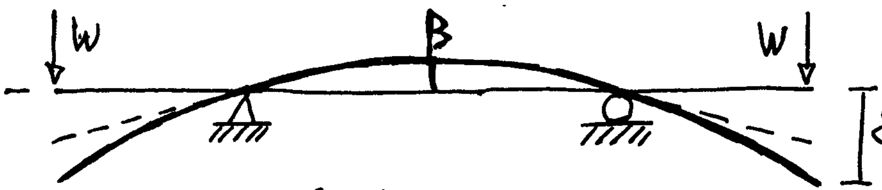
1A Engineering Paper 2
STRUCTURAL MECHANICS
Solutions to Examples Paper 5

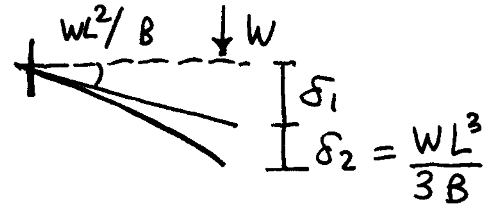
1. 

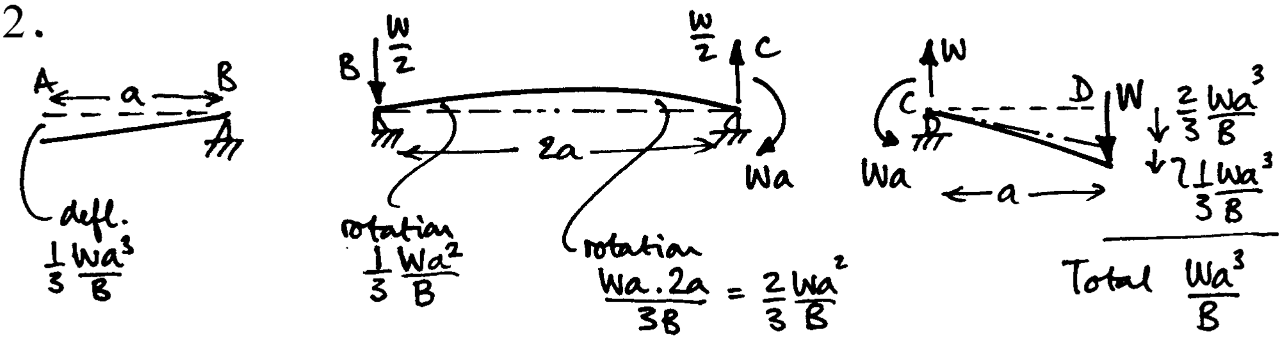
Central deflection (up):

$\beta = WL^3/2B$

Tip deflection (down):

$$\delta = \delta_1 + \delta_2 = \frac{WL^2}{B}L + \frac{WL^3}{3B} = \frac{4}{3}\,WL^3/B$$

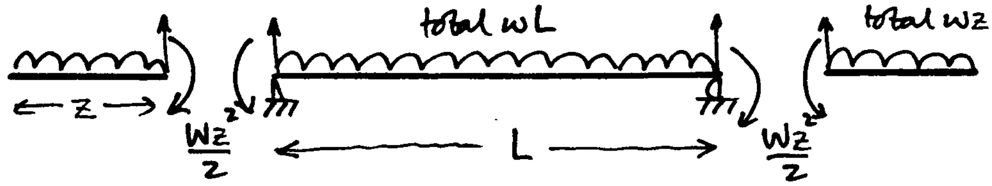


2. 

First, split up into 3 parts.
Then analyse statics. First for CD get force and couple at C. Opp. couple at C acts on middle S.S. beam [Not opposite vertical force, because support provides reaction]. This requires reactions at ends B, C of BC. Part AB has no forces.
Part BC is a data-book case: get the 2 end rotations. Continuity requires equal rotation at B in the 2 parts; hence defl. of A. Deflection of D in 2 parts: (i) if CD were rigid, like AB; then (ii) because it deflects as a cantilever – another data-book case.

Note that answer to Q 1 could be obtained by superposition of this case + its opposite.

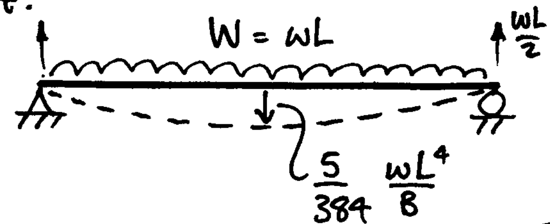
3.



Central beam on simple supports has 2 loadings:
(i) u.d. load, total wL
(ii) end couples $wz^2/2$, found by analysing equilibrium of overhanging parts.

Use principle of superposition to find central deflection of central part:

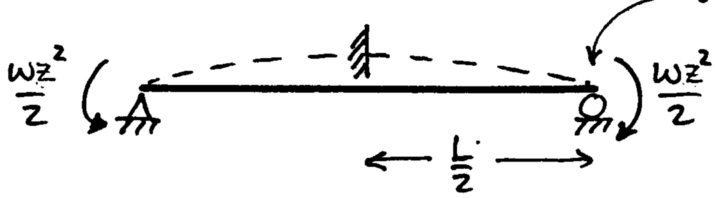
case (i)



(DataBook p.6)

no vertical reaction at ends

case (ii)



Treat one half as cantilever clamped at centre, length $L/2$, couple applied at tip.

Downward defl. of tip, from Data Book

$$= \left(\frac{wz^2}{2}\right)\left(\frac{L}{2}\right)^2 \cdot \frac{1}{2B} = \frac{1}{16}\frac{wz^2L^2}{B}$$

= upward deflection of centre of actual beam.

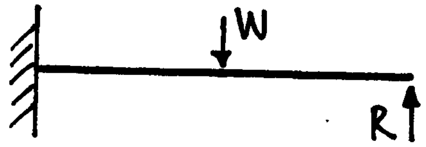
Since actual deflection is to be zero, these 2 magnitudes are equal:

$$\text{So} \quad \frac{5}{384}\frac{\cancel{w}L^{\cancel{4}\,2}}{\cancel{B}} = \frac{1}{16}\frac{\cancel{w}z^2\cancel{L^2}}{\cancel{B}} \quad \therefore \left(\frac{z}{L}\right)^2 = \frac{5\times 16}{384} = \frac{5}{24}$$

$$z/L \text{ req}^d = \sqrt{\frac{5}{24}} = 0.456; \quad z = 0.456\,L$$

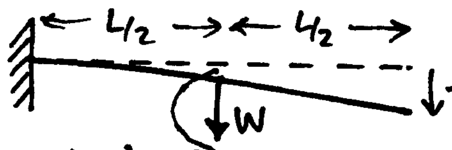
Final shape:

4.



Take out end support, apply (unknown) force R there instead.
Find total tip deflection by superposing 2 cases.

(i)



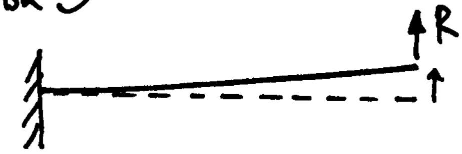
defl $= \frac{W(L/2)^3}{3B} = \frac{WL^3}{24B}$

slope $= \frac{W(L/2)^2}{2B} = \frac{WL^2}{8B}$

Data Bk

defl $= \underbrace{\frac{WL^3}{24B}}_{\text{mid-pt.}} + \underbrace{\frac{WL^3}{16B}}_{\text{effect of rotation of "pointer"}} = \cdots = \frac{5}{48}\frac{WL^3}{B}.$

(ii)

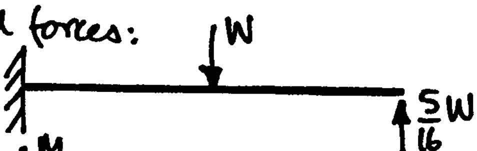


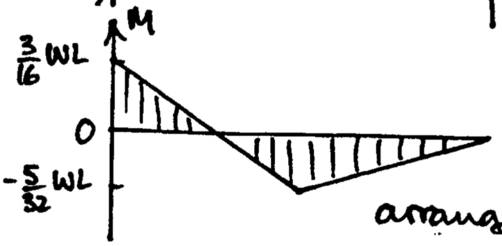
defl $= \frac{RL^3}{3B}$

Sum of deflections at tip $= 0$ $\therefore \frac{RL^3}{3B} = \frac{5}{48}\frac{WL^3}{B}$

$\boxed{\therefore R = \frac{5}{16}W}$

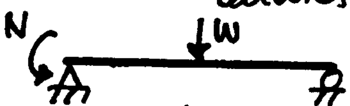
Final forces:



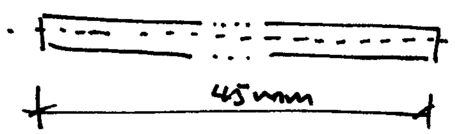


Notes (i) Similar to example in lectures

(ii) Could turn into SS beam loaded also by unknown couple at 1 end, and arrange for rotation there $= 0$.



5.



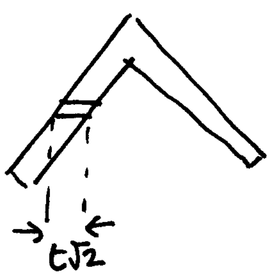
$I = \frac{1}{12} \times 45 \times 2^3 \ \mathrm{mm}^4$

$E : 210 \times 10^3 \ \mathrm{N/mm^2}$

$B : EI = \frac{1}{12} \times 45 \times 2^3 \times 210 \times 10^3 = 6.3 \times 10^6 \ \mathrm{N\,mm^2}$

$= \underline{6.3 \ \mathrm{Nm^2}}$

6. Considering disposition of material w.r.t. x-axis



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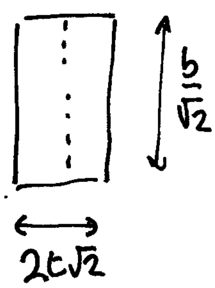


$I = \frac{1}{12}\left(\frac{b}{\sqrt{2}}\right)^3 \cdot (2t\sqrt{2})$

$= \frac{1}{12} bt^3$

7. (a) $I = \frac{\pi R^4}{4}$ for "solid" circle [lecture-notes; Mechanics Data Bk].

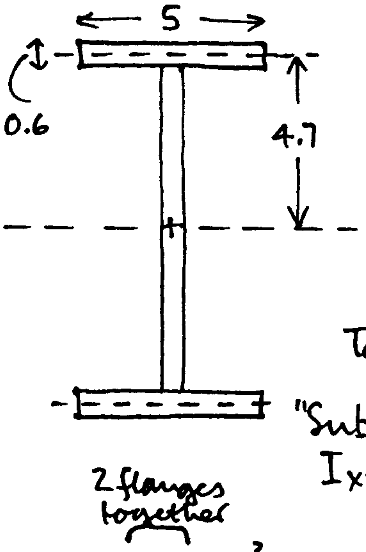
Here $R_{outer} = 24.15$ mm, $R_{inner} = 19.15$ mm $\Big\}$ $\therefore I = \frac{\pi}{4}(24.15^4 - 19.15^4)$

$= 161.5 \times 10^3$ mm^4

$= \underline{16.15 \text{ cm}^4}$.

(b) from Data Book (p.15) $I = 16.2$ cm^4

8.



For 1 flange:

$I_{own axis} = \frac{1}{12} \times 5 \times 0.6^3 = 0.09$ cm^4

$Ay^2 = 5 \times 0.6 \times 4.7^2 = \underline{66.27 \text{ cm}^4}$

Total contribution to I_{xx} = 66.36 cm^4

For web: $I_{xx} = \frac{1}{12} \times 0.6 \times 8.8^3 = 34.07$ cm^4

Total $I_{xx} = 34.07 + 2 \times 66.36 = \underline{166.79 \text{ cm}^4}$

"Subtraction" method:

$I_{xx} = \frac{1}{12} \times 5 \times 10^3 - \frac{1}{12} \times 4.4 \times 8.8^3 = 416.67 - 249.87$

$= \underline{166.79 \text{ cm}^4}$ ✓

$I_{yy} = \frac{1}{12} \times 1.2 \times 5^3 + \frac{1}{12} \times 8.8 \times 0.6^3 = 12.5 + 0.16 = \underline{12.66 \text{ cm}^4}$

Note: (i) I of these rectangles about own "minor" axes is negligible.
(ii) $I_{yy} < I_{xx}/10$ ... which is typical of practical I beams.

9. $\frac{\sigma}{y} = \frac{M}{I} = EK$. Here $E = 210 \times 10^9$ N/m^2 = 210,000 N/mm^2

$K = \frac{\text{angle}}{\text{length}} = \frac{60}{57.3} \times \frac{1}{250}$ mm^{-1}

angle in radians

$= 0.00419$ mm^{-1}

[radius of arc $= (0.00419)^{-1} = \underline{239 \text{ mm}}$]

Here $y = \frac{1}{2}$ thickness $= 0.4$ mm

$\therefore \sigma = 0.4 \times 210{,}000 \times 0.00419 \quad \cancel{\text{mm}} . \frac{\text{N}}{\text{mm}^2} \times \frac{1}{\cancel{\text{mm}}} = \underline{352 \text{ N/mm}^2}$

10. Application of $\sigma = \frac{My}{I}$.

NB the actual depth (and width) are a bit different from the nominal values

From tables, $I_{xx} = 16077$ cm^4 $= 16077 \times 10^4$ mm^4

$y = 358.6/2$ mm.

$M = 150 \times 10^6$ N mm

try working in mm here

$\sigma = \frac{My}{I} = \frac{150 \times 10^6}{16077 \times 10^4} \times \frac{358.6}{2} \quad \frac{\text{N mm . mm}}{\text{mm}^4} = \underline{167 \text{ N/mm}^2}$ (to 3 sig. figs)

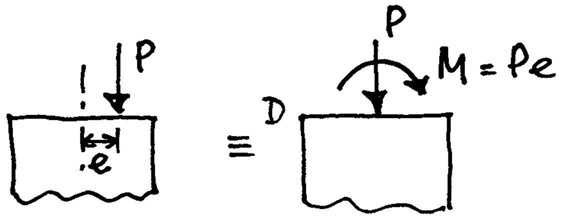
For minor-axis bending, $I = 1109 \times 10^4$ mm^4, $y = 172/2$ mm

$\therefore M = \frac{\sigma I}{y} = \frac{167 \times 1109 \times 10^4}{172/2} \quad \frac{\text{N}}{\text{mm}^2} \times \frac{\text{mm}^4}{\text{mm}} = 21.57 \times 10^6$ N mm

$= \underline{21.6 \text{ kNm}}$ ($\ll 150$)

Note: we could have used $Z = I/y$ values from the tables.

11. Let axial compressive load be P, at eccentricity e. This is equivalent (by statics) to a central force + pure bending moment M. At side D the bending moment causes tension; so the condition for there to be no tensile stress anywhere is critical at D.



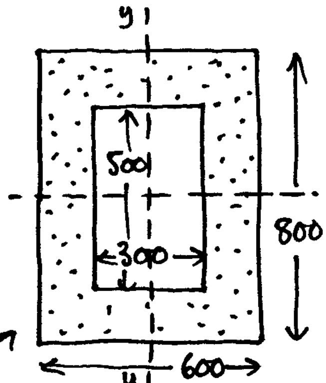
$$\sigma_D = -\frac{P}{A} + \frac{My}{I}$$

where A = cross-sectional area, πR^2
I = 2nd moment of area, $\frac{\pi R^4}{4}$
y = "extreme fibre" distance, R

So $\sigma_D = -\frac{P}{\pi R^2} + \frac{PeR}{\pi R^4/4} < 0$ provided

$$\frac{4\cancel{P}e}{\cancel{\pi}R^3} < \frac{\cancel{P}}{\cancel{\pi}R^2} \quad \therefore \quad e < \frac{R}{4}. \qquad \text{Q.E.D.}$$

12.



cross-section of chimney - dimensions in mm.

$$I_{yy} = \frac{8\times 6^3\times 10^8}{12} - \frac{5\times 3^3\times 10^8}{12}$$
$$= (144 - 11.2)\times 10^8 = 133\times 10^8\,\text{mm}^4.$$

Uniform compressive stress due to self-weight $= \rho g h$ (independent of cross-sectional shape)

$$= \frac{2000}{10^9}\times 9.81\times 5000 \quad \frac{\text{kg}}{\text{mm}^3}\cdot\frac{\text{m}}{\text{s}^2}\cdot\text{mm}$$
$$= 0.0981\ \text{N/mm}^2 \text{ at bottom.}$$

This gives us the allowable tensile bending stress at base.

So Allowable M at base $= \frac{\sigma I}{y} = \frac{0.0981\times 133\times 10^8}{300} \quad \frac{\text{N}}{\text{mm}^2}\times\frac{\text{mm}^4}{\text{mm}}$

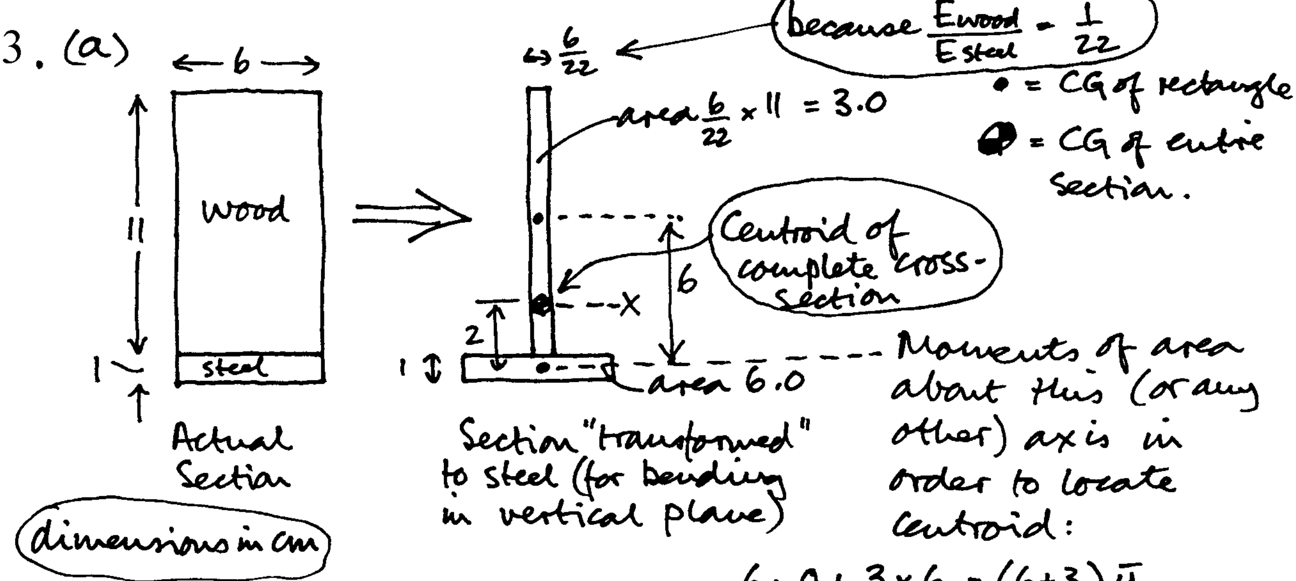
$$= 435\times 10^4\ \text{Nmm} = 4350\,\text{Nm}.$$

If pressure p (N/m²) acts on one face, 5 m high × 0.8 wide, overturning moment at base $= p\times 5\times 0.8\times \underbrace{2.5}_{\text{height of resultant force}} = 10p \quad \frac{\text{N}}{\text{m}^2}\cdot\text{m}^3$

$$\therefore\ p = \frac{M}{10} = \underline{435\ \text{N/m}^2}$$

We have assumed here that the bottom cross-section is the most critical. It is easy to demonstrate this formally.

13. (a)



Moments of area about this (or any other) axis in order to locate centroid:

$$6 \times 0 + 3 \times 6 = (6+3)\bar{y}$$

$$\therefore \bar{y} = 2$$

$$I_{xx} = \underbrace{\frac{6 \times 1^3}{12}}_{\text{I of lower rectangle about own centroidal axis}} + \underbrace{6 \times 2^2}_{Ay^2} + \underbrace{\frac{6}{22} \times \frac{11^3}{12}}_{\text{I of upper rectangle about own centroidal axis}} + \underbrace{3 \times 4^2}_{Ay^2} = \begin{array}{r} 0.5 \\ 24.0 \\ 30.25 \\ 48.00 \\ \hline 102.75\,\text{cm}^4 \end{array}$$

I_{xx}: for transformed steel section, about axis through centroid

$$= \frac{102.75}{10^8}\ \text{m}^4$$

$$B = EI = 210 \times 10^9 \times \frac{102.75}{10^8}\ \frac{\text{N}}{\text{m}^2}.\text{m}^4 = 215775\ \text{Nm}^2 = \underline{215.8\ \text{kNm}^2}.$$

(b) Find M at which $\sigma = 120 \times 10^6\ \text{N/m}^2$ in steel at "extreme fibre", 2.5 cm from centroidal axis:

$$M = \frac{\sigma I}{y} = \frac{120 \times 10^6 \times 102.75}{2.5 \times 10^{-2} \quad 10^8}\ \frac{\text{N}}{\text{m}^2}.\frac{\text{m}^4}{\text{m}} = 4932\ \text{Nm}$$

$$= \underline{4.93\ \text{kNm}}.$$

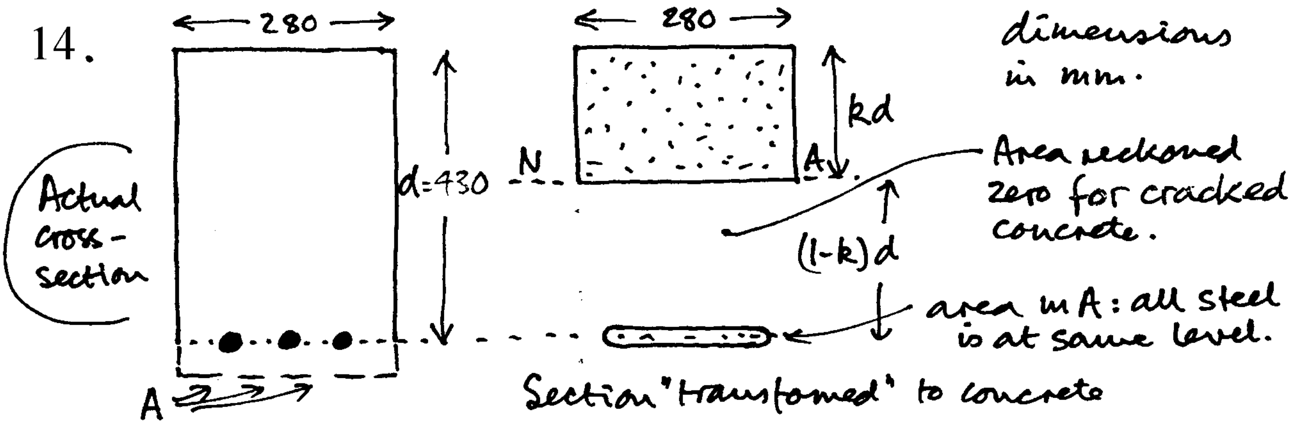
Now find M at which $\sigma = 10 \times 10^6\ \text{N/m}^2$ in wood at "extreme fibre" 9.5 cm from centroidal axis:

$$M = \frac{\sigma I_w}{y} = \frac{10 \times 10^6 \times 102.75 \times 22}{9.5 \times 10^{-2} \times 10^8} = 2379\ \text{Nm} = \underline{\underline{2.38\ \text{kNm}}}$$

I_w = I of section "transformed to wood"
= 22 × I of section " " steel.

So wood governs, because max. allowable stress in wood is reached for a smaller bending moment.

14.



Cross-sectional area of steel: $A = 3 \times \frac{\pi}{4} \times 25^2 = 1470\ \text{mm}^2 = 14.7\ \text{cm}^2$

Let depth of neutral axis be kd, as in lectures.
"See-saw" condition for neutral axis (see lecture notes)
reduces to $k^2\left[\frac{bd}{2mA}\right] = 1-k$ – a quadratic eqⁿ in k.

Here, $\left.\begin{array}{l} b = 280\ \text{mm} \\ d = 430\ \text{mm} \\ m = 15 \\ A = 1470\ \text{mm}^2 \end{array}\right\}$ $\frac{bd}{2mA} = \frac{280 \times 430}{2 \times 15 \times 1470} = 2.73$

So $2.73k^2 = 1-k$, or $2.73k^2 + k - 1 = 0$.
Roots of this are $k = \frac{-1 \pm \sqrt{1 + 4 \times 2.73}}{2 \times 2.73} = 0.449$ (take +)
(so $kd = 0.449 \times 430 = 193\ \text{mm}$)

For given curvature, $\varepsilon = \kappa y$, where y is distance from neutral axis, by geometry of bending ("railway track").
Let ε_c be (tensile) strain at top of cross-section
ε_s be " " in steel.

Then $\frac{\varepsilon_c}{\varepsilon_s} = \frac{-k}{1-k} = \frac{-0.449}{1-0.449} = -0.814$.

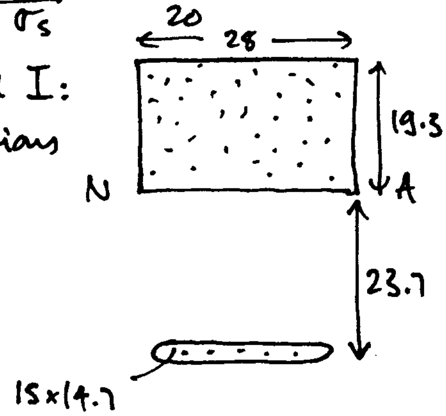
N.B. Lecture-notes say to ignore I of this piece about its own (centroidal) axis. Make sure that students understand why.

Use elastic law:
$\frac{\sigma_c}{\sigma_s} = \frac{E_c}{E_s} \cdot \frac{\varepsilon_c}{\varepsilon_s} = -\frac{0.814}{15} = -0.0543$

$\frac{\text{Allowable } \sigma_c}{\text{Allowable } \sigma_s} = -\frac{1}{20} = -0.050$

∴ Concrete governs (just).

Now find I: dimensions in cm.



$$\begin{aligned} I = {} & 15 \times 14.7 \times 23.7^2 && = 123\,852 \\ & + \tfrac{1}{12} \times 19.3^3 \times 28 && 16\,774 \\ & + 19.3 \times 28 \times \frac{19.3^2}{4} && 50\,323 \\ & && \overline{190\,949\ \text{cm}^4} \\ & && = 1.9 \times 10^5\ \text{cm}^4 \end{aligned}$$

$$M = \frac{\sigma I}{y} = \frac{7 \times 10^6}{0.193} \times \frac{1.9 \times 10^5}{10^8} = 68900\ \text{Nm}$$
$$= \underline{68.9\ \text{kNm}}$$

SDG
February 2007