Examples crib E/m 3/1

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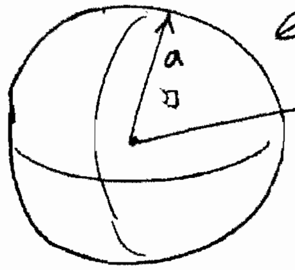


uniform sphere of charge, density ρ_v Cm^{-3}

Flux of D conserved across the boundary

Inside sphere $0 \le r \le a$ spherical gaussian surface

Gauss' Law $\Rightarrow$ Flux of D $D \times 4\pi r^2 = Q_{total}$

$$Q_{total} = \underbrace{4/3\,\pi r^3}_{\text{Vol of sphere}} \times \rho_v \quad C$$

$$\Rightarrow \quad D\,\cancel{4\pi}\,\cancel{r^2} = \frac{\cancel{4}}{3}\cancel{\pi}\, r^{\cancel{3}}\rho_v \qquad D = \frac{\rho_v}{3} r$$

$(D \propto r)$

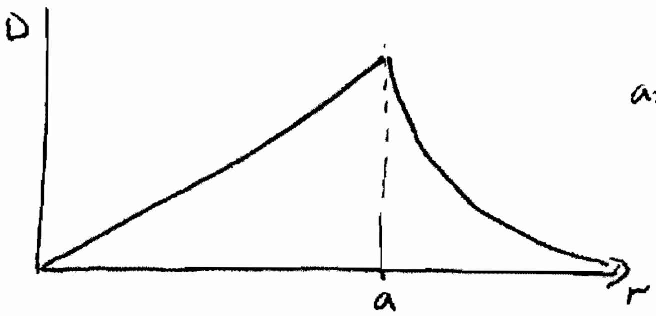
Outside sphere $a < r \le \infty$ spherical gaussian surface

Gauss' Law Flux of D $D \times 4\pi r^2 = Q_{tot}$

$$Q_{tot} = \frac{4}{3}\pi a^3 \rho_v$$

$$\Rightarrow \quad D\,\cancel{4\pi}\, r^2 = \cancel{4}/3\,\cancel{\pi} a^3 \rho_v \qquad D = \frac{a^3\rho_v}{3}\frac{1}{r^2} \quad (D \propto 1/r^2)$$

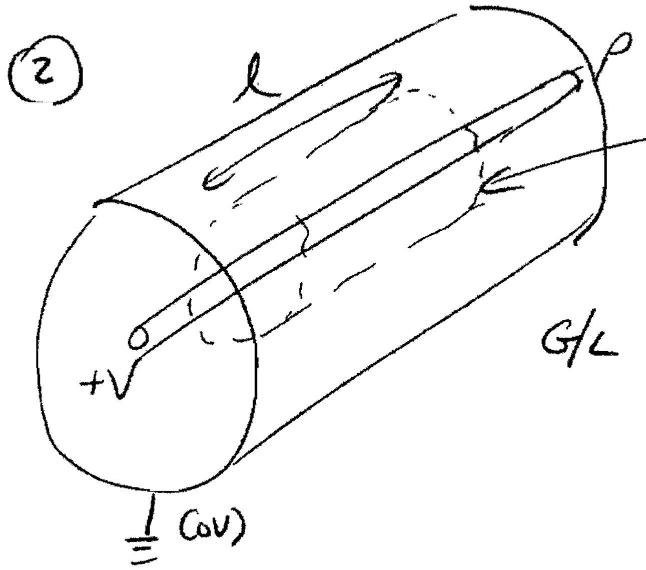
Plot of D vs r



at $r = a$ $D = \frac{a\rho_v}{3}$

$\Rightarrow$ continuity.

(2)



Gaussian surface to enclose charge on line

G/L flux of $D = Q$

$$D \cdot 2\pi r \cancel{l} = \rho \cancel{l}$$

$$\underline{D = \frac{\rho}{2\pi r}}$$

$$E = \varepsilon_0 \varepsilon_r D \quad \Rightarrow \quad E = \frac{\rho}{2\pi \varepsilon_0 \varepsilon_r r}$$

$$\delta V = -\int E\,dr \quad 0 - V = -\int_{r_1}^{r_2} \frac{\rho}{2\pi \varepsilon_0 \varepsilon_r r}\,dr$$

$$V = \frac{+\rho}{2\pi \varepsilon_0 \varepsilon_r}\Big[\ln(r)\Big]_{r_1}^{r_2} = \frac{\rho}{2\pi \varepsilon_0 \varepsilon_r}\ln(r_2/r_1)$$

Capacitance $Q = CV$ $(\rho = C_l V)$

$$\Rightarrow C_l = \frac{\rho}{V} = \frac{2\pi \varepsilon_0 \varepsilon_r}{\ln(r_2/r_1)} \quad \mathrm{Cm}^{-1}$$

* The maximum E-field will be at $r = r_1$

$$\Rightarrow E_{max} = \frac{\rho}{2\pi \varepsilon_0 \varepsilon_r r_1} \qquad \frac{\rho}{2\pi \varepsilon_0 \varepsilon_r} = \frac{V}{\ln(r_2/r_1)}$$

$$\Rightarrow E_{max} = \frac{V}{r_1 \ln(r_2/r_1)}$$

We want the minimum E_{max} as a function of inner radius r_1. This is equivalent to finding the maximum of $1/E_{max}$ in terms of r_1

$$\Rightarrow \quad \frac{\partial(1/E_{max})}{\partial r_1} = 0$$

$\ln(r_2/r_1) = \ln(r_2) - \ln(r_1)$

$$\frac{\partial(1/E_{max})}{\partial r_1} = \frac{\partial}{\partial r_1}\left(\frac{r_1 \ln(r_2/r_1)}{V} \right)$$

$$\Rightarrow \quad \frac{\ln(r_2/r_1)}{V} + \frac{\cancel{r_1}(0 - 1/\cancel{r_1})}{V} = \frac{1}{\cancel{V}}\left[\ln\left(\frac{r_2}{r_1}\right) - 1\right] = 0$$

$$\Rightarrow \quad \ln\left(\frac{r_2}{r_1}\right) = 1 \qquad \frac{r_2}{r_1} = e$$

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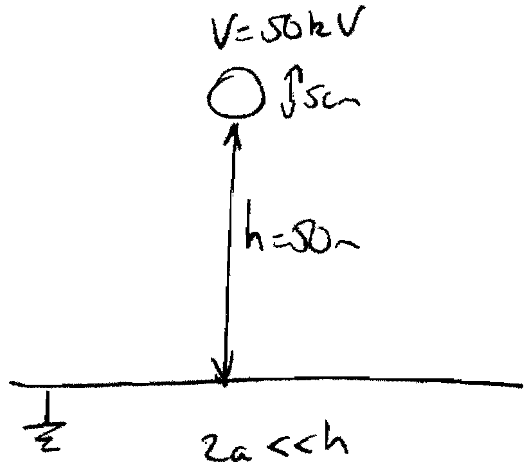
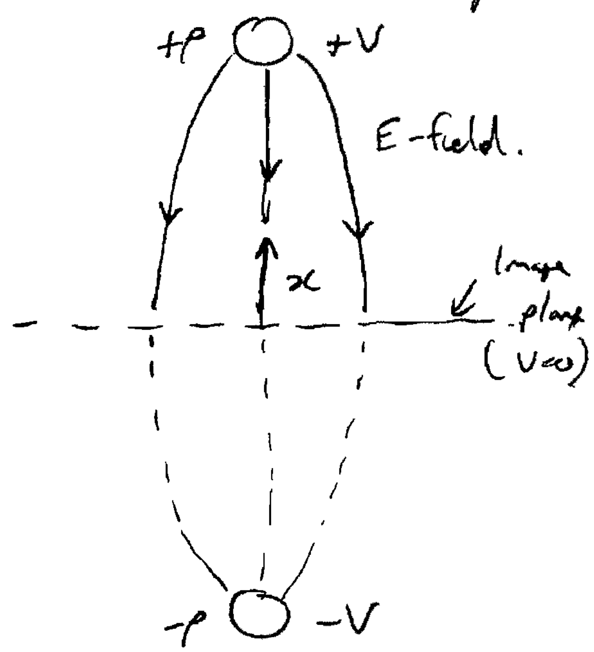


Method of Images



E-field at point x is given by the sum of the 2 charges $+\rho$ & $-\rho$ in the mirror system

From previous example G/L $E = \dfrac{\rho}{2\pi\varepsilon_0 r}$ $(\varepsilon_r = 1)$

$$\Rightarrow \quad E = \frac{+\rho}{2\pi\varepsilon_0(h-x)} - \frac{-\rho}{2\pi\varepsilon_0(h+x)}$$ (minus x direction)

$$= \frac{\rho}{2\pi\varepsilon_0}\left[\frac{1}{h-x} + \frac{1}{h+x}\right]$$

Integrate along E-field (minus ~~E~~ direction) from $x=0$ to obtain potential w/r earth plane (0V)

$$\Rightarrow \quad V - 0 = -\int_0^{h-a} -E\,dx$$

$$= \frac{\rho}{2\pi\varepsilon_0}\int_0^{h-a}\left[\frac{1}{h-x} + \frac{1}{h+x}\right]dx$$

$$= \frac{\rho}{2\pi\varepsilon_0}\Big[-\ln(h-x) + \ln(h+x)\Big]_0^{h-a}$$

$$= \frac{\rho}{2\pi\varepsilon_0}\ln\left(\frac{2h-a}{a}\right) \approx \frac{\rho}{2\pi\varepsilon_0}\ln\left(\frac{2h}{a}\right) \quad (a \ll h)$$

We want E-field at $x=0$ $\Rightarrow$ $\dfrac{\rho}{2\pi\varepsilon_0} = \dfrac{V}{\ln(2h/a)}$

$$E(x=0) = \frac{\rho}{2\pi\varepsilon_0}\left[\frac{1}{h} + \frac{1}{h}\right] = \frac{\rho}{2\pi\varepsilon_0}\frac{2}{h} = \frac{2V}{h\ln(2h/a)} = 241\ \mathrm{Vm^{-1}}$$

$$C_l = \frac{\rho}{V} = \frac{2\pi\varepsilon_0}{\ln(2h/a)} = 6.7\,pF\,m^{-1}$$

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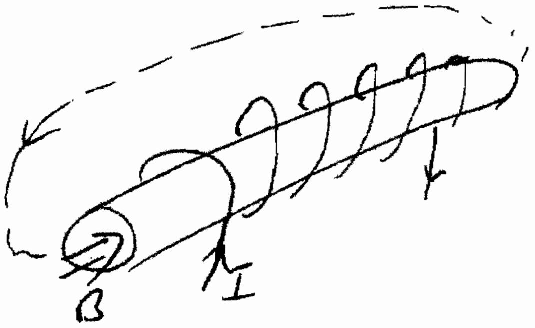

Ampère's law for an infinitely long solenoid. Circulation is along the length of the solenoid from $+\infty$ to $-\infty$. Or n per unit length

For length l $\quad \underbrace{B.l}_{\text{circulation along length } l} = \mu_0 (Nl) I$ $\quad$ No. of turns in length l

$$\Rightarrow \quad B = \mu_0 N I$$

Flux of B $\quad \phi = \mu_0 N I \pi r^2$ $\quad$ area of cross-section

We require $\phi = 10^{-3}$ Wb $\qquad r = 5 \times 10^{-2}$ m

$N = 1000$ Turns m^{-1}

$$\Rightarrow \quad I = 101\ A$$

$$B = \frac{\phi}{\pi r^2} = 0.13\ T$$

If the solenoid is full of soft iron then we must use Ampère for H.

$$H\cancel{l} = (N\cancel{l}) I \qquad B = \mu_0 \mu_r H$$

Assuming that the soft iron does not saturate then ~~the~~ B will be μ_r times larger in the solenoid. If μ_r saturates then we must plot an B-H curve to solve for non linear μ_r

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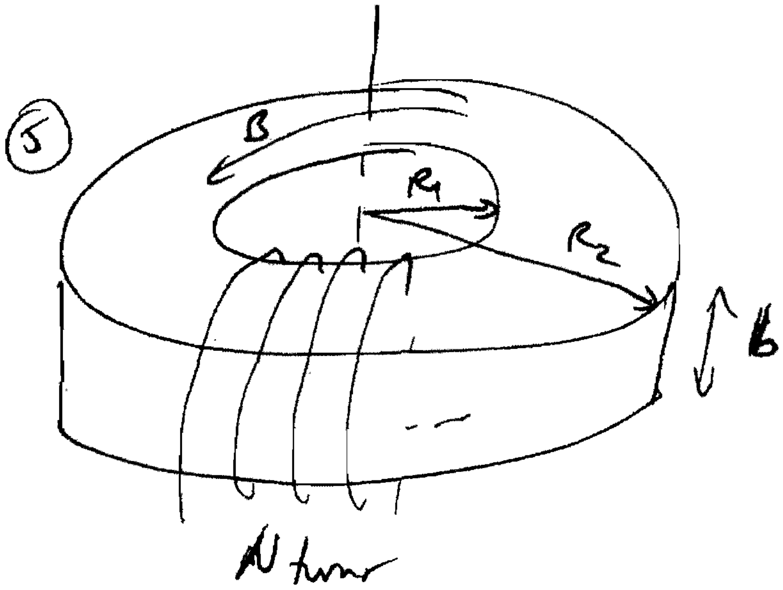
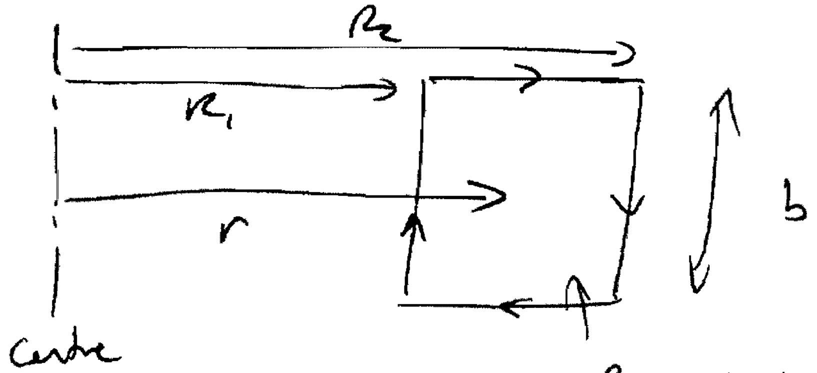


Consider a x section through the toroid passing through the centre

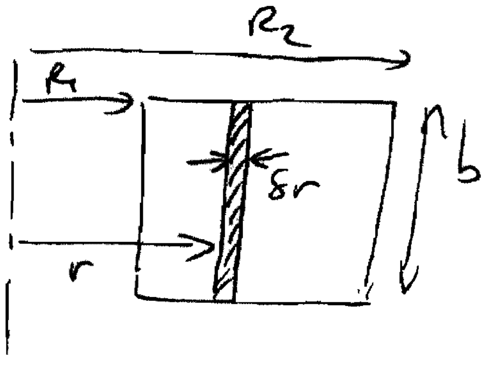


B is into the page
is a function of r

Apply Ampère's law around the toroid circulation through N turns of path length $2\pi r$

$$\Rightarrow \quad 2\pi r B = \mu_0 N I$$

$$B = \frac{\mu_0 N I}{2\pi r}$$



Now we consider the flux of B through a thin strip of area

$\delta\phi = \delta r \; b \; B$

$$\delta\phi = \frac{\delta r\, b \mu_0 N I}{2\pi r}$$ integrate to get total flux ϕ

$$\phi = \int_{R_1}^{R_2} \frac{\mu_0 N I b}{2\pi r}\, dr$$

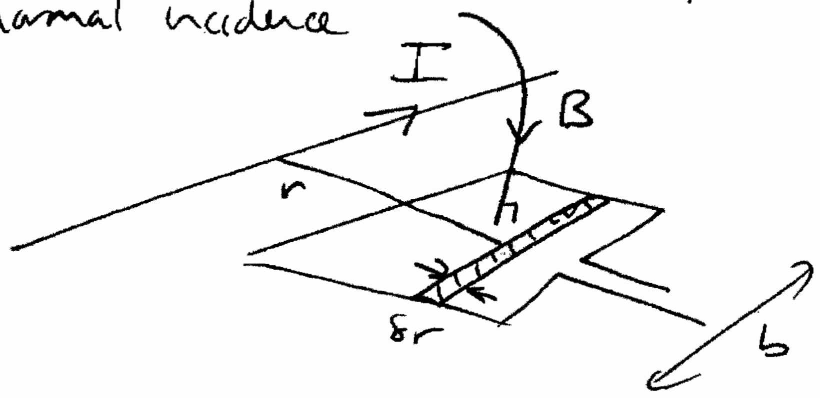
$$= \frac{\mu_0 N I b}{2\pi} \ln(R_2/R_1)$$

N turns $\Rightarrow$ total flux linkage $\phi' = N\phi$

$$= \frac{\mu_0 N^2 I b}{2\pi} \ln(R_2/R_1)$$

Self inductance $L = \dfrac{\phi'}{I} = \dfrac{\mu_0 N^2 b}{2\pi} \ln(R_2/R_1)$

(6) The flux density B due to the current I in the conductor intersects the plane of the coil at normal incidence



Single wire $\Rightarrow N=1$

Ampère's Law ($N=1$ for wire)

$$2\pi r B = \mu_0 I \qquad B = \frac{\mu_0 I}{2\pi r}$$

We can calculate the flux of B through the coil by finding an element of flux $\delta\phi$ passing through a strip of the coil of width δr

$$\Rightarrow \delta\phi = \delta r\, b\, B \qquad \text{integrate to get } \phi$$

$$\Rightarrow \phi = \int_{s}^{s+a} \frac{\mu_0 I}{2\pi r}\, b\, dr$$

$$= \frac{\mu_0 I b}{2\pi} \ln\left(\frac{s+a}{s}\right)$$

For N turns of the coil the flux linkage ϕ' will be $N\phi$

$$\Rightarrow \phi' = \frac{\mu_0 N I b}{2\pi} \ln\left(\frac{s+a}{a}\right)$$

Mutual inductance $M = \dfrac{\phi'}{I} = \dfrac{\mu_0 N b}{2\pi} \ln\left(\dfrac{s+a}{s}\right)$

$$= 8.75 \times 10^{-7}\ H$$

The current I induces an emf in the coil by Faraday's Law

$$V(t) = \frac{d\phi'}{dt} = M\frac{dI}{dt}$$

$V_{rms} = M\omega I_{rms}$ $\qquad I_{rms} = \dfrac{V_{rms}}{M 2\pi f} = 247A\ rms$

comes from the rms average of I $(= I_0 \cos(\omega t))$