Engineering **FIRST YEAR**

Part IA Paper 4: Mathematical Methods
EXAMPLES PAPER 10
Partial Differentiation

(Elementary exercises are marked †, problems of Tripos standard ∗)

Partial Derivatives, Chain Rule, Perfect Differentials

†1. Evaluate $\frac{\partial f}{\partial x}$; $\frac{\partial f}{\partial y}$; $\frac{\partial^2 f}{\partial x^2}$; $\frac{\partial^2 f}{\partial y^2}$; $\frac{\partial^2 f}{\partial x \partial y}$; $\frac{\partial^2 f}{\partial y \partial x}$ for the functions: (i) x^2y^5; (ii) $x \sin y$.

2. Figure 1 shows values of a continuous function $z = f(x, y)$ at points in the vicinity of the point $x = 2$, $y = 5$. Deduce approximate values for $\frac{\partial^2 f}{\partial x^2}$; $\frac{\partial^2 f}{\partial y^2}$; $\frac{\partial^2 f}{\partial x \partial y}$; and $\frac{\partial^2 f}{\partial y \partial x}$ at the point $x = 2$, $y = 5$.

   Show that this type of approximation always gives the same value for $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$.

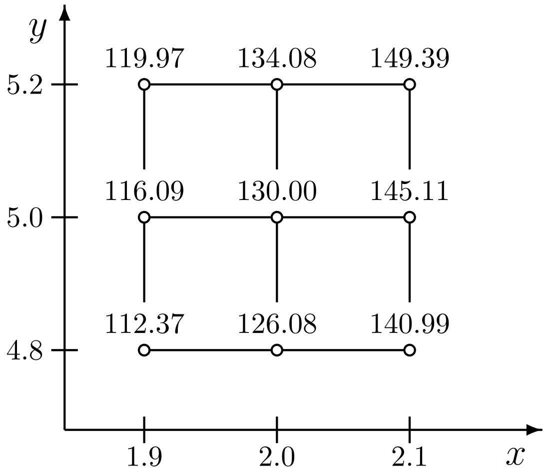

Figure 1: Values of z in the vicinity of $x = 2$, $y = 5$.

3. For what value of n is $\theta = t^n \exp\left(\frac{-r^2}{4t}\right)$ a solution of the equation $\frac{\partial}{\partial r}\left[r^2 \frac{\partial \theta}{\partial r}\right] = r^2 \frac{\partial \theta}{\partial t}$?

4. Given that $w = xyz$, where $x = \cos\theta \sin\phi$, $y = \sin\theta \sin\phi$ and $z = \cos\phi$, evaluate $\left(\frac{\partial w}{\partial \theta}\right)_\phi$

   (a) by substitution before differentiation;

   (b) from the chain rule $\left(\frac{\partial w}{\partial \theta}\right)_\phi = \left(\frac{\partial w}{\partial x}\right)_{y,z}\left(\frac{\partial x}{\partial \theta}\right)_\phi + \left(\frac{\partial w}{\partial y}\right)_{z,x}\left(\frac{\partial y}{\partial \theta}\right)_\phi + \left(\frac{\partial w}{\partial z}\right)_{x,y}\left(\frac{\partial z}{\partial \theta}\right)_\phi$.

5. Explain the term *perfect differential* (also known as an *exact* differential).

   Given that $dh = T\, ds + v\, dp$ is a perfect differential, show by considering $\frac{\partial^2 h}{\partial s \partial p}$ that

$$\left(\frac{\partial T}{\partial p}\right)_s = \left(\frac{\partial v}{\partial s}\right)_p .$$

   Define $g = h - Ts$ and derive in the same way the relation

$$\left(\frac{\partial v}{\partial T}\right)_p = -\left(\frac{\partial s}{\partial p}\right)_T .$$

Gradients, Normals, Maxima and Minima

6. The temperature in a given region of the x-y plane is determined by $T(x, y) = 2x^2 - 3xy$. Find
   (a) the temperature gradient at $(1, 1)$ at $30°$ to the x-axis;
   (b) the temperature gradient at $(1, 1)$ along the curve $y = x^3$ in the direction of increasing x;
   (c) the value and direction of the largest temperature gradient at $(1, 1)$.

†7. Find the gradient of the function $w = 2xz^2 - 3xy - 4x$. Determine the equation of the tangent plane to a level surface which passes through the point $(1, -1, 2)$.

8. Draw a rudimentary contour map of the function $z = xy(2 - x - 2y)$. (Hint: first draw lines for $z = 0$ and then think about the values of z in the regions between these contours.) Check your answer by plotting the contour map in Python. (See hint below.)

   Find the stationary points of the function. Determine from the sketch which are maxima, minima or saddle points. Check your answer by applying the test on p.5 of the maths data book.

9.* A surface is defined in terms of two parameters u and v by the vector relation $\mathbf{r} = \mathbf{F}(u, v)$, where $\mathbf{r}$ is the position vector of a point on the surface with respect to a fixed origin $\mathbf{0}$.

   Explain carefully why the vectors $\frac{\partial \mathbf{r}}{\partial u}$ and $\frac{\partial \mathbf{r}}{\partial v}$ are tangential to the surface.

   If $\mathbf{r} = (u^2 + v)\,\mathbf{i} + 2uv\mathbf{j} + (u + v^2)\mathbf{k}$, find $\frac{\partial \mathbf{r}}{\partial u}$ and $\frac{\partial \mathbf{r}}{\partial v}$ at the point $u = -1, v = -1$. Deduce the unit surface normal vector $\mathbf{n}$ at this point, and show that the surface lines u=constant and v=constant intersect orthogonally at this point.

Answers

1. (i) $2xy^5,\ 5x^2y^4,\ 2y^5,\ 20x^2y^3,\ 10xy^4,\ 10xy^4$.
   (ii) $\sin y,\ x\cos y,\ 0,\ -x\sin y,\ \cos y,\ \cos y$.
2. 120, 4, 10, 10.
3. $n = -1.5$.
4. $\cos 2\theta \sin^2\phi \cos\phi$.
6. (a) $(\sqrt{3} - 3)/2$; (b) $-8/\sqrt{10}$; (c) $\sqrt{10}$ in direction $(\mathbf{i} - 3\mathbf{j})/\sqrt{10}$.
7. $\nabla w = (2z^2 - 3y - 4)\mathbf{i} - 3x\mathbf{j} + 4xz\mathbf{k}$. Tangent plane: $7x - 3y + 8z = 26$.
8. saddle points at (0,0), (2,0), (0,1); maximum at $\left(\frac{2}{3}, \frac{1}{3}\right)$.
9. $\mathbf{n} = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$.

Hint for Python component of Question 8.

First, we need to sample z on a two-dimensional grid, just like the grid in Figure 1. Suppose we are interested in x values in the range -2 to 4, and we want to sample z at intervals of 0.1. First of all, in a Jupyter notebook, we set up a suitable vector of x and y values that we wish to sample using:

```
import numpy as np
x = np.arange(-2, 4, 0.1)
y = np.arange(-2, 4, 0.1)
```

Next, we stretch these values over the two-dimensional grid using:

```
X, Y = np.meshgrid(x, y)
```

This produces matrices `X` and `Y` containing the x and y coordinates respectively of every point on the grid. Finally, we produce the grid of z values using:

```
Z = X*Y*(2 - X - 2*Y)
```

Note that in Python/NumPy these operations are element-by-element (and not matrix multiplication). Next we want to plot the function using Matplotlib

```
import matplotlib.pyplot as plt
%matplotlib inline
plt.contour(X, Y, Z)
```

If fine control is required over the number of contours shown then could specify the range and step. For example,

```
plt.contour(X, Y, Z, np.arange(-1, 1, 0.1))
```

will show all contours just in in the range -1 to 1, at intervals of 0.1. Of course you can alternatively explicitly specify the points to be evaluated. The command

```
plt.contour(X, Y, Z, 20)
```

will draw contours up to 20 automatically-chosen levels. To label the legend on each contour, you can use:

```
p = plt.contour(X, Y, Z)
plt.clabel(p, inline=1, fontsize=10)
```

Note that you may find that you want to change the global default size of the figures produced by Matplotlib. The global default figure size can be set to 10×10 by

```
plt.rc('figure', figsize=(10,10))
```

Note that there are also Python libraries to draw 3D surface plots: if interested you can explore the use of `mplot3d` surface plots.

Suitable past Mathematics IA Tripos questions: Q10 1996-2000, Q9 2001-4, Q7 2005-6, Q10 2007-8, Q6 2009, Q7 2010, Q10 2011-13, Q9 2014-16, Q6 2017, Q10 2018.

PCW, Easter 2019