying capacity in 2017.

Maximum Sustainable Yield

The logistic curve can also be used to introduce another useful concept in population biology called the *maximum sustainable yield* of an ecosystem. The maximum sustainable yield is the maximum rate that individuals can be harvested (removed) without reducing the population size. Imagine, for example, harvesting fish from a pond. If the pond is at its carrying capacity, there will be no population growth. Any fish removed will reduce the population. Therefore, the maximum sustainable yield will correspond to some population less than the carrying capacity. In fact, because the yield is the same as dN/dt, the maximum yield will correspond to the point on the logistic curve where the slope is a maximum (its inflection point). Setting the derivative of the slope equal to zero, we can find that point. The slope of the logistic curve is given by (21):

$$\text{Yield} = \text{Slope} = \frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) \tag{21}$$

Setting the derivative of the slope equal to zero gives

$$\frac{d}{dt}\left(\frac{dN}{dt}\right) = r\,\frac{dN}{dt} - \frac{r}{K}\left(2N\,\frac{dN}{dt}\right) = 0$$

Letting N^* be the population at the maximum yield point gives

$$1 - \frac{2N^*}{K} = 0$$

so that

$$N^* = \frac{K}{2} \qquad \text{(for maximum sustainable yield)} \tag{28}$$

That is, if population growth is logistic, then the maximum sustainable yield will be obtained when the population is half the carrying capacity. The yield at that point can be determined by substituting (28) into (21):

$$\text{Maximum yield} = \left(\frac{dN}{dt}\right)_{max} = r\,\frac{K}{2}\left(1 - \frac{K/2}{K}\right) = \frac{rK}{4} \tag{29}$$

Using (26) lets us express the maximum yield in terms of the current growth rate R_0 and current size N_0:

$$\left(\frac{dN}{dt}\right)_{max} = \left(\frac{R_0}{1 - N_0/K}\right) \times \frac{K}{4} = \frac{R_0 K^2}{4(K - N_0)} \tag{30}$$

EXAMPLE 9 Harvesting Fish

Observations of a pond newly stocked with 100 fish shows their population doubles in the first year, but after many years, their population stabilizes at 4,000 fish. Assuming a logistic growth curve, what would be the maximum sustainable yield from this pond?

Solution In the first year, with no harvesting, the population doubles. One approach is to estimate the initial growth rate constant, R_0, from the doubling time equation (5):

$$R_0 = \frac{\ln 2}{t_d} = \frac{\ln 2}{1\text{ yr}} = 0.693\text{ yr}^{-1}$$

Since R_0 is an instantaneous rate, it conceptually only applies for a single value of N, not for all values of N (100–200 fish) occurring during the first year. A good estimate of the appropriate population size corresponding to the value of R_0 calculated would be the midpoint population for the year, 150 fish. However, as is often done, N_0 could also be taken as 100 fish (the initial population for the period in which the growth rate was measured) without significant error. So from (30) the maximum yield would be

$$\left(\frac{dN}{dt}\right)_{\max} = \frac{R_0K^2}{4(K - N_0)} = \frac{0.693\ \text{yr}^{-1} \times (4{,}000\ \text{fish})^2}{4(4{,}000 - 150\ \text{fish})} = 720\ \text{fish/yr}$$

If we had used 100 fish for N_0, the maximum sustainable yield would have been calculated as 710 fish/yr, showing there will only be a small error in the final answer as long as the initial population is small relative to the carrying capacity. This suggests a more intuitive way we could have gone about this because the initial population, N_0, is much lower than the carrying capacity, K. In this case, the instantaneous growth rate constant, R_0, is approximately equal to the growth rate constant, r, when there is no environmental resistance. That is, $r \approx R_0 = 0.693/\text{yr}$. Using this approximation along with the fact that population size when yield is highest is $K/2 = 2{,}000$ fish, from (21) we have

$$\text{Maximum yield} = rN\left(1 - \frac{N}{K}\right) \approx 0.693 \times 2{,}000\left(1 - \frac{2{,}000}{4{,}000}\right)$$
$$= 693\ \text{fish/yr}$$

5 Human Population Growth

The logistic growth equations just demonstrated are frequently used with some degree of accuracy and predictive capability for density-dependent populations, but they are not often used to predict human population growth. More detailed information on fertility and mortality rates and population age composition are usually available for humans, increasing our ability to make population projections. Use of such data enables us to ask such questions as: What if every couple had just two children? What if such replacement level fertility were to take 40 years to achieve? What would be the effect of reduced fertility rates on the ratios of retired people to workers in the population? The study of human population dynamics is known as *demography,* and what follows is but a brief glimpse at some of the key definitions and most useful mathematics involved.

The simplest measure of fertility is the *crude birth rate,* which is just the number of live births per 1,000 population in a given year. It is called crude because it does not take into account the fraction of the population that is physically capable of giving birth in any given year. The crude birth rate is typically in the range of 30 to 40 per 1,000 per year for the poorest, least developed countries of the world, whereas it is typically around 10 for the more developed ones. For the world as a whole, in 2006, the crude birth rate was 21 per 1,000. That means the 6.56 billion people alive then would have had about $6.56 \times 10^9 \times (21/1{,}000) = 138$ million live births in that year. (These statistics, and most others in this section, are taken from data supplied by the Population Reference Bureau in a most useful annual publication called the *2006 World Population Data Sheet*.)

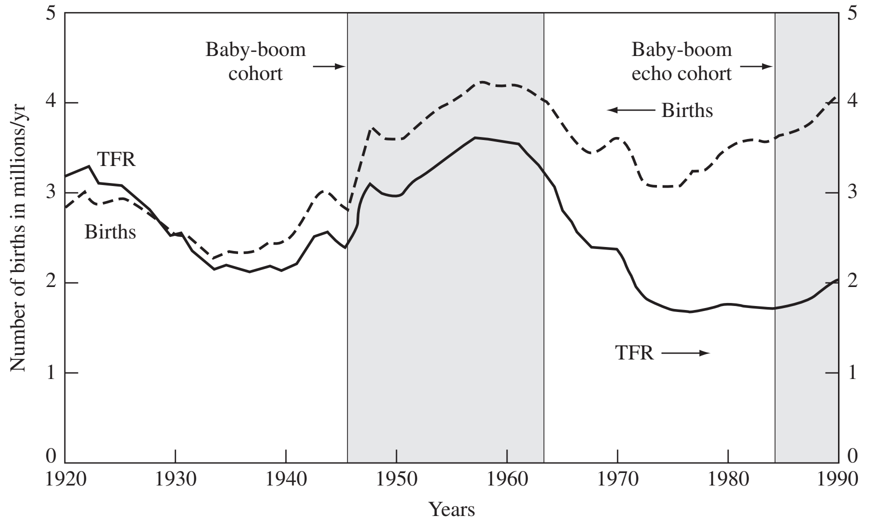

FIGURE 12 Annual births and total fertility rate in the United States.
(*Source:* Bouvier and De Vita, 1991.)

The *total fertility rate* (TFR) is the average number of children that would be born alive to a woman, assuming current age-specific birth rates remain constant through the woman's reproductive years. It is a good measure of the average number of children each woman is likely to have during her lifetime. In the developed countries of the world in 2006, the total fertility rate was 1.6 children per female, whereas in the less developed countries, the average was 2.9. In some of the poorest countries, the TFR is more than 6 children per woman. Figure 12 shows the history of the TFR for the United States. During the depression years of the 1930s, TFR was barely above 2; during the baby-boom era, from 1946 to 1964, it was well over 3 children per woman; in the mid-1970s, it dropped back to a level close to 2.

The number of children that a woman must have, on the average, to replace herself with one daughter in the next generation is called *replacement level fertility*. Replacement level fertility accounts for differences in the ratio of male to female births as well as child mortality rates. For example, in the United States, replacement level fertility is 2.11 children per female. At that fertility level, 100 women would bear 211 children on the average. Statistically, 108 of the children would be boys and 103 would be girls. About 3 percent of the girls would be expected to die before bearing children, leaving a net of 100 women in the next generation. Notice that differing mortality rates in other countries cause the replacement level of fertility to vary somewhat compared to the 2.11 just described. In many developing countries, for example, higher infant mortality rates raise the level of fertility needed for replacement to approximately 2.7 children.

As will be demonstrated later, having achieved replacement level fertility does not necessarily mean a population has stopped growing. When a large fraction of the population is young people, as they pass through their reproductive years, the total births will continue to grow despite replacement fertility. Figure 12 illustrates this concept for the United States. Even though replacement level fertility had been achieved by the mid-1970s, the total number of births rose as the baby boomers had their children. The cohorts born between 1981 and 1995 are often referred to as the baby boom echo. The continuation of population growth, despite achievement of replacement level fertility, is a phenomenon known as *population momentum,* and it will be described more carefully a bit later in the chapter.

The simplest measure of mortality is the *crude death rate,* which is the number of deaths per 1,000 population per year. Again, caution should be exercised in interpreting *crude* death rates because the age composition of the population is not accounted for. The United States, for example, has a higher crude death rate than Guatemala, but that in no way indicates equivalent risks of mortality. In Guatemala, only 4 percent of the population is over 65 years of age (and hence at greater risk of dying), whereas in the United States, a much large fraction of the population, 12 percent, is over 65.

An especially important measure of mortality is the *infant mortality rate,* which is the number of deaths to infants (under 1 year of age) per 1,000 live births in a given year. The infant mortality rate is one of the best indicators of poverty in a country. In some of the poorest countries of the world, infant mortality rates are over 140, which means 1 child in 7 will not live to see his or her first birthday. In the more developed countries, infant mortality rates are around 6 per 1,000 per year.

EXAMPLE 10 Birth and Death Statistics

In 2006, over 80 percent of the world's population, some 5.3 billion people, lived in the less developed countries of the world. In those countries, the average crude birth rate was 23, crude death rate was 8.5, and the infant mortality rate was 53. What fraction of the total deaths is due to infant mortality? If the less developed countries were able to care for their infants as well as they are cared for in most developed countries, resulting in an infant mortality rate of 6, how many infant deaths would be avoided each year?

Solution To find the number of infant deaths each year, we must first find the number of live births and then multiply that by the fraction of those births that die within the first year:

Now:

$$\begin{aligned}
\text{Infant Deaths} &= \text{Population} \times \text{Crude Birth Rate} \times \text{Infant Mortality Rate} \\
&= 5.3 \times 10^9 \times (23/1{,}000) \times (53/1{,}000) = 6.5 \times 10^6 \text{ per year} \\
\text{Total Deaths} &= \text{Population} \times \text{Crude Death Rate} \\
&= 5.3 \times 10^9 \times (8/1{,}000) = 42.4 \times 10^6 \text{ per year} \\
\text{Fraction Infants} &= 6.5/42.4 = 0.15 = 15 \text{ percent}
\end{aligned}$$

With lowered infant mortality rate:

$$\begin{aligned}
\text{Infant Deaths} &= 5.3 \times 10^9 \times (23/1{,}000) \times (6/1{,}000) = 0.73 \times 10^6 \text{ per year} \\
\text{Avoided Deaths} &= (6.5 - 0.73) \times 10^6 = 5.8 \text{ million per year}
\end{aligned}$$

It is often argued that reductions in the infant mortality rate would eventually result in reduced birth rates as people began to gain confidence that their offspring would be more likely to survive. Hence, the reduction of infant mortality rates through such measures as better nutrition, cleaner water, and better medical care is thought to be a key to population stabilization.

The difference between crude birth rate b and crude death rate d is called the *rate of natural increase* of the population. While it can be expressed as a rate per 1,000 of population, it is more common to express it either as a decimal fraction or as a percentage rate. As a simple, but important equation,

$$r = b - d \tag{31}$$

where r is the rate of natural increase. If r is treated as a constant, then the exponential relationships developed earlier can be used. For example, in 2006, the crude birth rate for the world was 21 per 1,000, and the crude death rate was 9 per 1,000, so the rate of natural increase was $(21 - 9)/1{,}000 = 12/1{,}000 = 0.012 =$ 1.2 percent. If this rate continues, then the world would double in population in about $70/1.2 = 58$ years.

Although it is reasonable to talk in terms of a rate of natural increase for the world, it is often necessary to include effects of migration when calculating growth rates for an individual country. Letting m be the *net migration rate,* which is the difference between immigration (in-migration) and emigration (out-migration), we can rewrite the preceding relationship as follows:

$$r = b - d + m \tag{32}$$

At the beginning of the industrial revolution, it is thought that crude birth rates were around 40 per 1,000, and death rates were around 35 per 1,000, yielding a rate of natural increase of about 0.5 percent for the world. As parts of the world began to achieve the benefits of a better and more assured food supply, improved sanitation, and modern medicines, death rates began to drop. As economic and social development has proceeded in the currently more developed countries, crude birth rates have fallen to about 12 per 1,000 and crude death rates to about 10 per 1,000. These countries have undergone what is referred to as the *demographic transition,* a transition from high birth and death rates to low birth and death rates, as shown in Figure 13.

The less developed countries of the world have also experienced a sizable drop in death rates, especially during the last half century. Imported medicines and better control of disease vectors have contributed to a rather sudden drop in death rates to their 2006 level of 8 per 1,000. In general, birth rates have not fallen nearly as fast, however. They are still up around 23 per 1,000, which yields a growth rate of 1.5 percent for over 80 percent of the world's population. The rapid (and historically speaking, extraordinary) population growth that the world is experiencing now is almost entirely due to the drop in death rates in developing countries without a significant corresponding drop in birth rates. Many argue that decreases in fertility depend on economic growth, and many countries today face the danger that economic growth may not be able to exceed population growth. Such countries may be

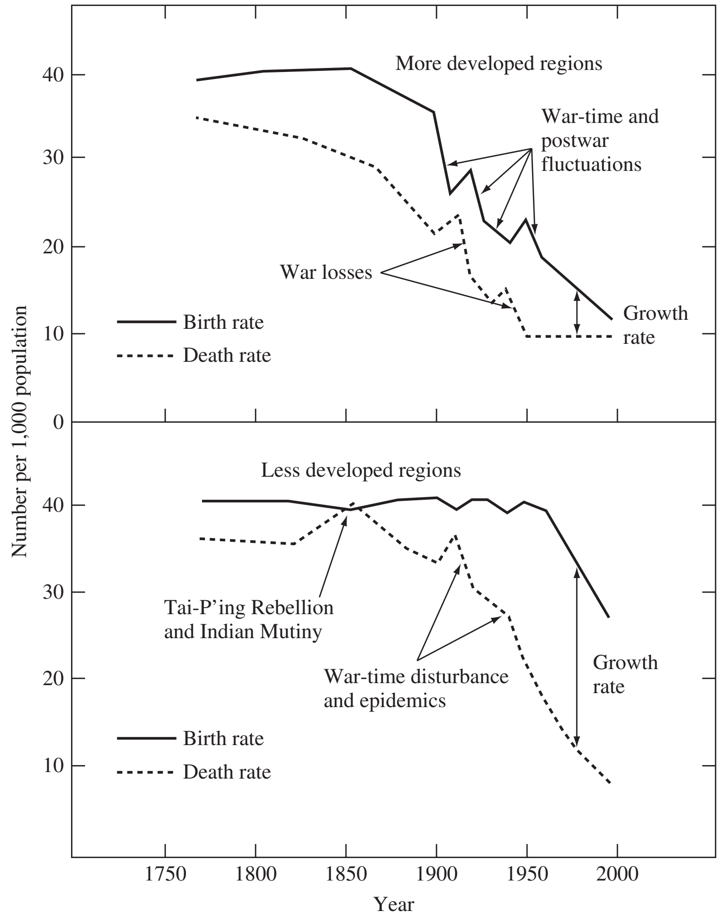

FIGURE 13 The demographic transition to low birth and death rates took over 100 years in the more developed nations. The less developed regions are in the middle of the process.
(*Source:* United Nations, 1971.)

temporarily stuck in the middle of the demographic transition and are facing the risk that population equilibrium may ultimately occur along the least desirable path, through rising death rates.

One developing country that stands out from the others is China. After experiencing a disastrous famine from 1958 to 1961, China instituted aggressive family planning programs coupled with improved health care and living standards that cut birth rates in half in less than 2 decades. Figure 14 shows China's rapid passage through the demographic transition, which was unprecedented in speed (the developed countries took more than 100 years to make the transition).

Some of these data on fertility and mortality are presented in Table 3 for the world, for more developed countries, for China, and for the less developed countries excluding China. China has been separated out because, in terms of population control, it is so different from the rest of the less developed countries and because its size so greatly influences the data.

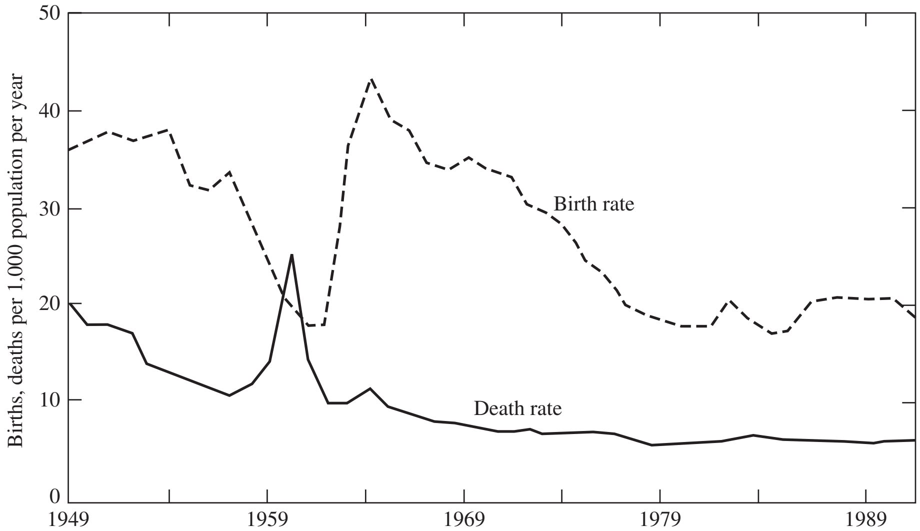

FIGURE 14 Birth and death rates in China, 1949 to 1991.
(*Source:* Tien et al., 1992.)

TABLE 3

Some Important Population Statistics (2006)

	World	More Developed Countries	Less Developed Countries (Excluding China)	China	USA
Population (millions)	6,555	1,216	4,028	1,311	299
% of world population	100	19	61	20	4.6
Crude birth rate, *b*	21	11	27	12	14
Crude death rate, *d*	9	10	9	7	8
Natural increase, *r* %	1.2	0.1	1.8	0.5	0.6
% Population under age 15	29	17	35	20	20
Total fertility rate	2.7	1.6	3.4	1.6	2.0
Infant mortality rate	52	6	61	27	6.7
% of total added 2006 to 2050	41	4	63	10	40
Per capita GNI[a] (US$)	9,190	27,790	4,410	6,600	41,950
% urban	48	77	42	37	79
Est. population 2025 (millions)	7,940	1,255	5,209	1,476	349
Added pop. 2006 to 2025 (millions)	1,385	39	1,181	165	50

[a]Gross National Income per capita 2005.
Source: Population Reference Bureau, 2006.

Age Structure

A great deal of insight and predictive power can be obtained from a table or diagram representing the age composition of a population. A graphical presentation of the data, indicating numbers of people (or percentages of the population) in each

FIGURE 15 A rapidly growing, expansive population (Morocco), compared with a population that is nearing stationary condition (Sweden).
(*Source:* Haupt and Kane, 1985.)

age category, is called an *age structure* or a *population pyramid*. An age structure diagram is a snapshot of a country's population trends that tells a lot about the recent past as well as the near future. Developing countries with rapidly expanding populations, for example, have pyramids that are triangular in shape, with each cohort larger than the cohort born before it. An example pyramid is shown in Figure 15 for Morocco. It is not uncommon in such countries to have nearly half of the population younger than 15 years old. The age structure shows that even if replacement fertility is achieved in the near future, there are so many young women already born who will be having their children that population size will continue to grow for many decades.

The second pyramid in Figure 15 is an example of a population that is not growing very rapidly, if at all. Notice that the sides of the structure are much more vertical. There is also a pinching down of the age structure in the younger years, suggesting a fertility rate below replacement level, so that if these trends continue, the population will eventually begin to decrease.

Two important terms are used by demographers to describe the shape of an age structure. A *stable population* has had constant age-specific birth and death rates for such a long time that the percentage of the population in any age category does not change. That is, the shape of the age structure is unchanging. A population that is stable does not have to be fixed in size; it can be growing or shrinking at a constant rate. When a population is both stable and unchanging in size it is called a *stationary population*. Thus all stationary populations are stable, but not all stable populations are stationary.

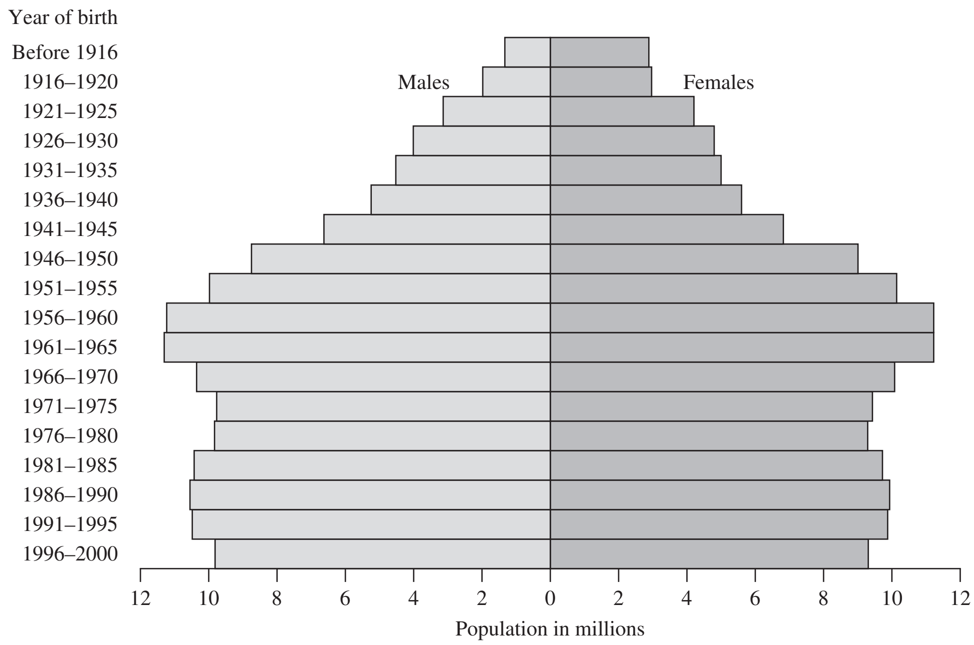

FIGURE 16 The age structure for the United States in 2000.
(*Source:* U.S. Census Bureau.)

The population pyramid for the United States in 2000 is shown in Figure 16. The nearly vertical sides on the lower part of the pyramid suggest that this is a population that is well underway toward a stationary condition. The dominant feature, however, is the bulge in the middle corresponding to the baby-boom cohorts born between 1946 and 1964. Thinking of the age structure as a dynamic phenomenon, the bulge moving upward year after year suggests an image of a python that has swallowed a pig, which slowly works its way through the snake. As the boomers pass through the age structure, resources are stretched to the limit. When they entered school, there were not enough classrooms and teachers, so new schools were built and teacher training programs expanded. When they left, schools closed, and there was an oversupply of teachers. Universities found themselves competing for smaller and smaller numbers of high school graduates. During their late teens and early twenties, arrest rates increased since those are the ages that are, statistically speaking, responsible for the majority of crimes. As they entered the workforce, competition was fierce and unemployment rates were persistently high, housing was scarce, and the price of real estate jumped. The baby-boom generation is now nearing retirement, and there is a boom in construction of retirement communities coupled with growing concern about the capability of the social security system to accommodate so many retirees.

By contrast, children born in the 1930s, just ahead of the baby-boom generation, are sometimes referred to as the "good times cohorts." Since there are so few of them, as they sought their educations, entered the job market, bought homes, and are now retiring, the competition for those resources has been relatively modest. Children born in the 1970s also benefit by being part of a narrow portion of the age

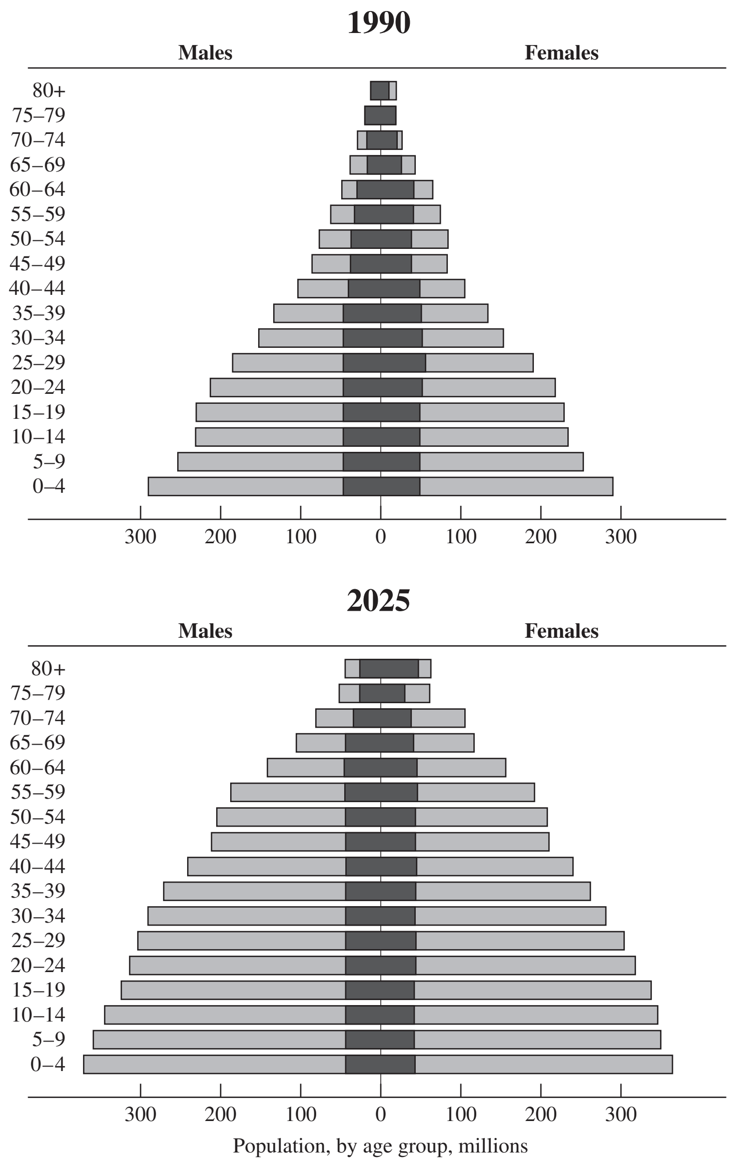

FIGURE 17 Age structures for the world in 1990 and projections for 2025, showing the developed countries in darker tones than the developing countries.
(*Source: The Economist,* January 27, 1996.)

structure. However, their future is less rosy since they will be part of the working population that will face the responsibility of supporting the more elderly baby-boom generation. At the bottom of the age structure are the baby-boom echo cohorts born in the 1980s and 1990s.

Figure 17 shows age structures for the world in 1990 and projections for 2025. The developed countries essentially have stationary populations in both years, but the developing countries are still growing rapidly.

Human Population Projections

If the age structure for a population is combined with data on age-specific birth and death rates, it is possible to make realistic projections of future population size and composition. The techniques that we will explore now are especially useful for predicting the effects and implications of various changes in fertility patterns that might be imagined or advocated. For example, suppose we want to determine the effects of replacement-level fertility. If replacement fertility were achieved today, how long would it take to achieve a stationary population? What would the age composition look like in the interim? What fraction of the population would be young and looking for work, and what fraction would be retired and expecting support? Or suppose a policy is established that sets a target for maximum size of the population (as is the case for China). How many children would each couple need to have to achieve that goal, and how would the number change with time?

Not long ago, the calculations required to develop these scenarios were tedious and best left to professional demographers. Now, however, with the widespread use of simple spreadsheet programs on personal computers, it is possible to work them out with relative ease.

The starting point in a population projection is the current age structure combined with mortality data obtained from *life tables*. Life tables are the mainstay of insurance companies for predicting the average number of years of life remaining as a function of the age of their clients. A life table is developed by applying a real population's age-specific death rates (the fraction of the people in a given age category who will die each year) to *hypothetical* stable and stationary populations having 100,000 live births per year, evenly distributed through the year, with no migration. As the 100,000 people added each year get older, their ranks are thinned in accordance with the age-specific death rates. It is then possible to calculate the numbers of people who would be alive within each age category in the following year.

Table 4 presents a life table for the United States that has been simplified by broadening the age intervals to 5-year increments rather than the 1-year categories used by insurance companies and demographers. These data, remember, are for a hypothetical population with 100,000 live births each year (and 100,000 deaths each year as well since this is a stationary population). The first column shows the age interval (e.g., 10–14 means people who have had their tenth birthday but not their fifteenth). The second column is the number of people who would be alive at any given time in the corresponding age interval and is designated L_x, where x is the age at the beginning of the interval. The third column, L_{x+5}/L_x, is the ratio of number of people in the next interval to the number in the current interval; it is the probability that individuals will live 5 more years (except in the case of those 80 and older, where the catch-all category of 85+ years old modifies the interpretation).

If we assume that the age-specific death rates that were used to produce Table 4 remain constant, we can use the table to make future population projections at five-year intervals for all but the 0–4 age category. The 0–4 age category will depend on fertility data, to be discussed later.

If we let $P_x(0)$ be the number of people in age category x to $x + 5$ at time $t = 0$, and $P_{x+5}(5)$ be the number in the next age category five years later, then

$$P_{x+5}(5) = P_x(0)\frac{L_{x+5}}{L_x} \tag{33}$$

TABLE 4

Age Distribution for a Hypothetical, Stationary Population with 100,000 Live Births Per Year in the United States[a]

Age Interval x to $x + 5$	Number in Interval L_x	$\frac{L_{x+5}}{L_x}$
0–4	494,285	0.9979
5–9	493,247	0.9989
10–14	492,704	0.9973
15–19	491,374	0.9951
20–24	488,966	0.9944
25–29	486,228	0.9940
30–34	483,311	0.9927
35–39	479,783	0.9898
40–44	474,889	0.9841
45–49	467,338	0.9745
50–54	455,421	0.9597
55–59	437,068	0.9381
60–64	410,013	0.9082
65–69	372,374	0.8658
70–74	322,401	0.8050
75–79	259,533	0.7163
80–84	185,904	0.9660
85 and over	179,583	0.0000

[a]Age-specific death rates are U.S. 1984 values.
Source: Abstracted from data in U.S. Dept. of Health and Human Services (1987).

That is, five years from now, the number of people in the next five-year age interval will be equal to the number in the interval now times the probability of surviving for the next five years (L_{x+5}/L_x).

For example, in the United States in 1985, there were 18.0 million people in the age group 0–4 years; that is, $P_0(1985) = 18.0$ million. We would expect that in 1990, the number of people alive in the 5–9 age category would be

$$\begin{aligned} P_5(1990) &= P_0(1985)(L_5/L_0) \\ &= 18.0 \times 10^6 \times 0.9979 = 17.98 \text{ million} \end{aligned}$$

Let's take what we have and apply it to the age composition of the United States in 1985 to predict as much of the structure as we can for 1990. This involves application of (33) to all of the categories, giving us a complete 1990 age distribution except for the 0–4 year olds. The result is presented in Table 5.

To find the number of children ages 0–4 to enter into the age structure, we need to know something about fertility patterns. Demographers use *age-specific fertility rates,* which are the number of live births per woman in each age category, to make these estimates. To do this calculation carefully, we would need to know the number of people in each age category who are women as well as statistical data on child mortality during the first five years of life. Since our purpose here is to develop a tool for asking questions of the "what if" sort, rather than making actual population projections, we can use the following simple approach to estimate the number

TABLE 5

1990 U.S. Population Projection Based on the 1985 Age Structure (Ignoring Immigration)

Age Interval x to $x + 5$	$\frac{L_{x+5}}{L_x}$	P_x (Thousands) 1985	1990
0–4	0.9979	18,020	$P_0(1990)$
5–9	0.9989	17,000	17,982
10–14	0.9973	16,068	16,981
15–19	0.9951	18,245	16,025
20–24	0.9944	20,491	18,156
25–29	0.9940	21,896	20,376
30–34	0.9927	20,178	21,765
35–39	0.9898	18,756	20,031
40–44	0.9841	14,362	18,564
45–49	0.9745	11,912	14,134
50–54	0.9597	10,748	11,609
55–59	0.9381	11,132	10,314
60–64	0.9082	10,948	10,443
65–69	0.8658	9,420	9,943
70–74	0.8050	7,616	8,156
75–79	0.7163	5,410	6,131
80–84	0.9660	3,312	3,875
85 and over	0.0000	2,113	3,199
Total:		237,627	$227,684 + P_0(1990)$

Source: 1985 data from Vu (1985).

of children ages 0–4 to put into our age composition table:

$$P_0(n + 5) = b_{15}P_{15}(n) + b_{20}P_{20}(n) + \cdots b_{45}P_{45}(n) \tag{34}$$

where b_x is the number of the surviving children born per person in age category x to $x + 5$. Equation (34) has been written with the assumption that no children are born to individuals under 15 or over 49 years of age, although it could obviously have been written with more terms to include them. Notice that we can interpret the sum of b_x's (Σb_x) to be the number of children each person would be likely to have. The total fertility rate, which is the expected number of children per woman, is therefore close to $2\Sigma b_x$.

For example, these fertility factors and the numbers of people in their reproductive years for the United States in 1985 are shown in Table 6. The total fertility rate can be estimated as $2\Sigma b_x$, which is $2 \times 1.004 = 2.01$ children per woman.

With this 20,385,000 added to the 227,684,000 found in Table 5, the total population in the United States in 1990 would have been 248,069,000. Notice that, in spite of the fact that fertility was essentially at the replacement level in 1985, the population would still be growing by about 2 million people per year. This is an example of *population momentum,* a term used to describe the fact that a youthful

TABLE 6

Calculating Births in Five-Year Increments for the United States, 1985

Age Category	P_x(1985) (Thousands)	b_x (Births Per Person During Five-Year Period)	Births (Thousands in Five Years)
15–19	18,245	0.146	2,664
20–24	20,491	0.289	5,922
25–29	21,896	0.291	6,372
30–34	20,178	0.190	3,834
35–39	18,756	0.075	1,407
40–45	14,362	0.013	187
Totals		1.004	20,385

age structure causes a population to continue growing for several decades after achieving replacement fertility. To finish off the aforementioned estimate, we would have to add in net migration, which is currently 897,000 legal immigrants per year plus probably hundreds of thousands of immigrants entering the country illegally.

Population Momentum

Let us begin with an extremely simplified example, which does not model reality very well, but which does keep the arithmetic manageable. We will then see the results of more carefully done population scenarios.

Suppose we have an age structure with only three categories: 0–24 years, 25–49 years, and 50–74 years, subject to the following fertility and mortality conditions:

1. All births occur to women as they leave the 0–24 age category.
2. All deaths occur at age 75.
3. The total fertility rate (TFR) is 4.0 for 25 years; then it drops instantaneously to 2.0.

Suppose a population of 5.0 billion people has an age structure at time $t = 0$, as shown in Figure 18*a*. Equal numbers of men and women are assumed so that the number of children per person is half the TFR. During the first 25 years, the 2.5 billion people who start in the 0–24 age category all pass their twenty-fifth birthday, and all of their children are born. Because TFR is 4.0, these 2.5 billion individuals bear 5.0 billion children (2 per person). Those 2.5 billion people now are in the 25–49 age category. Similarly, the 1.5 billion people who were 25–49 years old are now ages 50–74. Finally, the 1.0 billion 50–74 year olds have all passed their seventy-fifth birthdays and, by the rules, they are all dead. The total population has grown to 9.0 billion, and the age structure is as shown in Figure 18*b*.

After the first 25 years have passed, TFR drops to the replacement level of 2.0 (1 child per person). During the ensuing 25 years, the 5.0 billion 0–24 year olds have 5.0 billion children and, following the logic given previously, the age structure at $t = 50$ years is as shown in Figure 18*c*. With the replacement level continuing, at $t = 75$ years, the population stabilizes at 15 billion. A plot of population versus time is shown in Figure 19. Notice that the population stabilizes 50 years after

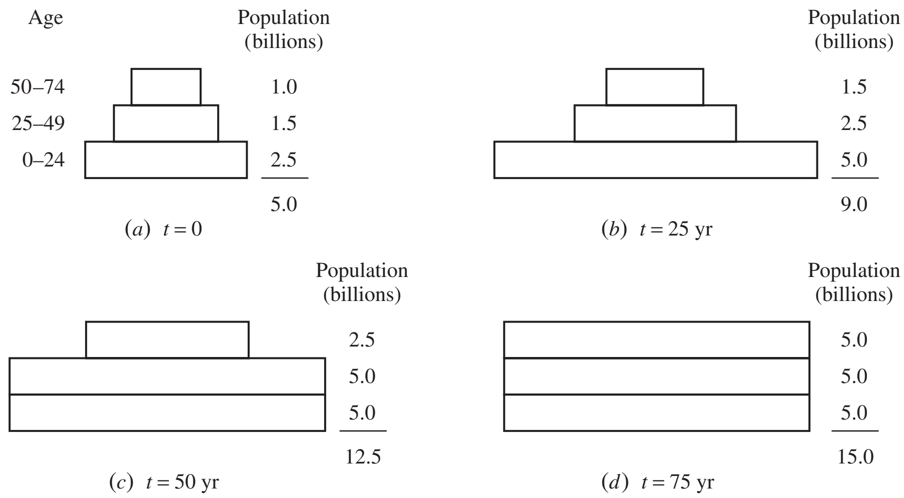

FIGURE 18 Age structure diagrams illustrating population momentum. For the first 25 years, TFR = 4; thereafter, it is at the replacement level of 2.0, yet population continues to grow for another 50 years.

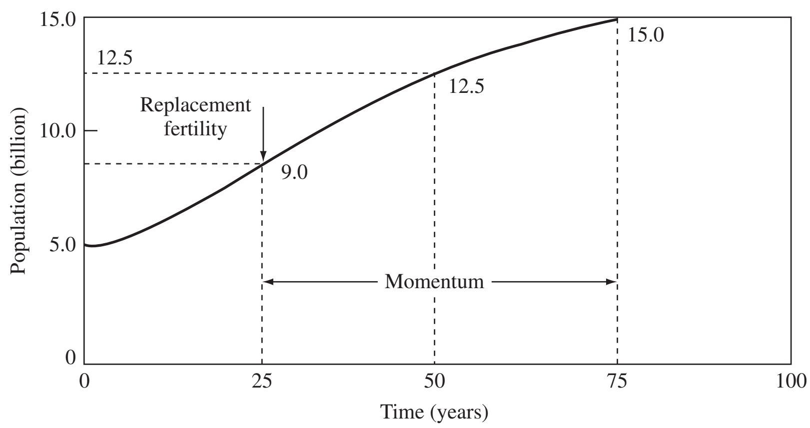

FIGURE 19 In the hypothetical example, it takes 50 years of replacement-level fertility before population stops growing.

replacement-level fertility is achieved, during which time it grows from 9 billion to 15 billion.

A more carefully done set of population scenarios is presented in Figure 20. In the constant-path scenario, population is assumed to continue growing at the 1980 growth rate into the indefinite future. The slow fertility-reduction path assumes that the world's fertility will decline to reach replacement level by 2065, which results in a stabilized population of about 14 billion by the end of the twenty-first century. The moderate fertility-reduction path assumes that replacement-level fertility will be reached by 2035, with the population stabilizing at about 10 billion. Finally, the

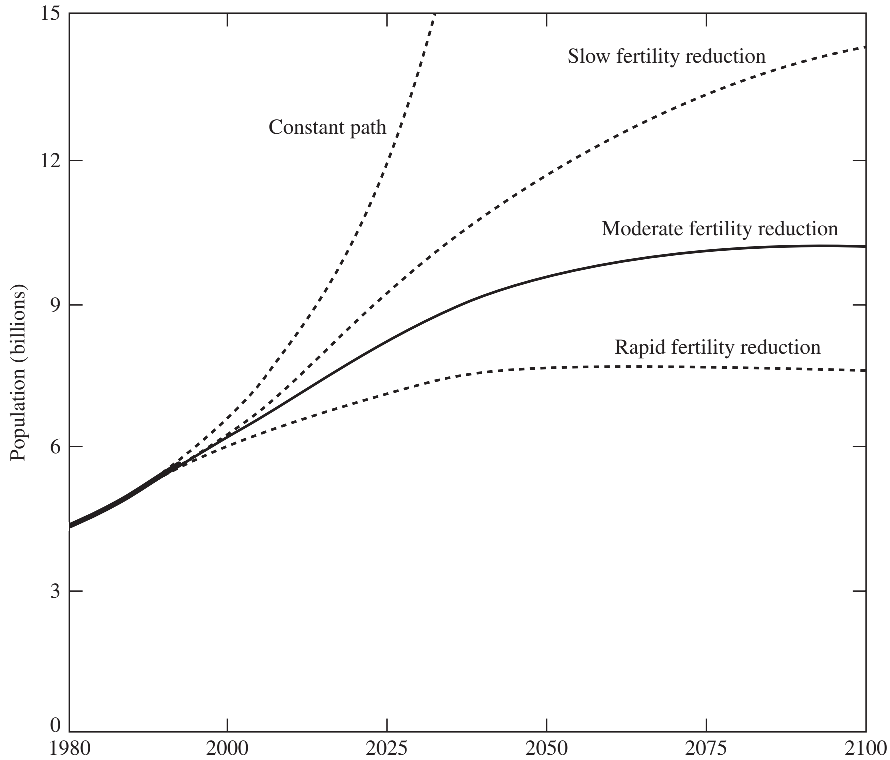

FIGURE 20 Four scenarios for the future of the world's population. The moderate path has fertility declining to replacement level by 2035, with momentum carrying the population to approximately 10 billion by the end of the twenty-first century. (*Source:* Haupt and Kane, 1985.)

rapid fertility-reduction path assumes that the world's fertility will decline to replacement level by 2010. Even under this most optimistic scenario, the world's population would still grow to approximately 7.5 billion. The implications of population momentum are obviously of tremendous importance.

PROBLEMS

1 World population in 1850 has been estimated at about 1 billion. World population reached 4 billion in 1975.

(a) Use the doubling time approximation (5) to estimate the exponential rate of growth that would produce those numbers.

(b) Use the exponential growth equation (6) to find the growth rate.

2 Tuition at a major university rose from $1,500/yr in 1962 to $20,000/yr in 1995.

(a) What exponential rate of growth characterized that period of time?

(b) If that growth rate were to continue for another 25 years (enough time for current students to have children in college!) what would the tuition be?

3 The world's population 10,000 years ago has been estimated at about 5 million. What exponential rate of growth would have resulted in the population in 1850, which is estimated at 1 billion? Had that rate continued, what would the population have been in the year 2000?

4 Suppose we express the amount of land under cultivation as the product of four factors:

$$\text{Land} = (\text{land/food}) \times (\text{food/kcal}) \times (\text{kcal/person}) \times (\text{population})$$

The annual growth rates for each factor are (1) the land required to grow a unit of food, −1 percent (due to greater productivity per unit of land); (2) the amount of food grown per calorie of food eaten by a human, +0.5 percent (because with affluence, people consume more animal products, which greatly reduces the efficiency of land use); (3) the per capita calorie consumption, +0.1 percent; and (4) the size of the population, +1.5 percent. At these rates, how long would it take to double the amount of cultivated land needed? At that time, how much less land would be required to grow a unit of food?

5 Suppose world carbon emissions are expressed as the following product:

$$\text{Carbon emissions} = (\text{energy/person}) \times (\text{carbon/energy}) \times (\text{population})$$

If per capita energy demand increases at 1.5 percent per year, fossil fuel emissions of carbon per unit of energy increase at 1 percent per year, and world population grows at 1.5 percent per year,

(a) How long would it take before we are emitting carbon at twice the current rate?

(b) At that point, by what fraction would per capita energy demand have increased?

(c) At that point, by what fraction would total energy demand have increased?

6 Under the assumptions stated in Problem 5, if our initial rate of carbon emission is 5×10^9 tonnes C/yr and if there are 700×10^9 tonnes of carbon in the atmosphere now,

(a) How long would it take to emit a total amount of carbon equal to the amount now contained in the atmosphere?

(b) If half of the carbon that we emit stays in the atmosphere (the other half being "absorbed" in other parts of the biosphere), how long would it take for fossil fuel combustion to double the atmospheric carbon concentration?

7 Consider the following disaggregation of carbon emissions:

$$\text{Carbon emissions (kg C/yr)} = \text{Population} \times \frac{\text{Energy (kJ/yr)}}{\text{Person}} \times \frac{\text{Carbon (kgC)}}{\text{Energy (kJ)}}$$

Using the following estimates for the United States and assuming that growth rates remain constant,

	Population	(kJ/yr)/Person	kg C/kJ
1990 amounts	250×10^6	320×10^6	15×10^{-6}
Growth, r (%/yr)	0.6	0.5	−0.3

(a) Find the carbon emission rate in 2020.

(b) Find the carbon emitted in those 30 years.

(c) Find total energy demand in 2020.

(d) Find the per capita carbon emission rate in 2020.

8 World reserves of chromium are about 800 million tons, and current usage is about 2 million tons per year. If growth in demand for chromium increases exponentially at a constant rate of 2.6 percent per year, how long would it take to use up current reserves? Suppose the total resource is five times current reserves; if the use rate continues to grow at 2.6 percent, how long would it take to use up the resource?

9 Suppose a Gaussian curve is used to approximate the production of chromium. If production peaks at six times its current rate of 2 million tons per year, how long would it take to reach that maximum if the total chromium ever mined is 4 billion tons (five times the current reserves)? How long would it take to consume about 80 percent of the total resource?

10 Suppose we assume the following:

(a) Any chlorofluorocarbons (CFCs) released into the atmosphere remain in the atmosphere indefinitely.

(b) At current rates of release, the atmospheric concentration of fluorocarbons would double in 100 years (what does that say about Q/P_0?).

(c) Atmospheric release rates are, however, not constant but growing at 2 percent per year. How long would it take to double atmospheric CFC concentrations?

11 Bismuth-210 has a half-life of 4.85 days. If we start with 10 g of it now, how much would we have left in 7 days?

12 Suppose some sewage drifting down a stream decomposes with a reaction rate coefficient k equal to 0.2/day. What would be the half-life of this sewage? How much would be left after 5 days?

13 Suppose human population grows from 6.3 billion in 2000 to an ultimate population of 10.3 billion following the logistic curve. Assuming a growth rate of 1.5 percent in 2000, when would the population reach 9 billion? What would the population be in 2050? Compare this to the moderate fertility reduction scenario of Figure 20.

14 Suppose a logistic growth curve had been used to project the world's population back in 1970, when there were 3.65 billion people, and the growth rate was 2.0 percent per year. If a steady-state population of 10.3 billion had been used (the moderate path in Figure 20), what would the projected population have been for 1995 (when it was actually 5.7 billion) and for 2025?

15 Suppose we stock a pond with 100 fish and note that the population doubles in the first year (with no harvesting), but after a number of years, the population stabilizes at what we think must be the carrying capacity of the pond, 2,000 fish. Growth seems to have followed a logistic curve.

(a) What population size should be maintained to achieve maximum yield, and what would be the maximum sustainable fish yield?

(b) If the population is maintained at 1,500 fish, what would be the sustainable yield?

16 What would be the sustainable yield from the pond in Example 9 if the population is maintained at 3,000 fish?

17 A lake has a carrying capacity of 10,000 fish. At the current level of fishing, 2,000 fish per year are taken with the catch uniformly distributed throughout the year. It is seen that the fish population holds fairly constant at about 4,000. If you wanted to maximize the sustainable yield, what would you suggest in terms of population size and yield?

18 Suppose we stock an island with 250 rabbits and find that the population is 450 after one year (with no harvesting). After a number of years, the population stabilizes at what we

think must be the carrying capacity of the island, 3,500 rabbits. Assume growth has followed a logistic curve.

(a) We now want to start harvesting the rabbits to sell, with the objective of harvesting the most rabbits possible per year from the island. What population size should be maintained to achieve the maximum sustainable yield and what would be the maximum sustainable rabbit yield?

(b) If the population is maintained at 1,200 rabbits, what would be the sustainable yield?

19 The Upset National Forest has a carrying capacity of 7,000 deer. At the current level of recreational hunting, 300 deer per year are taken during a 2-week hunting season. After the season, the deer population is always about 2,200 deer. You want to maximize the sustainable yield of deer from the forest.

(a) What should be the population of deer in the forest for maximum sustainable yield?

(b) What will be the maximum sustainable yield of deer from the forest?

(c) If hunting were stopped so no further deer were taken from the forest, how long would it take for the population size to reach the population (the population you calculated in part (a)) that is necessary for maximum sustainable yield?

20 The following statistics are for India in 1985: population, 762 million; crude birth rate, 34; crude death rate, 13; infant mortality rate, 118 (rates are per thousand per year). Find (a) the fraction of the total deaths that are infants less than 1 year old; (b) the avoidable deaths, assuming that any infant mortality above 10 could be avoided with better sanitation, food, and health care; and (c) the annual increase in the number of people in India.

21 The following statistics are for India in 1995: population, 931 million; crude birth rate, 29; crude death rate, 9; infant mortality rate, 74 (rates are per thousand per year). Find (a) the fraction of the total deaths that are infants less than 1 year old; (b) the avoidable deaths, assuming that any infant mortality above 10 could be avoided with better sanitation, food, and health care; and (c) the annual increase in the number of people in India. (For comparison, in 1985, the population was growing at 16 million per year, and the avoidable deaths were 2.8 million per year.)

22 Consider a simplified age structure that divides a population into three age groups: 0–24, with 3.0 million; 25–49, with 2.0 million; and 50–74, with 1.0 million. Suppose we impose the following simplified fertility and mortality constraints: All births occur just as the woman leaves the 0–24 age category, and no one dies until their seventy-fifth birthday, at which time they all die. Suppose we have replacement-level fertility starting now. Draw the age structure in 25 years, 50 years, and 75 years. What is the total population size at each of these times?

23 Use the same initial population structure given in Problem 22, with all births on the mothers twenty-fifth birthday. Draw the age structure in 25, 50, and 75 years under the following conditions: No deaths occur until the fiftieth birthday, at which time 20 percent die; the rest die on their seventy-fifth birthday. For the first 25 years, the total fertility rate is 4 (2 children/person), and thereafter it is 2.

24 Consider the following simplified age structure: All births are on the mothers twentieth birthday and all deaths are on the sixtieth birthday. Total population starts at 290,000 (half males, half females) and is growing at a constant rate of 3.5 percent per year (see Figure P24). (*Hint:* what does that say about the doubling time?)

Draw the age structure in 20 years. If the total fertility rate is a single, constant value during those 20 years, what is it?

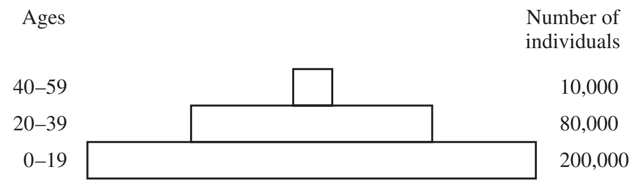

FIGURE P24

25 The following age structure and survival data are for China in 1980. Suppose the birth factors (corresponding to a total fertility rate of 1.0) are as shown. Estimate the population of China in 1990.

Age	Population (millions)	L_{x+10}/L_x	b_x
0–9	235	0.957	0
10–19	224	0.987	0.25
20–29	182	0.980	0.25
30–39	124	0.964	0
40–49	95	0.924	0
50–59	69	0.826	0
60–69	42	0.633	0
70–79	24	0.316	0
80–89	6	0	0
Total:	*1,001*		

26 Use a spreadsheet to project the China population data given in Problem 25 out to the year 2030. What is the population at that time?

27 Use a spreadsheet to project the China population data given in Problem 25 out to the year 2030 but delay the births by one 10-year interval (that is, $b_{10} = 0$, $b_{20} = 0.25$, and $b_{30} = 0.25$). Compare the peak population in Problem 25 to that obtained by postponing births by 10 years.

28 The birth-rate data for China in Problem 25 were for the very optimistic one-child per family scenario (TFR = 1). In reality, the TFR has been closer to 2. Assuming that each woman had 2 children while she was in the 20 to 30 age group (and none at any other age),

(a) Repeat this population projection for China for 1990.

(b) Continuing that birth rate, find the population in 2000.

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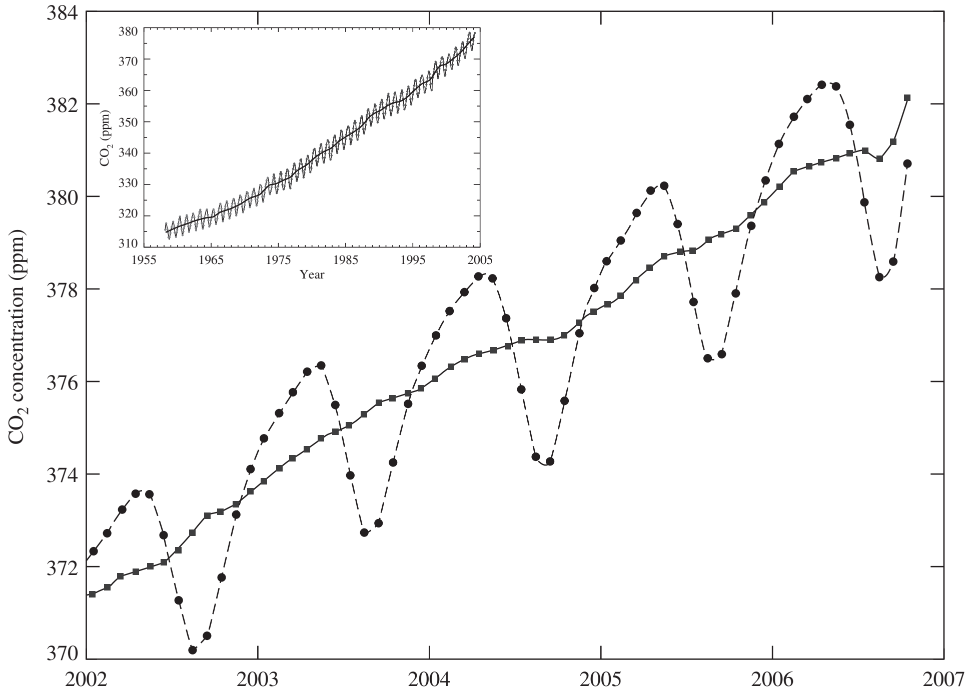


FIGURE 13 Recent global CO_2 concentrations. The oscillations are month-by-month mean values; the smoothed line is a moving average over 10 adjacent months. (*Source:* NOAA Web site http://www.cmdl.noaa.gov/ccgg/trends/index.php#mlo, 2006.)

Atmospheric carbon dioxide concentrations inferred from Antarctic ice cores and other evidence over the past 1,000 years, combined with more recent direct measurements, are shown in Figure 14. Over most of that time period, the concentration of carbon dioxide hovered at close to 280 ppm, and that is the value that is commonly used as a reference point for comparison with current readings and future projections. Carbon dioxide concentrations are now more than one-third higher than they were just before the industrial revolution.

The Carbon Cycle

The atmosphere contains about 800 GtC, where 1 GtC means 1 gigaton of carbon (10^9 metric tons or 10^{15} g). Since almost all of that carbon is stored in the form of CO_2 (less than 1 percent is in other carbon-containing compounds such as methane and carbon monoxide), in most circumstances it is reasonable to assume that atmospheric carbon is entirely CO_2. The amount of carbon locked up in terrestrial vegetation (610 GtC) is of the same order of magnitude as that in the atmosphere, but both of these amounts are dwarfed by the 39,000 GtC stored in the oceans.

Natural processes continuously transport enormous amounts of carbon back and forth among the atmosphere, biosphere, and the oceans. The carbon flux into and out of the atmosphere during photosynthesis and respiration is on the order of 60 GtC/yr. The oceans absorb around 90 Gt/C/yr and store almost all of it in the

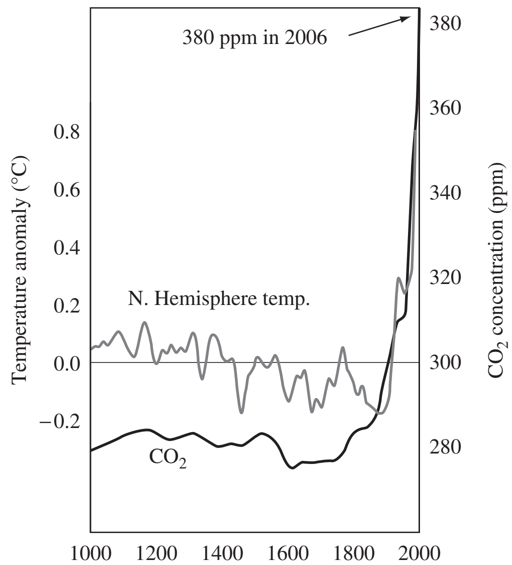


FIGURE 14 Carbon dioxide concentration and Northern Hemisphere temperature over the past 1,000 years.

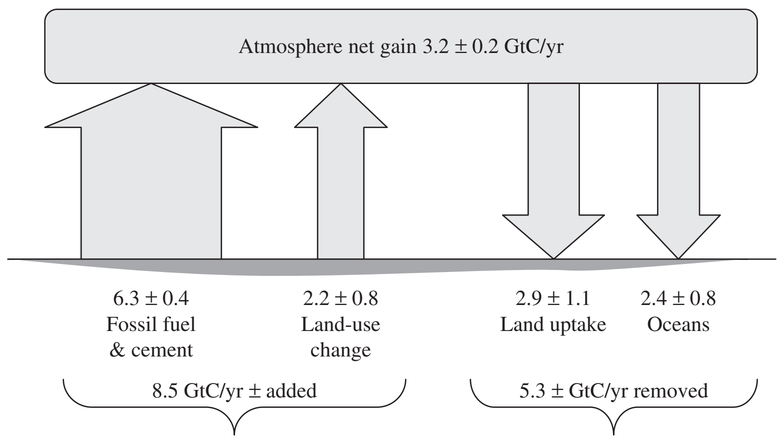


FIGURE 15 Human perturbations to the global carbon cycle during the 1990s.
(*Source:* Based on data from Houghton, 2003.)

form of bicarbonate ions (HCO_3^-), but some becomes part of the marine food chain. A similar quantity is returned to the atmosphere. A very small portion of nonliving organic matter each year ends up in sediments. The slow, historical accumulation of that organic carbon is the source of our fossil fuels—oil, natural gas, and coal. When these are burned, ancient carbon is returned to the atmosphere.

By comparison with the natural fluxes of carbon, the additional amounts added to the atmosphere by combustion, cement production, and changes in land use are modest, but they are enough to cause a significant response by the climate system. Figure 15 summarizes the impact of human perturbations to the carbon fluxes into and out of the atmosphere during the 1990s. Fossil fuel combustion and cement production deliver 6.3 GtC/yr to the atmosphere, while land use changes, such as biomass burning and harvesting of forests, add another 2.2 GtC/yr. Not all of that 8.5 GtC/yr remains in the atmosphere, however. Greater carbon uptake through purposeful reforestation efforts, as well as stimulated plant growth caused by higher CO_2 levels and increased nitrogen deposition on the soils from fossil fuel combustion, removes around 2.9 GtC/yr. Finally, higher atmospheric CO_2 increases the oceans' absorption of carbon, providing an additional 2.4 GtC/yr sink.

As Figure 15 suggests, fossil fuel combustion and land-use changes in the 1990s added about 8.5 GtC/yr to the atmosphere. Of that amount, 5.3 GtC/yr was returned to the oceans or other terrestrial sinks, leaving about 3.2 GtC/yr remaining in the atmosphere. The ratio of the amount of anthropogenic carbon emitted to the amount that remains in the atmosphere is known as the *airborne fraction*. Using these data, the airborne fraction has been:

$$\text{Airborne fraction} = \frac{3.2 \text{ GtC/yr remaining in atmosphere}}{8.5 \text{ GtC/yr anthropogenic additions}} = 0.38 = 38\% \qquad (17)$$

Thus, roughly speaking, somewhat less than half of the carbon we emit stays in the atmosphere. But the airborne fraction is not necessarily a fixed quantity. For example, if large areas of land are deforested, the ability of the biosphere to absorb carbon would be reduced, and the atmospheric fraction would increase. Likewise, CO_2 fertilization of terrestrial biomass can stimulate plant growth, which increases the rate of removal of atmospheric carbon, so the airborne fraction could get smaller. The airborne fraction also depends on how fast carbon is being added to the atmosphere. For scenarios with little or no growth in emissions or even declining emissions, the oceans and plants have more time to absorb carbon, so the atmospheric fraction could be lower. On the other hand, for rapidly increasing emission rates, carbon sinks cannot keep up and the fraction remaining in the atmosphere may be higher.

The following example develops another useful relationship, this time between the concentration of CO_2 and the tons of carbon in the atmosphere. This ratio, coupled with an estimate of the airborne fraction, provides the key to predicting future CO_2 concentrations for various carbon emission scenarios.

EXAMPLE 2 Carbon Content of the Atmosphere

Find a relationship between the concentration of carbon dioxide and the total amount of carbon in the atmosphere. The total mass of the atmosphere is estimated to be 5.12×10^{21} g.

Solution We will need to know something about the density of air. That, of course, varies with altitude, but finding it under some particular conditions will work. From Table 1 we know the concentration of each gas in air. Recall from Section 1.2 that 1 mole of each gas occupies 22.414×10^{-3} m^3 at Standard

Temperature and Pressure (0°C and 1 atmosphere), which is 44.61 mol/m^3. The following table organizes the calculation:

Gas	$\frac{m^3\ gas}{m^3\ air}$	$\times$	g/mol	$\times$	$\frac{mol}{m^3\ gas}$	$=$	$\frac{g}{m^3\ air}$
N_2	0.7808		28		44.61		975.3
O_2	0.2095		32		44.61		299.1
Ar	0.0093		40		44.61		16.6
CO_2	0.00038		44		44.61		0.75
						Total	1291.8 g/m^3

If all of the atmosphere were at standard temperature and pressure, it would have a density of 1291.8 g/m^3, and its mass would still be 5.12×10^{21} g. Putting these together gives

$$1\text{ ppm} = \frac{1\text{ m}^3\text{ CO}_2}{10^6\text{ m}^3\text{ air}} \cdot 44.61\frac{\text{mole}}{\text{m}^3\text{ CO}_2} \cdot 12\frac{\text{g C}}{\text{mole}} \cdot \frac{5.12 \times 10^{21}\text{ g air}}{1291.8\dfrac{\text{g air}}{\text{m}^3\text{ air}}} \cdot 10^{-15}\frac{\text{GtC}}{\text{g C}} = 2.12\text{ GtC}$$

Notice this calculation has taken advantage of the fact that volumetric concentrations (ppm) are independent of temperature or pressure.

From Example 2, we have the following very useful relationship:

$$1\text{ ppm CO}_2 = 2.12\text{ GtC} \tag{18}$$

For example, knowing the concentration of CO_2 in 2006 was 380 ppm, the total amount of carbon in the atmosphere can be estimated to be

$$380\text{ ppm} \times 2.12\text{ GtC/ppm} = 806\text{ GtC}$$

EXAMPLE 3 Estimating the Rate of Change of CO_2

Suppose global fossil fuel combustion emits 7.4 GtC/yr and cement production adds another 0.5 GtC. Assuming the an airborne fraction of 0.38 and assuming no change in emissions associated with land use, what rate of change in CO_2 concentration would you expect?

Solution Including the 2.2 GtC/yr land-use emissions from Figure 17 gives a total emission rate of 2.2 + 7.4 + 0.5 = 10.1 GtC/yr. Using the 0.38 airborne fraction along with the 2.12 GtC/ppm ratio gives

$$\Delta CO_2 = \frac{10.1\text{ GtC/yr} \times 0.38}{2.12\text{ GtC/ppm CO}_2} = 1.8\text{ ppm CO}_2\text{/yr}$$

Carbon Emissions from Fossil Fuels

Use of energy and the resulting emissions of carbon vary considerably from country to country. The United States, with less than 5 percent of the world's population, emits 22 percent of global energy-related CO_2. The second largest emitter is China,

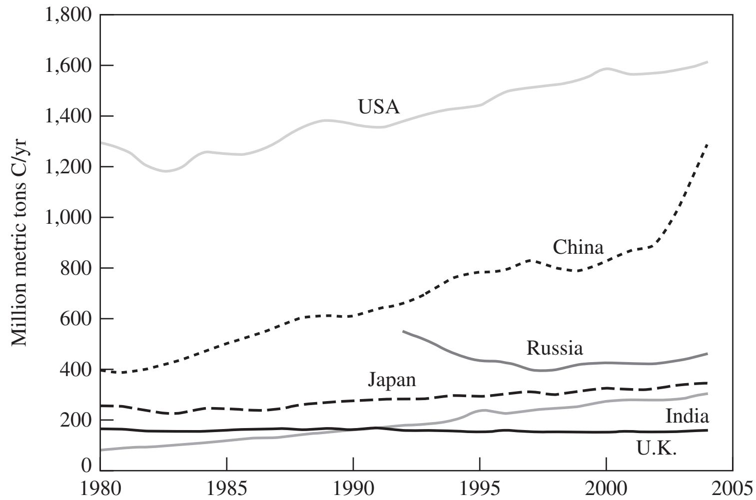


FIGURE 16 Fossil fuel carbon emissions.
(*Source:* EIA data, 2006.)

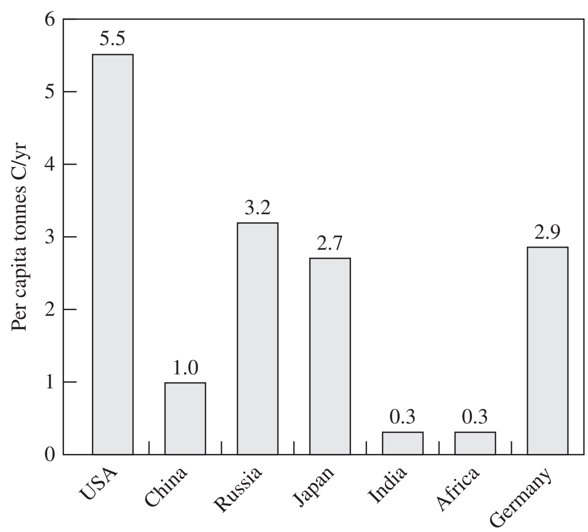


FIGURE 17 Per capita carbon emissions.
(*Source:* EIA data, 2006.)

and as Figure 16 indicates, it is very rapidly closing the gap with the United States. In fact, it is projected that China's emissions will exceed those of the United States by 2010. While China's emissions growth is certainly worrisome, two other measures provide some insight into the root cause of our global carbon problem. On a per capita basis, the United States emits far more carbon than any other country—roughly double that of most other advanced countries and more than five times as much as China (Figure 17). Moreover, it is the accumulated emissions from developed countries that are overwhelmingly responsible for the rising CO_2 concentrations in the atmosphere (Figure 18).

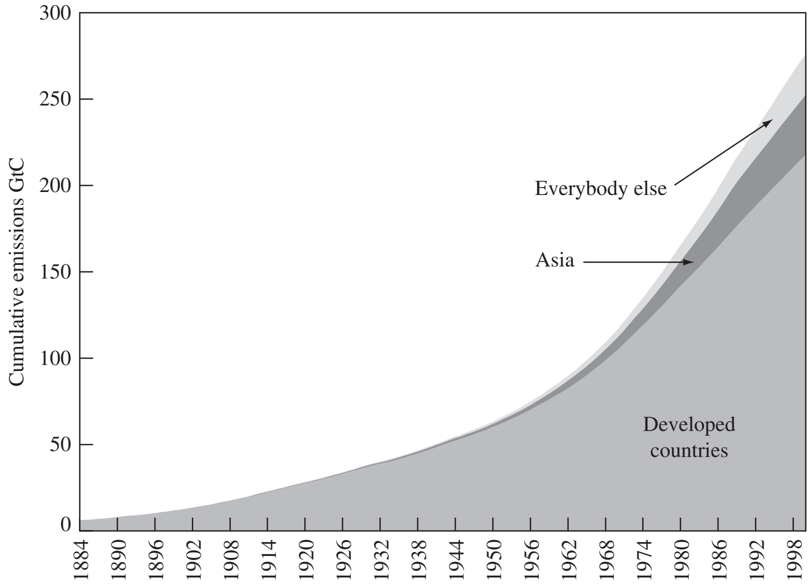


FIGURE 18 Cumulative carbon emissions by region.
(*Source:* E. Wanless with data from G. Marland, Oak Ridge National Labs.)

Carbon Emissions from Industrial Processes

A number of manufacturing processes result in carbon emissions on site that are not included in the usual accounting for fossil fuel combustion. The primary source of these industrial emissions is the calcination of limestone ($CaCO_3$) to create lime (CaO). These two compounds are basic materials in the production of cement, iron and steel, and glass. Other industrial emissions result from the production and use of soda ash (Na_2CO_3), the manufacture of carbon dioxide, and the production of aluminum.

The largest single industrial CO_2 source, however, results from cement production. Concrete, which is probably the most important building material in the world, is made up of a mixture of Portland cement, fine aggregate (sand), coarse aggregate (crushed stone), and water. Portland cement, which is the binding agent for concrete, derived its name in the early nineteenth century from its similarity to a particular type of stone found on the Isle of Portland in Dorset, England. It is typically on the order of 12 to 15 percent by weight of the concrete mix.

The major raw material needed for the manufacture of Portland cement is limestone ($CaCO_3$), along with a source of silicon such as clay or sand. Processing of those materials is done in high-temperature kilns, usually fired with fossil fuels whose combustion leads to CO_2 emissions. In addition, the calcination of $CaCO_3$ into lime (CaO) that occurs in those kilns emits its own CO_2 as the following reaction suggests:

$$\text{Limestone } (CaCO_3) + \text{Clay or Sand } (Si) + \text{Heat} \rightarrow \text{Cement } (CaO, SiO_2) + CO_2 \tag{19}$$

The average intensity of carbon emissions from cement production is about 0.222 tons of carbon per ton of cement, with about half of that being the result of calcinations and half released during combustion (Worrell et al., 2001). In addition, power plants supplying electricity for plant operations emit their own carbon, but that is usually not included in the category of industrial process emissions. All told, cement manufacturing globally contributes close to 5 percent of all anthropogenic carbon emissions. The total industrial emissions category accounts for about 0.8 GtC/yr.

As it turns out, fly ash from coal-fired power plants can be used as a replacement for some of the cement in concrete. This *fly-ash concrete* not only reduces carbon emissions by roughly one ton of CO_2 per ton of replaced cement, but it also results in a concrete that is stronger and more durable than its conventional counterpart. It also recycles a relatively useless waste product that would otherwise have to be disposed of. Concrete mixtures with more fly ash than cement are now becoming popular in the emerging green building industry.

Carbon Intensity of Fossil Fuels

The amount of carbon released per unit of energy delivered is called the *carbon intensity*. Some fuels have high carbon intensity, such as coal, and some conversion systems release no direct carbon at all, such as wind turbines or nuclear power. Interestingly, biomass fuels may also be used in ways that emit little or no carbon when new plants are grown to replace the ones that were burned. No energy system is likely to have zero carbon emissions, however, since it is hard to avoid such emissions during the mining, materials processing, and construction of any energy facility.

EXAMPLE 4 Carbon Intensity of Methane

Find the carbon intensity of methane based on its higher heating value (HHV) of 890 kJ/mol (which includes the energy of condensation of the water vapor formed. Then find the carbon intensity based on the lower heating value (LHV) of 802 kJ/mol.

Solution First, write a balanced chemical reaction for the oxidation of methane:

$$CH_4 + 2\,O_2 \rightarrow CO_2 + 2\,H_2O$$

So, burning 1 mol of CH_4 liberates 890 kJ of energy while producing 1 mol of CO_2. Since 1 mol of CO_2 has 12 g of carbon, the HHV carbon intensity of CH_4 is

$$\text{HHV carbon intensity} = \frac{12 \text{ g C}}{890 \text{ kJ}} = 0.0135 \text{ gC/kJ} = 13.5 \text{ gC/MJ}$$

Similarly, the LHV carbon intensity would be

$$\text{LHV carbon intensity} = \frac{12 \text{ gC}}{802 \text{ kJ}} = 0.015 \text{ gC/kJ} = 15.0 \text{ gC/MJ}$$

The LHV carbon intensity assumes the latent heat of the water vapor produced is not available as usable energy.

TABLE 3

LHV and HHV Carbon Intensities and Emissions for Typical Fossil Fuels

Fuel	LHV Carbon Intensity (gC/MJ)	HHV Carbon Intensity (gC/MJ)	2004 Global Carbon Emissions (GtC/yr)
Natural gas	15.3	13.8	1.58 (21.2%)
Petroleum	20.0	19.7	2.96 (39.8%)
Coal	25.8	24.2	2.89 (38.9%)

Sources: Carbon intensities from NAS, 1992; emissions from EIA, 2006.

We can get a quick estimate of the carbon intensity of other fuels by using the fact that the energy released during combustion of carbon-based fuels is approximately proportional to the amount of oxygen they consume (Baird, 1995). For example, if we consider coal to be purely carbon and oil to be approximately CH_2, we can write the following oxidation reactions:

Coal: $$C + O_2 \rightarrow CO_2 \tag{20}$$

Oil: $$CH_2 + 1.5\ O_2 \rightarrow CO_2 + H_2O \tag{21}$$

Natural gas: $$CH_4 + 2\ O_2 \rightarrow CO_2 + 2\ H_2O \tag{22}$$

The same amount of carbon is released for each of these reactions, but since energy is roughly proportional to oxygen consumption, per unit of carbon emitted we would expect about 1.5 times as much energy from oil as coal and twice as much energy from natural gas compared to coal. Turning that around, coal is more carbon-intensive than oil, and oil emits more carbon than natural gas.

The actual LHV and HHV carbon intensities for typical fossil fuels, combined with estimates of total carbon emissions, are given in Table 3. Note the slight difference between the carbon intensity of natural gas and the intensity of methane itself found in Example 4. Natural gas is mostly methane, but it also includes other hydrocarbons, which alters the calculation slightly. Since the proportions of methane and other hydrocarbons vary from one source to another, the carbon intensities given in Table 3 are "typical," and you are likely to find other, slightly different, values in the literature. The same goes for variations in types of coal. Another comment worth making is that energy consumption figures in the United States are often based on the higher heating value of a fuel, while most of the rest of the world, including the IPCC, uses LHV values. The LHV value makes more sense for power plants since they never recover the latent heat of their water vapor emissions, but HHV values are more appropriate for high-efficiency condensing furnaces and water heaters, which do capture that heat. To be consistent with the IPCC, in this chapter, all energy data and carbon intensities will be based on the LHV values.

The carbon intensity data given in Table 3 suggest that sizable reductions in carbon emissions are possible by switching from coal to natural gas. It is unfortunately the case, however, that most of the world's fossil fuel resources are in the form of coal. It is interesting to note that almost 90 percent of the world recoverable resources of coal are in just three regions: the United States, the former USSR, and China.

Table 4 presents data on the world's fossil fuel resources. These resources are shown as a resource base, which consists of already identified reserves plus an estimate at the 50-percent probability level of remaining undiscovered resources. The

TABLE 4

Energy Content of Global Fossil Fuel Resources and Occurrences, in Exajoules (EJ)

Fuel	Conventional Resources	Unconventional Resources	Total Resource Base	Additional Occurrences
Natural gas	9,200	26,900	36,100	>832,000
Petroleum	8,500	16,100	24,600	>25,000
Coal	25,200	100,300	125,500	>130,000
Totals	42,900	143,300	186,200	>987,000

Source: Nakicenovic, 1996.

resources are also described as coming from conventional sources of the type now being exploited, as well as unconventional sources that might be usable in the future. Unconventional sources of oil include oil shale, tar sands, and heavy crude; unconventional natural gas sources include gas in Devonian shales, tight sand formations, geo-pressurized aquifers, and coal seams. An additional column in Table 4 is labeled "additional occurrences." These are additional resources with unknown certainty of occurrence and/or with unknown or no economic significance in the foreseeable future. Enormous amounts of methane locked in methane hydrates ($CH_4 \cdot 6H_2O$) under the oceans (estimated at over 800,000 EJ) are the most important of these.

Example 5 suggests what might happen to the CO_2 concentration in the atmosphere if we were to burn all of the world's coal resource base.

EXAMPLE 5 Burning the World's Coal

Estimate the increase in atmospheric CO_2 if the 125,500 EJ of coal were to be burned. Assume a constant airborne fraction of 38 percent.

Solution We can first estimate the carbon content using the LHV value from Table 3:

$$125{,}500 \text{ EJ} \times 25.8 \text{ gC/MJ} \times 10^{12} \text{ MJ/EJ} \times 10^{-15} \text{ GtC/gC} = 3{,}238 \text{ GtC}$$

which is roughly four times as much carbon as currently exists in the atmosphere. Converting this to CO_2 and including the 0.38 airborne fraction gives

$$\Delta CO_2 = \frac{3{,}238 \text{ GtC} \times 0.38}{2.12 \text{ GtC/ppm } CO_2} = 580 \text{ ppm } CO_2$$

That would result in 2.5 times as much CO_2 in the atmosphere as we have today. In fact, it would likely be higher than that if the airborne fraction increases due to the oceans ceasing to be such a good carbon sink.

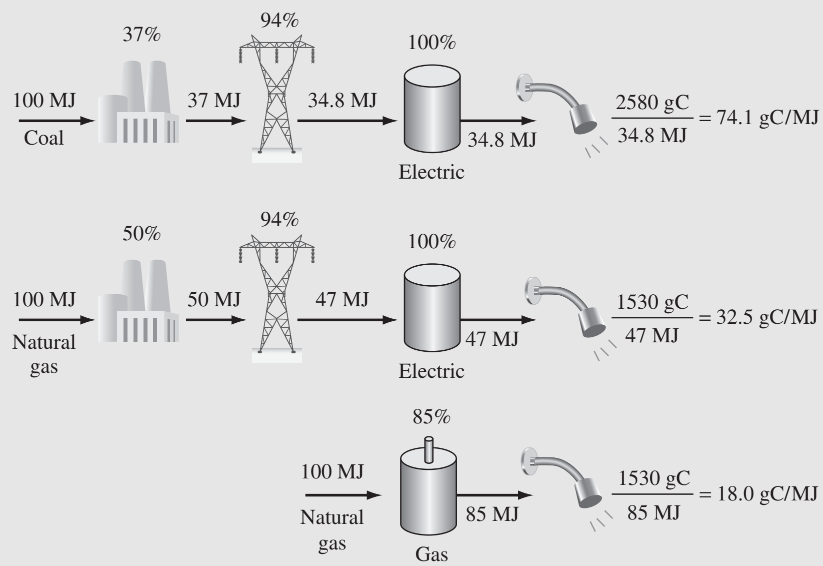
It is interesting to note that a similar calculation to that shown in Example 5 suggests that the oil and gas resource base has the potential to add only about one-third as much CO_2 to the atmosphere as would burning all of the world's coal resources. That calculation is less certain, however, because of the relatively unknown carbon emission factors that would be appropriate for the unconventional oil and gas resources. It is also complicated by the fact that the principal component of natural gas, methane, is a much more potent greenhouse gas than CO_2, which means methane leakage can amplify its potential global warming.

The carbon-intensity factors given in Table 3 suggest that switching from coal to oil or natural gas would reduce emissions significantly. For example, it would appear that switching from coal to natural gas would reduce carbon emissions by about 41 percent while delivering the same amount of energy. Example 6 shows how efficiency advantages as well as carbon advantages associated with using natural gas can substantially increase that advantage.

EXAMPLE 6 Efficiency and Carbon Intensity Combined

Compare the carbon emissions to heat household water using the following three energy systems: 1) a very good, 37 percent-efficient coal-fired power plant delivering electricity to a 100 percent-efficient electric water heater; 2) a new, 50 percent-efficient natural-gas-fired combined-cycle power plant for that same electric water heater; and 3) an 85 percent-efficient gas-fired water heater. Assume 6 percent losses in electrical transmission lines.

Solution Let us base our comparison on 100 MJ of energy provided to each system. Using the LHV values given in Table 3, burning 100 MJ of coal releases 2,580 g of carbon, while 100 MJ of natural gas releases 1,530 gC. As suggested here, the coal-plant system delivers $100 \times 0.37 \times 0.94 = 34.8$ MJ to heat water; the more efficient gas-fired power plant delivers $100 \times 0.50 \times 0.94 = 47$ MJ to heat water; and using gas in the water heater delivers 85 MJ of heat.



The carbon intensities for each system are shown in the figure. Switching from coal to gas in the power plant reduces carbon emissions for the electric water heater by 56 percent. Using gas directly in the water heater reduces carbon emissions by 75 percent (much more than the 41 percent improvement expected based just on the carbon intensity of natural gas versus coal).

Estimating Emissions: The Kaya Identity

Predicting future concentrations of carbon dioxide depends on numerous assumptions about population growth, economic factors, energy technology, and the carbon cycle itself. The usual approach involves developing a range of emission scenarios that depend on those factors and then using those scenarios to drive mathematical models of how the atmosphere and climate system will react to those inputs. At the level of treatment given in this short section, we can't begin to approach the complexity of those models; however, we can make a few simple calculations to at least give a sense of some of the important factors.

One way to build simple models of environmental problems is to start with the notion that impacts are driven by population, affluence, and technology, which is sometimes referred to as the *IPAT identity* (Ehrlich and Holdren, 1971).

$$\text{Environmental Impact} = (\text{Population}) \times (\text{Affluence}) \times (\text{Technology}) \tag{23}$$

The following application of IPAT to carbon emissions from energy sources is often referred to as the *Kaya identity* (Kaya, 1990).

$$C = \text{Population} \times \frac{\text{GDP}}{\text{Person}} \times \frac{\text{Primary Energy}}{\text{GDP}} \times \frac{\text{Carbon}}{\text{Primary Energy}} \tag{24}$$

where

$$C = \text{carbon emission rate (GtC/yr)}$$

$$\frac{\text{GDP}}{\text{Person}} = \frac{\text{GDP}}{\text{P}} = \text{per capita gross domestic product (\$/person-yr)}$$

$$\frac{\text{Primary Energy}}{\text{GDP}} = \frac{\text{PE}}{\text{GDP}} = \text{primary energy intensity, (EJ/\$)}$$

$$\frac{\text{Carbon}}{\text{Primary Energy}} = \frac{\text{C}}{\text{PE}} = \text{carbon intensity, (GtC/EJ)}$$

Equation (24) incorporates the key quantities that drive our energy-related carbon emissions. It includes economic and population scenarios plus two factors that are central to energy: energy intensity and carbon intensity. Carbon intensity has already been introduced. *Energy intensity* is the amount of energy required to create a unit of economic activity as measured by gross domestic product (GDP). It is usually thought of as a surrogate for the country's energy efficiency. For example, Japan, which only needs half the energy to produce a unit of GDP, is often considered to be roughly twice as energy efficient as the United States. While there is some truth to that assertion, it sometimes masks differences in the standard of living and the climate in each country. For example, the United States has larger houses that are kept warmer in more severe winters, so if more energy is required it may have more to do with those factors than whether or not homes are better insulated in one country or the other. Japan is also a small, densely populated country with relatively short travel distances, so transportation energy would likely be less as well, even with an equivalent level of transportation efficiency.

For example, the Kaya identity for the year 2010 looks something like this:

$$\begin{aligned} C &= 6.9 \times 10^9 \text{ people} \times \$4{,}605/\text{person-yr} \times 14.9 \text{ EJ}/\$10^{12} \times 0.016 \text{ GtC/EJ} \\ &= 7.6 \text{ GtC/yr} \end{aligned}$$

TABLE 5

1990 to 2020 Average Annual Growth Rates (%/yr) Used in the IPCC IS92a Scenario for Energy-Related Carbon Emissions

Region	Population	$\frac{\text{GDP}}{\text{Person}}$	$\frac{\text{PE}}{\text{Person}}$	$\frac{\text{Carbon}}{\text{PE}}$
China and centrally planned Asia	1.03	3.91	−1.73	−0.32
Eastern Europe and ex-USSR	0.43	1.49	−0.66	−0.24
Africa	2.63	1.25	0.26	−0.21
United States	0.57	2.33	−1.81	−0.26
World	1.40	1.53	−0.97	−0.24

Equation (24) expresses the carbon emission rate as the product of four terms: population, GDP, carbon intensity, and energy intensity. Recall from Section 3.2 that if each of the factors in a product can be expressed as a quantity that is growing (or decreasing) exponentially, then the overall rate of growth is merely the sum of the growth rates of each factor. That is, assuming each of the factors in (23) is growing exponentially, the overall growth rate of carbon emissions r is given by

$$r = r_{\mathrm{P}} + r_{\mathrm{GDP/P}} + r_{\mathrm{PE/GDP}} + r_{\mathrm{C/PE}} \tag{25}$$

By adding the individual growth rates as has been done in (25), an overall growth rate is found, which can be used in the following emission equation:

$$\mathrm{C} = \mathrm{C}_0 e^{\mathrm{rt}} \tag{26}$$

where

C = carbon emission rate after t yrs (GtC/yr)
C_0 = initial emission rate (GtC/yr)
r = overall exponential rate of growth (yr^{-1})

Table 5 shows population, economic growth, carbon intensity, and energy intensity values that have been used in one of the most-cited IPCC emission scenarios (IS92a) for energy. For the world as a whole, energy intensity and carbon intensity are both improving over time, which helps offset population and economic growth.

The cumulative emissions from a quantity growing exponentially at a rate r, over a period of time T is given by

$$\mathrm{C}_{\mathrm{tot}} = \int_0^T \mathrm{C}_0 e^{\mathrm{rt}}\, dt = \frac{\mathrm{C}_0}{r}(e^{\mathrm{rT}} - 1) \tag{27}$$

Combining (27) with an estimate of the atmospheric fraction, along with our 2.12 GtC/ppm CO_2 conversion factor, lets us make simple estimates of future CO_2 concentrations in the atmosphere, as Example 7 demonstrates.

EXAMPLE 7 Kaya Estimate of Future Carbon Emissions

Emissions from fossil-fuel combustion in 2010 are estimated to be 7.6 GtC/yr. In the same year, atmospheric CO_2 concentration is estimated to be 390 ppm. Assume the atmospheric fraction remains constant at 0.38.

a. Assuming the energy growth rates shown in Table 5 don't change, estimate the energy-related carbon-emission rate in 2050.

b. Estimate the cumulative energy-related carbon added to the atmosphere between 2010 and 2050.

c. Add into your scenario carbon emissions from industrial processes (especially cement) of 0.7 GtC/yr in 2010 and growing at 1.3%/yr. Also add a constant 0.9 GtC/yr from land-use changes. Estimate the CO_2 concentration in 2050.

Solution

a. The overall growth rate in energy-related carbon emissions is just the sum of the individual growth rates:

$$r = 1.40\% + 1.53\% - 0.97\% - 0.24\% = 1.72\% = 0.0172/\text{yr}$$

With 40 years of growth at 1.72% per year, the emission rate in 2050 would be

$$C_{2050} = C_{2010}e^{rT} = 7.6\, e^{0.0172\times 40} = 15.1 \text{ GtC/yr}$$

b. Over those 40 years, the cumulative energy emissions would be

$$C_{tot} = \frac{C_0}{r}(e^{rT} - 1) = \frac{7.6}{0.0172}(e^{0.0172\times 40} - 1) = 437 \text{ GtC}$$

c. The cumulative carbon emissions from industrial processes and land-use changes is

$$\text{Industrial } C_{tot} = \frac{C_0}{r}(e^{rT} - 1) = \frac{0.7}{0.013}(e^{0.013\times 40} - 1) = 37 \text{ GtC}$$

$$\text{Land Use } C_{tot} = 0.9 \text{ GtC/yr} \times 40 \text{ yrs} = 36 \text{ GtC}$$

Using the 2.12 GtC/ppm CO_2 ratio and a 0.38 atmospheric fraction makes our estimate of CO_2 in 2050

$$CO_2 = 390 + \frac{(437 + 37 + 36)\text{ GtC} \times 0.38}{2.12 \text{ GtC/ppm } CO_2} = 481 \text{ ppm}$$

One of the weakest aspects of the Kaya identity as expressed in (24) is its use of primary energy per GDP as a measure of energy efficiency. Primary energy, which is in essence energy as it is taken out of the ground, can be reduced by efficiency improvements on the supply side (e.g., more efficient electric power plants) as well as on the demand side (e.g., by making light bulbs more efficient). One way to address the two different efficiency improvements is to introduce another factor, call it *final energy FE,* which is the energy purchased by consumers (e.g., gasoline, natural gas, electricity). Another extension is based on prospects for carbon capture and

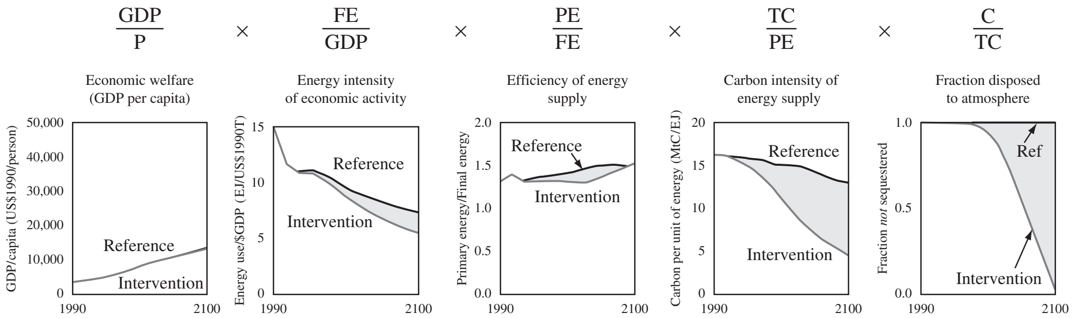


FIGURE 19 An expanded Kaya analysis showing a reference scenario and an intervention scenario designed to stabilize atmospheric CO_2.
(*Source:* Hummel, 2007 based on the IPCC A2-4.5 W/m^2 scenario.)

storage in the future. Letting *TC* be total carbon in the fuel and *C* be carbon actually emitted to the atmosphere, the Kaya identity can be expanded as follows:

$$C = P \times \frac{GDP}{P} \times \frac{FE}{GDP} \times \frac{PE}{FE} \times \frac{TC}{PE} \times \frac{C}{TC} \tag{28}$$

This much more useful disaggregation breaks the overall carbon emissions into much tighter packages for analysis. One such analysis showing a comparison between a more or less business-as-usual carbon scenario with one designed to ultimately stabilize CO_2 is shown in Figure 19.

A Climate Sensitivity Parameter

As we pump more and more CO_2 into the atmosphere, the marginal impact of each additional ton decreases as its absorption bands approach saturation. That suggests a nonlinear relationship between CO_2 and the resulting global warming that it causes. One commonly used representation of this phenomenon is given in (29).

$$\Delta T_e = \frac{\Delta T_{2X}}{\ln 2} \ln\left[\frac{(CO_2)}{(CO_2)_0}\right] \tag{29}$$

where

ΔT_e = the equilibrium, global, mean surface temperature change
ΔT_{2X} = the equilibrium temperature change for a doubling of atmospheric CO_2
$(CO_2)_0$ = the initial concentration of CO_2
(CO_2) = the concentration of CO_2 at another time

The increase in surface temperature that results from a doubling of CO_2 in the atmosphere is called the *climate sensitivity,* ΔT_{2X}. Notice what happens to (29) when CO_2 is double the initial amount. The change in surface temperature is what it should be, that is,

$$\Delta T_e = \frac{\Delta T_{2X}}{\ln 2} \ln\left[\frac{2(CO_2)_0}{(CO_2)_0}\right] = \frac{\Delta T_{2X}}{\ln 2} \ln 2 = \Delta T_{2X}$$

If the concentration of CO_2 is quadrupled,

$$\Delta T_e = \frac{\Delta T_{2X}}{\ln 2} \ln\left[\frac{4(CO_2)_0}{(CO_2)_0}\right] = \frac{\Delta T_{2X}}{\ln 2} \ln (2^2) = 2\Delta T_{2X}$$