Engineering Tripos **FIRST YEAR**

Part IA Paper 2: Structures and Materials

MATERIALS

Examples Paper 2 – Elasticity of Materials

Mechanics of Elastic Deformation,
Microstructural Origin and Manipulation of Elastic Properties,
Stiffness-limited Design

Straightforward questions are marked with a †
Tripos standard questions are marked with a *
You will need to use the Materials Databook.

<u>*Mechanics of Elastic Deformation*</u>

† **1.** Figure 1 shows a typical element of material with Young's modulus E and Poisson's ratio ν, which is subjected to normal stresses σ_1, σ_2 and σ_3. Consider the state of *hydrostatic pressure*, $\sigma_1 = \sigma_2 = \sigma_3 = -p$.

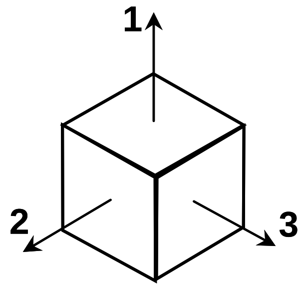

Figure 1

(a) Use the simultaneous equations for Hooke's Law in 3D to find the volumetric strain (or dilatation), Δ, under hydrostatic pressure.

(b) The *bulk modulus*, K, is defined under this loading as: $K = \dfrac{\text{Hydrostatic stress}}{\text{Volumetric strain}}$

(i) Find the expression relating K to E and ν, and check your answer in the Materials Databook.

(ii) Find an approximate relationship between K and E for metals ($\nu \approx 0.3$).

(iii) For rubber, with $\nu \approx 0.5$, find the dilatation Δ under hydrostatic pressure, and the bulk modulus, K? What do these results mean physically?

2. Figure 2a shows a sample of section area A loaded in uniaxial tension with a stress σ_1. On a plane inclined at an angle θ, the stresses acting are a normal stress σ and a shear stress τ.

(a) By considering the equilibrium of the *forces* acting in the axial and transverse directions on the shaded part of the sample, derive expressions for σ and τ in terms of σ_1 and θ.

(b) Sketch or plot the variation of the shear stress as a function of θ. In what direction is the shear stress a maximum, and what is then its value?

(c) For a square sample loaded in biaxial tension and compression, as in Figure 2b, use superposition to infer *by inspection* what the normal and shear stresses will be on the diagonal planes shown dashed. Sketch these stresses acting on a square element oriented parallel to these dashed lines, and comment on the result.

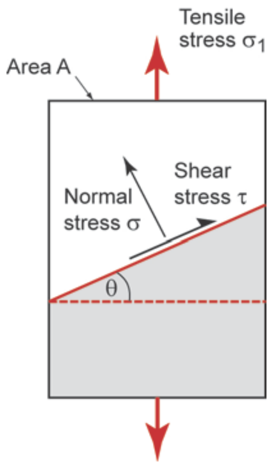

Figure 2a

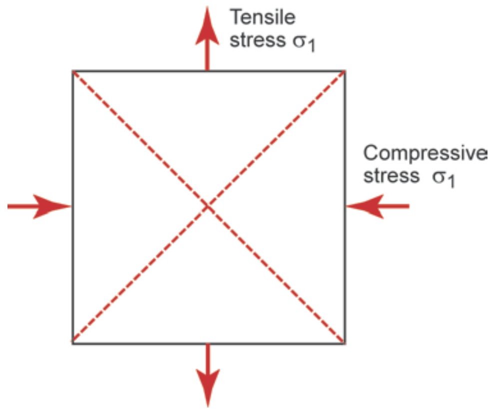

Figure 2b

3. A butyl rubber mounting is designed to cushion a sensitive device against shock loading. It consists of a cube of the rubber of side-length $L = 40\text{mm}$ located in a slot in a rigid plate, as shown in Figure 3. The slot and cube have the same cross-sectional dimensions, so that the strain in one direction is constrained to be zero. The slot surfaces are lubricated so that friction on the cube is negligible. The compressive load is $F = 50$ N.

Find the deflection δ of the block in the direction of loading, and hence the stiffness (F/δ) in this constrained condition. By what factor is the stiffness increased, compared to loading the block of rubber without transverse constraint? (Use a mid-range value of E for butyl rubber from the table in the Materials Databook, and assume $\nu = 0.5$.)

[Hint: use the expression derived in lectures for the "effective modulus" (σ_1/ε_1) in terms of Young's modulus E and Poisson's ratio ν, for the case of constraint in one transverse direction.]

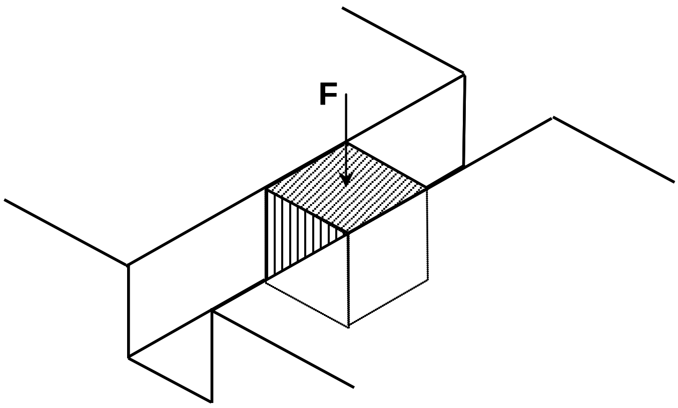

Figure 3

* **4**. A cube of linear elastic material (as in Figure 1) is again subjected to a compressive stress σ_1 in the 1-direction, but is now constrained ($\varepsilon = 0$) in both the 2 and 3 directions.

(a) Derive an expression for the "effective modulus" (σ_1/ε_1) in this case.
[Hint: first find σ_2 and σ_3 in terms of σ_1.]

(b) Sketch the variation of effective modulus with ν and comment on the limiting values when $\nu = 0$ (foam) and $\nu \approx 0.5$ (rubber).

(c) A variant is proposed in the design of the rubber mounting (Figure 3). The block is to be replaced with an axially loaded cylinder of rubber completely filling a cylindrical hole in the steel plate. Use the result in (a) to explain whether this is a good idea.

(d) Explain why the rubber soles of running shoes are designed with some combination of air or gel pockets, partially foamed rubber, and a tread.

* **5.** Square porcelain tiles are to be manufactured with a uniform layer of glaze, which is thin compared to the thickness of the tile, as shown in section in Figure 4. They are fired at a high temperature T_1 and then cooled slowly to room temperature T_o (= 20°C). The relevant coefficients of thermal expansion and Young's moduli for the glaze and the porcelain are defined in the figure.

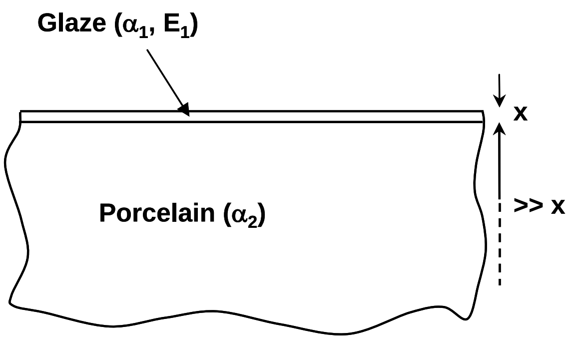

Figure 4

(a) Derive an expression for the thermal strain in the glaze, assuming first that it is unconstrained by the porcelain. Use superposition to derive an expression for the stress in the glaze after cooling. What properties determine whether the stress is tensile or compressive?
[Hint: note that the induced thermal stress is biaxial, with the same value in two perpendicular directions].

(b) Two tile types are to be manufactured from porcelain. The first type is decorative, using tension in the surface to form a mosaic of cracks. For the second type, compression in the surface is desirable (to resist surface cracking and give improved strength). Use the data below to select a suitable glaze (A, B or C) for each of these applications.

Firing temperature = 700°C
Young's modulus of glaze = 90 GPa
Poisson's ratio of glaze = 0.2
Tensile failure stress of glaze = 10 MPa

Coefficients of thermal expansion:

	$\alpha \times 10^{-6}$ (K^{-1})
Porcelain	2.2
Glaze A	1.5
Glaze B	2.3
Glaze C	2.5

6. (a) Iron has a BCC crystal structure (known as the α phase, or ferrite) at room temperature (20°C). Its theoretical density at this temperature was found in Examples Paper 1 (Exercise B6a). Given that the linear thermal coefficient of expansion of iron = 12×10^{-6} K^{-1}, determine the density of BCC iron at 910°C.

(b) At 910°C BCC iron can transform to an FCC form (the γ phase, or austenite). What is its density at this temperature after the transformation? (Hint: consider the atomic packing fraction). Assuming that there is no change in phase or in expansion coefficient from 910 to 1300°C, sketch how the length of a sample of iron would change with temperature from 20°C to 1300°C. (Note: dimensional changes of this type have been used extensively as a simple method for quantifying phase changes in iron alloys).

Microstructural Origin and Manipulation of Elastic Properties

7. (a) Use the modulus-density property chart in the Materials Databook to identify the ranges of Young's modulus for ceramics, metals and polymers. Explain briefly how the nature of the bonding in these material classes accounts for the relative magnitude of the elastic properties.

(b) Below the glass-transition temperature T_g, most polymers have a modulus of order 1-2 GPa. Above T_g, the different classes of polymers show wide variations in elastic response: some are barely affected, in others the modulus falls to ≈1 MPa or less. Sketch the variation of Young's modulus with temperature for the different classes of polymer, clearly indicating the glass transition temperature, and explain the following in terms of bonding and microstructure:

(i) the elastic response of a typical amorphous polymer;
(ii) the effect of crystallinity on the response;
(iii) the effect of cross-linking.

Explain why some polymers are much easier to recycle than others.

8. Assuming the relationship $E_{foam} = E_{solid}\left(\frac{\rho_{foam}}{\rho_{solid}}\right)^2$, estimate the range of modulus-density combinations which might be achievable by foaming the following metals to relative densities in the range 0.1 – 0.3: Ni alloys, Mg alloys. [Use mid-range data from the Materials Databook]. Locate approximate "property bubbles" for these new metallic foams on the E- ρ chart in the Materials Databook. With which existing materials would these foams appear to compete, in terms of modulus and density? Think of a property for which the metallic foams will differ significantly from the competition.

9. Medical prosthetic implants such as hip replacements have traditionally been made of metals such as stainless steel or titanium. These are not ideal as they are much stiffer than the bone, giving relatively poor transfer of load into the bone. New composite materials are being developed to provide a much closer stiffness match between the implant material and the bone. One possibility uses high density polyethylene (HDPE) containing particulate hydroxyapatite (HA, the natural mineral in bone, which can be produced artificially). Data for some experimental composites are provided below.

(a) Plot the upper and lower bounds for the Young's modulus of HDPE-HA composites (formulae in the Materials Databook) together with the experimental data.

(b) Which of the bounds is closer to the data for the particulate composite? Extrapolate the experimental data to estimate the volume fraction of hydroxyapatite particulate that is required to match the stiffness of bone (using the shape of the nearest bound as a guide). Is this volume fraction practical?

[Young's modulus (typical values): Bone 7 GPa, Hydroxyapatite 80 GPa, HDPE 0.65 GPa]

Volume fraction of HA	Young's modulus (GPa)
0	0.65
0.1	0.98
0.2	1.6
0.3	2.73
0.4	4.29

*** 10.** (a) Derive an expression for the density ρ_c of a composite containing a volume fraction V_f of reinforcement, in terms of the densities of the matrix ρ_m and of the reinforcement ρ_f. Does the result depend on the form of the reinforcement (fibres or particulate)?

(b) A composite material consists of parallel fibres of Young's modulus E_f in a matrix of Young's modulus E_m. The volume fraction of fibres is V_f. Derive an expression for E_c (Young's modulus of the composite along the direction of the fibres) in terms of E_f, E_m and V_f.

(c) Use the material property data in the Table below to find ρ_c and E_c for the following composites: (i) carbon-fibre/epoxy resin ($V_f = 0.5$), (ii) glass-fibre/polyester resin ($V_f = 0.5$).

Material	Density (Mg m^{-3})	Young's modulus (GPa)
Carbon fibre	1.90	390
Glass fibre	2.55	72
Epoxy resin	1.15	3
Polyester resin	1.15	3

(d) A magnesium company has developed an experimental metal-matrix composite (MMC), by casting magnesium containing 20% (by volume) of particulate SiC. Use the lower bound estimate for Young's modulus (Materials Databook) to estimate E_c for this MMC. Mid-range data for E and ρ of Mg alloys and SiC should be taken from the Materials Databook.

(e) Find the *specific stiffness* E/ρ for the three composites in (c,d). Compare the composites with steels, for which $E/\rho \approx 28$ GPa.Mg^{-1}m^3.

<u>Material Selection: Stiffness-limited Design</u>

Questions 11-13 introduce material selection in design. All can be solved using the Materials Databook or the *Cambridge Engineering Selector (CES)* software (though it is usually easier in software). Try both methods, at least for some of the questions.

CES may be downloaded from the CUED Website (PCs running Windows, or Mac with good emulator; instructions on Moodle 1P2 Materials). Familiarise yourself with CES using the *CES Tutorial* (on Moodle) or follow the *CES* > *Help* > *Video Tutorials* (in the CES software).

11. (a) A component is made from brass (a copper alloy). Identify three alternative classes of alloy which offer higher Young's modulus for this component.

(b) Find materials with E > 40 GPa and ρ < 2 Mg/m^3, and identify the cheapest.

(c) Find metals and composites that are both stiffer <u>and</u> lighter than: (i) steels; (ii) Ti alloys; (iii) Al alloys.

(d) Compare the specific stiffness, E/ρ, of steels, Ti alloys, Al alloys, Mg alloys, GFRP, CFRP and wood (parallel to the grain).

(e) Comment on the usefulness of the approaches in (c) and (d) for seeking improved performance in lightweight, stiffness-limited design.

(CES tips: in (d), either plot E-ρ and use a line of slope unity, as in the Databook, or use the "Advanced" feature in a graph stage to plot a bar chart of E/ρ).

12. Polymer ropes and lines for use on water are often designed to float, to aid in their retrieval and to avoid applying a downwards load to an object or person attached to them in the water. Excessive stretch is undesirable, so a lower limit of 0.5 GPa is also imposed on Young's modulus.

(a) Identify suitable polymers (in the Databook, refer to the tables of data rather than the charts, which do not show individual polymers).

(b) Look at the environmental resistance of these polymers, and comment on any possible weaknesses for this application.

(c) Use the information for material applications to see if the polymers identified are used in practice.

(CES tips: (a) requires a limit or graph stage; (b) requires a limit stage, or browsing on the results of (a); (c) requires browsing, or use of the Search facility).

13. A sailing enthusiast is seeking materials for lightweight panels to use in a sea-going yacht. The panels are of rectangular cross-section and will be loaded in bending, as shown in Figure 5. The span L and width b of the panels are fixed, but the thickness d may vary (up to a maximum specified value). The required stiffness is specified as a maximum allowable deflection δ under a given central load W, for a simply supported span. The designer is interested in two scenarios: (i) minimum mass; (ii) minimum material cost. In either case, the environmental resistance to sea water must be above average.

NB: The central deflection of a simply supported span under a point load is given by:

$$\delta = \frac{W L^3}{48 E I} \quad \text{where } I = \frac{b d^3}{12} \quad \text{(from Structures Databook).}$$

(a) Show that the stiffness and geometric constraints lead to the relationship $E d^3 = \text{constant}$, where E is Young's modulus. Explain why the design specification leads to a minimum allowable value for E.

(b) Write down an expression for the first objective to be minimised (mass) and use the stiffness constraint to eliminate the free variable (thickness). Hence define the material performance index to *maximise* for a minimum mass design.

(c) On a Young’s modulus – density property chart, how are materials with the best values of the performance index identified? Use this method to identify a short-list of candidate materials (excluding ceramics and glasses - why is this?) Comment on the influence on your candidate materials of the other factors in the design specification:

(i) environmental resistance to salt water;

(ii) maximum allowable thickness (for which assume $E > 5$ GPa is required).

(d) Modify your material performance index to describe the alternative objective of minimum material cost. Use a suitable material property chart (in CES) to identify a short-list of materials (excluding ceramics and glasses as before, and noting materials which may be excluded on grounds of inadequate resistance to sea water or excessive thickness).

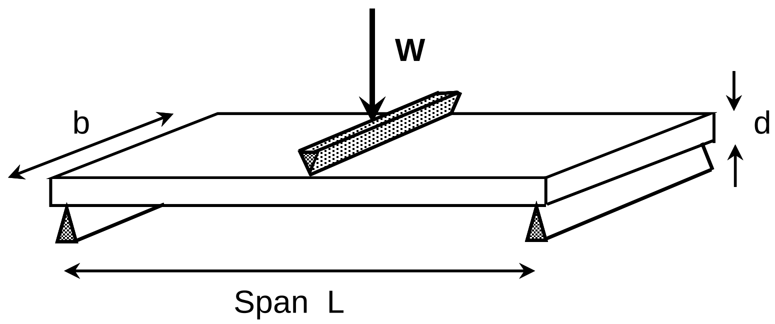

Figure 5

Answers

1. (a) $\Delta = -3\,p\,(1-2\nu)/E$
   (b) (i) see Materials Databook; (ii) $K \approx E$; (iii) $\Delta \approx 0$; $K \to \infty$
2. (a) $\sigma = \sigma_1 \cos^2\theta$, $\tau = \sigma_1 \sin\theta \cos\theta$
   (b) Maximum τ at 45°
   (c) $\sigma = 0$, $\tau = \sigma_1$
3. (F/δ) = 80 N/mm; 33% stiffer
4. $\dfrac{\sigma_1}{\varepsilon_1} = \dfrac{E(1-\nu)}{(1-\nu-2\nu^2)}$
5. (a) Thermal strain = $(\alpha_1-\alpha_2)(T_1-T_o)$; Thermal stress = $\dfrac{E}{(1-\nu)}(\alpha_1-\alpha_2)(T_1-T_o)$
   (b) Glaze C, Glaze A.
6. (a) BCC iron at 910°C: ρ = 7629 kg m^{-3}
   (b) FCC iron at 910°C: ρ = 8300 kg m^{-3}
8. Ni alloy foam: 2.05 – 18.5 GPa
   Mg alloy foam: 0.45 – 4.0 GPa
9. Approx. 50% HA
10. (a) $\rho_c = \rho_f V_f + (1-V_f)\rho_m$
    (b) $E_c = E_f V_f + (1-V_f)E_m$
    (c) Carbon-fibre epoxy: ρ_c = 1.53 Mg m^{-3}, E_c = 197 GPa
        Glass-fibre polyester: ρ_c = 1.85 Mg m^{-3}, E_c = 37.5 GPa
    (d) ρ_c = 2.1 Mg m^{-3}, E_c = 54 GPa
13. (b) $E^{1/3}/\rho$

Suggested Tripos Questions

2011 Q7
2012 Q8(b), 10(a,c), 11(a)
2013 Q11
2014 Q7
2015 Q7, 9, 12(a)
2016 Q10, 12
2017 Q10, 12(b)
2018 Q7

H.R. Shercliff
Lent Term 2019