Paper 1: Mechanical Engineering

Examples Paper 1

Question 1

1 Draw a suitable Free Body Diagram for each of the following examples, thinking carefully about the direction of the forces and where they act:

(a) A car driving at constant speed up a hill of constant slope;

(b) A coffee cup on a seat table in an aeroplane during take-off;

(c) A cyclist travelling at constant speed around a corner, taking the cyclist and bike as one body (draw two diagrams, one from behind and one from the side);

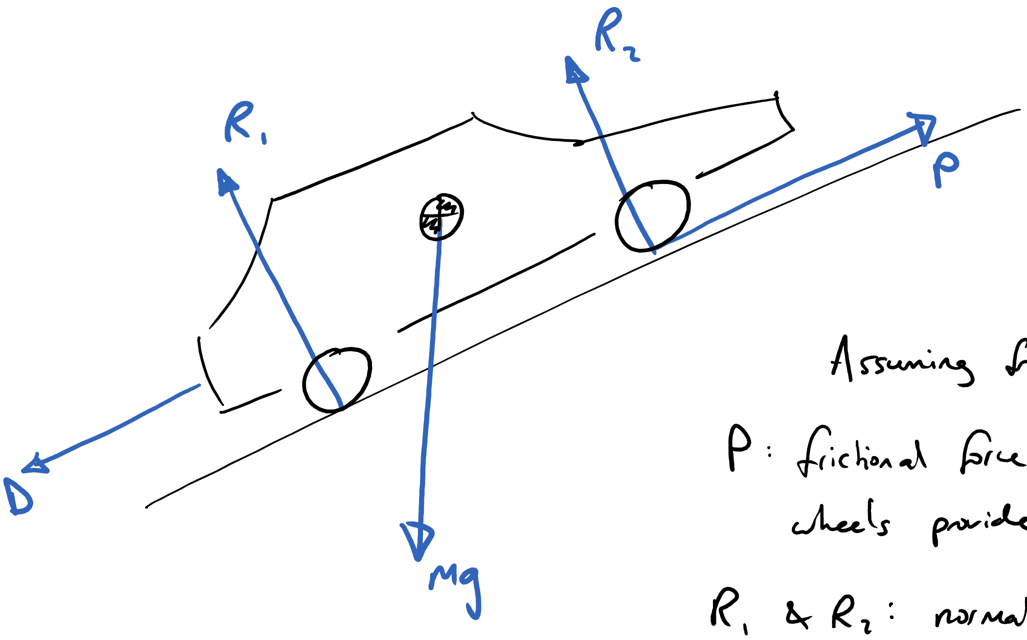
(d) The oar of a rowing boat during a stroke;

(e) A rowing boat including the oars within the Free Body Diagram;

(f) A sailing yacht travelling at 90 degrees to the apparent wind, as illustrated in Figure 1. Note: the interaction of the wind and sails generates a force that has a *lift* component that is orthogonal to the wind, and a *drag* component that is inline with the wind. The boat itself experiences drag as it travels through the water, and there is a keel under the boat that acts in a similar way to wheels: producing very little resistance to motion longitudinally, but a large reaction force laterally.

Note: there is not a unique solution for these diagrams. Please discuss with supervisors.

(a) A car driving at constant speed up a hill of constant slope;



Assuming front wheel drive:

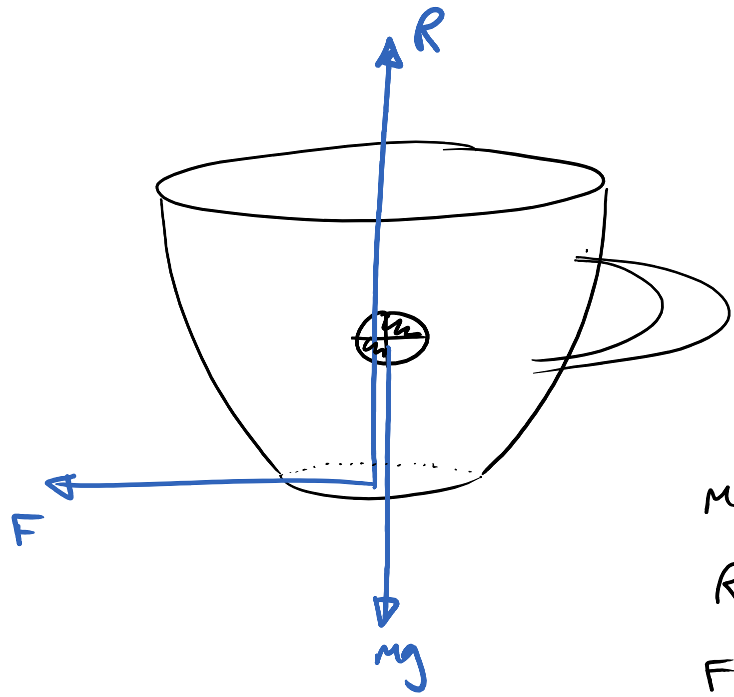
P: frictional force acting on drive wheels provide forward force

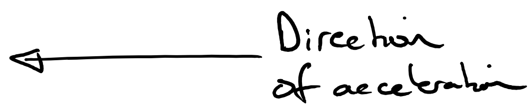
R_1 & R_2: normal reaction forces on front and back wheels.

mg: weight of vehicle acting through centre of mass.

D: drag force from air resistance. This is a complicated distribution of force so its centre of action has not been specified in diagram.

(b) A coffee cup on a seat table in an aeroplane during take-off;





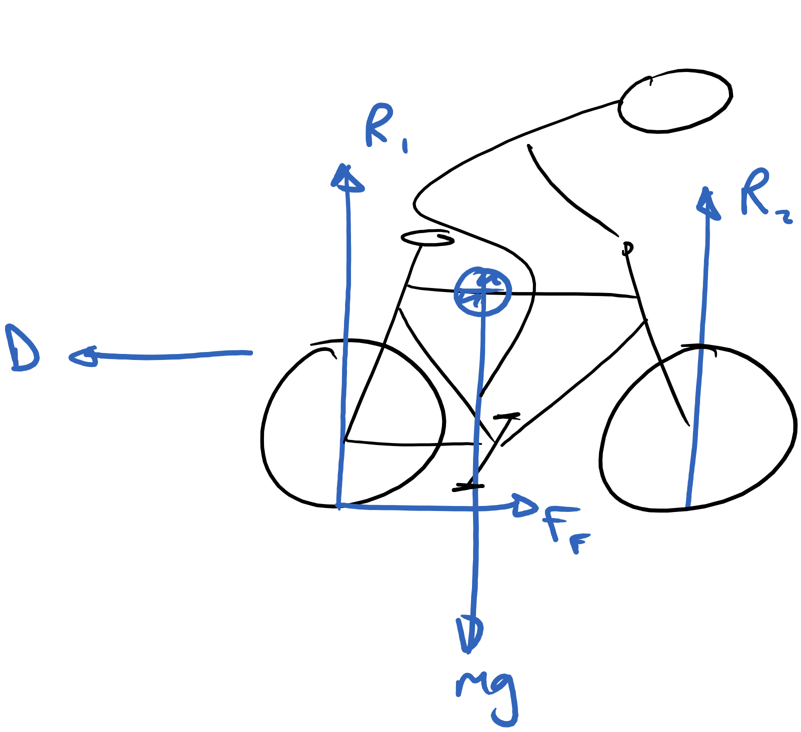
mg: weight of coffee cup.

R: normal reaction force

F: friction force of table accelerating cup.

(c) A cyclist travelling at constant speed around a corner, taking the cyclist and bike as one body (draw two diagrams, one from behind and one from the side);

side:



Similar to car. In this case rear wheels are driven. Person is included in FBD so internal forces (eg on pedals) not needed.

R_1 & R_2: normal reaction forces

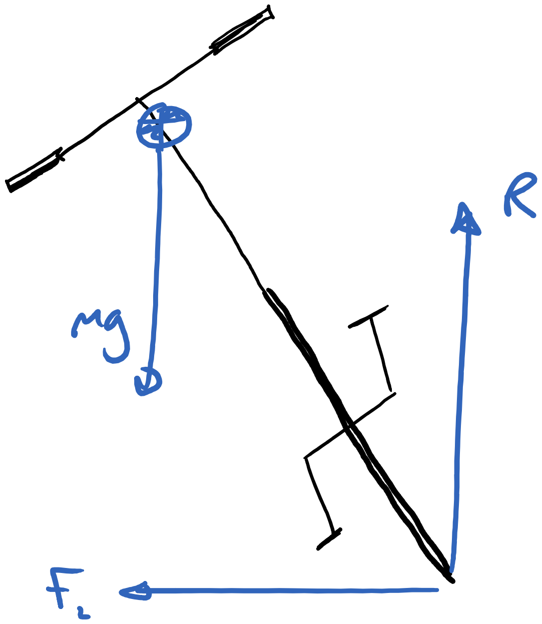
mg: weight of bike + cyclist

D: drag force from air resistance.

F_F: friction force driving bike forwards against D. (forward component).

(c) A cyclist travelling at constant speed around a corner, taking the cyclist and bike as one body (draw two diagrams, one from behind and one from the side);

behind: (cyclist omitted for clarity).
cornering left.

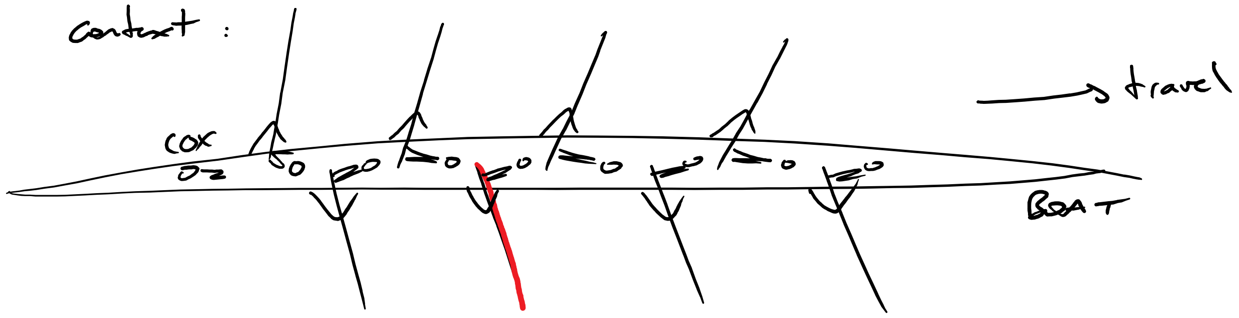


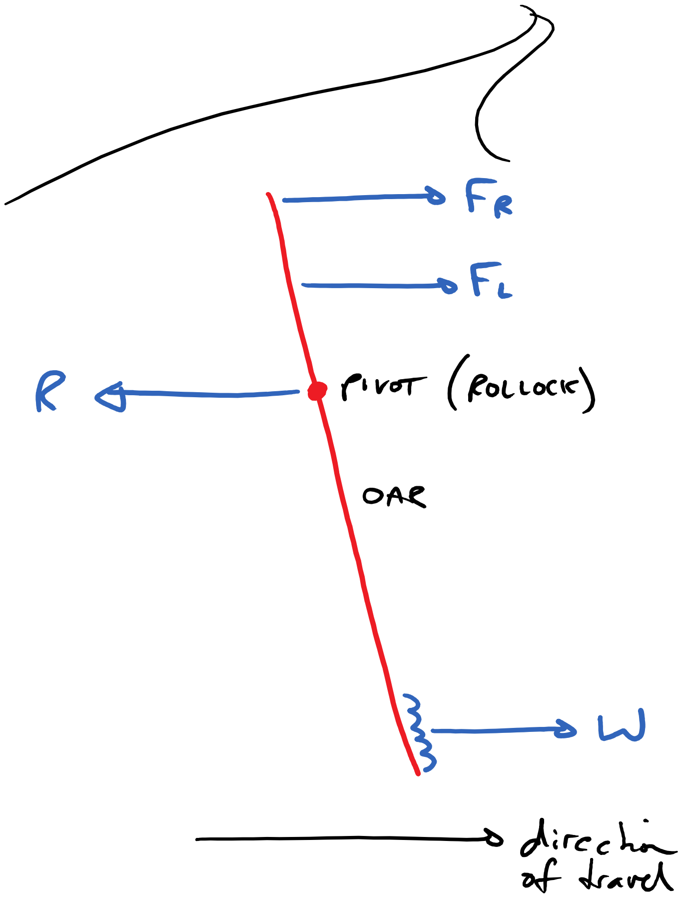
mg: weight of bike + cyclist.

R: normal reaction force.

F_L: lateral component of friction force that causes centripetal acceleration towards centre of curvature of path.

(d) The oar of a rowing boat during a stroke;





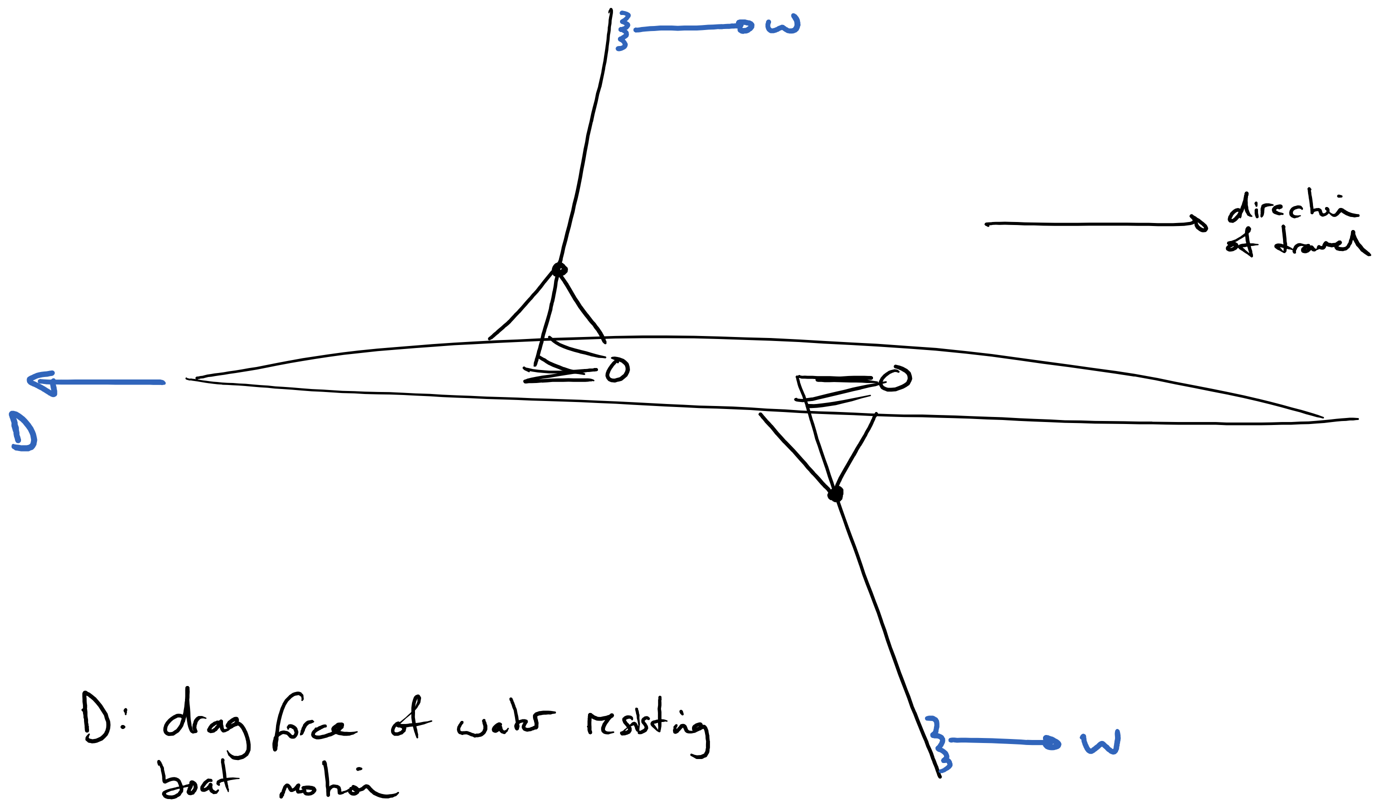
F_R & F_L : left and right arm forces on oar

R : reaction force of rollock on oar

W : distributed force of water on blade of oar

(e) A rowing boat including the oars within the Free Body Diagram;

for simplicity a pair rather than an eight:

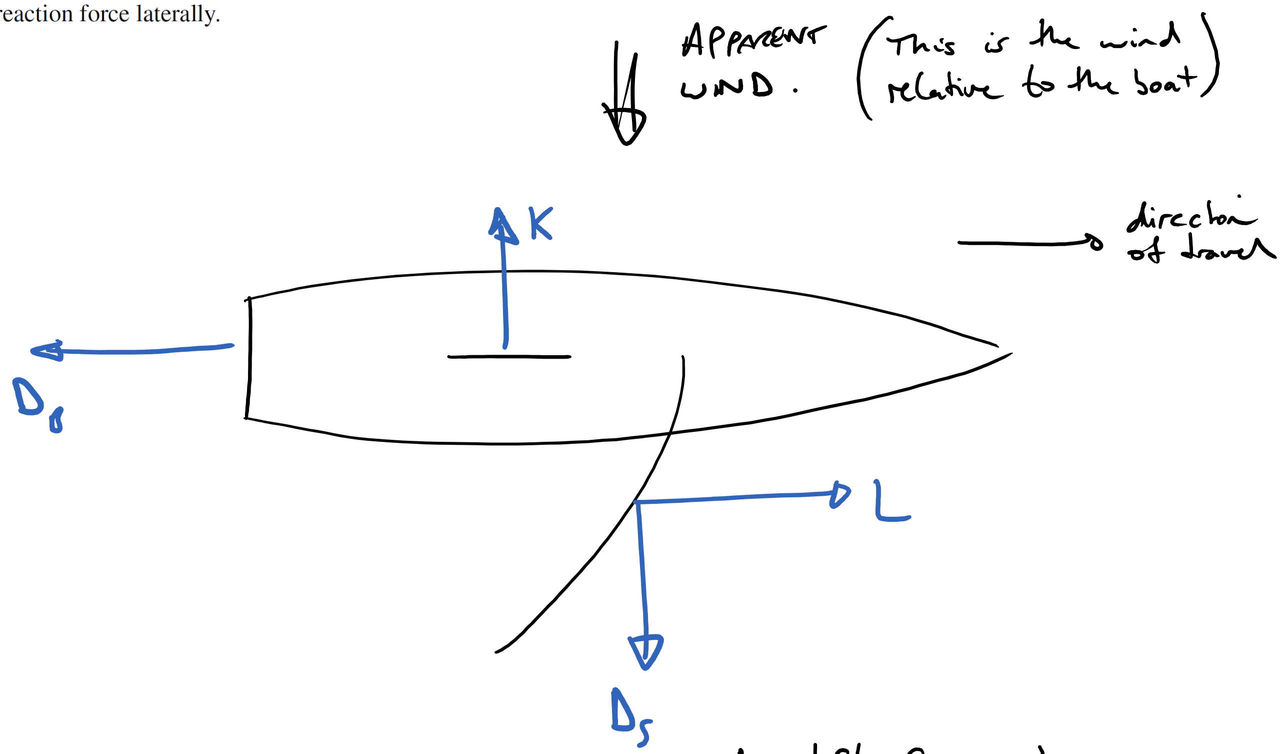


D: drag force of water resisting boat motion

W: distributed force of water acting on blades as pulled through water, note direction

Note: internal forces not part of Free Body Diagram.

(f) A sailing yacht travelling at 90 degrees to the apparent wind, as illustrated in Figure 1. Note: the interaction of the wind and sails generates a force that has a *lift* component that is orthogonal to the wind, and a *drag* component that is inline with the wind. The boat itself experiences drag as it travels through the water, and there is a keel under the boat that acts in a similar way to wheels: producing very little resistance to motion longitudinally, but a large reaction force laterally.



L: lift from sail
D_S: drag from sail
D_B: drag on boat
K: keel reaction force

Note: boat moves forward because the force on the sail has a component in forward direction.
Try re-drawing with the boat pointing 45° into the wind: this demonstrates how yachts can sail upwind.

Question 2

2 Which of the following can reasonably be approximated as inertial frames of reference:

(a) Reference frame fixed to a car during a 0-60 mph test?

(b) Reference frame fixed to a car travelling at 60 mph in a straight line on a smooth road?

(c) Reference frame fixed to a wind turbine blade rotating at constant speed with z-axis always aligned to one of the blades?

(d) Reference frame fixed to a wind turbine blade rotating at constant speed, with z-axis of reference frame always point vertically upwards?

(a, c, d) are all accelerating reference frames so are not inertial.

(b) is the only inertial frame.

For (b) if the road was rough, the car would be bouncing up and down, so the frame would again be non-inertial.

Question 3

3 What is the weight in Newtons of a person whose mass is 75 kg? If the person were to jump out of an aeroplane what would be their weight during free fall?

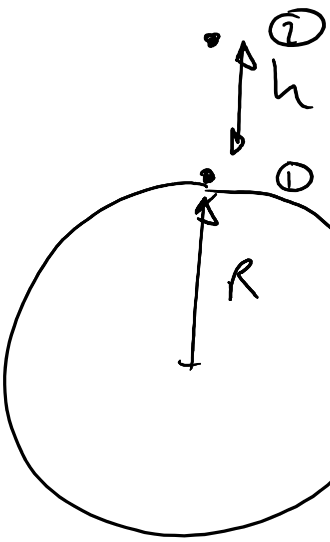
$$W = mg = 75 \times 9.81$$
$$= 735.75 \text{ N}$$
$$\simeq 736 \text{ N}$$

(appropriate level of accuracy)

Weight unchanged in free-fall (except for a very small change in the value g due to altitude of jump).

Nevertheless, the person would <u>feel</u> 'weightless' at the outset because the internal forces of the person would be nearly zero.

4 At what altitude h above the north pole is the weight of an object reduced to one half of its value on the earth's surface? Assume the earth is a sphere of radius R and express h as a fraction of R.



$$W_2 = \frac{GMm}{(R+h)^2}$$

$$W_1 = \frac{GMm}{R^2}$$

$$W_2 = \frac{1}{2} W_1$$

$$\frac{GMm}{(R+h)^2} = \frac{GMm}{2R^2}$$

$$\Rightarrow (R+h)^2 = 2R^2$$

ie $R+h = \sqrt{2}\,R$.

so $h = R(\sqrt{2}-1)$

$h/R \simeq 0.41$ //

Question 5

5 A particle P moves around a circle having a fixed centre C, radius R, and origin O on the circle's circumference as illustrated in Figure 2.

(a) Derive an expression for the Cartesian coordinates (x, y) of P in terms of R and ψ.

(b) Using Cartesian coordinates and associated unit vectors, find an expression for the position $\mathbf{r}$, velocity $\dot{\mathbf{r}}$ and acceleration $\ddot{\mathbf{r}}$ of the particle in terms of R, ψ and its derivatives $\dot{\psi}$ and $\ddot{\psi}$.

(c) If $\dot{\psi}$ is constant, what can you say about the speed of the particle? Show the direction of travel along the path if $\dot{\psi} > 0$.

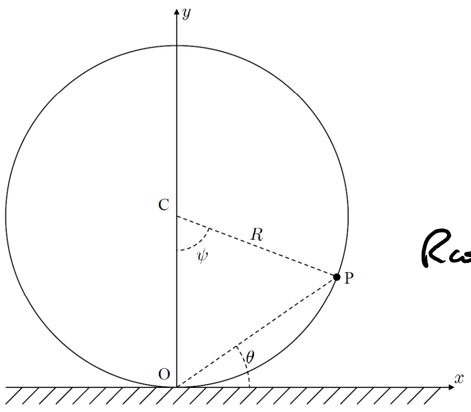
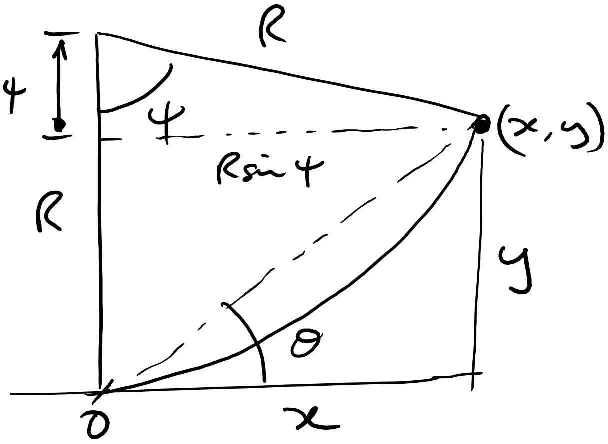


Figure 2



(a) $x = R\sin\psi$

$y = R(1-\cos\psi)$.

(b) $\underline{r}_p = x\,\underline{i} + y\,\underline{j} = R\sin\psi\,\underline{i} + R(1-\cos\psi)\,\underline{j}$

$\dot{\underline{r}}_p = \dfrac{d\underline{r}_p}{dt} = \dfrac{d\underline{r}_p}{d\psi}\dfrac{d\psi}{dt} = R\dot{\psi}\cos\psi\,\underline{i} + R\dot{\psi}\sin\psi\,\underline{j}$

$\ddot{\underline{r}}_p = \dfrac{d\dot{\underline{r}}_p}{dt} = R\ddot{\psi}\cos\psi\,\underline{i} - R\dot{\psi}^2\sin\psi\,\underline{i}$

$\quad + R\ddot{\psi}\sin\psi\,\underline{j} + R\dot{\psi}^2\cos\psi\,\underline{j}$

$= (R\ddot{\psi}\cos\psi - R\dot{\psi}^2\sin\psi)\,\underline{i}$

$\quad + (R\ddot{\psi}\sin\psi + R\dot{\psi}^2\cos\psi)\,\underline{j}$

(c) if $\dot{\psi} = \text{constant} = \Omega$: $u = \sqrt{\dot{x}^2 + \dot{y}^2}$

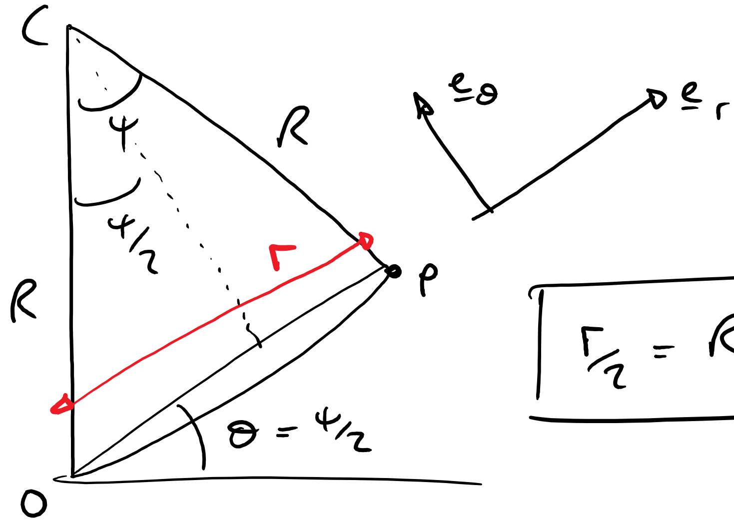
$= \dot{\psi}R$

$= \Omega R$.

Constant speed, circular motion ANTICLOCKWISE for $\dot{\psi} > 0$.

Question 6

6 For the particle moving around the circle shown in Figure 2 find the vector expressions for the position $\mathbf{r}$, velocity $\dot{\mathbf{r}}$ and acceleration $\ddot{\mathbf{r}}$ in polar coordinates (r, θ) with unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$. Take the origin for polar coordinates at O so that r is the distance from O to P and θ is the anticlockwise angle between the x-axis and OP.



$$\boxed{\frac{r}{2} = R \sin\frac{\psi}{2}}$$

$$\underline{r}_p = 2x\,\underline{e}_r = 2R\sin\frac{\psi}{2}\,\underline{e}_r .$$

$$\dot{\underline{r}}_p = \frac{d\underline{r}_p}{dt} = 2R\,\frac{\dot\psi}{2}\cos\frac{\psi}{2}\,\underline{e}_r + 2R\sin\frac{\psi}{2}\,\dot{\underline{e}}_r$$

now $\dot{\underline{e}}_r = \dot\theta\,\underline{e}_\theta = \frac{\dot\psi}{2}\,\underline{e}_\theta$

so: $\dot{\underline{r}}_p = R\dot\psi\cos\frac{\psi}{2}\,\underline{e}_r + R\dot\psi\sin\frac{\psi}{2}\,\underline{e}_\theta$

$$\ddot{\underline{r}}_p = \frac{d\dot{\underline{r}}_p}{dt} = R\ddot\psi\cos\frac{\psi}{2}\,\underline{e}_r - R\frac{\dot\psi^2}{2}\sin\frac{\psi}{2}\,\underline{e}_r + R\frac{\dot\psi^2}{2}\cos\frac{\psi}{2}\,\underline{e}_\theta$$
$$+ R\ddot\psi\sin\frac{\psi}{2}\,\underline{e}_\theta + R\frac{\dot\psi^2}{2}\cos\frac{\psi}{2}\,\underline{e}_\theta - R\frac{\dot\psi^2}{2}\sin\frac{\psi}{2}\,\underline{e}_r$$

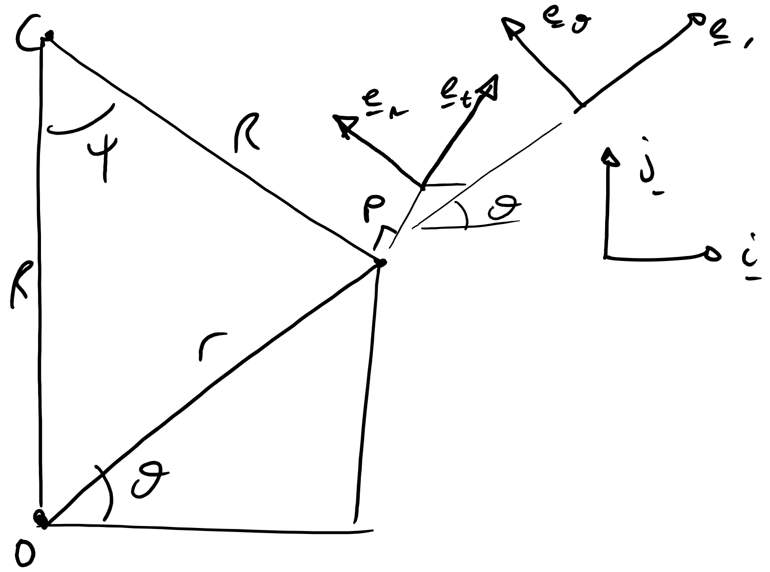
$$\left[\dot{\underline{e}}_\theta = -\dot\theta\,\underline{e}_r = -\frac{\dot\psi}{2}\,\underline{e}_r\right]$$

$$\ddot{\underline{r}}_p = \left[R\ddot\psi\cos\frac{\psi}{2} - R\dot\psi^2\sin\frac{\psi}{2}\right]\underline{e}_r$$
$$+ \left[R\ddot\psi\sin\frac{\psi}{2} + R\dot\psi^2\cos\frac{\psi}{2}\right]\underline{e}_\theta$$

Question 7

7 (a) For the particle moving around the circle shown in Figure 2 find the vector expressions for the velocity $\dot{\mathbf{r}}$ and acceleration $\ddot{\mathbf{r}}$ in intrinsic coordinates (s, ψ) with unit vectors $\mathbf{e}_t$ and $\mathbf{e}_n$.

(a)



Position not obvious in intrinsic coordinates: but not asked for.

Velocity is by definition in $\underline{e}_t$ direction, so:

$$\dot{\underline{r}}_P = R\dot{\psi}\,\underline{e}_t \quad // \qquad \text{as speed of } P \;\; v_P = R\dot{\psi}$$

Now differentiate: $\ddot{\underline{r}}_P = \dfrac{d\dot{\underline{r}}_P}{dt} = \dfrac{d\dot{\underline{r}}_P}{d\psi}\dfrac{d\psi}{dt}$

$$= R\ddot{\psi}\,\underline{e}_t + R\dot{\psi}\,\dot{\underline{e}}_t$$

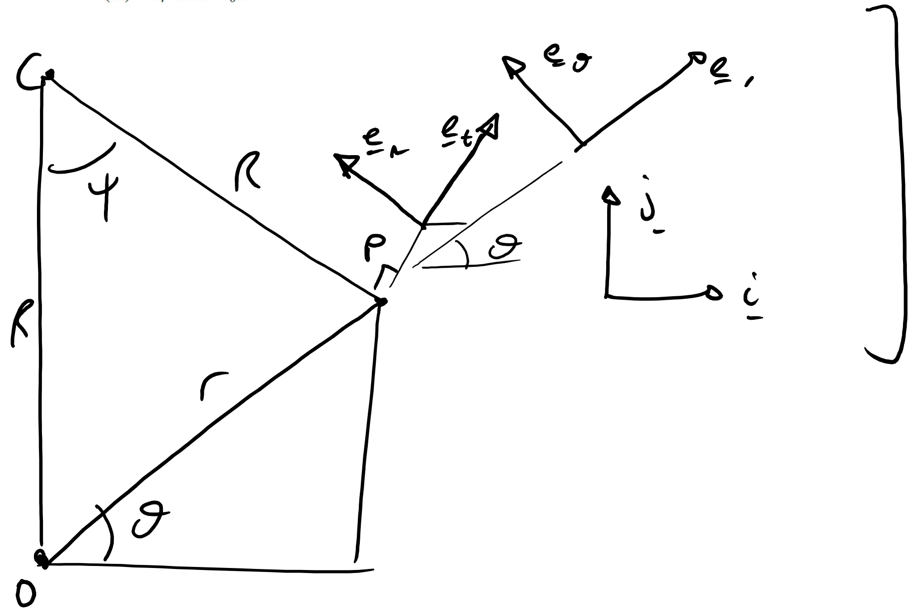
$\underline{e}_t$ rotates with CP, ie at rate $\dot{\psi}$, so $\dot{\underline{e}}_t = \dot{\psi}\,\underline{e}_n$

$$\text{so } \ddot{\underline{r}}_P = R\ddot{\psi}\,\underline{e}_t + R\dot{\psi}^2\,\underline{e}_n \quad //$$

Question 7 (continued)

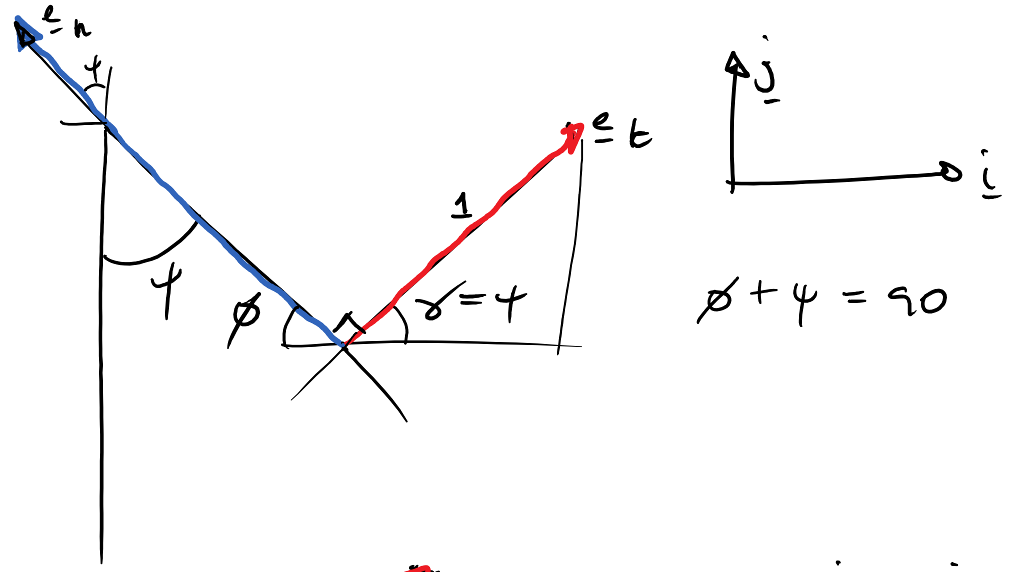
(b) By considering the relationship between the three pairs of unit vectors ($\mathbf{e}_t$, $\mathbf{e}_n$), ($\mathbf{e}_r$, $\mathbf{e}_\theta$), and ($\mathbf{i}, \mathbf{j}$), express $\mathbf{e}_t$ and $\mathbf{e}_n$ in terms of:

(i) i and j;
(ii) $\mathbf{e}_r$ and $\mathbf{e}_\theta$.



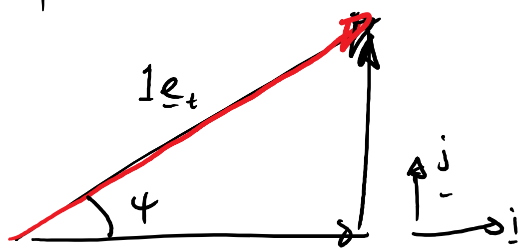
shows all unit vectors together

(i) Now consider $\underline{e}_t$ & $\underline{e}_n$ more carefully:

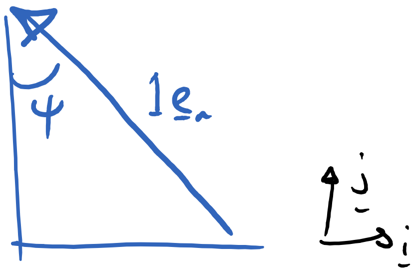


$\phi + \psi = 90 \Rightarrow \gamma = \psi.$

so



$\underline{e}_t = \cos\psi\,\underline{i} + \sin\psi\,\underline{j}$ //



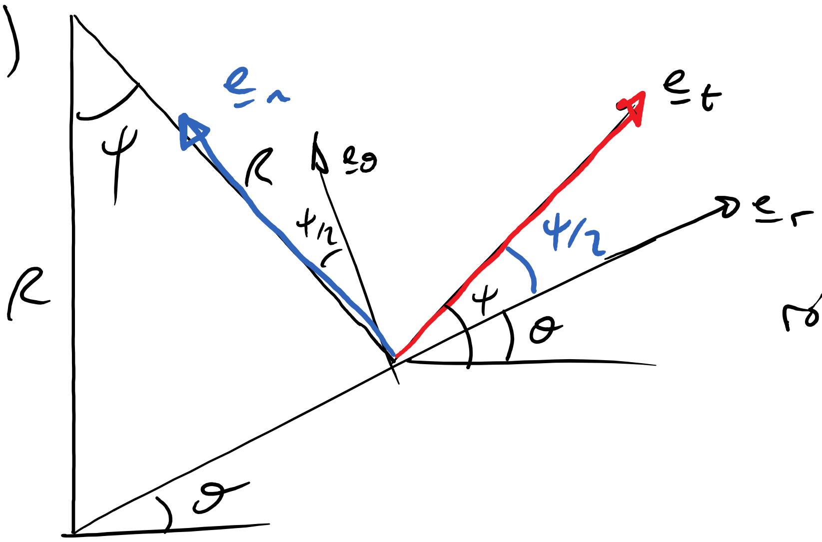
$\underline{e}_n = -\sin\psi\,\underline{i} + \cos\psi\,\underline{j}$ //

(b) By considering the relationship between the three pairs of unit vectors $(\mathbf{e}_t, \mathbf{e}_n)$, $(\mathbf{e}_r, \mathbf{e}_\theta)$, and $(\mathbf{i}, \mathbf{j})$, express $\mathbf{e}_t$ and $\mathbf{e}_n$ in terms of:

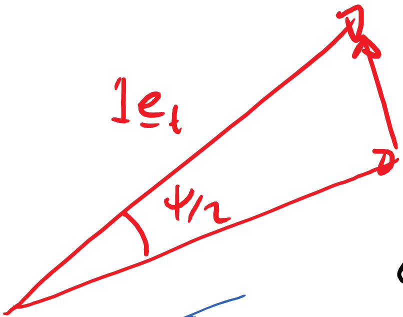
(i) $\mathbf{i}$ and $\mathbf{j}$;

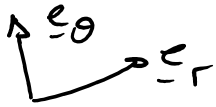
(ii) $\mathbf{e}_r$ and $\mathbf{e}_\theta$.

(ii)

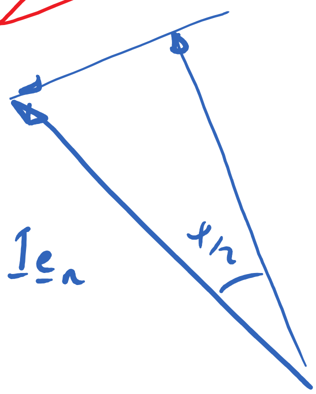


rotation is $\psi - \theta = \psi/2$

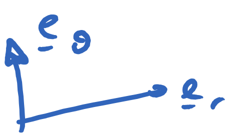




$$\underline{e}_t = \cos\psi/2 \, \underline{e}_r + \sin\psi/2 \, \underline{e}_\theta$$



$$\underline{e}_n = -\sin\psi/2 \, \underline{e}_r + \cos\psi/2 \, \underline{e}_\theta$$



(c) Three coordinate systems have been used to describe the same particle's velocity and acceleration (Cartesian in Question 5, polar in Question 6 and intrinsic in this question). To demonstrate the equivalence, substitute the expressions for $\mathbf{e}_t$ obtained above into the expression for the velocity of the particle obtained in intrinsic coordinates above to confirm that you obtain the velocity expressions found in Cartesian coordinates in Question 5 and in polar coordinates in Question 6.

Note: Similar substitutions can be made into the expression for the acceleration above to demonstrate equivalence.

$$\dot{\underline{r}}_p = R\dot{\psi}\,\underline{e}_t$$

cartesian:

$$\underline{e}_t = \cos\psi\,\underline{i} + \sin\psi\,\underline{j}$$

$$\dot{\underline{r}}_p = R\dot{\psi}\cos\psi\,\underline{i} + R\dot{\psi}\sin\psi\,\underline{j}$$ // as before in Q5

polar:

$$\underline{e}_t = \cos\psi/2\,\underline{e}_r + \sin\psi/2\,\underline{e}_\theta$$

$$\dot{\underline{r}}_p = R\dot{\psi}\cos\psi/2\,\underline{e}_r + R\dot{\psi}\sin\psi/2\,\underline{e}_\theta$$ // as before in Q6.

8 A taut string CP is unwrapped from a fixed drum, centre O and radius R, with a uniform angular velocity $\dot{\psi}$. The end of the string P is initially in contact with the drum at A, then traces out a planar curved path AB as shown in Figure 3.

(a) Find an expression for the position vector of P relative to O in terms of the intrinsic coordinate unit vectors $\mathbf{e}_t$ and $\mathbf{e}_n$ as shown (note that this is a rare case for which the position vector is readily found in intrinsic coordinates).

(b) By differentiation find the velocity $\dot{\mathbf{r}}$ and acceleration $\ddot{\mathbf{r}}$ of P in this coordinate system.

(c) From the acceleration verify that the radius of curvature of the path of P is equal to CP. Would you expect this?

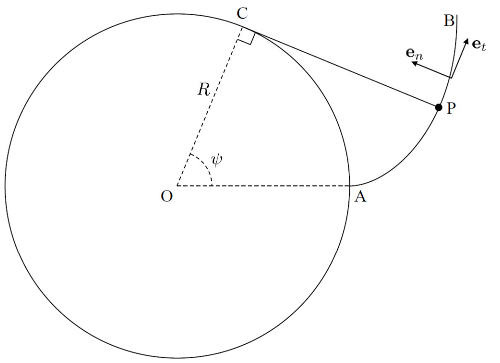


Figure 3

(a) $\underline{r} = R\underline{e}_t - |CP|\underline{e}_n$

$|CP| = |CA| = R\psi$

$\underline{r} = -R\psi\,\underline{e}_n + R\underline{e}_t$

(b) $\dot{\underline{r}} = -(R\dot{\psi}\,\underline{e}_n + R\psi\,\dot{\underline{e}}_n) + R\dot{\underline{e}}_t$

$= -(R\dot{\psi}\,\underline{e}_n - R\psi\dot{\psi}\,\underline{e}_t) + R\dot{\psi}\,\underline{e}_n$

$= R\psi\dot{\psi}\,\underline{e}_t$

$\ddot{\underline{r}} = R\dot{\psi}^2\underline{e}_t + R\psi\overset{0}{\cancel{\ddot{\psi}}}\,\underline{e}_t + R\psi\dot{\psi}\,\dot{\underline{e}}_t$

$= R\dot{\psi}^2\underline{e}_t + R\psi\dot{\psi}^2\underline{e}_n$

(c) $\dfrac{\dot{s}^2}{\rho} = R\psi\dot{\psi}^2$, i.e. $\rho = \dfrac{\dot{s}^2}{R\psi\dot{\psi}^2} = \dfrac{(R\psi\dot{\psi})^2}{R\psi\dot{\psi}^2}$

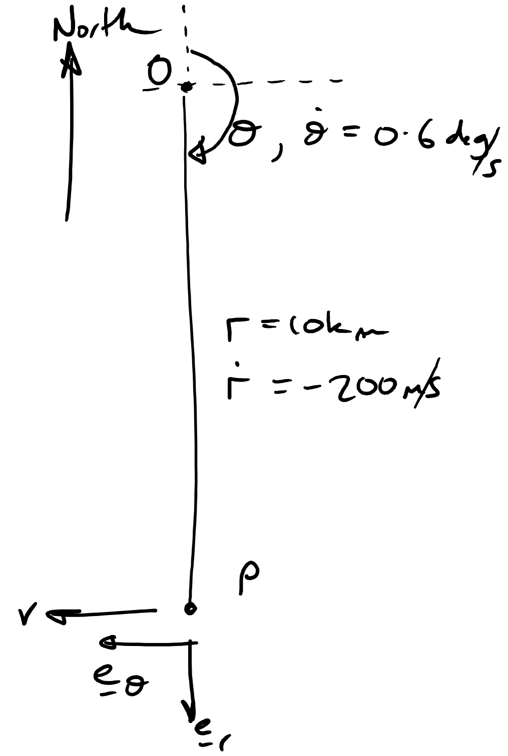
$= R\psi = |CP|$

Expected, as C is instantaneous centre of motion

9 A tracking radar detects an aircraft due south at a range of 10 km. The radar points continually at the aircraft. It measures a range that is decreasing at a constant rate of 200 ms^{-1} while the radar is rotating clockwise (viewed from above) at a constant rate of 0.6 deg s^{-1}. Take θ to measure the angle clockwise from North.

(a) What is the course (direction) and speed of the aircraft?

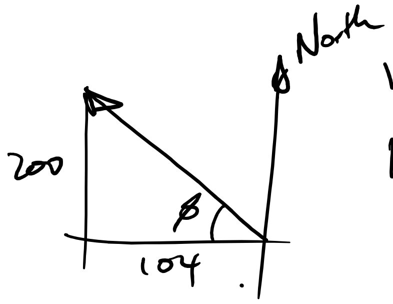
SETUP PROBLEM WITH A DIAGRAM:



(a) $\underline{r} = r\underline{e}_r$

$\dot{\underline{r}} = \dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta = -200\underline{e}_r + 10k \cdot 0.6\pi/180\ \underline{e}_\theta$

$= -200\underline{e}_r + 104.72\,\underline{e}_\theta \quad \mathrm{ms}^{-1}$



$v = 225.76\ \mathrm{ms}^{-1}$

heading $\phi = \tan^{-1}\frac{200}{104} = 62.53°$

$\theta = 270 + 62.53 = 332.5°$

(North West)

(b) What is the component of the aircraft's acceleration along its path?

Find acceleration: this time using databook expression in polar coordinates:

$$\ddot{\underline{r}} = \left(\cancelto{0}{\ddot{r}} - r\dot{\theta}^2\right)\underline{e}_r + \left(\cancelto{0}{r\ddot{\theta}} + 2\dot{r}\dot{\theta}\right)\underline{e}_\theta$$

$$= -10k\left(\frac{0.6\pi}{180}\right)^2 \underline{e}_r + 2.(-200)\left(\frac{0.6\pi}{180}\right)\underline{e}_\theta$$

$$= -1.1\,\underline{e}_r - 4.2\,\underline{e}_\theta$$

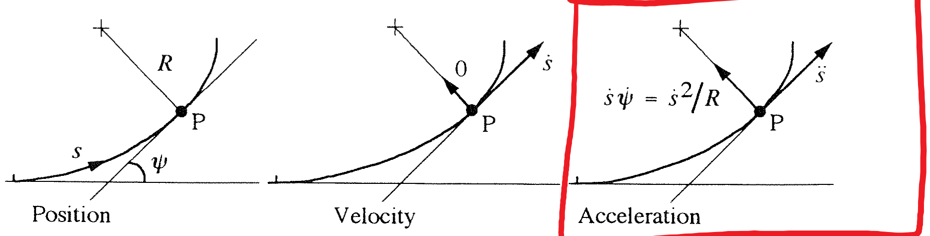
Need component in direction of path, so dot product with unit vector in direction of velocity:

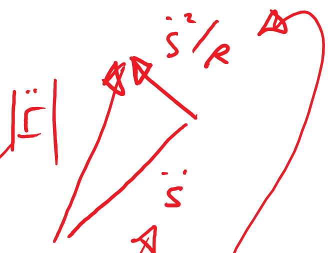
$$\ddot{\underline{r}} \cdot \hat{\dot{\underline{r}}} = \left(-1.1\,\underline{e}_r - 4.2\,\underline{e}_\theta\right) \cdot \frac{\left(-200\,\underline{e}_r + 104.72\,\underline{e}_\theta\right)}{225.76}$$

$$= -0.97\ ms^{-2}$$

(c) What is the value of the instantaneous radius of curvature of the path of the aircraft?

1.2: Velocity and acceleration in intrinsic coordinates





d/b: $\frac{\dot{s}^2}{\rho}$ = acceleration towards instantaneous centre.

$|\ddot{\underline{r}}| = 4.33\ ms^{-2}$

$|\ddot{\underline{r}}|$ along path $= 0.97\ ms^{-2}$

$|\ddot{\underline{r}}|$ to centre $= \sqrt{4.33^2 - 0.97^2} = 4.22\ ms^{-2} = \dot{s}^2/\rho$

$$\rho = \frac{225.76^2}{4.22} = 12.078\,km$$

(d) Use the Python template `p1q9_template.ipynb` to produce a plot of the path of the aircraft. It is mostly complete and only requires a few changes to make it work, and it can be run online without installing Python. See computing help at the end of this examples paper for more information

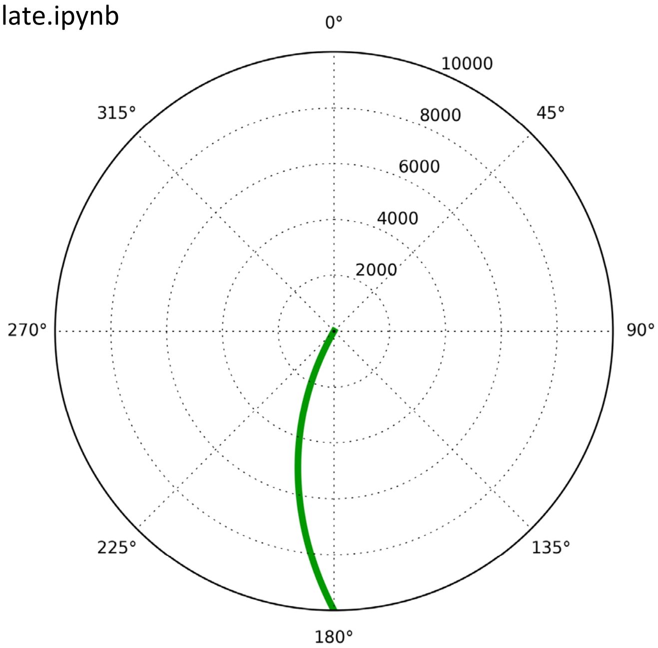
$$\underline{r} = r\underline{e}_r$$

$$\dot{\underline{r}} = \dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta$$

numerically :
$$r(t) = r(0) + \int_0^t \dot{r}\,dt = \overset{10k}{r(0)} + \underset{\text{const.}}{\overset{-260}{\dot{r}}}\,t$$

$$\theta(t) = \theta(0) + \int_0^t \dot{\theta}\,dt = \underset{\pi}{\theta(0)} + \underset{\frac{0.6\pi}{180}}{\overset{}{\dot{\theta}}}\,t \quad \text{const.}$$

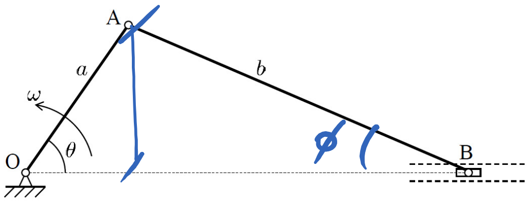
p1q9_template.ipynb

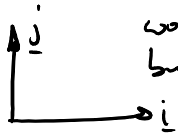


Question 10

10 $*$ A crank OA is driven by a piston AB such that it rotates at a constant angular speed ω as shown in Figure 4. Point B is constrained to move horizontally.

(a) Find an expression for the position vector $\mathbf{r}_A$, $\mathbf{r}_{B/A}$, and $\mathbf{r}_B$ as a function of ω;





cartesian most suitable here (polar would be fine for A but hard for B).

$$\underline{r}_A = a\cos\theta\,\underline{i} + a\sin\theta\,\underline{j} \longrightarrow \theta = \omega t \text{ if } \theta = 0 \text{ when } t=0.$$

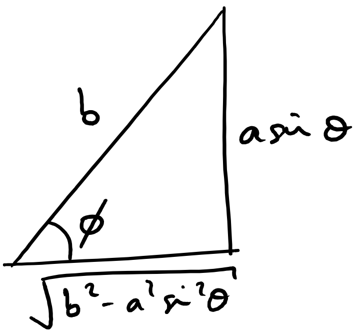
so $$\underline{r}_A = a\cos\omega t\,\underline{i} + a\sin\omega t\,\underline{j}$$

$$\underline{r}_{B/A} = b\cos\phi\,\underline{i} - b\sin\phi\,\underline{j}$$

How to determine ϕ? It is due to constraint that B is on horizontal line. So:

$$a\sin\theta = b\sin\phi, \text{ giving } \sin\phi = \frac{a\sin\theta}{b}$$

also need $\cos\phi$: construct triangle from $\sin\phi$



$$\cos\phi = \frac{\sqrt{b^2 - a^2\sin^2\theta}}{b}$$

hence $$\underline{r}_{B/A} = \sqrt{b^2 - a^2\sin^2\theta}\,\underline{i} - a\sin\theta\,\underline{j}$$

& $$\underline{r}_B = \left(a\cos\theta + \sqrt{b^2 - a^2\sin^2\theta}\right)\underline{i}$$

$$\underline{r}_B = \left(a\cos\omega t + \sqrt{b^2 - a^2\sin^2\omega t}\right)\underline{i}$$

Question 10 (continued)

(b) Find the velocity of A and B;

$$\underline{r}_B = \left(a\cos\omega t + \sqrt{b^2 - a^2\sin^2\omega t}\right)\underline{i}$$

$$\dot{\underline{r}}_B = -a\omega\sin\omega t - \frac{\frac{1}{\cancel{2}}a^2 \cdot \cancel{2}\omega\sin\omega t\cos\omega t}{\sqrt{b^2 - a^2\sin^2\omega t}}$$

$$= \left(-a\omega\sin\omega t - \frac{a^2\omega\sin\omega t\cos\omega t}{\sqrt{b^2 - a^2\sin^2\omega t}}\right)\underline{i}$$

$$\underline{r}_A = a\cos\omega t\,\underline{i} + a\sin\omega t\,\underline{j}$$

$$\dot{\underline{r}}_A = -a\omega\sin\omega t\,\underline{i} + a\omega\cos\omega t\,\underline{j}$$

Question 10 (continued)

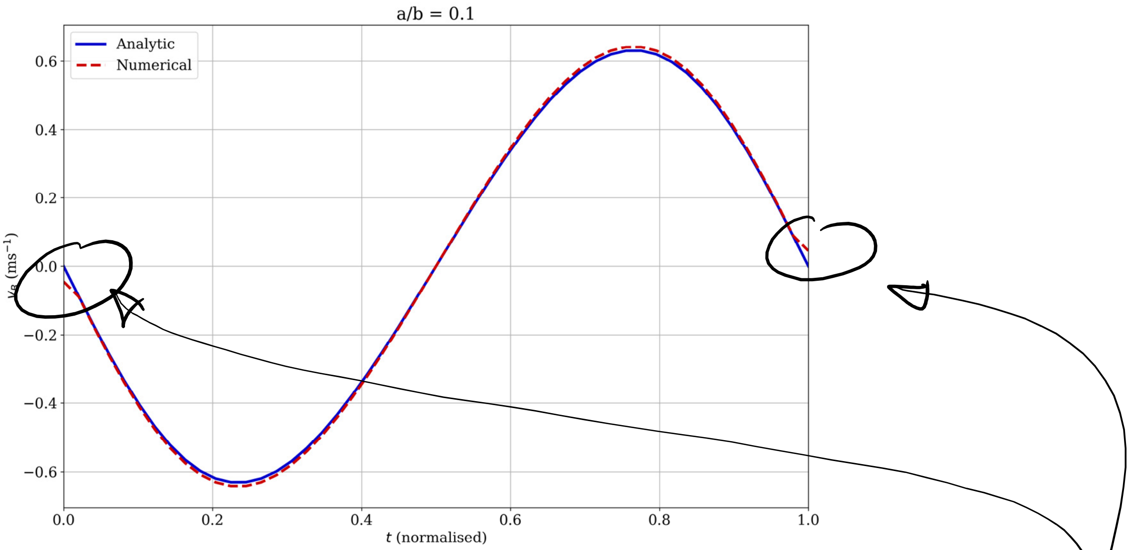
(c) Write down an expression that would give a numerical approximation of the velocity of A and B using your analytical expression for the position vectors;

- Subdivide time into intervals of time T.
- Consider estimate of velocity $\hat{v}$ at time $t = kT$
- Calculate average velocity over time T:

$$\hat{v}_A(kT) = \frac{r_A(kT) - r_A([k-1]T)}{T}$$

$$\hat{v}_B(kT) = \frac{r_B(kT) - r_B([k-1]T)}{T}$$

(d) Using the template Python file `p1q10_template.ipynb` numerically differentiate the position vector and plot the position and velocity of B over one complete revolution of the crank OA, and compare this with a plot of your analytic solution from (b);



- Reasonably good agreement overall.
- Some edge effects: artefact of 'gradient' function in numpy
- Reducing step size would improve approximation

Question 10 (continued)

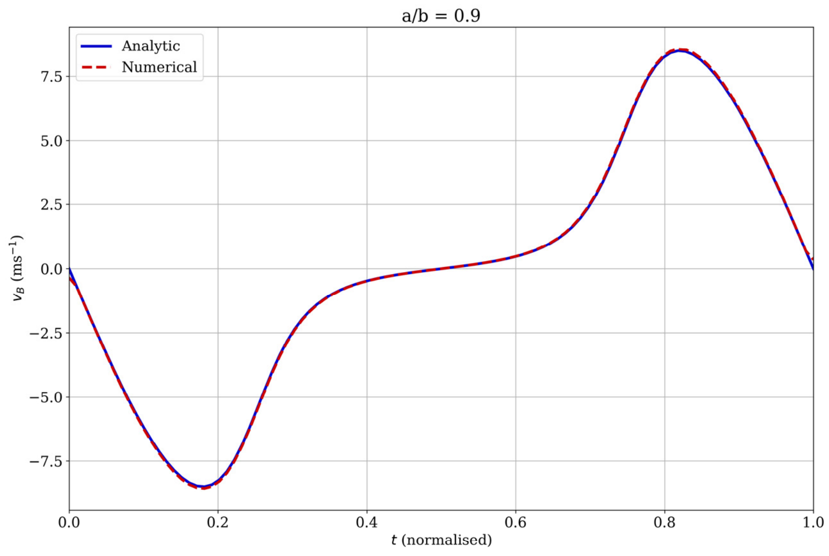
(e) What factors affect the accuracy of your numerical approximation?

- time step (smaller => more accurate)
- differentiation method (more accurate as possible, eg central difference, see maths databook).

(f) For the ratios $a/b = 0.1, 0.5, 0.9$ identify the time in the cycle at which the maximum velocity occurs. What happens as $a \to b$?

a/b	t_min (s)	t_max (s)
0.1	0.23	0.77
0.5	0.19	0.81
0.9	0.18	0.82

Cycle becomes increasingly distorted as compared to a sinusoid:



& velocity starts to have sudden jumps, which would give spikes in acceleration.