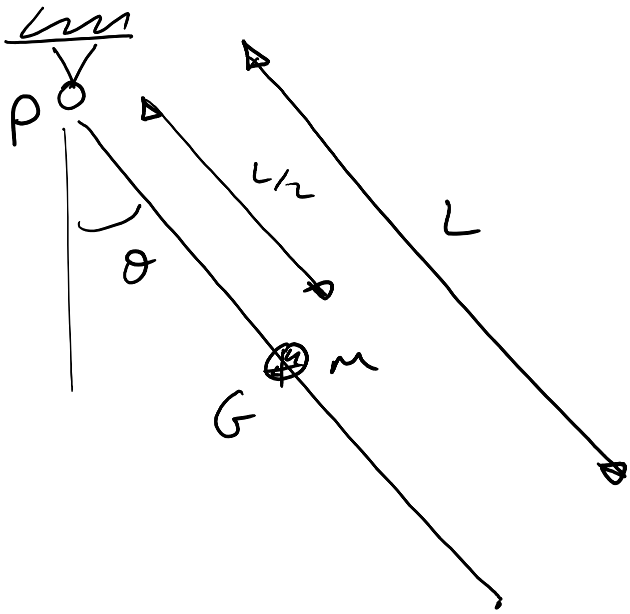
Paper 1: Mechanical Engineering

Examples Paper 4

Question 1

1 A uniform rod is used as a pendulum for a clock: the rod is connected to a frictionless pivot at one end and can swing in a vertical plane. The angular position is defined such that $\theta = 0$ represents the pendulum hanging vertically downwards.

(a) What is the moment of inertia of the pendulum about the pivot?



about centre of mass: $I_G = \frac{mL^2}{12}$ (from q3).

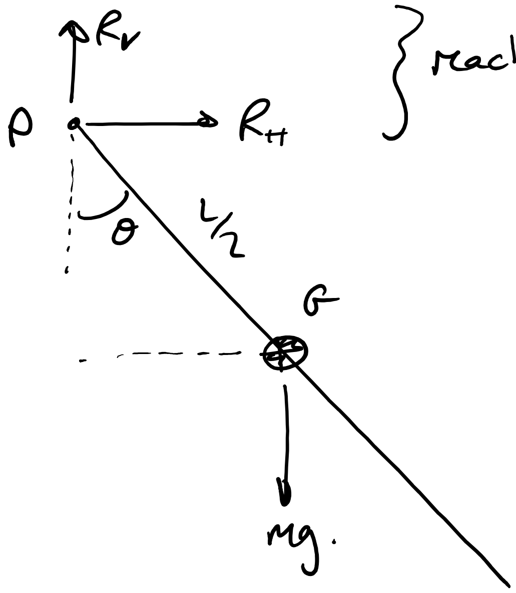
about pivot: $I_P = \frac{mL^2}{12} + m\left(\frac{L}{2}\right)^2$ by // axis theorem.

$$= \frac{mL^2}{3}$$

(b) Derive the equation of motion for the pendulum;

The pendulum axis is fixed, so it's easiest to work directly in terms of torque & I_P rather than separating out the translational & rotational motion (which are not independent).

FBD



} reaction components.

$\overset{+}{\circlearrowleft}$ $\sum$ moments about P = inertia about P × angular acceleration.

$$mg\frac{L}{2}\sin\theta = \frac{mL^2}{3} \times (-\ddot{\theta})$$

giving $\quad \frac{mL^2}{3}\ddot{\theta} + \frac{mgL}{2}\sin\theta = 0.$

$$\rightarrow \quad \frac{L}{3}\ddot{\theta} + \frac{g}{2}\sin\theta = 0.$$

(c) What is the general solution if θ is small and what is its physical interpretation?

$$2L\ddot{\theta} + 3g\sin\theta = 0$$

small θ: $\sin\theta \simeq \theta$.

$$\Rightarrow \quad 2L\ddot{\theta} + 3g\theta = 0.$$

2^{nd} order differential equation has general solution

$$\theta = A\sin\omega t + B\cos\omega t$$

$$\rightarrow \quad 2L(-A\omega^2\sin\omega t - B\omega^2\cos\omega t) + 3g(A\sin\omega t + B\cos\omega t) = 0$$

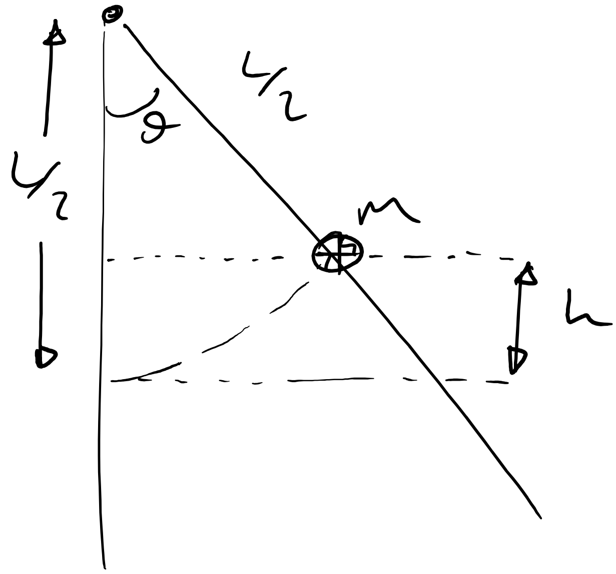
equating sin & cos terms gives:

$$\omega = \sqrt{3g/2L}$$

This means that the rod will continue to oscillate sinusoidally with frequency $\omega = \sqrt{3g/2L}$ if released from a small angle θ.

Question 1 (continued)

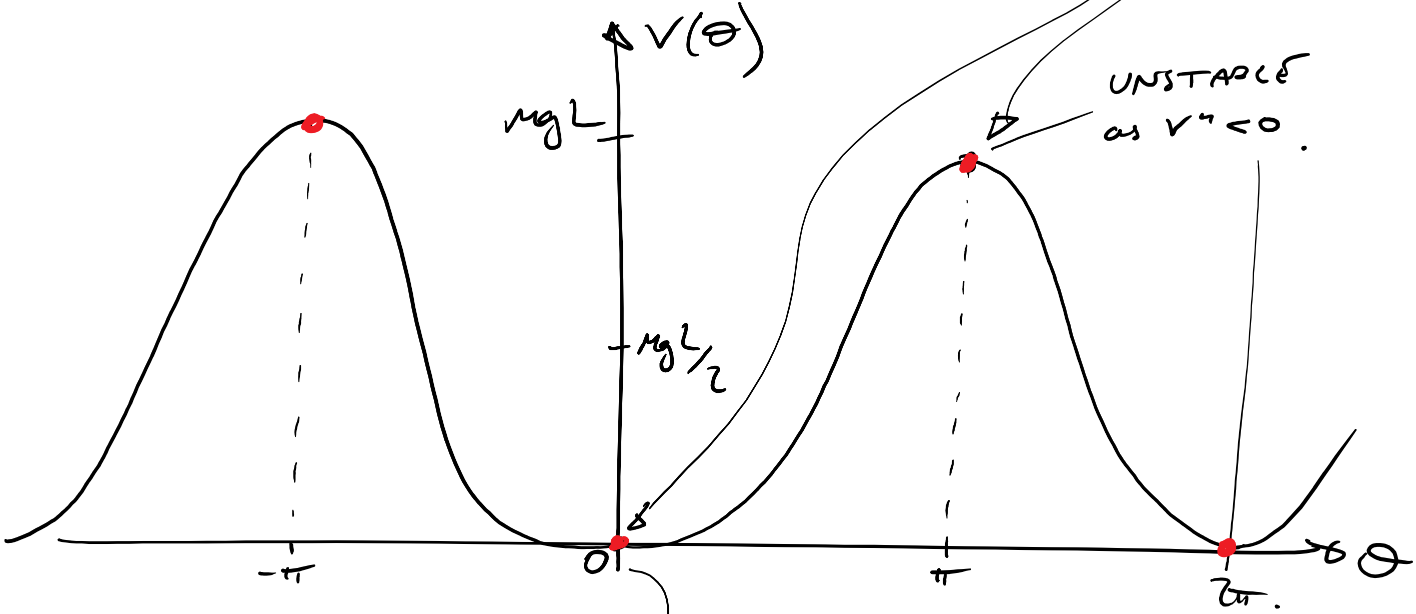
(d) Find the potential energy as a function of its angular position θ;



$$V = mgh = mg\frac{L}{2}(1-\cos\theta)$$

(e) How many equilibrium positions are there and which are stable?

Equilibria when $V' = 0$ at red dots:



UNSTABLE as $V'' < 0$.

STABLE as $V'' > 0$.

Seemingly infinite number of equilibria, but only two, as $\theta = \theta_0 + n.2\pi$ represents same place.

$\theta = 0$ is stable as $V'' > 0$.

$\theta = \pi$ is unstable as $V'' < 0$.

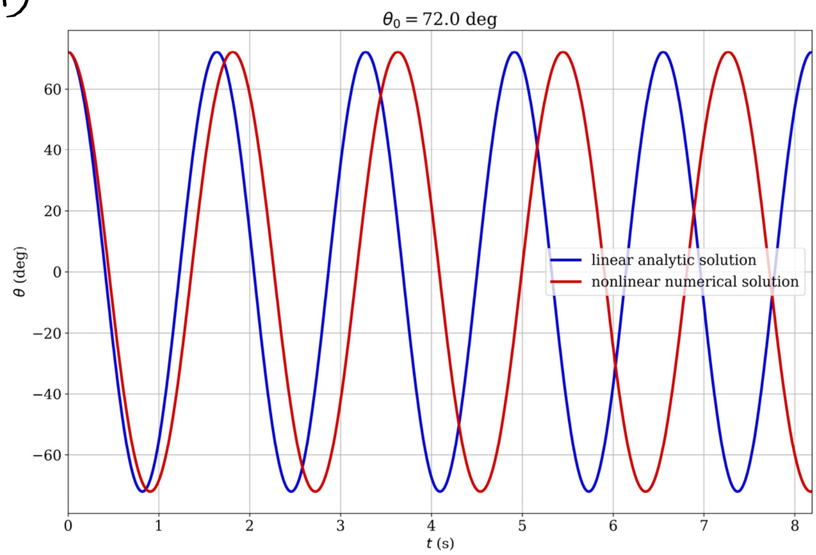
(f) Use the Python template file `p4q1_template.ipynb` to numerically solve the equation of motion when the pendulum is released from a given initial angle $\theta = \theta_0$, and plot:

(i) the angular position as a function of time;

(ii) the angular position on the x-axis and the angular velocity on the y-axis for a range of initial release angles (this kind of plot is known as the 'phase portrait' of a dynamical system).

(g) Compare your analytic solution in (b) with your numerical solution in (e). What happens as the release angle increases?

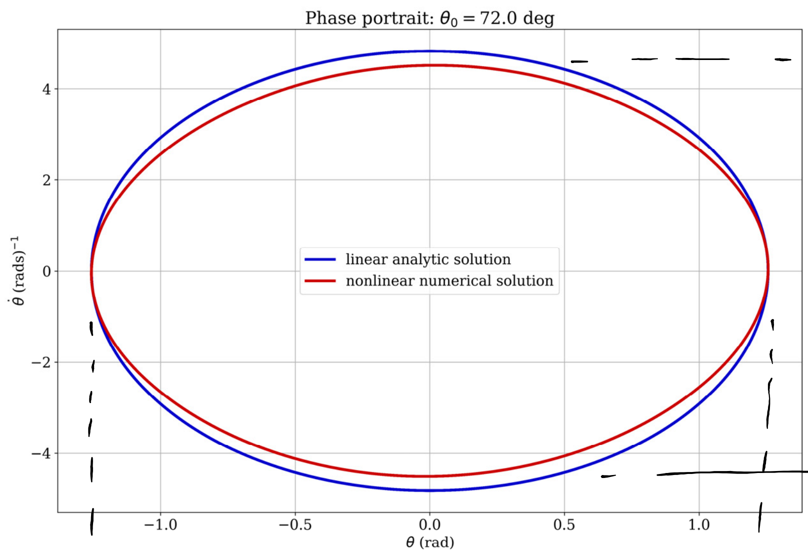
(i)



One example shown: $\theta_0 = 72°$
$= 0.4\pi$ rad.

Small angle approximation not valid: actual oscillation period longer than linear prediction.

(ii)



$\Delta\dot{\theta}$ is smaller for nonlinear solution, consistent with above.

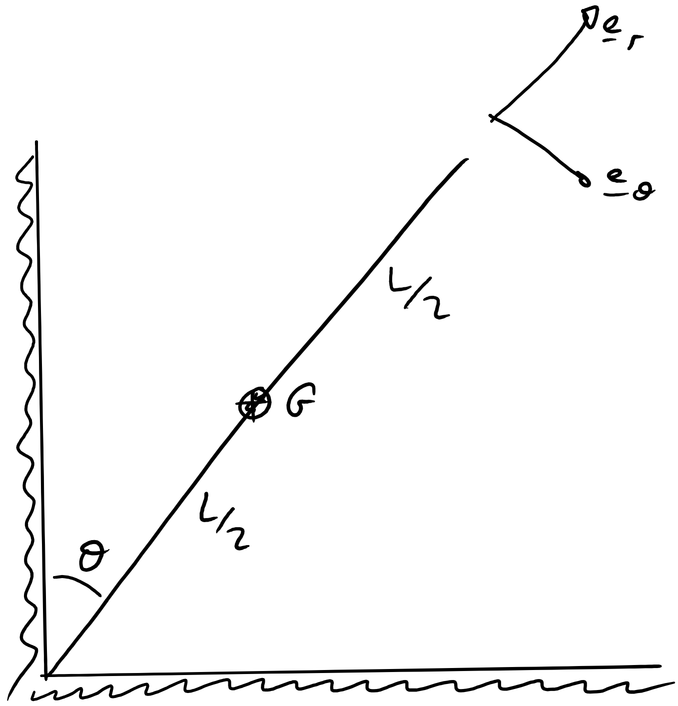
$\Delta\theta$ is the same

Question 2

2 A uniform ladder is held vertically against a wall standing with its foot in the angle between the wall and the floor. The top of the ladder is moved a short distance away from the wall and then released. The angular position is defined such that the ladder is vertical when $\theta = 0$.

(a) Assuming that the floor is rough so that the foot of the ladder does not slip:

(i) What is the acceleration of the centre of mass as a function of θ, $\dot{\theta}$, and $\ddot{\theta}$?



$$\underline{r}_G = \frac{L}{2}\underline{e}_r$$

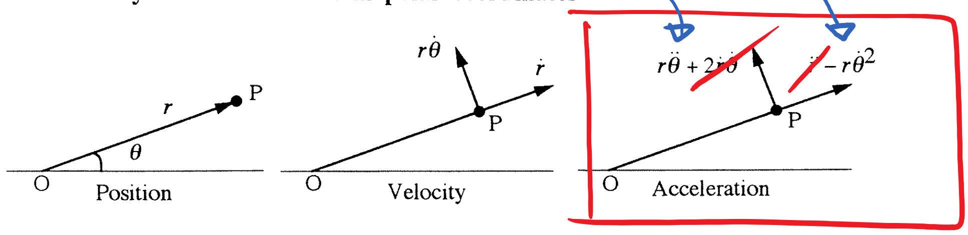
$$\dot{\underline{r}}_G = \frac{L}{2}\dot{\underline{e}}_r = \frac{L}{2}\dot{\theta}\,\underline{e}_\theta$$

$$\ddot{\underline{r}}_G = \frac{L}{2}\ddot{\theta}\,\underline{e}_\theta + \frac{L}{2}\dot{\theta}\,\dot{\underline{e}}_\theta$$

$$= \frac{L}{2}\ddot{\theta}\,\underline{e}_\theta - \frac{L}{2}\dot{\theta}^2\,\underline{e}_r \quad //$$

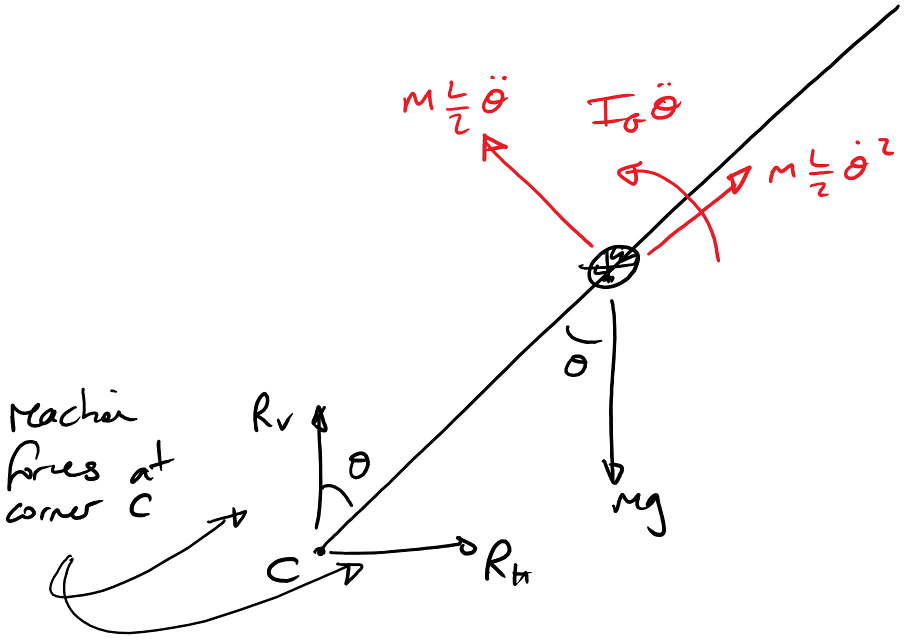
could also have obtained directly from databook

1.1: Velocity and acceleration in polar coordinates



Question 2 (continued)

(ii) Draw the Free Body Diagram of the ladder at an arbitrary angle θ during its fall, including D'Alembert forces and moments;



$$\left[I_G = \frac{mL^2}{12} \right]$$

(iii) Find the equation of motion of the ladder and integrate it to find the angular velocity as a function of θ [Hint: recall that $\ddot{\theta} = \dot{\theta}\mathrm{d}\dot{\theta}/\mathrm{d}\theta$];

Notice: no slip, so ladder is effectively pivoted at C. Enough to consider rotation alone:

$$\sum M_C \overset{+}{\curvearrowright} \Rightarrow \underbrace{mg\frac{L}{2}\sin\theta}_{\text{weight moment}} - \underbrace{m\frac{L}{2}\ddot{\theta}\cdot\frac{L}{2}}_{\text{centre of mass D'Alembert moment}} - \underbrace{\frac{mL^2}{12}\ddot{\theta}}_{\text{rotational D'Alembert moment}} = 0.$$

(sum moments about C +ve clockwise)

$$g/2 \sin\theta - L/3\,\ddot{\theta} = 0.$$

(same as pendulum in Q4 but sign different because definition of θ different)

Equation of motion:

$$\ddot{\theta} = \frac{3g}{2L}\sin\theta$$

$$\dot{\theta}\frac{d\dot{\theta}}{d\theta} = \ddot{\theta} = \frac{3g}{2L}\sin\theta$$

$$\dot\theta \frac{d\dot\theta}{d\theta} = \frac{3g}{2L}\sin\theta$$

initial condition $\dot\theta = 0$ at $\theta = 0$.

$$\int_0^{\dot\theta} \dot\theta \, d\dot\theta = \int_0^{\theta} \frac{3g}{2L}\sin\theta \, d\theta$$

$$\left[\dot\theta^2/2\right]_0^{\dot\theta} = \left[-\frac{3g}{2L}\cos\theta\right]_0^{\theta}$$

$$\dot\theta^2/2 = \frac{3g}{2L}(1-\cos\theta)$$

$$\dot\theta^2 = \frac{3g}{L}(1-\cos\theta) \quad //$$

alternative helpful identity:

$$\ddot\theta = \frac{d}{d\theta}\left(\tfrac{1}{2}\dot\theta^2\right)$$

$$\Rightarrow \frac{d}{d\theta}\left(\tfrac{1}{2}\dot\theta^2\right) = \frac{3g}{2L}\sin\theta$$

$$\tfrac{1}{2}\dot\theta^2 = C - \frac{3g}{2L}\cos\theta$$

$$\theta = 0,\ \dot\theta = 0 \Rightarrow C = \frac{3g}{2L}$$

$$\Leftrightarrow \dot\theta^2/2 = \frac{3g}{2L}(1-\cos\theta)$$

leading to same result.

(iv) Calculate the magnitude and direction of the horizontal force acting on the foot of the ladder just before the ladder strikes the ground.

$$\sum F_H \Rightarrow R_H = \frac{mL}{2}\ddot\theta\cos\theta - \frac{mL}{2}\dot\theta^2\sin\theta$$

sum forces horizontally

$$\left[\ \ddot\theta = \frac{3g}{2L}\sin\theta \quad \& \quad \dot\theta^2 = \frac{3g}{L}(1-\cos\theta)\ \right] \text{ from above.}$$

$$\text{So } R_H = \frac{mL}{2}\cdot\frac{3g}{2L}\sin\theta\cos\theta - \frac{mL}{2}\frac{3g}{L}(1-\cos\theta)\sin\theta$$

$$= \frac{3mg}{4}\sin\theta\cos\theta - \frac{3mg}{2}(1-\cos\theta)\sin\theta$$

$$= \frac{9mg}{4}\sin\theta\cos\theta - \frac{3mg}{2}\sin\theta \quad //$$

at $\theta = 90^\circ$, $R_H = -\frac{3mg}{2}$ // ie in $-\underline{i}$ direction.

(b) Now assume that the floor is smooth. Through what angle does the ladder turn before its foot starts to come away from the wall?

$$R_H = \frac{9mg}{4}\sin\theta\cos\theta - \frac{3mg}{2}\sin\theta$$

If floor smooth R_H can only be +ve.
Leaves wall when $R_H = 0$.

$$\frac{3mg}{2}\sin\theta\left(\frac{3}{2}\cos\theta - 1\right) = 0.$$

when $\sin\theta = 0$ $\quad (0, 180°)$

at start $\quad$ gone through floor.

OR $\frac{3}{2}\cos\theta - 1 = 0.$

$$\cos\theta = 2/3$$

$$\theta = 48.2°$$

More than required for this question:
– need to check R_V also remains +ve:

$$R_V = mg - \frac{mL}{2}\ddot{\theta}\sin\theta - \frac{mL}{2}\dot{\theta}^2\cos\theta$$

$$\left[\quad \ddot{\theta} = \frac{3g}{2L}\sin\theta \quad \& \quad \dot{\theta}^2 = \frac{3g}{L}(1-\cos\theta) \right] \text{ from above}$$

$$= mg - \frac{mL}{2}\frac{3g}{2L}\sin^2\theta - \frac{mL}{2}\frac{3g}{L}(1-\cos\theta)\cos\theta$$

$$= mg - \frac{3mg}{4}\sin^2\theta + \frac{3mg}{2}\cos^2\theta - \frac{3mg}{2}\cos\theta.$$

$$= mg - \frac{3mg}{4}(1-\cos^2\theta) + \frac{3mg}{2}\cos^2\theta - \frac{3mg}{2}\cos\theta$$

$$= mg - \frac{3mg}{4}\left(1-\cos^2\theta\right) + \frac{3mg}{2}\cos^2\theta - \frac{3mg}{2}\cos\theta$$

$$= mg - \frac{3mg}{4} + \frac{9mg}{4}\cos^2\theta - \frac{3mg}{2}\cos\theta.$$

$$= \frac{mg}{4} + \frac{9mg}{4}\cos^2\theta - \frac{3mg}{2}\cos\theta.$$

$$\frac{4R_v}{mg} = 1 + 9\cos^2\theta - 6\cos\theta = \left(3\cos\theta - 1\right)^2$$

[which is always true
so assumptions are ok.]

Question 3

3 A slice of toast of width $2L$ is held with its centre of gravity projecting a distance a beyond the edge P of a horizontal table, as shown in Figure 1. The toast is released from its horizontal position and initially rotates about edge P without slipping. During this motion, its inclination to the horizontal is θ.

(a) Calculate the angular velocity and acceleration $\dot{\theta}$ and $\ddot{\theta}$ during this initial rotation;

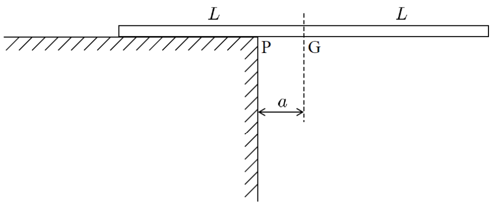
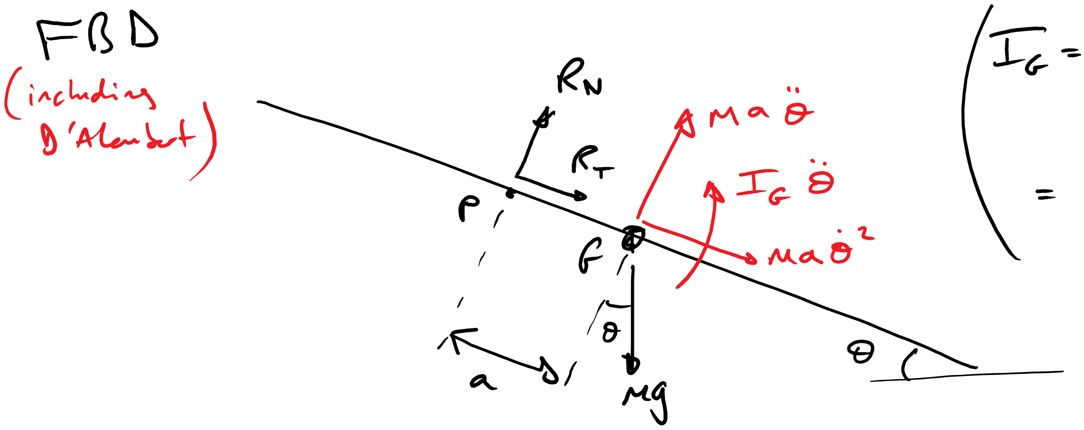


Figure 1



$$\left(\begin{aligned} I_G &= \frac{m(2L)^2}{12} \\ &= \frac{mL^2}{3} \end{aligned} \right)$$

As purely rotating, can obtain $\ddot{\theta}$ by moment equations alone:

$$\sum M_P^{\circlearrowleft} \Rightarrow \quad mg.a\cos\theta - ma^2\ddot{\theta} - I_G\ddot{\theta} = 0.$$

$$3\cancel{m}ga\cos\theta - 3\cancel{m}a^2\ddot{\theta} - \frac{\cancel{m}L^2}{\cancel{3}}\ddot{\theta} = 0.$$

$$\ddot{\theta} = \frac{3ga\cos\theta}{L^2 + 3a^2} \quad // \quad = \dot{\theta}\frac{d\dot{\theta}}{d\theta}$$

integrate:
$$\int_0^{\dot{\theta}} \dot{\theta}\, d\dot{\theta} = \int_0^{\theta} \frac{3ga}{L^2+3a^2} \cos\theta\, d\theta$$

$$\dot{\theta}^2/2 = \frac{3ga}{L^2+3a^2}\left[\sin\theta\right]_0^{\theta}$$

$$\dot{\theta}^2/2 = \frac{3ga}{L^2+3a^2}\sin\theta$$

$$\dot{\theta}^2 = \frac{6ga}{L^2+3a^2}\sin\theta \quad //$$

(b) Show that the toast will rotate about P until θ reaches the value given by:

$$\tan\theta = \frac{L^2}{9a^2+L^2}\mu$$

where μ is the coefficient of friction between the slice of toast and the table.

Slipping will start if $\left|\frac{R_T}{R_V}\right| > \mu$.

Need to find R_T & R_V

$$\sum F_T \Rightarrow \quad R_T = -ma\dot{\theta}^2 - mg\sin\theta$$

$$\sum F_N \Rightarrow \quad R_V = mg\cos\theta - ma\ddot{\theta}$$

$$\frac{R_T}{R_V} = \frac{-\cancel{m}a\dot{\theta}^2 - \cancel{m}g\sin\theta}{\cancel{m}g\cos\theta - \cancel{m}a\ddot{\theta}}$$

$$= \frac{-a\dfrac{6ga}{L^2+3a^2}\sin\theta - g\sin\theta}{g\cos\theta - a\dfrac{3ga\cos\theta}{L^2+3a^2}}$$

$$= \frac{-6ga^2\sin\theta - (L^2+3a^2)g\sin\theta}{-3ga^2\cos\theta + (L^2+3a^2)g\cos\theta}$$

Question 3 (continued)

$$= \frac{-6ga^2\sin\theta - (L^2+3a^2)g\sin\theta}{-3ga^2\cos\theta + (L^2+3a^2)g\cos\theta}$$

$$= \frac{-9ga^2\sin\theta - L^2 g\sin\theta}{L^2 g\cos\theta}$$

$$= -9\frac{a^2}{L^2}\cdot\tan\theta - \tan\theta$$

$$= -\tan\theta\left(1 + 9a^2/L^2\right)$$

$\left|\frac{R_T}{R_N}\right| = \mu$ at slipping limit, so :

$$\left(1 + 9a^2/L^2\right)\tan\theta = \mu.$$

$$\tan\theta = \frac{L^2}{9a^2+L^2}\,\mu \quad // \text{ as required.}$$

4 An 'inerter' is a device that has been used in Formula 1 suspension systems, that gives a force proportional to the *relative* acceleration between two connection points. A simple inerter design is shown in Figure 2. Each connection point A and B is joined to a row of gear teeth ('rack'). A circular gear ('pinion') connects the two racks. The rack and pinion mechanism can be assumed to be light. The pinion is fixed to the centre of a light rod and a mass m is attached to each end of the rod. The effective radius of the central pinion is r.

(a) What is the mass moment of inertia of the rotor (the light rod and masses) about its centre of mass?

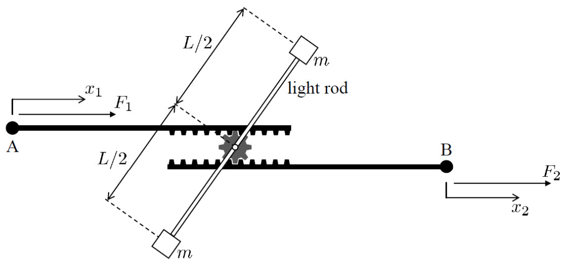
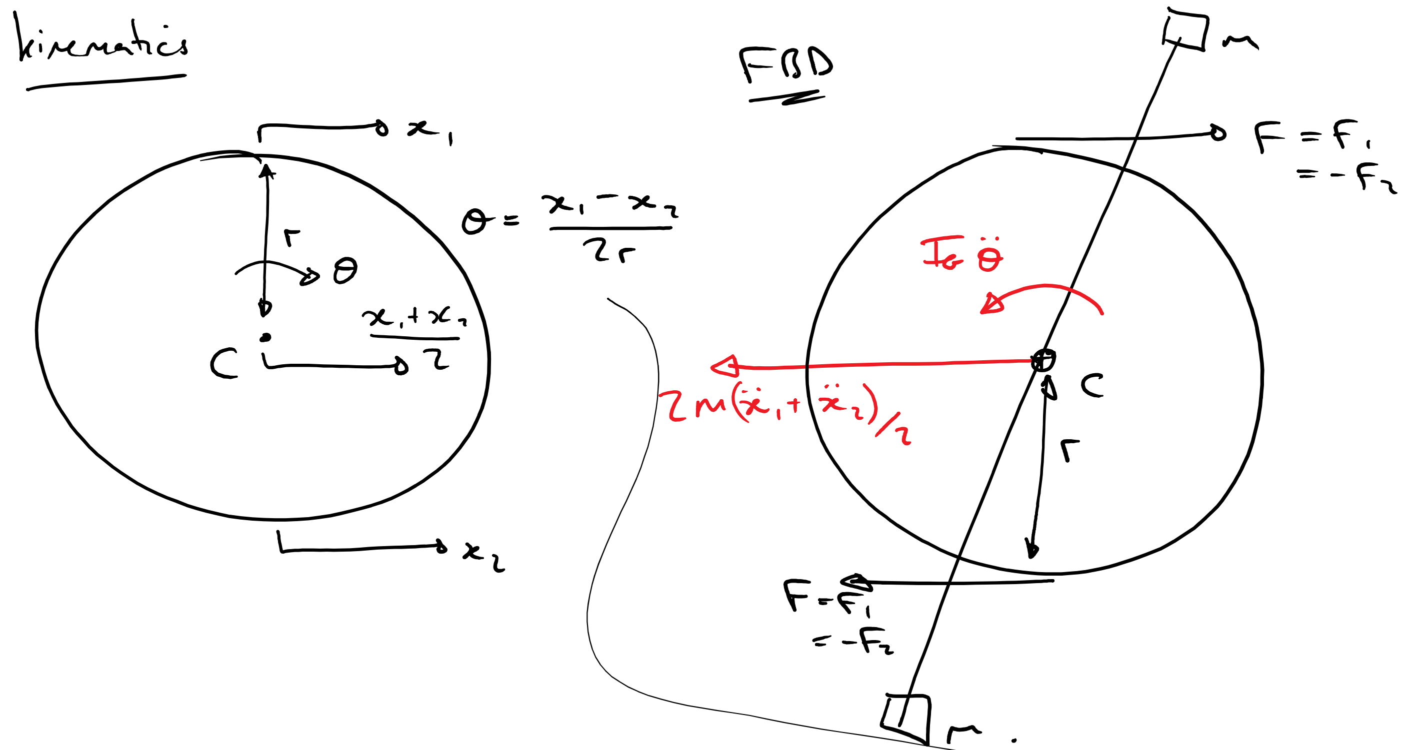


Figure 2

$$I_G = \int r^2 dm = m\left(\frac{L}{2}\right)^2 \times 2 = \frac{mL^2}{2}$$

mass concentrated at ends.

(b) Assuming that $F_1 = -F_2 \equiv F$ derive an expression for F in terms of the relative acceleration of A and B;



$$\sum M_C \circlearrowright \Rightarrow \qquad 2Fr - I_G \ddot{\theta} = 0.$$

$$2Fr = \frac{mL^2}{2} \frac{(\ddot{x}_1 - \ddot{x}_2)}{2r}$$

$$F = \frac{mL^2}{8r^2} (\ddot{x}_1 - \ddot{x}_2) \quad //.$$

n/b force at connections is proportional to relative acceleration.

(c) Under what circumstances is it reasonable to assume that $F_1 = -F_2$?

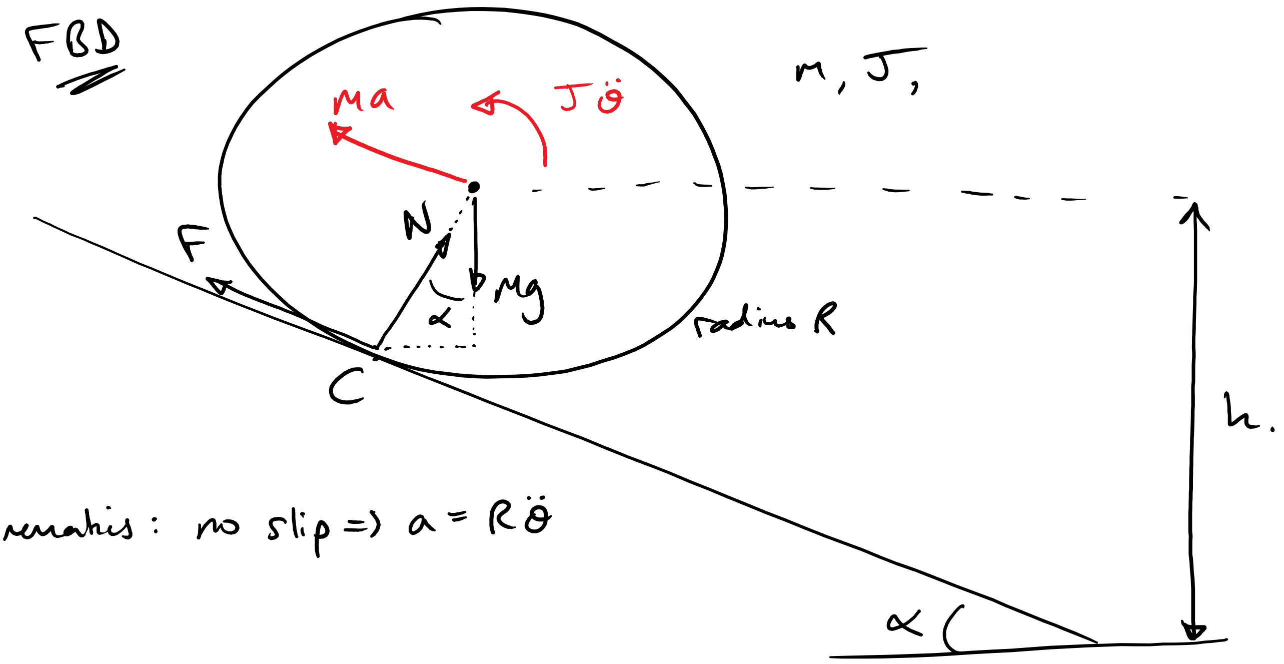
Assumption implies actual mass of device is small relative to $\left(\frac{mL^2}{8r^2}\right)$, true if L/r is large in this case. //

(d) What are some of the practical design challenges of this kind of device?

- Minimising actual device mass,
- minimising friction & "backlash" in gears
- designing compact device. //

5 A cylinder of mass m and polar moment of inertia J is released from rest at a height h, then rolls down a rough slope of angle α.

(a) Assuming no slipping occurs, derive an expression for the linear acceleration of the centre of mass after it has been released;



Kinematics : no slip $\Rightarrow a = R\ddot{\theta}$

Unknown reaction forces so take moments about contact

$$\sum M_c \circlearrowleft \Rightarrow \quad mgR\sin\alpha - maR - J\ddot{\theta} = 0.$$

$$mgR\sin\alpha - maR - J\,a/R = 0.$$

$$a = \frac{mgR\sin\alpha}{mR + J/R} = \frac{mgR^2\sin\alpha}{mR^2 + J}$$

(b) Will a hollow or solid cylinder reach the bottom of the slope first?

$J_{hollow} = mR^2$ (all mass concentrated at $r=R$)

$$J_{solid} = \int r^2\,dm = \underbrace{\frac{m}{\pi R^2}}_{\text{mass per area}} \int_0^R r^2 . \underbrace{2\pi r\,dr}_{dA}$$

$$= \frac{m.2\pi}{\pi R^2}\left[\frac{r^4}{4}\right]_0^R = \frac{m}{2R^2}.R^4 = \frac{mR^2}{2}$$

$$J_{hollow} > J_{solid}$$

$$\text{so} \quad a_{hollow} < a_{solid}$$

Solid cylinder will reach bottom first: note that this result holds even if the actual masses are different!

(c) What will the kinetic energy of the hollow and solid cylinders be at the bottom of the slope?

$$KE = \Delta PE = mgh. \quad \text{for both}$$

(d) What will the translational velocities of the hollow and solid cylinders be at the bottom of the slope?

$$KE = \frac{1}{2}mv^2 + \frac{1}{2}J\dot{\theta}^2 \qquad \& \qquad v = R\dot{\theta}$$

$$= \frac{1}{2}mv^2 + \frac{1}{2}J\left(\frac{v}{R}\right)^2 = mgh.$$

$$v^2\left(m + J/R^2\right) = 2mgh$$

$$v = \sqrt{\frac{2mgh}{m + J/R^2}}$$

i) hollow: $J = mR^2$, $\quad v = \sqrt{\dfrac{2mgh}{m + m}}$

$$= \sqrt{gh}$$

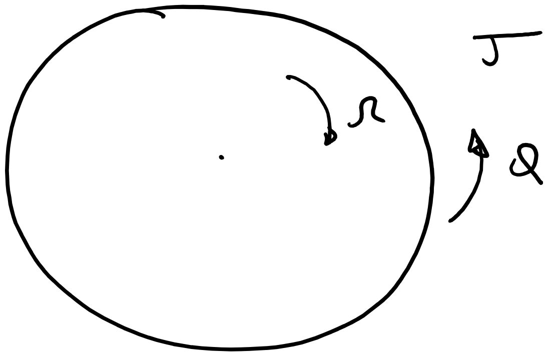
ii) solid: $J = mR^2/2$, $\quad v = \sqrt{\dfrac{2mgh}{m + m/2}}$

$$= \sqrt{\frac{4gh}{3}} = 2\sqrt{gh/3}$$

Question 6

6 A rotor with polar moment of inertia J is spinning in frictionless bearings and initially has angular velocity Ω. A constant torque Q is then applied to bring the rotor to rest.

(a) Determine from energy considerations the angle that the rotor turns through before it comes to rest;



$KE_1 = \frac{1}{2} J \Omega^2$.

Work done $= Q\theta$.

$Q\theta = \frac{1}{2} J \Omega^2$ to bring to rest.

$\theta = \frac{1}{2} J \Omega^2 / Q$ //

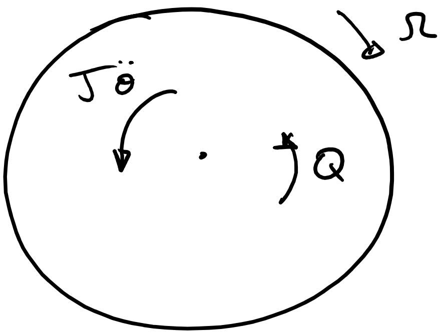
(b) Determine from momentum considerations the time the rotor takes to come to rest;

$\int Q \, dt = \Delta$ momentum.

$Q.\tau = J\Omega$

$\tau = J\Omega / Q$ //

(c) Use D'Alembert's principle to write down the equation of motion for the braking period. Show that the solution to this equation is consistent with your answers to (a) and (b).



$Q = -J\ddot{\theta}$

Integrate w.r.t. time:

$Qt = -J\dot{\theta} + C$

at $t = 0$, $\dot{\theta} = \Omega$, $\Rightarrow C = J\Omega$.

$Qt = J(\Omega - \dot{\theta})$

when $\dot{\theta} = 0$, $t = J\Omega / Q$ // as (b)

$$Q = -J\ddot{\theta} = -J\dot{\theta}\frac{d\dot{\theta}}{d\theta}$$

$$\int_0^{\theta} Q\,d\theta = \int_{\Omega}^{0} -J\dot{\theta}\,d\dot{\theta}$$

limits express start & end conditions.

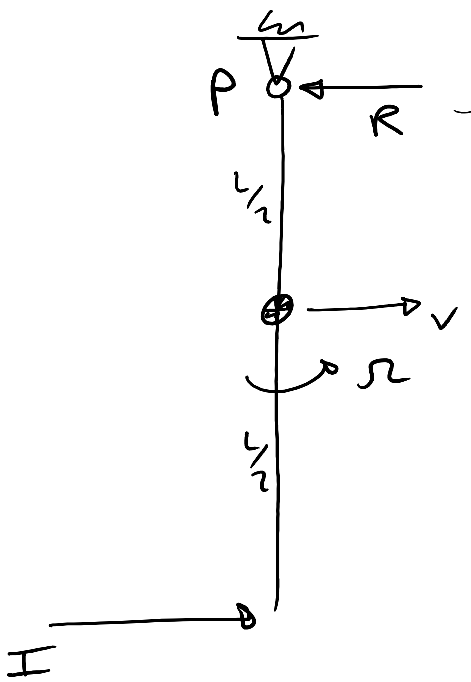
$$\left[Q\theta\right]_0^{\theta} = \left[-J\dot{\theta}^2/2\right]_{\Omega}^{0}$$

$$Q\theta = J\Omega^2/2$$

$$\theta = \tfrac{1}{2}J\Omega^2/Q \quad // \text{ as (a)}$$

7 A rigid uniform bar of mass M and length L hangs vertically under gravity from a frictionless hinge. A short impulse I is applied transversely to the bar at its lower end.

(a) What is the angular velocity with which the bar starts to move after the impulse?



reaction impulse.

(Impulse force larger than gravitational forces.)

$$I_G = \frac{ML^2}{12}$$

Angular momentum about P before : $H_1 = 0$

Angular momentum about P after : $H_2 = Mv L/2 + I_G \Omega$

Impulse moment about P = change in angular momentum.

$$Mv\frac{L}{2} + \frac{ML^2}{12}\Omega = IL, \quad \& \quad \Omega\frac{L}{2} = v.$$

$$M\Omega\left(\frac{L}{2}\right)^2 + \frac{ML^2}{12}\Omega = IL$$

$$\Omega = \frac{IL}{ML^2/3} = \frac{3I}{ML}$$

(b) What is the impulsive reaction at the hinge?

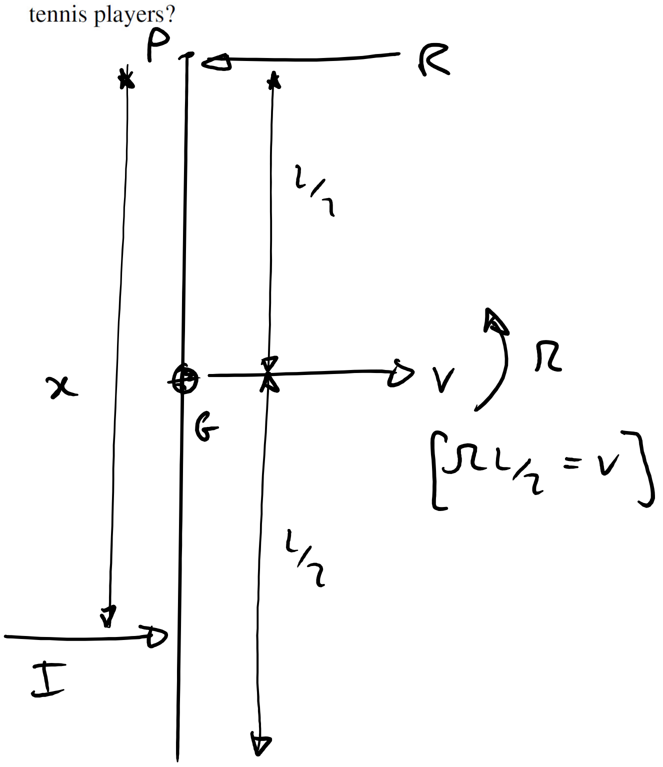
$$v = \Omega L/2 = \frac{3I}{ML}\cdot\frac{L}{2} = \frac{3I}{2M}$$

impulse = change in linear momentum

$$I - R = \cancel{M}\frac{3I}{2\cancel{M}}$$

$$R = I\left(1 - 3/2\right) = -I/2$$ ie to right

(c) How far down the bar from the hinge should the impulse be applied if there is to be no impulse reaction at the hinge? What is the implication of this result for cricket, baseball and tennis players?



Moments about P:

$$Ix = Mv\frac{L}{2} + \frac{ML^2}{12}\Omega$$

$$= M\Omega\left(\frac{L}{2}\right)^2 + \frac{ML^2}{12}\Omega$$

$$= M\Omega L^2/3$$

$$\Omega = \frac{3Ix}{ML^2}$$

Linear momentum:

$$I - R = Mv = M\Omega L/2$$

now want $R = 0$,

so: $$I = M\Omega L/2 = \cancel{M} \cdot \frac{3Ix}{\cancel{M}L^2} \cdot \frac{L}{2}$$

$$1 = \frac{3x}{2L}, \quad x = \frac{2L}{3}$$

ie 1/3 up from bottom.

Implication for sport: no impulse will be felt at handle if strike ball at 'sweet spot' of racket. Exact location will depend on racket design.

8 A frisbee is thrown such that it lands on a platform at position A which is at a distance d from the centre of the platform, as illustrated in Figure 3. The frisbee has mass m_F and mass moment of inertia I_F about its centre, and is released with translational velocity U_0 and angular velocity ω_0 anticlockwise. The platform is initially at rest but is free to rotate about its centre C and has a mass moment of inertia I_P. After the frisbee has landed it does not slip relative to the platform.

(a) What is the total initial angular momentum about point C?

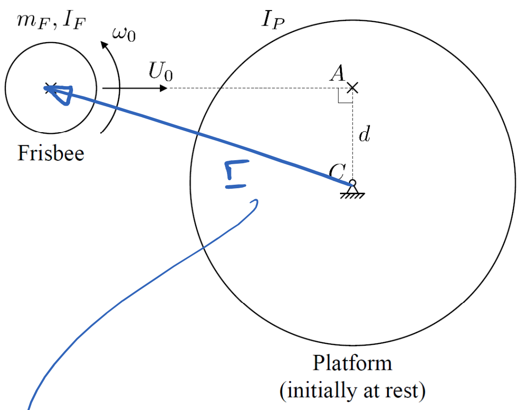
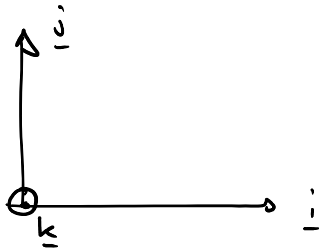


Figure 3



Initial angular momentum $$\underline{h}_{C1} = \underbrace{\underline{h}_F}_{\text{angular momentum of frisbee about its centre of mass}} + \underbrace{\underline{r} \times m_F \underline{v}_F}_{\text{contribution due to translation of centre of mass}} + \underbrace{0}_{\text{angular momentum of platform}}$$

$$= (I_F \omega_0 - m_F U_0 d)\,\underline{k} \quad //$$ ie +ve anticlockwise.

(b) What is the mass moment of inertia about point C of the combined system after the frisbee has landed?

$$I_{combined} = \underbrace{I_P}_{\text{platform}} + \underbrace{I_F}_{\text{frisbee about its CoM}} + \underbrace{m_F d^2}_{\text{by parallel axis theorem}} \quad //$$

(c) What is the angular velocity (including its direction) of the platform after the frisbee has landed?

Angular momentum about C conserved because no external torque about point C during landing. There will be reaction forces at point C but these do not apply any moment about C.

$$\underline{h}_{C1} = \underline{h}_{C2}$$

Take angular velocity after landing $\Omega \underline{k}$, ie +ve anticlockwise

$$\left(I_F \omega_0 - m_F U_0 d\right)\underline{k} = I_{combined}\, \Omega \underline{k}$$

$$\Rightarrow \Omega = \frac{I_F \omega_0 - m_F U_0 d}{I_p + I_F + m_F d^2}\, \underline{k}$$

(d) Under what circumstances does the platform rotate anticlockwise, and how is this consistent with the principle of conservation of linear momentum?

anticlockwise if Ω +ve,

ie $I_F \omega_0 > m_F U_0 d$

Seemingly counterintuitive case: frisbee lands on platform then reverses linear translation direction.

But notice: although no external torque is applied about point C, there are external reaction forces so we do not expect linear momentum to be conserved.

Question 9

9 A rigid uniform plank of length $3a$ is initially at rest and held at a height h above a horizontal rail as shown in Figure 4. When released it falls with zero angular velocity before hitting the rail at a distance a from one end.

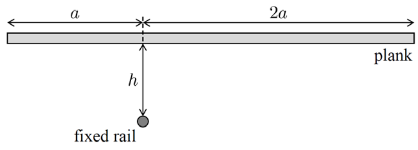


Figure 4

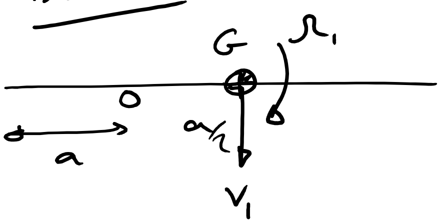
(a) Find the velocity v_1 of the plank immediately prior to its first impact with the rail;

$$PE = mgh$$

$$KE = \frac{1}{2} m v_1^2 = mgh \Rightarrow \qquad v_1 = \sqrt{2gh}$$

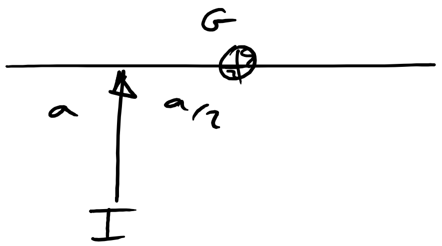
(b) Hence find an expression for the angular velocity of the plank immediately after the impact in terms of the pre-impact velocity v_1, given that the coefficient of restitution between the plank and the rail is e (where $0 \le e \le 1$);

BEFORE



$v_1 = \sqrt{2gh}$

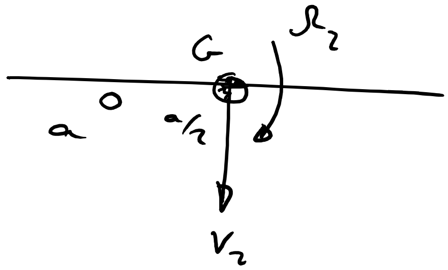
IMPACT



gravity insignificant during impulse.

I_L only present if +ve (left bar is not 'sticky').

AFTER



define sign convention here: this choice is deliberately same sign as 'before' to avoid confusion.

Moment of momentum conserved about rail during impact:

$$m v_1 a/2 + I_G \underbrace{\Omega_1}_{=0} = m v_2 a/2 + I_G \Omega_2$$

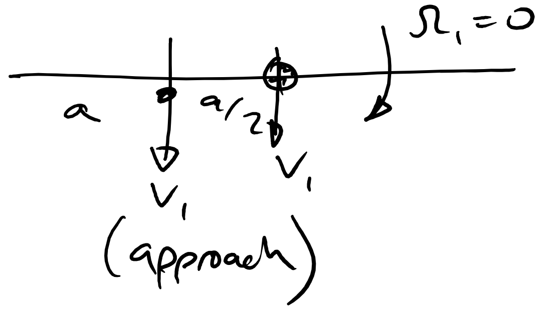
$$I_G = \frac{m(3a)^2}{12} = \frac{3ma^2}{4}$$

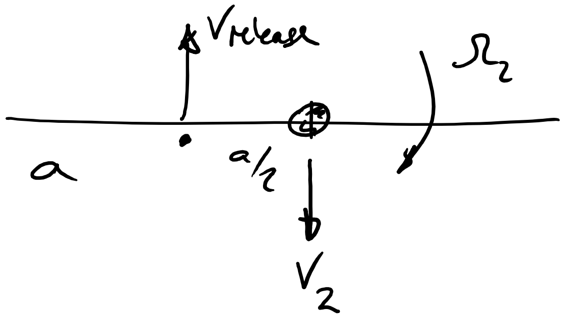
$$m v_1 a/2 = m v_2 a/2 + \frac{3ma^2}{4} \Omega_2 \qquad (1)$$

two unknowns, need second equation

Coefficient of restitution $= e$.

$$V_{release} = e \text{ approach}$$





$$V_{release} = eV_1$$
$$= -V_2 + \Omega_2(L-x)$$

$$eV_1 = -V_2 + \Omega_2 a/2 \quad \text{——— (2)}$$

$$V_2 = \Omega_2 a/2 - eV_1$$

combine with (1):

$$mV_1 a/2 = mV_2 a/2 + \frac{3ma^2}{4}\Omega_2 \quad \text{——— (1)}$$

$$mV_1 a/2 = m\left(\Omega_2 a/2 - eV_1\right) a/2 + \frac{3ma^2}{4}\Omega_2$$

$$\cancel{m}a^{\cancel{2}}\Omega_2 = \cancel{m}V_1 \cancel{a}/2 + \cancel{m}\cancel{a}eV_1/2$$

$$\Omega_2 = \frac{V_1(1+e)}{2a}$$

$$V_2 = \frac{V_1(1+e)}{2\cancel{a}} \cdot \frac{\cancel{a}}{2} - eV_1 = \frac{V_1}{4} + \frac{eV_1}{4} - eV_1$$
$$= V_1\left(\frac{1}{4} - 3e/4\right)$$

(c) For the cases when $e = 1/3$ and $e = 1$, find the greatest height reached by the midpoint of the plank during its subsequent behaviour. Is any energy lost in the impact when $e = 1$?

$$v_2 = \frac{v_1}{4}(1-3e)$$

Rotational contribution to kinetic energy is constant after impact, so only need translational contribution to compute height:

$$\Delta PE = \Delta KE$$

$$\cancel{m}gh_2 = \frac{1}{2}\cancel{m}v_2^2$$

this is KE change if v_2 upwards, ie $1-3e < 0$, $e > 1/3$

$$h_2 = v_2^2/2g = \frac{1}{2g} \cdot \frac{v_1^2}{16}(1-3e)^2$$

$$= \frac{v_1^2}{32g}(1-3e)^2, \quad v_1 = \sqrt{2gh}$$

$$= \frac{2gh}{32\cancel{g}}(1-3e)^2$$

$$= \frac{h}{16}(1-3e)^2.$$

if $e = 1/3$, $h_2 = 0$;

if $e = 1$, $h_2 = \frac{h}{16} \cdot 4 = h/4$ //

Question 9 (continued)

Is any energy lost in the impact when $e = 1$?

$$KE \text{ after} = \frac{1}{2} m v_1^2 + \frac{1}{2} I_G \Omega_2^2$$

$$v_2 = \frac{v_1}{4}(1 - 3e) = -\frac{v_1}{2}$$

$$\Omega_2 = \frac{v_1(1+e)}{2a} = v_1/a \, .$$

so $$KE = \frac{1}{2} m v_1^2/4 + \frac{1}{2} \frac{3m\cancel{a^2}}{4} \cdot v_1^2/\cancel{a^2} \, .$$

$$= \frac{1}{2} m v_1^2 \left(\frac{1}{4} + \frac{3}{4}\right) = \frac{1}{2} m v_1^2$$

same as initial KE so no energy loss.