Paper 1: Mechanical Engineering

Examples Paper 2

Elementary exercises are marked †, problems of Tripos standard $*$.
Answers can be found at the back of the paper.

Equations of motion

1 † A ball of mass m is thrown from the ground with initial horizontal velocity u_0, and initial vertical velocity v_0 at time $t = 0$.

(a) If there is no resistance to motion, what is the vector equation of motion of the ball after it has been released?

(b) Show that the height y of the ball above the ground at time t is given by:

$$y = v_0 t - \frac{gt^2}{2}$$

(c) Find the position vector of the ball as a function of time t;

(d) If $v_0/g = 0.8\,\mathrm{s}$, find the time t at which the ball first strikes the ground;

2 A particle of mass m is given an initial speed u_0 at the bottom of a frictionless loop-the-loop of radius r. The loop-the-loop is circular and is in a vertical plane.

(a) Draw a Free Body Diagram of the particle at an arbitrary angular position θ around the loop;

(b) By differentiating the position vector in polar coordinates (taking the origin to be the centre of the loop), derive a vector expression for the acceleration of the particle and verify that your answer agrees with the Mechanics Databook;

(c) Find an expression for the normal reaction force acting on the particle;

(d) Derive the equation of motion of the particle;

(e) Use the Python template `p2q2_template.ipynb` to compute a numerical solution to the equation of motion. See computing help at the end of the examples paper for more information.

(f) How could you check if your numerical simulation is giving a trustworthy solution?

Potential energy and stability

3 A proposed design for a bistable spring is shown in Figure 1: the aim is to design a mechanical device that can be at rest in one of two positions. It consists of a light inextensible rod AB connected to a pivot A via a torsional spring. The spring applies a torque T proportional to the angle of rotation such that $T = K\theta$, where K is the torsional stiffness of the spring. The other end of the rod B is connected via a linear spring to point O, such that the tension force in the spring is $F = kx$, where k is the linear spring stiffness and x is the extension of the spring OB (assuming its natural length is zero).

(a) Find an expression that relates x and θ;

(b) Find an expression for the potential energy of the system as a function of θ [hint: the potential energy of a torsional spring is given by $V = \frac{1}{2}K\theta^2$];

(c) How many equilibria are there and what does your answer depend on?

(d) Under what conditions is the switch unstable at $\theta = 0$?

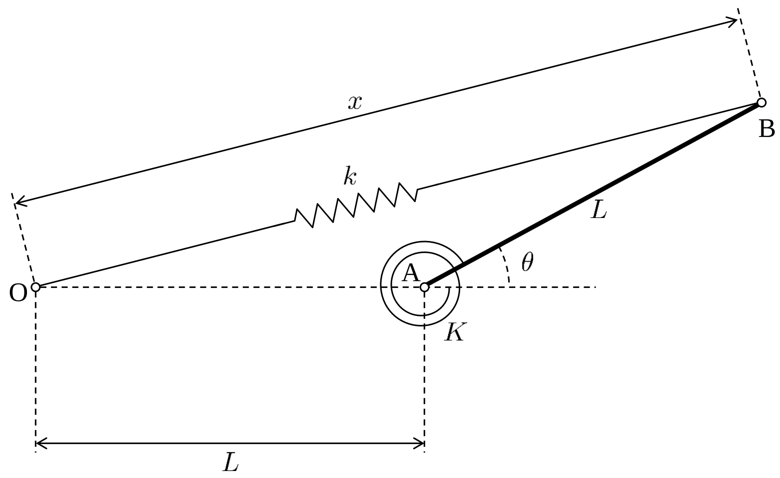

Figure 1

Momentum and Energy

4 A particle of mass m_a and initial velocity u_a collides with a second particle m_b that is initially at rest.

(a) Write down an expression for the velocities of the two particles after the collision. Why can the individual velocities not be determined from momentum considerations alone?

(b) Is it possible for the first particle to have a larger velocity after the collision than before? Why or why not?

(c) If the collision is perfectly elastic (i.e. the coefficient of restitution $e = 1$), what ratio of masses m_b/m_a results in the second particle having a large velocity and what is the limiting velocity? [Hint: recall the velocity result for elastic 1D collisions]

5 Two balls are released from rest at the same time from height h: the upper ball has mass m while the lower ball has mass M: the upper ball is much lighter than the lower ball, i.e. $m \ll M$, and there is a small gap between the balls at the time of release. Assume all collisions are elastic (with coefficient of restitution $e = 1$). Find an approximate expression for the maximum height reached by the lighter ball (treating the balls as particles of negligible size compared to h).

Variable mass dynamics

6 The Space Shuttle has a mass of 2000 tonnes on take-off. It ejects fuel at a constant rate of 9000 $\mathrm{kgs^{-1}}$ for the first 30 s of its flight. The initial acceleration of the Shuttle at take-off is 1.2 g, where $g = 9.81\,\mathrm{ms^{-1}}$.

(a) What is the speed relative to the Shuttle that the fuel is being ejected?

(b) Assuming the mass of the Shuttle remains constant, what is its acceleration and speed after 30 s?

(c) $*$ Calculate the acceleration and speed of the Shuttle after 30 s, assuming the mass does not remain constant.

7 $*$ Coffee beans are being dropped from rest onto the scale pan of an electronic balance from fixed height h and at a constant rate $\dot{m}$. They do not bounce. The flow of beans is cut off at source when the balance reads the exact weight required. Show that the weight of beans in the air at that instant will exactly compensate for the false reading of the balance caused by the change in momentum of the falling beans.

Angular Momentum

8 Figure 2 shows a frictionless horizontal plane upon which a particle A of mass m slides. This particle is attached to a similar particle B by a light inextensible string, which passes through a small frictionless hole. The string is of sufficient length that particle B always remains clear of the plane. Particle B moves in a vertical line at all times.

Particle A is a distance r from the hole when it is projected with a speed V_1 perpendicular to the string. Some time later the particle has reached a distance of $2r$ from the hole and its speed is now V_2, again perpendicular to the string.

(a) † Determine an expression in terms of V_1 for the speed V_2 of the particle in the new position;

(b) Find an expression for V_1 in terms of the gravitational acceleration g and r;

(c) For these two positions do you expect the string tension to be equal to mg?

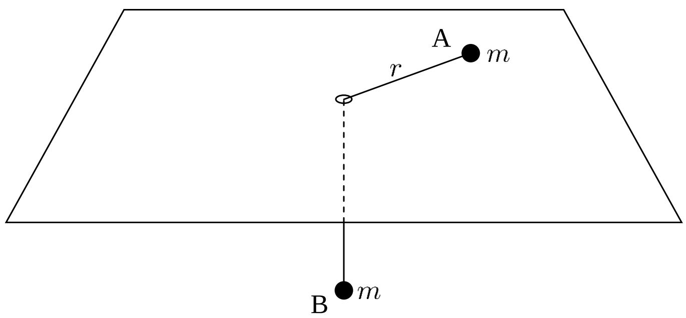

Figure 2

9 (a) In the motion of a particle, under what circumstances is:

(i) total mechanical energy conserved?

(ii) moment of momentum conserved about an axis?

Note that moments taken about an axis and about a point are different: see the lecture notes, or discuss with your supervisor.

(b) A smooth conical vessel with a cone angle of $\theta = 70°$ and a height of $h_2 = 2$ m is fixed with its vertex downwards and its axis vertical, as shown in Figure 3. A particle is projected with horizontal velocity v_1 on the inner surface at a height $h_1 = 1$ m measured vertically above the vertex. Show that if v_1 is approximately equal to 5.1 ms^{-1} the particle will *just* remain inside the vessel. Will v_1 be different if a cone with a different cone angle is used?

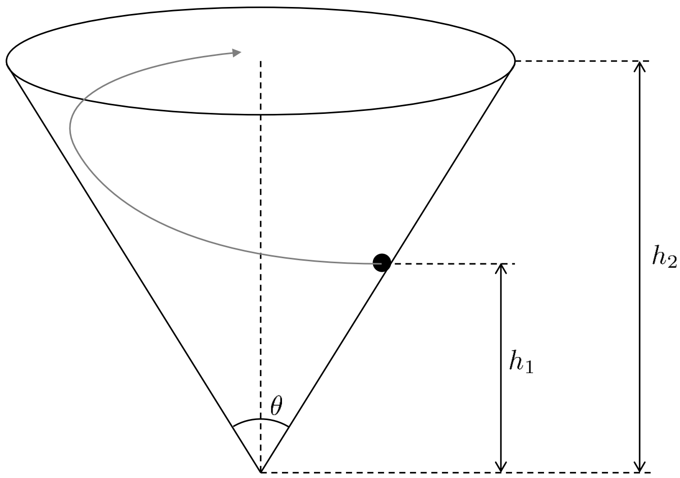

Figure 3

Satellite motion

10 A satellite of mass 2000 kg is in elliptical orbit about the earth. At its perigee (point of closest approach) it has an altitude of 1100 km and a speed of 7900 ms^{-1}. The earth's radius is 6400 km and g at the earth's surface is $g = 9.81\,\mathrm{ms}^{-2}$.

(a) What is the energy of the satellite? How much work must be done on the satellite to put it into orbit?

(b) If the 'burn' of the launching rocket is 10 minutes, at what rate (i.e. power) does the satellite gain energy?

(c) What is the altitude and speed of the satellite at its apogee (the point on its orbit furthest from the earth)?

(d) What is the eccentricity of its orbit and the length of the minor axis of its elliptical path?

11 An artificial satellite of mass m is transferred from one circular orbit at speed v_P to another at speed $v_P/7$ by exerting a short-duration thrust (impulse I_P) tangentially at P, as illustrated in Figure 4 (not to scale).

(a) Using the equations of motion for the satellite in its two circular orbits, determine the ratio of the radii r_2/r_1 of the two orbits;

(b) The impulse I_P at P increases the speed of the satellite from v_p to v_1. By considering the motion from P to A, determine the speed of arrival v_2 at A in terms of v_p;

(c) Determine the magnitude of the impulse I_P in terms of m and v_p.

(d) Another short duration thrust (impulse I_A) is required at A to enter a circular orbit of radius r_2. Determine the magnitude and direction of this impulse in terms of m and v_p.

(e) What is the shape of the path traced from P to A? What happens to the satellite if no impulse is delivered at A?

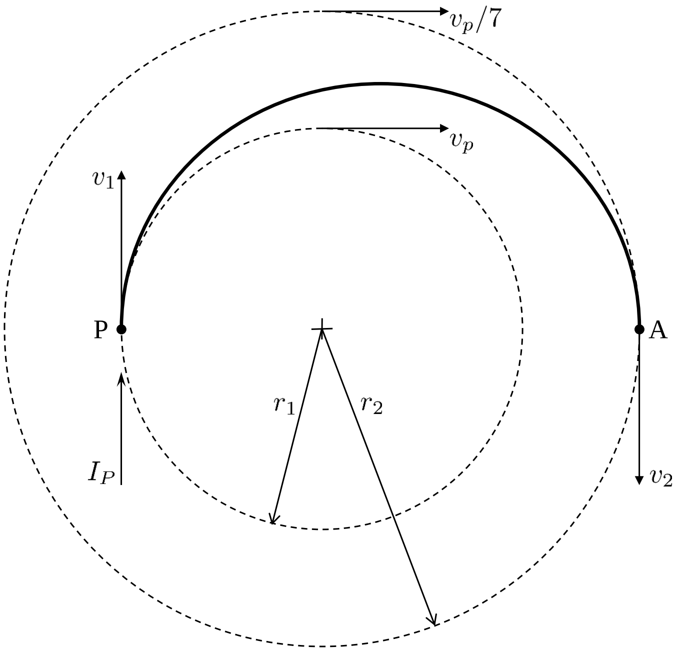

Figure 4

Computing Help

The Python examples are very easy to run online without any installation:

1. Go to: https://notebooks.azure.com/torebutlin/libraries/ia-mechanics
2. click on the relevant template file;
3. click on the 'clone' button (near top left);
4. if needed: log in to Azure using your Raven account;
5. agree to creating a clone when prompted;
6. click on the relevant template file again: this will start a working iPython Notebook that you can run and edit.

You can also run the files locally by installing Python. The most straightforward way is to download 'Anaconda' from: https://www.anaconda.com/download/. Once installed, then open the 'Jupyter Notebook' app from the start menu found inside the Anaconda folder. You can navigate to the folder where you are keeping your `*.ipynb` files and open the templates.

Suitable past Tripos questions

Equations of motion: IA 2015 Q8
Potential energy and stability: IA 2018 Q8
Momentum and Energy: IA 2016 Q8
Variable mass dynamics: IA 2014 Q9
Angular Momentum: IA 2017 Q11
Satellite motion: IA 2018 Q10; IA 2016 Q10; IA 2013 Q9

Answers

1(a). $m\ddot{\mathbf{r}}_P + mg\mathbf{j} = 0$

1(c). $\mathbf{r}_P = u_0 t\mathbf{i} + (v_0 t - gt^2/2)\mathbf{j}$

1(d). $t \approx 1.6\,\text{s}$

2(b). $\ddot{\mathbf{r}}_P = -r\dot{\theta}^2\mathbf{e}_r + r\ddot{\theta}\mathbf{e}_\theta$

2(c). $R = m\left(g\cos\theta + r\dot{\theta}^2\right)$

2(d). $r\ddot{\theta} = -g\sin\theta$

3(a). $x^2 = 2L^2(1+\cos\theta)$

3(b). $V = \frac{1}{2}K\theta^2 + kL^2(1+\cos\theta)$

3(c). HINT: equate the two terms of the potential energy expression and sketch them, considering the effect of changing the stiffness K.

3(c). One equilibrium if $K \geq kL^2$, or $\{3, 5, 7, \dots\}$ equilibria if $K < kL^2$.

3(d). $\theta = 0$ equilibrium unstable if $K < kL^2$.

4(a). $m_a u_a = m_a v_a + m_b v_b$

4(b). Yes.

4(c). Second particle has high velocity if $m_b \ll m_a$, limiting velocity is $v_b \to 2u_a$.

5. Maximum height $H = 9h$.

6(a). Relative velocity $u = 4796\,\text{ms}^{-1}$

6(b). Acceleration $a = 1.2g$, speed $v = 353\,\text{ms}^{-1}$

6(c). Acceleration $a = 15.14\,\text{ms}^{-2}$, speed $v = 401\,\text{ms}^{-1}$

8(a). $V_2 = V_1/2$

8(b). $V_1 = \sqrt{8gr/3}$

8(c). No.

10(a). Work Done $W = 8.08 \times 10^{10}\,\text{J}$.

10(b). $P = 135\,\text{MW}$

10(c). Altitude $A = 4062\,\text{km}$

10(d). Eccentricity $e = 0.165$; Length of minor axis $2b = 17.7 \times 10^3\,\text{km}$.

11(a). $r_2/r_1 = 49$

11(b). $v_2 = v_P/35$

11(c). $I_P = 2mv_P/5$

11(d). $I_A = +4mv_P/35$ in same direction as v_2.

11(e). Ellipse.