Paper 1: Mechanical Engineering

Examples Paper 3

Elementary exercises are marked †, problems of Tripos standard ∗.
Answers can be found at the back of the paper.

Kinematics of a rigid body

1 † A 3D rigid body is rotating about an axis at an angular rate of $10\,\mathrm{rads}^{-1}$. The positive direction of the axis is defined by $\mathbf{n} = 3\mathbf{i} + 2\mathbf{j}$. If a fixed point O in the rigid body lies on the axis, determine the velocity of a point P in the body when the instantaneous vector from O to P is $\mathbf{r}_{P/O} = (20\mathbf{i} + 10\mathbf{j})$ mm.

2 † A playground roundabout of radius 2 m shown in Figure 1 rotates in an anticlockwise direction about O at $1.5\,\mathrm{rads}^{-1}$. A child holds on to the roundabout at A, a distance of 2 m from O. A second child holds on to the roundabout at B, a distance of 1.5 m from O.

(a) Express the angular velocity of the roundabout as a vector;

(b) What is the angular velocity of each child?

(c) What is the angular velocity of the vector $\mathbf{r}_{B/A}$?

(d) Determine the velocities of A and B;

(e) Determine the velocity of A relative to B ($\dot{\mathbf{r}}_{A/B}$) and the velocity of B relative to A ($\dot{\mathbf{r}}_{B/A}$);

(f) Show that the velocity of B is equal to the velocity of A plus the velocity of B relative to A.

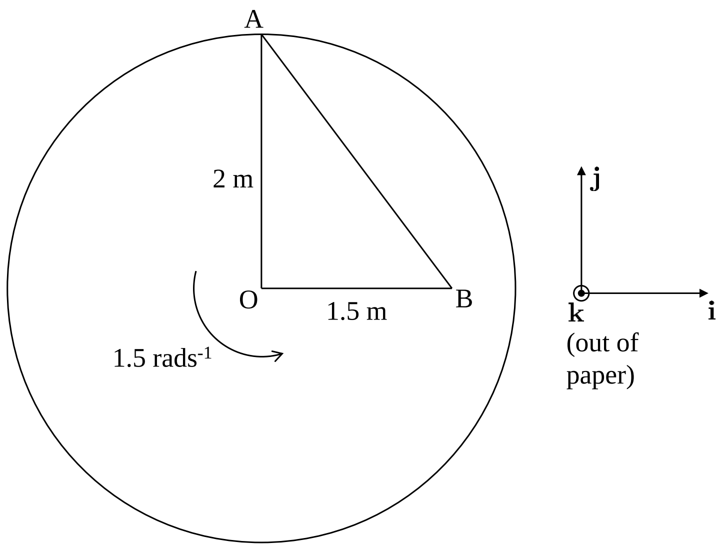

Figure 1

3 A person throws a uniform rigid stick AB, which is 0.8 m long. It goes spinning across the horizontal surface of a frozen lake. At a particular instant the velocity of end A is as shown in Figure 2a. The direction of the velocity of end B is as shown in Figure 2a and is known to be correct. The magnitude of the velocity at B is *thought* to be 8 ms^{-1}.

(a) Is the magnitude of the velocity of the end B correct? If not, what should it be?

(b) Determine the angular velocity of the stick as a vector.

(c) Find the velocity of the centre of gravity of the stick:

(i) using $\mathbf{v}_G = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{G/A}$;

(ii) by taking the average of $\mathbf{v}_A$ and $\mathbf{v}_B$.

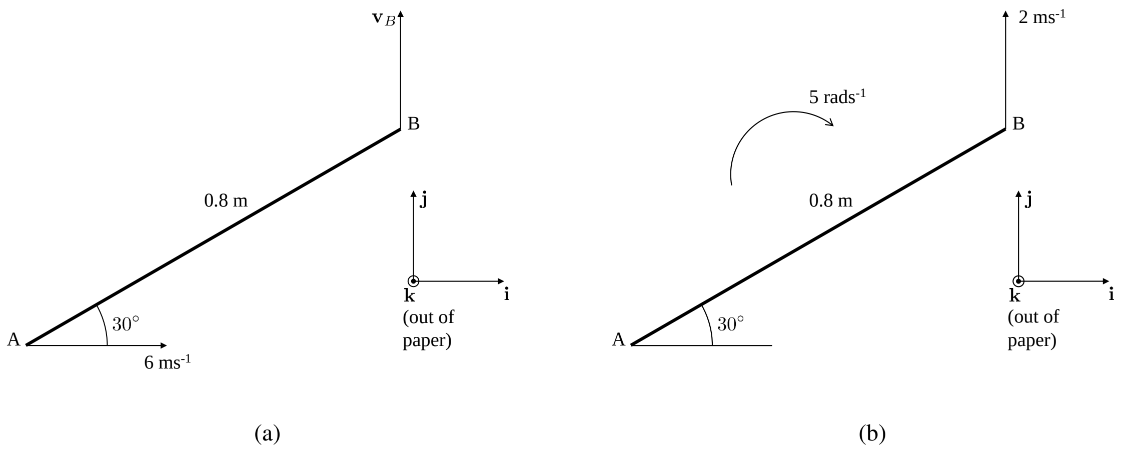

Figure 2

Instantaneous centres

4 (a) Locate the instantaneous centre for the stick in Figure 2a and use it to confirm that the stick's angular velocity is $\boldsymbol{\omega} = 15\mathbf{k}$ rads^{-1}.

(b) The same stick is shown in Figure 2b but the motion is different. Locate the instantaneous centre at the instant shown.

5 A plank 4 m long has one end on horizontal ground and rests against the top corner of a vertical wall 2.5 m high. The bottom end is sliding away from the wall towards the right at a rate of 1.5 ms^{-1}. Locate the instantaneous centre for the plank at the instant when the bottom end is 2 m from the wall and determine:

(a) the angular velocity of the plank;

(b) the velocity of the top end of the plank;

(c) the point on the plank which has the smallest speed.

Rotating reference frames

6 † A wheel of radius R is rolling without slip at an angular speed Ω along a flat surface such that it moves forward at a constant overall speed V_0. Unit vectors $\mathbf{i}$ and $\mathbf{j}$ are defined with respect to a fixed reference frame, while unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are fixed to the wheel (i.e. they rotate with the wheel).

(a) Identify the instantaneous centre of the wheel;

(b) Express the angular velocity of the wheel as a vector in terms of the forward speed V_0;

(c) What are the velocities of A, B, D and E relative to the centre C in terms of the rotating unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$?

(d) What are the velocities of points A, B, C, D and E on the wheel with respect to the fixed reference frame: why is the velocity zero for one of these points?

(e) Find vector expressions for the accelerations of these points: why is the acceleration zero for one of these points?

(f) Use the Python template `p3q6_template.ipynb` to obtain the position of an arbitrary point on the wheel during one complete revolution, and numerically differentiate the results to approximate the velocity and acceleration. Check to what extent your numerical estimates agree with your results above by plotting:
(i) the path traced out by a chosen point;
(ii) the absolute velocity of that point as a function of time;
(iii) the absolute acceleration of that point as a function of time.

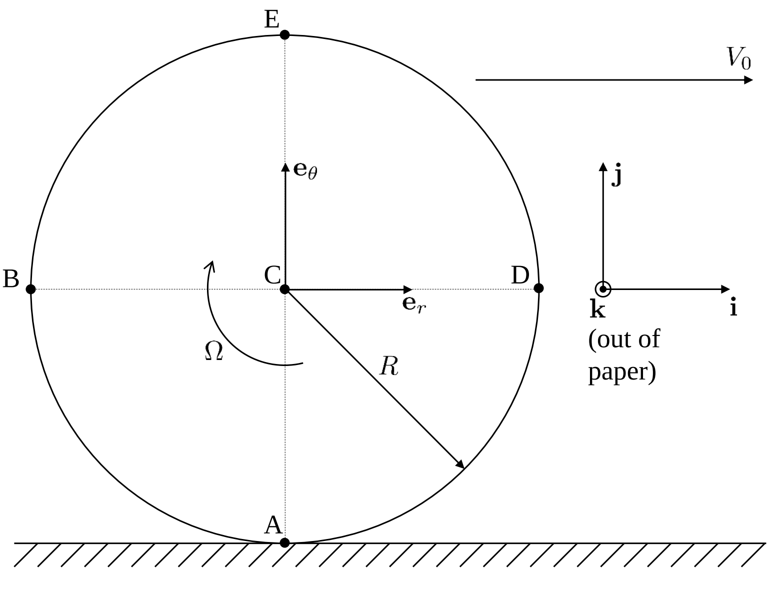

Figure 3

7 Point A has coordinates (1,1,0) m, and point B has coordinates (2,0,1) m. They are fixed within a 3D rigid body and have absolute velocities $\dot{\mathbf{r}}_A = (4\mathbf{i} + 2\mathbf{k})\,\mathrm{ms}^{-1}$ and $\dot{\mathbf{r}}_B = (9\mathbf{i} - \mathbf{j} - 4\mathbf{k})\,\mathrm{ms}^{-1}$.

(a) Check that the given velocities are consistent with rigid body motion;

(b) Find the angular velocity $\boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$ of the body when the component ω_z in the $\mathbf{k}$ direction is:

(i) $\omega_z = 0\,\mathrm{rads}^{-1}$;

(ii) $\omega_z = 1\,\mathrm{rads}^{-1}$;

This leads to an apparent contradiction because we have found two *different* angular velocities giving the same relative velocity $\mathbf{v}_{B/A}$ for two points fixed in a rigid body.

(c) Subtract the two angular velocity vectors, and show that this difference gives an angular velocity vector that is parallel to $\mathbf{r}_{B/A}$. Use this to explain the apparent contradiction.

8 A coriolis flow meter measures the flow rate of a fluid through a pipe by using the coriolis effect. Figure 4 shows the path of a pipe in a basic design. The pipe is nominally vertical with fluid flowing downwards at a speed V_0. It then enters a semi-circular section of pipe that follows a path of radius R. The whole pipe is set in rotation about the z-axis at a constant angular velocity $\omega = \Omega\mathbf{k}$. The unit vectors in Figure 4 are defined within this rotating reference frame.

(a) Derive an expression for the acceleration of a fluid particle P at an arbitrary position θ along the semi-circular section of pipe;

(b) Identify the coriolis acceleration and show that it is proportional to the flow rate;

(c) Sketch the magnitude of the coriolis acceleration for $0 \leq \theta \leq \pi$ and use this to explain how the flow rate can be measured [hint: consider the force required to produce this component of acceleration and hence how the pipe may deform slightly].

In practice, coriolis flow meters do not fully rotate, but rather a small oscillation is imposed such that $\Omega = A \sin \omega_0 t$. The coriolis acceleration still produces an oscillating twist in the pipe that can be detected by external transducers. In addition, the shape of the pipe need not be semi-circular and only requires a small deviation to produce a measurable effect. You can try this at home by swinging a hanging section of hose pipe.

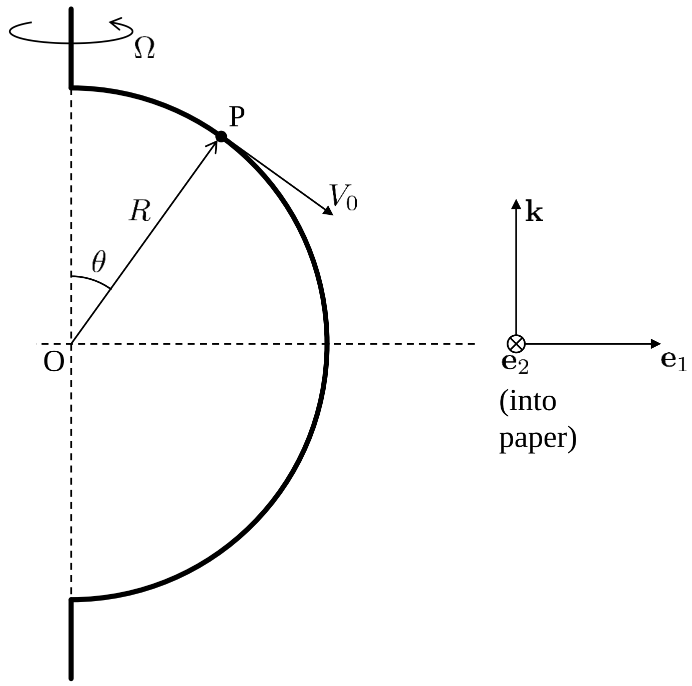

Figure 4

Centre of mass and moment of inertia

9 Find the centre of mass as a vector $\mathbf{r}_{G/O}$ for the objects shown in Figure 5:

(a) a solid cylinder;

(b) a triangular lamina;

(c) a rectangular lamina made up of two sections of equal dimensions: the right side having half the mass of the left side;

(d) A circular disc with a circular hole offset from the centre (assume $d < a/6$).

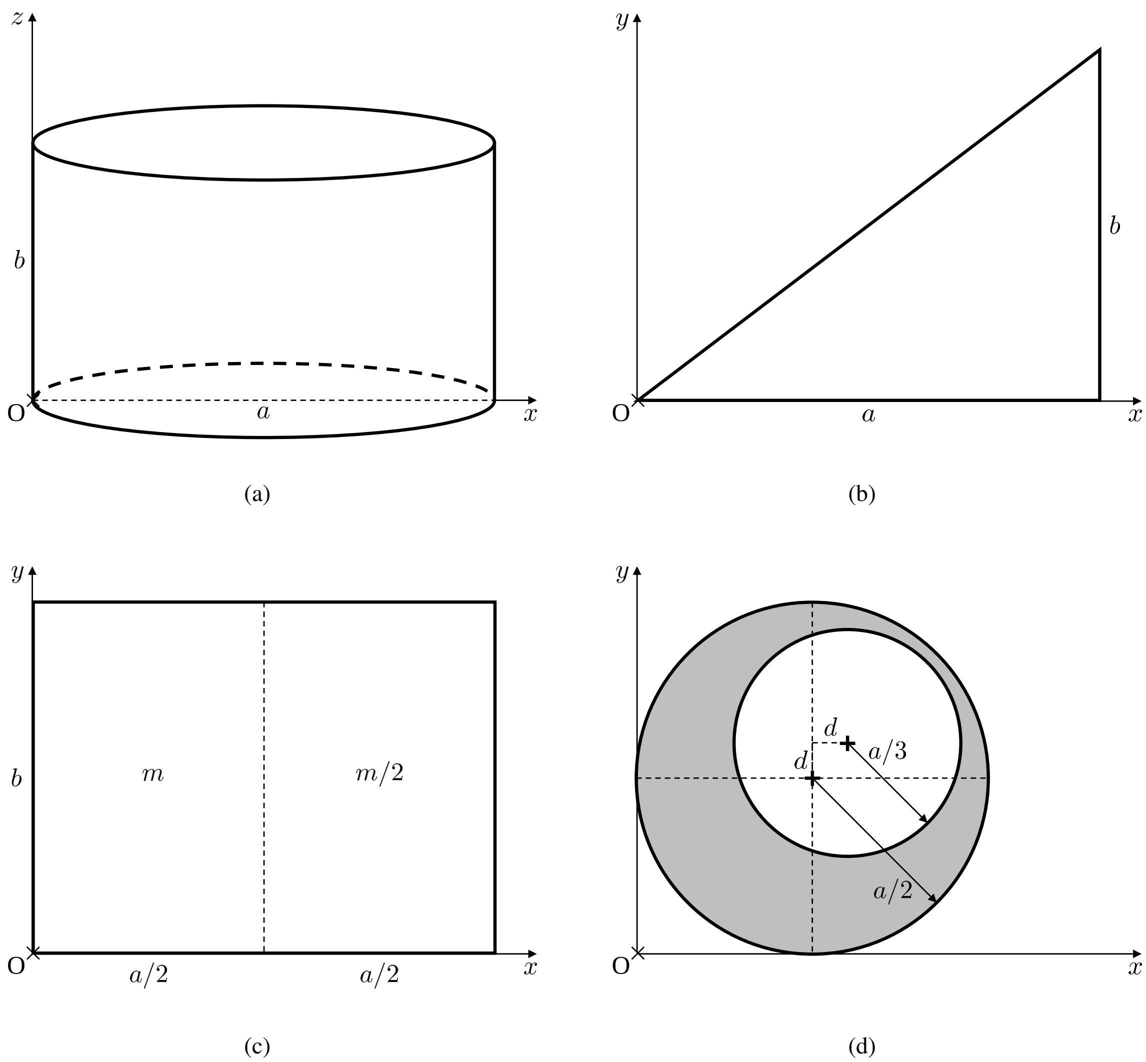

Figure 5

10 (a) Find from first principles the Mass Moment of Inertia I_{zz} of the triangular lamina in Question 9(b), about an axis passing through the origin O and parallel to the z-axis;

(b) For the triangular lamina find I_{xx} and I_{yy}, the mass moments of inertia about the x-axis and y-axis;

(c) Use the perpendicular axis theorem to confirm the result obtained in (a);

(d) Use the parallel axis theorem to find the moment of inertia I_G of the triangular lamina about an axis passing through its centre of mass G and parallel to the z-axis;

(e) To which objects in Question 9 can the parallel and perpendicular axis theorems be applied?

Computing Help

The Python examples are very easy to run online without any installation:

1. Go to: https://notebooks.azure.com/torebutlin/libraries/ia-mechanics
2. click on the relevant template file;
3. click on the 'clone' button (near top left);
4. if needed: log in to Azure using your Raven account;
5. agree to creating a clone when prompted;
6. click on the relevant template file again: this will start a working iPython Notebook that you can run and edit.

You can also run the files locally by installing Python. The most straightforward way is to download 'Anaconda' from: https://www.anaconda.com/download/. Once installed, then open the 'Jupyter Notebook' app from the start menu found inside the Anaconda folder. You can navigate to the folder where you are keeping your `*.ipynb` files and open the templates.

Suitable past Tripos questions

Kinematics of a rigid body: IB 2014 Q3a,b
Instantaneous centres: IA 2014 Q8a
Rotating reference frames: IB 2017 Q3; IB 2015 Q5
Centre of mass and moment of inertia: IA 2018 Q9; IA 2015 Q7; IB 2015 Q6a

Answers

1. $\mathbf{v}_P = -\dfrac{100}{\sqrt{13}}\mathbf{k}\,\mathrm{mms}^{-1}$

2(a). $\boldsymbol{\omega} = 1.5\mathbf{k}$

2(b). $\boldsymbol{\omega}_A = \boldsymbol{\omega}_B = 1.5\mathbf{k}$

2(c). $\boldsymbol{\omega}_{AB} = 1.5\mathbf{k}$

2(d). $\mathbf{v}_A = -3\mathbf{i}\,\mathrm{ms}^{-1}$; $\mathbf{v}_B = 2.25\mathbf{j}\,\mathrm{ms}^{-1}$

2(e). $\mathbf{v}_{A/B} = -3\mathbf{i} - 2.25\mathbf{j}\,\mathrm{ms}^{-1}$; $\mathbf{v}_{B/A} = +3\mathbf{i} + 2.25\mathbf{j}\,\mathrm{ms}^{-1}$

3(a). Velocity of B should be $v_B = 10.4\,\mathrm{ms}^{-1}$.

3(b). $\boldsymbol{\omega} = 15\mathbf{k}\,\mathrm{rads}^{-1}$

3(c). $\mathbf{v}_G = 3\mathbf{i} + 5.2\mathbf{j}\,\mathrm{ms}^{-1}$

4(a). IC is at intersection of horizontal line through B and vertical line through A.

4(b). IC is 0.4 m to right of B.

5(a). $\boldsymbol{\omega} = 0.37\mathbf{k}\,\mathrm{rads}^{-1}$

5(b). $|\mathbf{v}_B| = 1.0\,\mathrm{ms}^{-1}$

5(c). Point C.

6(a). IC is at Point A.

6(b). $\boldsymbol{\omega} = -\dfrac{V_0}{R}\mathbf{k}$

6(c). $\mathbf{v}_{A/C} = -V_0\mathbf{e}_r$; $\mathbf{v}_{B/C} = +V_0\mathbf{e}_\theta$; $\mathbf{v}_{D/C} = -V_0\mathbf{e}_\theta$; $\mathbf{v}_{E/C} = +V_0\mathbf{e}_r$

6(d). $\mathbf{v}_A = \mathbf{0}$; $\mathbf{v}_B = V_0(\mathbf{i}+\mathbf{j})$; $\mathbf{v}_C = V_0\mathbf{i}$; $\mathbf{v}_D = V_0(\mathbf{i}-\mathbf{j})$; $\mathbf{v}_E = 2V_0\mathbf{i}$

6(e). $\mathbf{a}_A = \dfrac{V_0^2}{R}\mathbf{j}$; $\mathbf{a}_B = \dfrac{V_0^2}{R}\mathbf{i}$; $\mathbf{a}_C = \mathbf{0}$; $\mathbf{a}_D = -\dfrac{V_0^2}{R}\mathbf{i}$; $\mathbf{a}_E = -\dfrac{V_0^2}{R}\mathbf{j}$

7(b)(i). $\boldsymbol{\omega} = \mathbf{i} + 5\mathbf{j}$

7(b)(ii). $\boldsymbol{\omega} = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$

8(a). $\ddot{\mathbf{r}}_P = \left[R\ddot{\theta}\cos\theta - R(\dot{\theta}^2 + \Omega^2)\sin\theta\right]\mathbf{e}_1 + \left[2R\Omega\dot{\theta}\cos\theta\right]\mathbf{e}_2 - \left[R\ddot{\theta} + R\dot{\theta}^2\cos\theta\right]\mathbf{k}$

8(c). Component in $\mathbf{e}_2$ direction.

9(a). $\mathbf{r}_G = \dfrac{a}{2}\mathbf{i} + \dfrac{b}{2}\mathbf{k}$

9(b). $\mathbf{r}_G = \dfrac{2a}{3}\mathbf{i} + \dfrac{b}{3}\mathbf{j}$

9(c). $\mathbf{r}_G = \dfrac{5a}{12}\mathbf{i} + \dfrac{b}{2}\mathbf{j}$

9(d). $\mathbf{r}_G = \left(\dfrac{a}{2} - \dfrac{4d}{5}\right)(\mathbf{i}+\mathbf{j})$

10(a). $I_{zz} = M\left(\dfrac{a^2}{2} + \dfrac{b^2}{6}\right)$

10(b). $I_{xx} = \dfrac{Mb^2}{6}$, $I_{yy} = \dfrac{Ma^2}{2}$

10(d). $I_G = M\left(\dfrac{a^2}{18} + \dfrac{b^2}{18}\right)$

10(e). Parallel axis theorem: all. Perpendicular axis theorem: (b,c,d).