Paper 1: Mechanical Engineering

Examples Paper 4

Elementary exercises are marked †, problems of Tripos standard ∗.
Answers can be found at the back of the paper.

Equations of motion and D'Alembert forces

1 A uniform rod is used as a pendulum for a clock: the rod is connected to a frictionless pivot at one end and can swing in a vertical plane. The angular position is defined such that $\theta = 0$ represents the pendulum hanging vertically downwards.

(a) What is the moment of inertia of the pendulum about the pivot?

(b) Derive the equation of motion for the pendulum;

(c) What is the general solution if θ is small and what is its physical interpretation?

(d) Find the potential energy as a function of its angular position θ;

(e) How many equilibrium positions are there and which are stable?

(f) Use the Python template file `p4q1_template.ipynb` to numerically solve the equation of motion when the pendulum is released from a given initial angle $\theta = \theta_0$, and plot:
(i) the angular position as a function of time;
(ii) the angular position on the x-axis and the angular velocity on the y-axis for a range of initial release angles (this kind of plot is known as the 'phase portrait' of a dynamical system).

(g) Compare your analytic solution in (b) with your numerical solution in (e). What happens as the release angle increases?

2 A uniform ladder is held vertically against a wall standing with its foot in the angle between the wall and the floor. The top of the ladder is moved a short distance away from the wall and then released. The angular position is defined such that the ladder is vertical when $\theta = 0$.

(a) Assuming that the floor is rough so that the foot of the ladder does not slip:

(i) What is the acceleration of the centre of mass as a function of θ, $\dot{\theta}$, and $\ddot{\theta}$?

(ii) Draw the Free Body Diagram of the ladder at an arbitrary angle θ during its fall, including D'Alembert forces and moments;

(iii) Find the equation of motion of the ladder and integrate it to find the angular velocity as a function of θ [Hint: recall that $\ddot{\theta} = \dot{\theta}\mathrm{d}\dot{\theta}/\mathrm{d}\theta$];

(iv) Calculate the magnitude and direction of the horizontal force acting on the foot of the ladder just before the ladder strikes the ground.

(b) Now assume that the floor is smooth. Through what angle does the ladder turn before its foot starts to come away from the wall?

Try this with a ruler on your desk.

3 A slice of toast of width $2L$ is held with its centre of gravity projecting a distance a beyond the edge P of a horizontal table, as shown in Figure 1. The toast is released from its horizontal position and initially rotates about edge P without slipping. During this motion, its inclination to the horizontal is θ.

(a) Calculate the angular velocity and acceleration $\dot{\theta}$ and $\ddot{\theta}$ during this initial rotation;

(b) Show that the toast will rotate about P until θ reaches the value given by:

$$\tan\theta = \frac{L^2}{9a^2 + L^2}\mu$$

where μ is the coefficient of friction between the slice of toast and the table.

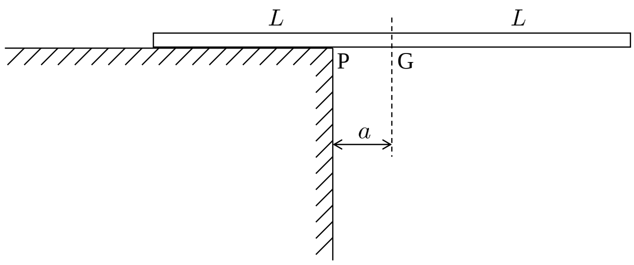

Figure 1

4 An 'inerter' is a device that has been used in Formula 1 suspension systems, that gives a force proportional to the *relative* acceleration between two connection points. A simple inerter design is shown in Figure 2. Each connection point A and B is joined to a row of gear teeth ('rack'). A circular gear ('pinion') connects the two racks. The rack and pinion mechanism can be assumed to be light. The pinion is fixed to the centre of a light rod and a mass m is attached to each end of the rod. The effective radius of the central pinion is r.

(a) What is the mass moment of inertia of the rotor (the light rod and masses) about its centre of mass?

(b) Assuming that $F_1 = -F_2 \equiv F$ derive an expression for F in terms of the relative acceleration of A and B;

(c) Under what circumstances is it reasonable to assume that $F_1 = -F_2$?

(d) What are some of the practical design challenges of this kind of device?

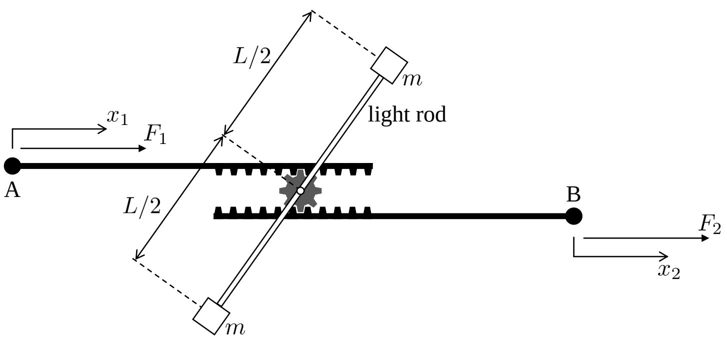

Figure 2

5 A cylinder of mass m and polar moment of inertia J is released from rest at a height h, then rolls down a rough slope of angle α.

(a) Assuming no slipping occurs, derive an expression for the linear acceleration of the centre of mass after it has been released;

(b) Will a hollow or solid cylinder reach the bottom of the slope first?

(c) What will the kinetic energy of the hollow and solid cylinders be at the bottom of the slope?

(d) What will the translational velocities of the hollow and solid cylinders be at the bottom of the slope?

Momentum and energy

6 A rotor with polar moment of inertia J is spinning in frictionless bearings and initially has angular velocity Ω. A constant torque Q is then applied to bring the rotor to rest.

(a) Determine from energy considerations the angle that the rotor turns through before it comes to rest;

(b) Determine from momentum considerations the time the rotor takes to come to rest;

(c) Use D'Alembert's principle to write down the equation of motion for the braking period. Show that the solution to this equation is consistent with your answers to (a) and (b).

7 A rigid uniform bar of mass M and length L hangs vertically under gravity from a frictionless hinge. A short impulse I is applied transversely to the bar at its lower end.

(a) What is the angular velocity with which the bar starts to move after the impulse?

(b) What is the impulsive reaction at the hinge?

(c) How far down the bar from the hinge should the impulse be applied if there is to be no impulse reaction at the hinge? What is the implication of this result for cricket, baseball and tennis players?

8 A frisbee is thrown such that it lands on a platform at position A which is at a distance d from the centre of the platform, as illustrated in Figure 3. The frisbee has mass m_F and mass moment of inertia I_F about its centre, and is released with translational velocity U_0 and angular velocity ω_0 anticlockwise. The platform is initially at rest but is free to rotate about its centre C and has a mass moment of inertia I_P. After the frisbee has landed it does not slip relative to the platform.

(a) What is the total initial angular momentum about point C?

(b) What is the mass moment of inertia about point C of the combined system after the frisbee has landed?

(c) What is the angular velocity (including its direction) of the platform after the frisbee has landed?

(d) Under what circumstances does the platform rotate anticlockwise, and how is this consistent with the principle of conservation of linear momentum?

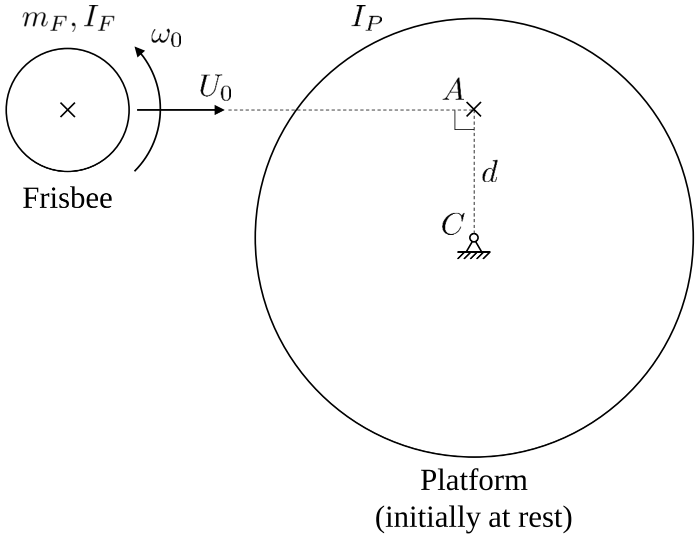

Figure 3

9 A rigid uniform plank of length $3a$ is initially at rest and held at a height h above a horizontal rail as shown in Figure 4. When released it falls with zero angular velocity before hitting the rail at a distance a from one end.

(a) Find the velocity v_1 of the plank immediately prior to its first impact with the rail;

(b) Hence find an expression for the angular velocity of the plank immediately after the impact in terms of the pre-impact velocity v_1, given that the coefficient of restitution between the plank and the rail is e (where $0 \leq e \leq 1$);

(c) For the cases when $e = 1/3$ and $e = 1$, find the greatest height reached by the midpoint of the plank during its subsequent behaviour. Is any energy lost in the impact when $e = 1$?

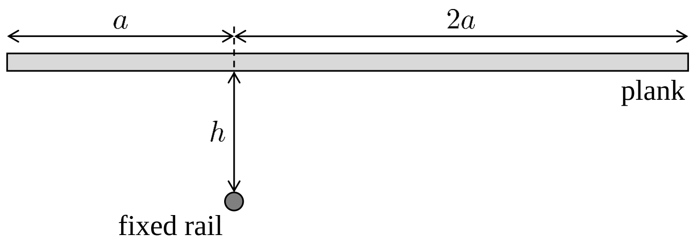

Figure 4

Computing Help

The Python examples are very easy to run online without any installation:

1. Go to: https://notebooks.azure.com/torebutlin/libraries/ia-mechanics
2. click on the relevant template file;
3. click on the 'clone' button (near top left);
4. if needed: log in to Azure using your Raven account;
5. agree to creating a clone when prompted;
6. click on the relevant template file again: this will start a working iPython Notebook that you can run and edit.

You can also run the files locally by installing Python. The most straightforward way is to download 'Anaconda' from: https://www.anaconda.com/download/. Once installed, then open the 'Jupyter Notebook' app from the start menu found inside the Anaconda folder. You can navigate to the folder where you are keeping your `*.ipynb` files and open the templates.

Suitable past Tripos questions

Equations of motion and D'Alembert forces: IA 2018 Q7; IA 2017 Q8; IA 2013 Q8; IB 2017 Q2; IB 2017 Q6; IB 2016 Q6; IB 2015 Q1; IB 2014 Q4; IB 2013 Q6
Momentum and energy: IA 2014 Q7; IB 2017 Q1; IB 2017 Q5; IB 2016 Q2; IB 2015 Q4; IB 2014 Q1

Answers

1(a). $I_P = \dfrac{mL^2}{3}$

1(b). Equation of motion: $2L\ddot{\theta} + 3g\sin\theta = 0$

1(c). General solutionfor small θ: $\theta = A\sin(\omega t) + B\cos(\omega t)$

1(d). $V = \dfrac{mgL}{2}(1-\cos\theta)$

1(e). Two distinct equilibria, $\theta = 0$ is stable.

2(a)(i). $\ddot{\mathbf{r}}_G = \dfrac{L}{2}\ddot{\theta}\mathbf{e}_\theta - \dfrac{L}{2}\dot{\theta}^2\mathbf{e}_r$

2(a)(iii). Equation of motion: $2L\ddot{\theta} - 3g\sin\theta = 0$. Angular velocity $\dot{\theta} = \sqrt{\dfrac{3g}{L}(1-\cos\theta)}$.

2(a)(iv). $R_H = -\dfrac{3mg}{2}$ (i.e. in $-\mathbf{i}$ direction)

2(b). $\theta \approx 48.2$ degrees

3(a). $\ddot{\theta} = \dfrac{3ga\cos\theta}{L^2+3a^2}$, $\dot{\theta} = \sqrt{\dfrac{6ga}{L^2+3a^2}\sin\theta}$

4(a). $I_G = \dfrac{mL^2}{2}$

4(b). $F = \dfrac{mL^2}{8r^2}(\ddot{x}_1 - \ddot{x}_2)$

5(a). Acceleration $a = \dfrac{mgR^2 \sin\alpha}{mR^2 + J}$

5(b). Solid.

5(c). Both cylinders will have the same kinetic energy $KE = mgh$.

5(d). Hollow: $v_h = \sqrt{gh}$; Solid: $v_s = 2\sqrt{gh/3}$

6(a). $\theta = \dfrac{J\Omega^2}{2Q}$

6(b). $T = \dfrac{J\Omega}{Q}$

7(a). $\Omega = \dfrac{3I}{ML}$

7(b). $R = \dfrac{I}{2}$

7(c). Distance from pivot $x = \dfrac{2L}{3}$

8(a). $h = I_F\omega_0 - m_F U_0 d$, positive anticlockwise

8(b). $I_{\text{combined}} = I_P + I_F + m_F d^2$

8(c). $\Omega = \dfrac{I_F\omega_0 - m_F U_0 d}{I_P + I_F + m_F d^2}$, positive anticlockwise

8(d). $I_F\omega_0 > m_F U_0 d$

9(a). $v_1 = \sqrt{2gh}$

9(b). $\Omega_2 = \dfrac{v_1(1+e)}{2a}$

9(c). For $e = 1/3$, $h_2 = 0$; for $e = 1$, $h_2 = h/4$.