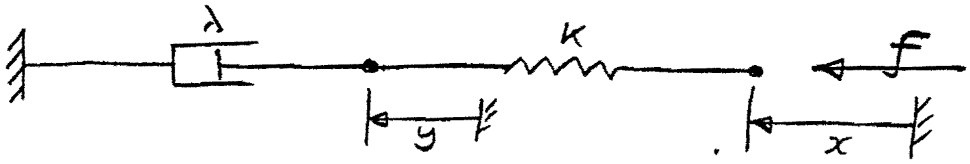
PART IA

Mechanical Vibrations
Examples Paper 1 Solutions

1.



Applied force $f = k(x-y) = \lambda \dot{y}$

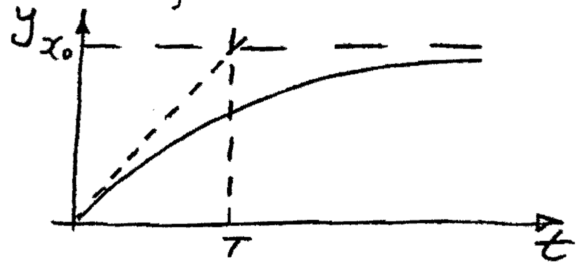
$\therefore \quad \underline{\underline{T\dot{y} + y = x}} \qquad$ where $T = \lambda/k$

For the step input as given, the general solution is

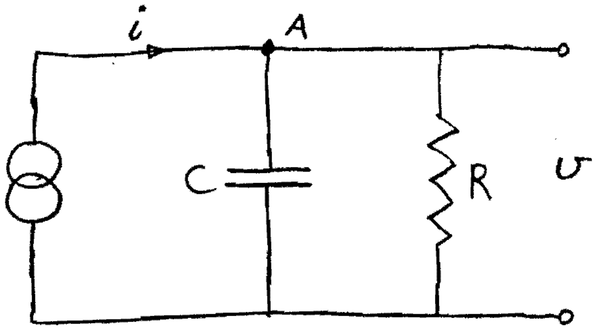
$$y(t) = \underbrace{Ae^{-t/T}}_{C.F.} + \underbrace{x_0}_{P.I.}$$

and to satisfy $y=0$ at $t=0$, $A = -x_0$

$\therefore \quad y = x_0\left(1 - e^{-t/T}\right)$



2.



Sum of currents at A

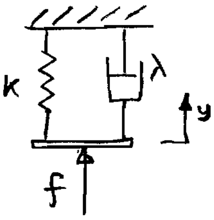
$\therefore \quad C\dot{v} + \frac{v}{R} - i = 0$

$\therefore \quad \underline{\underline{T\dot{v} + v = iR}}$

where $T = RC$

Using the result from Q1., the response to a step input current of magnitude i_0 is

$$\underline{\underline{v = i_0 R\left(1 - e^{-t/T}\right)}}$$

3. 

Sum of forces : $f = ky + \lambda\dot{y}$

$\therefore \; T\dot{y} + y = \frac{f}{k}$ where $T = \lambda/k$

From Q1, response to step input force of magnitude f_0

$$y = \frac{f_0}{k}\left(1 - e^{-t/T}\right)$$

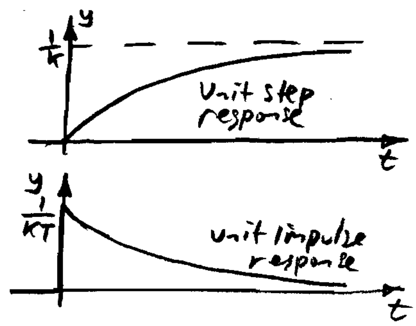
Unit impulse is the derivative of a unit step,

Unit step response : $y = \frac{1}{k}\left(1 - e^{-t/T}\right)$

Unit impulse response : $y = \frac{d}{dt}\left(\frac{1}{k}\left(1 - e^{-t/T}\right)\right)$

$$= \frac{1}{kT} e^{-t/T}$$

Impulse I $\therefore$ $y = \frac{I}{kT} e^{-t/T}$

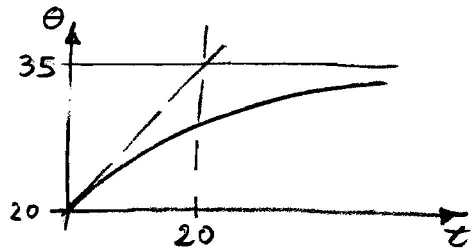


If $y(t)$ is a unit step function, then intuitively the dashpot will require an infinite force when the step occurs. Substitute in equation, and recall that $\frac{d}{dt}(\text{step}) = \delta(t)$

to find $f(t) = k\left[T\delta(t) + 1\right]$ for $t \geq 0$

4. $T\dot{\theta} + \theta = \theta_i$

$T \approx 20$ s from initial gradient intercept with asymptote.



Temperature rises at $\frac{9}{60}$ °c/s

$\therefore \; 20\dot{\theta} + \theta = \frac{9}{60}t + 20$ (input is a ramp)

Unit ramp response is the integral of the unit step response

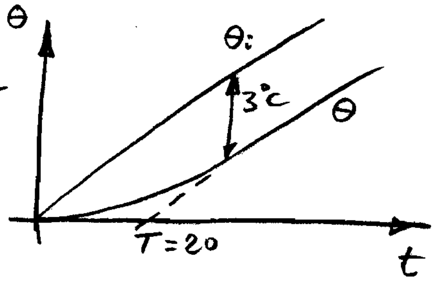
$$\therefore \; \theta = \frac{9}{60}\int\left(1 - e^{-t/20}\right)dt$$

$$= \frac{9}{60}\left(t + 20e^{-t/20}\right) + \text{const}$$

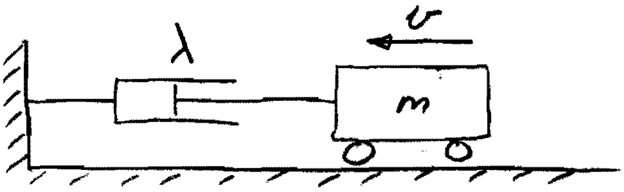
and $\theta = 20$ at $t = 0$

$$\therefore \; \theta = 17 + \frac{9}{60}t + 3e^{-t/20}$$

$$= \theta_i - 3\left(1 - e^{-t/20}\right)$$



5.



Sum of forces on the train $\therefore \quad m\dot{v} = -\lambda v$

$$\therefore \quad \underline{\underline{T\dot{v} + v = 0}} \qquad \text{where } T = \frac{m}{\lambda}$$

Either $\quad v = A e^{-t/T}$. At $t=0$, $v = v_0$

$$\therefore \quad v = v_0 e^{-t/T}$$

$$\therefore \quad x = \int v\,dt = -v_0 T e^{-t/T} + c$$

and take $x = 0$ at $t = \infty$ (final rest position)

$$\therefore \quad c = 0$$

$$\therefore \quad x = -v_0 T e^{-t/T}$$

$$\therefore \quad \underline{\underline{\text{distance moved} = v_0 T = \frac{v_0 m}{\lambda}}}$$

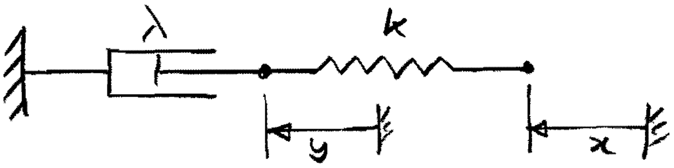
or $\quad$ use $\dot{v} = v\dfrac{dv}{dx} \qquad \therefore \quad T \cancel{v} \dfrac{dv}{dx} + \cancel{v} = 0$

$$\therefore \quad v = \int \frac{1}{T}\,dx = \frac{x}{T} + c$$

$$x = 0 \text{ when } v = 0 \quad \therefore \quad c = 0$$

$\therefore \quad x = vT \quad$ and for $v = v_0$, $x = v_0 T$ as above

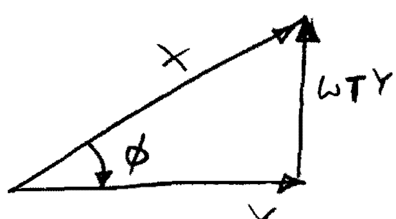
6.



From Q1, $\quad T\dot{y} + y = x \qquad$ where $T = \lambda/k$

Put $x = Xe^{i\omega t}$, $y = Ye^{i\omega t}$ $\quad \therefore (Ti\omega + 1)Y = X$

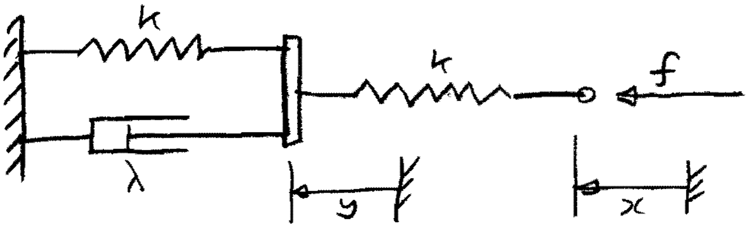
$$\therefore \quad Y = \frac{X}{1 + i\omega T} \qquad \therefore \quad \underline{\underline{|Y| = \frac{X}{\sqrt{1 + (\omega T)^2}}}} \qquad \text{and } \underline{\underline{\phi = \tan^{-1}\omega T}}$$



- As $\omega \to 0$, $Y \to X$ because at low frequencies the dashpot resistance is very low
- As $\omega \to \infty$ $Y \to 0$ because the dashpot is very stiff at high frequencies

phasor diagram $\quad (1 + i\omega T)Y = X \quad$ in the complex plane

7.



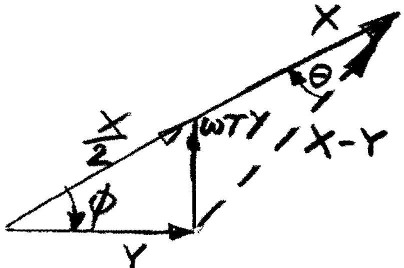
Sum of forces $\quad f = k(x-y) = ky + \lambda \dot{y}$

$$\therefore \quad \underline{\underline{T\dot{y} + y = \frac{x}{2}}} \quad \text{where } T = \frac{\lambda}{2k} = 0.0025 \text{ s}$$

$x = Xe^{i\omega t}$, $y = Ye^{i\omega t}$

$$\therefore \quad Y = \frac{X}{2(1+i\omega T)}, \quad |Y| = \frac{X}{2\sqrt{1+(\omega T)^2}}, \quad \phi = \tan^{-1}\omega T$$

At $\omega = 31.8 \times 2\pi = 200$ rad/s
and $X = 25$ mm , $|Y| = 11.2$ mm
$\phi = 26.6°$ (phase lag)



phasor diagram

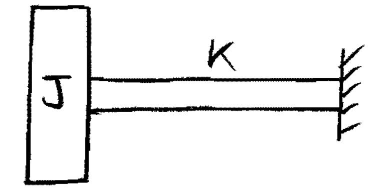
$f = k(x-y)$ and with $f = Fe^{i\omega t}$

$F = k(X-Y)$ and $X-Y$ is shown on the phasor diagram, hence F leads X by phase θ which can be obtained from the diagram or by using complex arithmetic

$$F = k(X-Y) = kX\left(1 - \frac{1}{2(1+i\omega T)}\right)$$

$$= \frac{kX}{2}\left(\frac{1+2i\omega T}{1+i\omega T}\right)$$

and for the values given, $|F| = 1.58$ N & $\theta = -18.4°$
(which is a phase lead)

8. 

$$J\ddot{\theta} = -K\theta$$

$$\therefore \frac{\ddot{\theta}}{\omega_n^2} + \theta = 0 \qquad \omega_n^2 = \frac{K}{J} = 7500\ s^{-2}$$

$$\therefore \omega_n = 86.6\ rad/s \equiv 13.8\ Hz$$

Solution: Undamped SHM

$$\theta = A \sin\omega_n t + B\cos\omega_n t$$

$\theta = 0$ at $t = 0$ $\therefore$ $B = 0$

$\dot{\theta} = 50$ rad/s at $t=0$ $\therefore$ $A\omega_n = 50$ $\therefore A = 0.577$

$$\therefore \underline{\theta = 0.577 \sin\omega_n t}$$

10% reduction after 10 cycles is light damping.

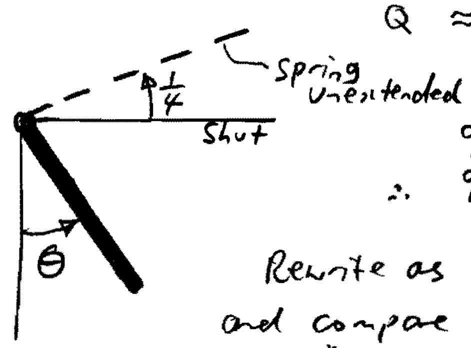
Logarithmic decrement $\delta = \ln\frac{\theta_1}{\theta_2}$ $\therefore$ $10\delta = \ln\frac{\theta_1}{\theta_{11}}$

$$\therefore \delta = \frac{1}{10}\ln\frac{1}{0.9} = \underline{0.0105}$$

Then $N\delta = \ln\frac{1}{0.002}$ $\therefore$ $N = 590$ cycles which, at 13.8 Hz take <u>43 seconds</u> to complete

Light damping: $\zeta \approx \frac{\delta}{2\pi} = \underline{0.00168}$

$Q \approx \frac{1}{2\zeta} = \underline{298}$

9. 

$$90\ddot{\theta} = 50\left(\frac{\pi}{2} + \frac{1}{4} - \theta\right) - 200\dot{\theta}$$

$$\therefore 90\ddot{\theta} + 200\dot{\theta} + 50\theta = 50\left[\frac{\pi}{2} + \frac{1}{4}\right]$$

Rewrite as $\frac{90}{50}\ddot{\theta} + 4\dot{\theta} + \theta = \frac{\pi}{2} + \frac{1}{4}$

and compare with data book page 4 which gives

$$\frac{\ddot{\theta}}{\omega_n^2} + \frac{2\zeta\dot{\theta}}{\omega_n} + \theta = \theta_0$$

$\therefore \omega_n = 0.745$ rad/s and $\zeta = 1.49$

Door closes when $\theta = \frac{\pi}{2}$, ie $\frac{\theta}{\theta_0} = \frac{\frac{\pi}{2}}{\frac{\pi}{2}+\frac{1}{4}} = 0.863$

From curve, this occurs at $\omega_n t = 5.3$ $\therefore$ $\underline{t = 7.1s}$

Gradient of curve $= \frac{d\frac{\theta}{\theta_0}}{d(\omega_n t)} = \frac{\dot{\theta}}{\left(\frac{\pi}{2}+\frac{1}{4}\right)\omega_n} = 0.057$

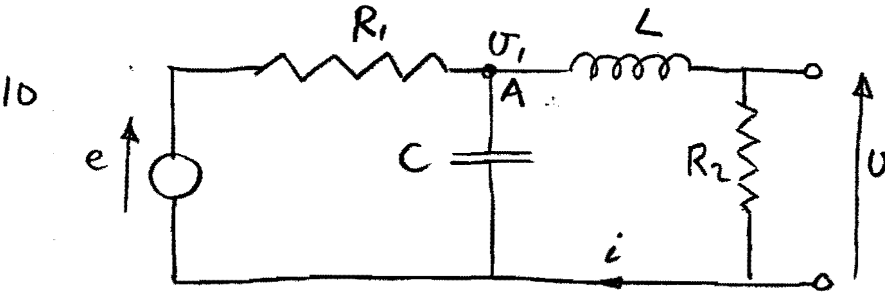
$\therefore$ angular velocity at closure $= \underline{\dot{\theta} = 0.075\ rad/s}$

Critical damping ($\zeta = 1$) is the lowest damping at which oscillations do not occur. $\zeta = \frac{\lambda}{2\sqrt{KJ}}$

To get $\zeta < 1$, increase J by a factor of $(1.49)^2$ or more

So $J > 200$ kg m², an increase of <u>110 kg m²</u>

10



Sum currents at A $\therefore$ $\dfrac{v_1 - e}{R_1} + C\dfrac{dv_1}{dt} + \dfrac{v}{R_2} = 0$ (1)

Rate of change of i $\therefore$ $v_1 - v = L\dfrac{di}{dt} = \dfrac{L}{R_2}\dfrac{dv}{dt}$ (2)

Differentiate (2) $\therefore$ $\dfrac{dv_1}{dt} = \dfrac{dv}{dt} + \dfrac{L}{R_2}\dfrac{d^2v}{dt^2}$

Substitute (2) & (3) into (1) to eliminate v_1

$$\therefore \quad LC\frac{R_1}{R_2}\frac{d^2v}{dt^2} + \left(CR_1 + \frac{L}{R_2}\right)\frac{dv}{dt} + \left(1 + \frac{R_1}{R_2}\right)v = e$$

$$\therefore \quad \frac{LC}{1+\frac{R_2}{R_1}}\frac{d^2v}{dt^2} + \frac{\left(CR_1 + \frac{L}{R_2}\right)}{1+\frac{R_1}{R_2}}\frac{dv}{dt} + v = \frac{e}{1+\frac{R_1}{R_2}}$$

Compare with data book page 6

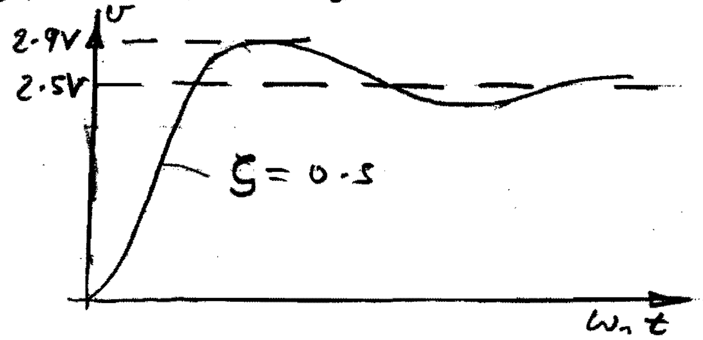
$$\frac{\ddot{v}}{\omega_n^2} + \frac{2\zeta\dot{v}}{\omega_n} + v = e$$

gives $\underline{\underline{\omega_n = \sqrt{\frac{1}{LC}\left(1 + \frac{R_2}{R_1}\right)}}}$ and $\underline{\underline{\zeta = \frac{1}{2}\left(\frac{R_2}{R_1}\right)\left(CR_1 + \frac{L}{R_2}\right)\Big/\sqrt{LC\left(1+\frac{R_2}{R_1}\right)}}}$

putting $\dfrac{L}{R_2} = CR_1 = \frac{1}{3}\times 10^{-3}\,\text{s}$ and $R_1 = 3R_2$, $\omega_n = 6000$ rad/s and $\zeta = 0.5$

For step change in e of 10V, steady state value of v is 2.5V

Response curve given in data book page 6



Maximum response $= 2.5\text{V} \times 1.17 = 2.9\text{V}$

D Cebon