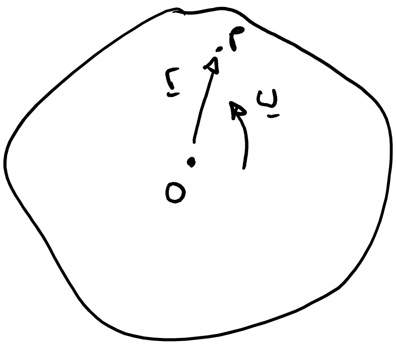
Paper 1: Mechanical Engineering

Examples Paper 3

Question 1

1 † A 3D rigid body is rotating about an axis at an angular rate of $10\,\text{rads}^{-1}$. The positive direction of the axis is defined by $\mathbf{n} = 3\mathbf{i} + 2\mathbf{j}$. If a fixed point O in the rigid body lies on the axis, determine the velocity of a point P in the body when the instantaneous vector from O to P is $\mathbf{r}_{P/O} = (20\mathbf{i} + 10\mathbf{j})$ mm.



$$\underline{\omega} = 10.\frac{(3\underline{i} + 2\underline{j})}{\sqrt{13}}$$

$$\underline{V}_P = \underline{V}_O + \underline{V}_{P/O}$$

$$= 0 + \underline{\omega} \times \underline{r}$$

$$= \frac{10}{\sqrt{13}} \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 2 & 0 \\ 20 & 10 & 0 \end{vmatrix}$$

$$= \frac{10}{\sqrt{13}} \begin{pmatrix} 0 \\ 0 \\ -10 \end{pmatrix} = \frac{-100}{\sqrt{13}} \underline{k} \text{ mms}^{-1}$$

2 † A playground roundabout of radius 2 m shown in Figure 1 rotates in an anticlockwise direction about O at 1.5 rads^{-1}. A child holds on to the roundabout at A, a distance of 2 m from O. A second child holds on to the roundabout at B, a distance of 1.5 m from O.

(a) Express the angular velocity of the roundabout as a vector;

(b) What is the angular velocity of each child?

(c) What is the angular velocity of the vector $\mathbf{r}_{B/A}$?

(d) Determine the velocities of A and B;

(e) Determine the velocity of A relative to B ($\dot{\mathbf{r}}_{A/B}$) and the velocity of B relative to A ($\dot{\mathbf{r}}_{B/A}$);

(f) Show that the velocity of B is equal to the velocity of A plus the velocity of B relative to A.

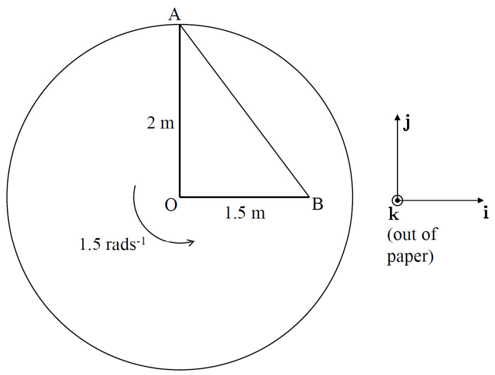


Figure 1

a) $\underline{\omega} = 1.5\underline{k}$

b) $\underline{\omega}_A = \underline{\omega}_B = 1.5\underline{k}$

c) $\underline{\omega}_{AB} = 1.5\underline{k}$

d) $\underline{V}_A = \underline{\omega} \times \underline{r}_A$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 0 & 1.5 \\ 0 & 2 & 0 \end{vmatrix}$$

$$= \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix} = -3\underline{i} \text{ ms}^{-1}$$

$\underline{V}_B = \underline{\omega} \times \underline{r}_B$

$$= 2.25\underline{j} \text{ ms}^{-1}$$

e) $\underline{V}_{A/B} = \underline{\omega} \times \underline{r}_{A/B} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 0 & 1.5 \\ -1.5 & 2 & 0 \end{vmatrix}$

$$\begin{pmatrix} -3 \\ -2.25 \\ 0 \end{pmatrix} = -3\underline{i} - 2.25\underline{j} \text{ ms}^{-1}$$

$\underline{V}_{B/A} = \underline{\omega} \times \underline{r}_{B/A} = -\underline{V}_{A/B} = 3\underline{i} + 2.25\underline{j}$

f) $\underline{V}_B = \underline{V}_A + \underline{V}_{B/A} = -3\underline{i} + 3\underline{i} + 2.25\underline{j} = 2.25\underline{j}$

as above.

Question 3

3 A person throws a uniform rigid stick AB, which is 0.8 m long. It goes spinning across the horizontal surface of a frozen lake. At a particular instant the velocity of end A is as shown in Figure 2. The direction of the velocity of end B is as shown in Figure 2 and is known to be correct. The magnitude of the velocity at B is *thought* to be 8 ms^{-1}.

(a) Is the magnitude of the velocity of the end B correct? If not, what should it be?

(b) Determine the angular velocity of the stick as a vector.

(c) Find the velocity of the centre of gravity of the stick:

(i) using $\mathbf{v}_G = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{G/A}$;

(ii) by taking the average of $\mathbf{v}_A$ and $\mathbf{v}_B$.

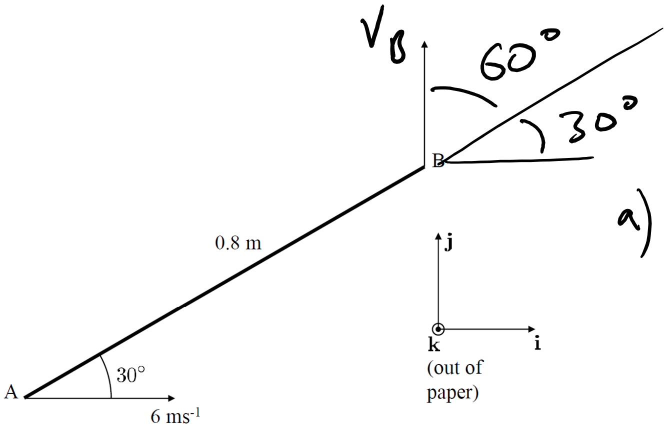


Figure 2

a) Components in direction along stick are equal:

$$6 \cos 30 = V_B \cos 60.$$

$$V_B = 6 \frac{\sqrt{3}}{2} \cdot 2$$

$$= 10.4 \text{ ms}^{-1}$$

$$\neq 8.$$

b) $$\underline{v}_B = \underline{v}_A + \underline{\omega} \times \underline{r}_{B/A}$$

unknown, except direction is $\underline{k}$

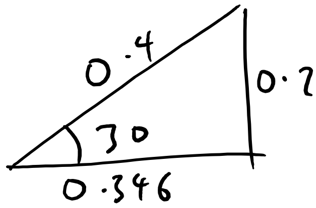
$$10.4\underline{j} = 6\underline{i} + \underbrace{(\omega \underline{k}) \times (0.8 \cos 30\, \underline{i} + 0.8 \sin 30\, \underline{j})}_{\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 0 & \omega \\ 0.693 & 0.4 & 0 \end{vmatrix} = \begin{bmatrix} -0.4\omega \\ 0.693\omega \\ 0 \end{bmatrix}}$$

$\underline{i}$ component: $0 = 6 - 0.4\omega$, $\omega = 6/0.4 = 15 \text{ rads}^{-1}$

so $\underline{\omega} = 15\underline{k} \text{ rads}^{-1}$

check $\underline{j}$-component: $10.4 = 0.693\omega \Rightarrow \omega = 15 \text{ rads}^{-1}$ ✓

c) i) $\underline{V}_G = \underline{V}_A + \underline{\omega} \times \underline{r}_{G/A}$



$$\underline{r}_{G/A} = 0.346\,\underline{i} + 0.2\,\underline{j}$$

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 0 & 15 \\ 0.346 & 0.2 & 0 \end{vmatrix} = \begin{pmatrix} -3 \\ 5.2 \\ 0 \end{pmatrix}$$

$$\underline{V}_G = 6\underline{i} - 3\underline{i} + 5.2\underline{j} = 3\,\underline{i} + 5.2\,\underline{j} \text{ ms}^{-1}$$

ii) $\underline{V}_G = (\underline{V}_A + \underline{V}_B)/2 = (6\underline{i} + 10.4\underline{j})/2 = 3\,\underline{i} + 5.2\,\underline{j}$ ms^{-1}

as before

Question 4

4 (a) Locate the instantaneous centre for the stick in Figure 2a and use it to confirm that the stick's angular velocity is $\omega = 15\mathbf{k}\,\mathrm{rad\,s^{-1}}$.

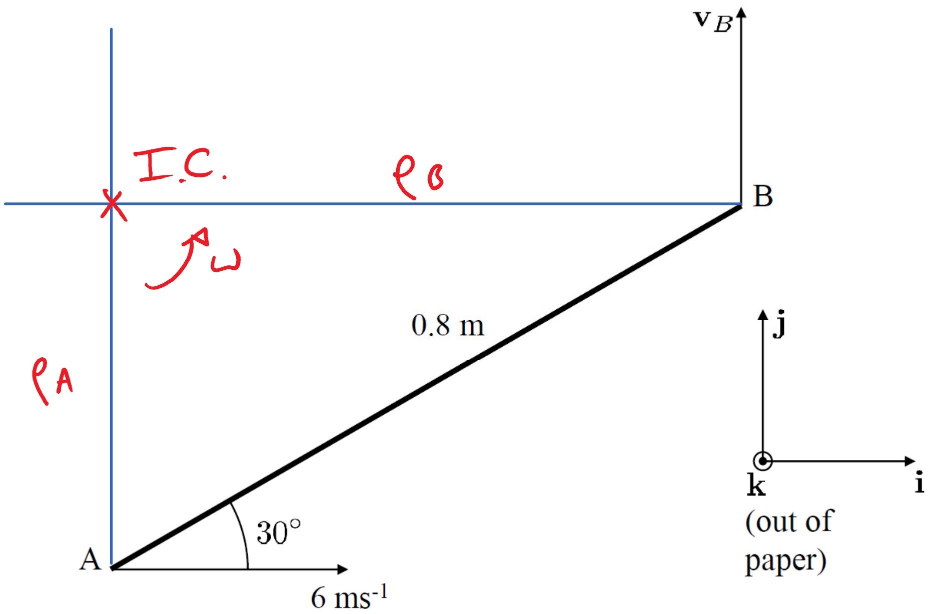


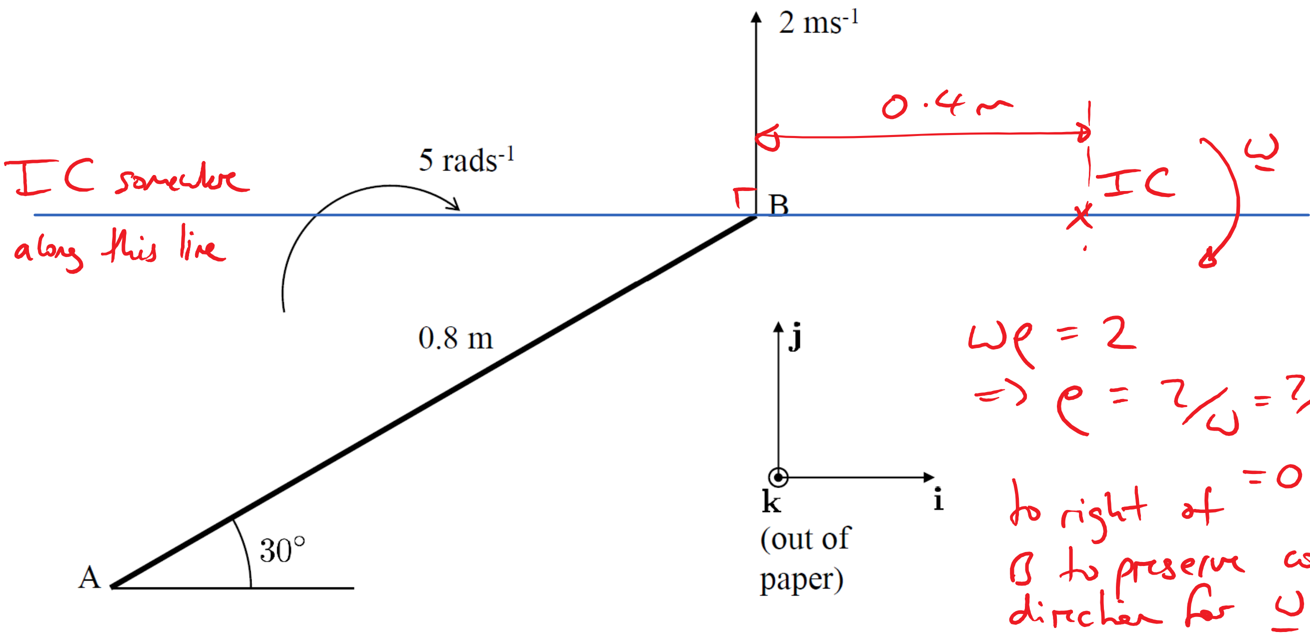
Figure 2

$\omega \rho_A = 6 \Rightarrow \omega = 6/0.8\sin 30 = 15\,\mathrm{rad\,s^{-1}}$

$\omega \rho_B = 10.4 \rightarrow \omega = 10.4/0.8\cos 30 = 15\,\mathrm{rad\,s^{-1}}$ ✓

as before.

(b) The same stick is shown in Figure 2b but the motion is different. Locate the instantaneous centre at the instant shown.



$\omega \rho = 2$

$\Rightarrow \rho = 2/\omega = 2/5 = 0.4\,m$

to right of B to preserve correct direction for $\underline{\omega}$

Question 5

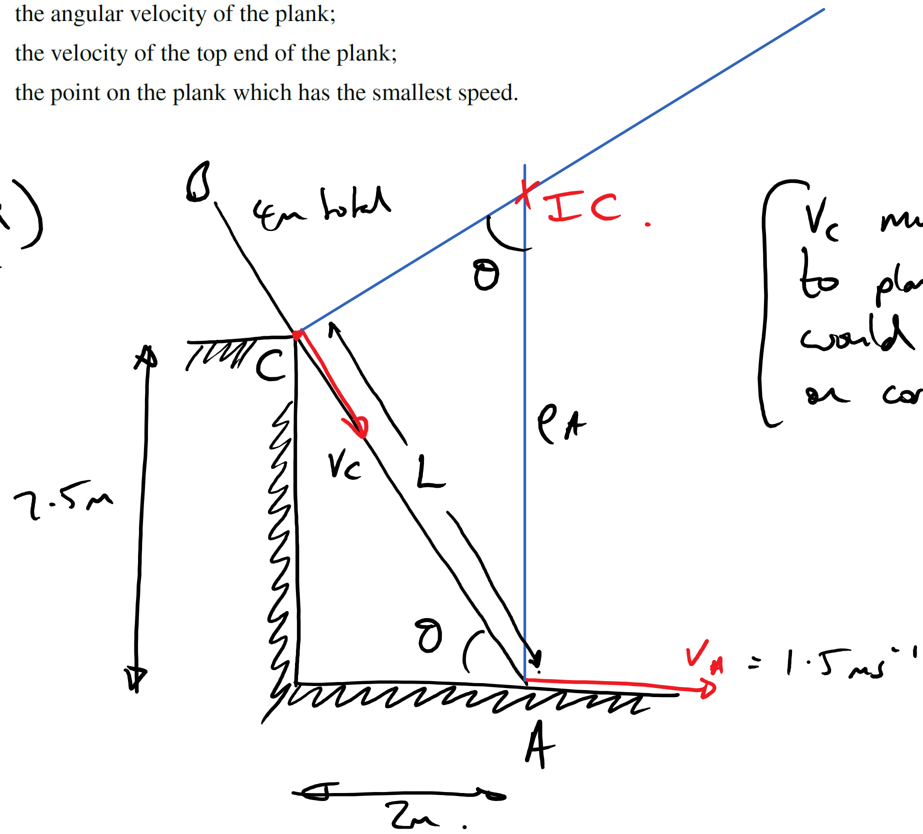
5 A plank 4 m long has one end on horizontal ground and rests against the top corner of a vertical wall 2.5 m high. The bottom end is sliding away from the wall towards the right at a rate of 1.5 ms^{-1}. Locate the instantaneous centre for the plank at the instant when the bottom end is 2 m from the wall and determine:

(a) the angular velocity of the plank;

(b) the velocity of the top end of the plank;

(c) the point on the plank which has the smallest speed.

(a)



[V_c must be parallel to plank, otherwise it would not be resting on corner C.]

$$L = \sqrt{2.5^2 + 2^2} = 3.2m.$$

$$\sin\theta = L/\rho_A$$

$$\& \quad \sin\theta = 2.5/3.2 = 0.78$$

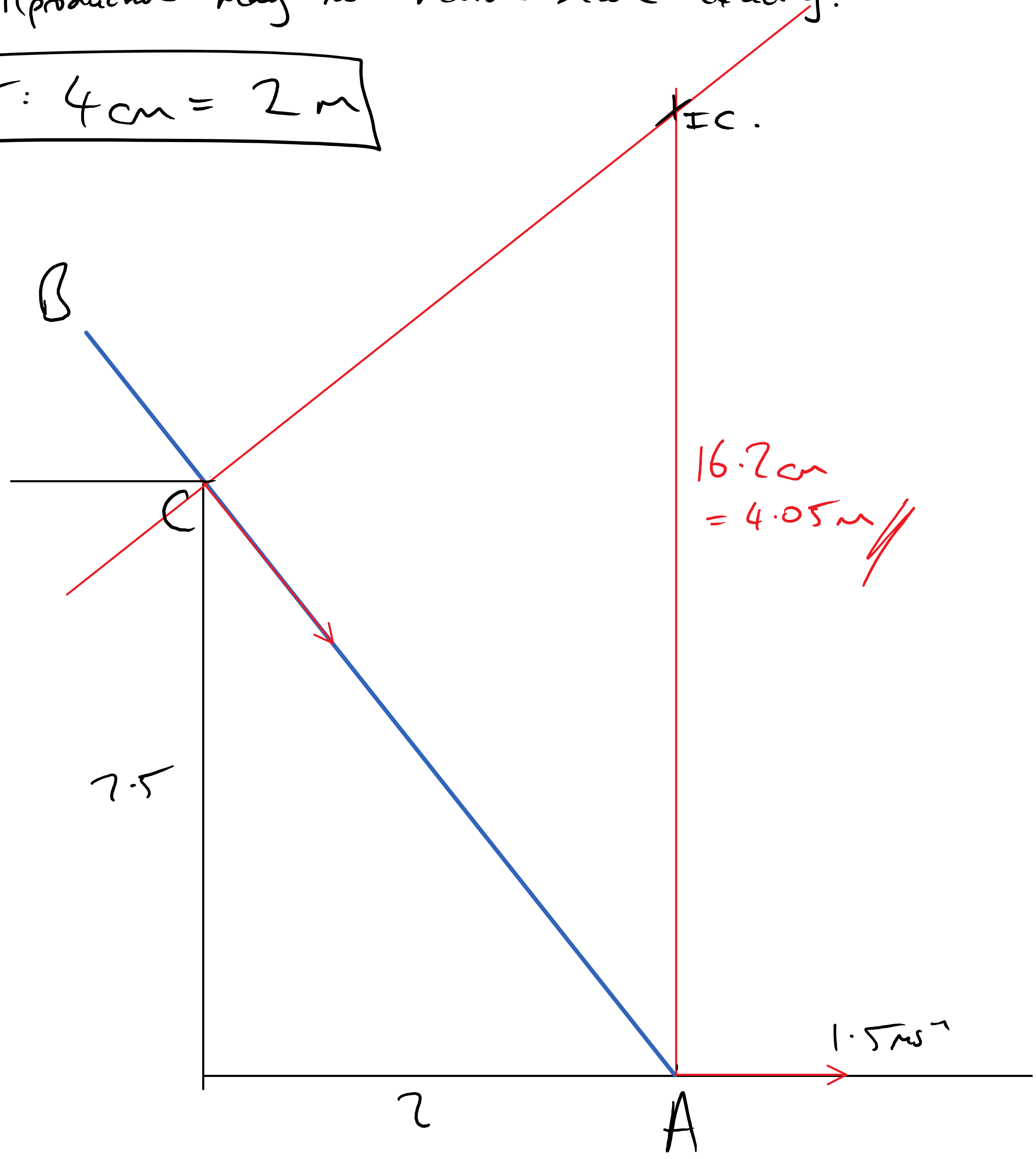
$$\text{so } \rho_A = 3.2/0.78 = 4.1m$$

So IC is 4.1m above A

$$\& \quad \underline{\omega} = \frac{1.5}{4.1}\underline{k} = 0.37\,\underline{k} \text{ rads}^{-1}$$

Could also do scale diagram & measure:
Note reproduction may not match scale exactly.

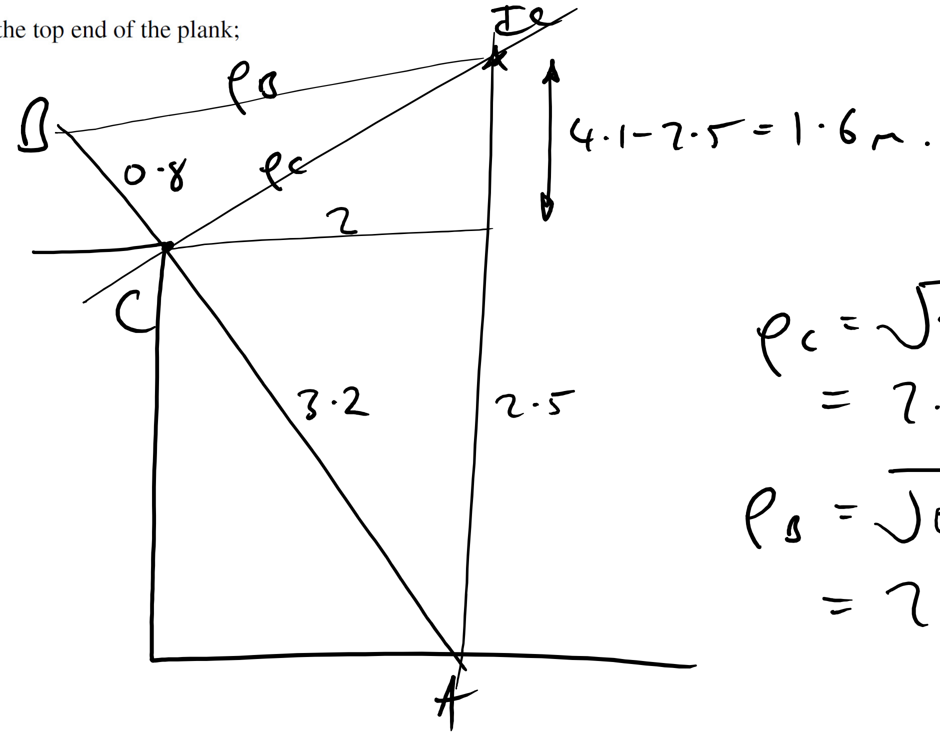
$$\boxed{\text{SCALE: } 4\,\text{cm} = 2\,\text{m}}$$



$$\underline{\omega} = \frac{1.5}{4.1}\underline{k} = 0.37\,\underline{k}\ \text{rad s}^{-1}$$

Question 5 (continued)

(b) the velocity of the top end of the plank;



$$\rho_C = \sqrt{2^2 + 1.6^2} = 2.56 \text{ m}$$

$$\rho_B = \sqrt{0.8^2 + 2.56^2} = 2.68 \text{ m}.$$

so $|\underline{V}_B| = \rho_B \omega$.
$= 2.68 \times 0.37$
$= 1.0 \text{ ms}^{-1}$

(c) the point on the plank which has the smallest speed.

$v = \omega \rho$ so need ρ to be small.

Look for smallest distance between plank and IC

$\Rightarrow$ point C.

$$\left[V_C = \omega \rho_C = 0.37 \times 2.56 = 0.95 \text{ ms}^{-1} \right]$$

Question 6

6 † A wheel of radius R is rolling without slip at an angular speed Ω along a flat surface such that it moves forward at a constant overall speed V_0. Unit vectors $\mathbf{i}$ and $\mathbf{j}$ are defined with respect to a fixed reference frame, while unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are fixed to the wheel (i.e. they rotate with the wheel).

(a) Identify the instantaneous centre of the wheel;

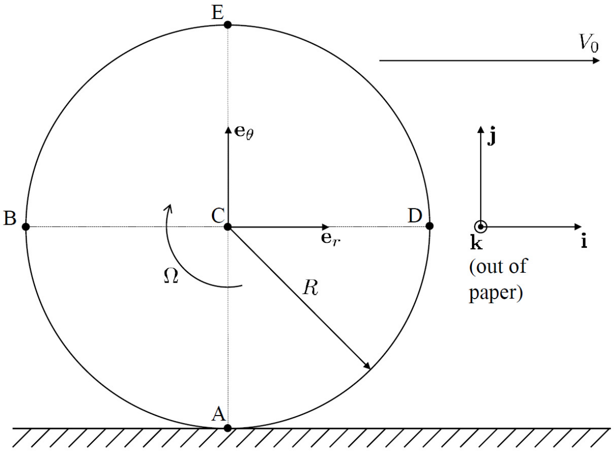


Figure 5

IC is at Point A, as velocity is instantaneously zero if not slipping.

Question 6 (continued)

(b) Express the angular velocity of the wheel as a vector in terms of the forward speed V_0;

No slip $\Rightarrow$ $\Omega . R = V_o$.

$\Omega = V_o / R$.

Vector: $\underline{\omega} = -\Omega \underline{k} = -\dfrac{V_o}{R} \underline{k}$

(c) What are the velocities of A, B, D and E relative to the centre C in terms of the rotating unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$?

$$\underline{v}_{A/C} = \underline{\omega} \times \underline{r}_{A/C} = \begin{vmatrix} \underline{e}_r & \underline{e}_\theta & \underline{k} \\ 0 & 0 & -\frac{V_o}{R} \\ 0 & -R & 0 \end{vmatrix}$$

$$= -V_o \underline{e}_r \,//$$

similarly:

$\underline{v}_{B/C} = V_o \underline{e}_\theta \,//$

$\underline{v}_{D/C} = -V_o \underline{e}_\theta \,//$

$\underline{v}_{E/C} = V_o \underline{e}_r \,//$

[n/b rather than evaluate full cross product, can see from geometry & symmetry what direction each relative velocity should be.]

(d) What are the velocities of points A, B, C, D and E on the wheel with respect to the fixed reference frame: why is the velocity zero for one of these points?

[n/b: at instant shown, $\underline{e}_r = \underline{i}$ & $\underline{e}_\theta = \underline{j}$]

$\underline{v}_A = \underline{v}_C + \underline{v}_{A/C} = V_o \underline{i} - V_o \underline{e}_r = V_o \underline{i} - V_o \underline{i} = 0 \,//$

$\underline{v}_B = \underline{v}_C + \underline{v}_{B/C} = V_o \underline{i} + V_o \underline{e}_\theta = V_o (\underline{i} + \underline{j}) \,//$

$\underline{v}_C = V_o \underline{i} \,//$

$\underline{v}_D = V_o (\underline{i} - \underline{j}) \,//$

$\underline{v}_E = V_o \underline{i} + V_o \underline{e}_r = 2 V_o \underline{i} \,//$

n/b: $\underline{v}_A = 0$ because it is the instantaneous centre of the wheel //

(e) Find vector expressions for the accelerations of these points: why is the acceleration zero for one of these points?

$$\underline{a}_A = \underline{a}_C + \underline{a}_{A/C}$$

$\underline{a}_C = 0$. as constant velocity V_0.

$$\underline{a}_{A/C} = \frac{d}{dt}\left(\underline{v}_{A/C}\right) = \frac{d}{dt}\left(-V_0\,\underline{e}_r\right)$$

$$= -\dot{V}_0\,\underline{e}_r - V_0\,\dot{\underline{e}}_r$$

$\dot{V}_0 = 0$

$$\dot{\underline{e}}_r = \underline{\omega} \times \underline{e}_r$$
$$= -\Omega\underline{k} \times \underline{e}_r$$
$$= -\Omega\underline{e}_\theta$$
$$= \frac{-V_0}{R}\underline{e}_\theta$$

so $\underline{a}_{A/C} = +\frac{V_0^2}{R}\underline{e}_\theta$

$$\underline{a}_A = \frac{V_0^2}{R}\underline{e}_\theta = \frac{V_0^2}{R}\underline{j} \quad //$$

Similarly, by geometry & symmetry:

$$\underline{a}_B = \frac{V_0^2}{R}\underline{e}_r = \frac{V_0^2}{R}\underline{i} \quad //$$

$\underline{a}_C = \underline{0}$. // Because constant velocity.

$$\underline{a}_D = -\frac{V_0^2}{R}\underline{i} \quad //$$

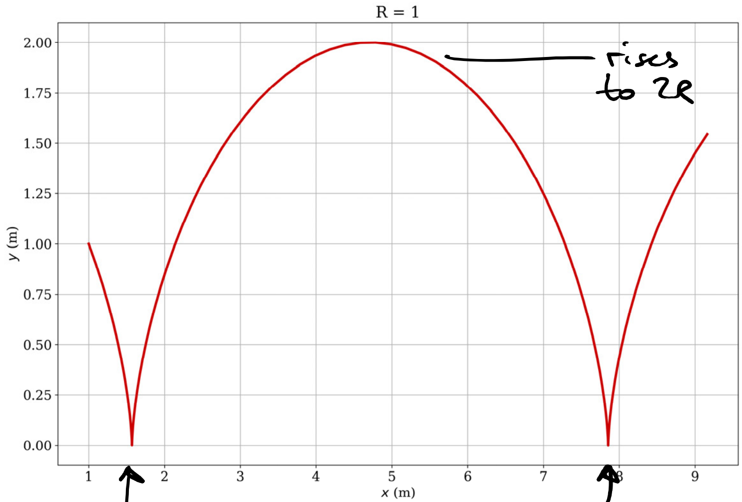
$$\underline{a}_E = -\frac{V_0^2}{R}\underline{j} \quad //$$

(f) Use the Python template `p3q6_template.ipynb` to obtain the position of an arbitrary point on the wheel during one complete revolution, and numerically differentiate the results to approximate the velocity and acceleration. Check to what extent your numerical estimates agree with your results above by plotting:

(i) the path traced out by a chosen point;
(ii) the absolute velocity of that point as a function of time;
(iii) the absolute acceleration of that point as a function of time.

(i)

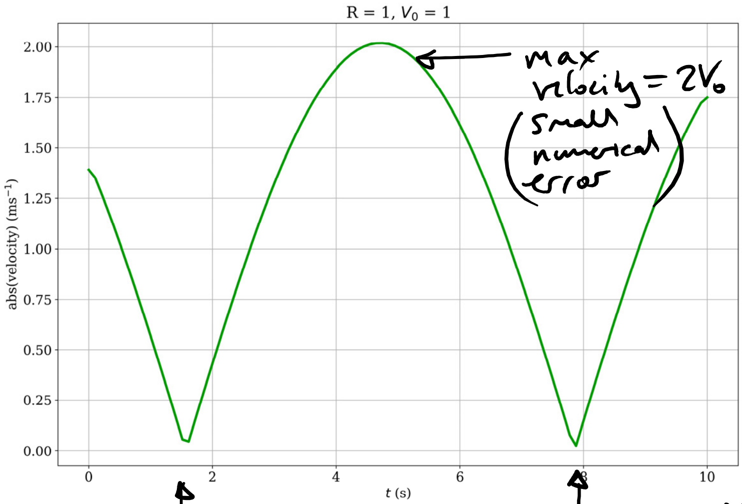
plot (x, y)



wheel touches ground

(ii)

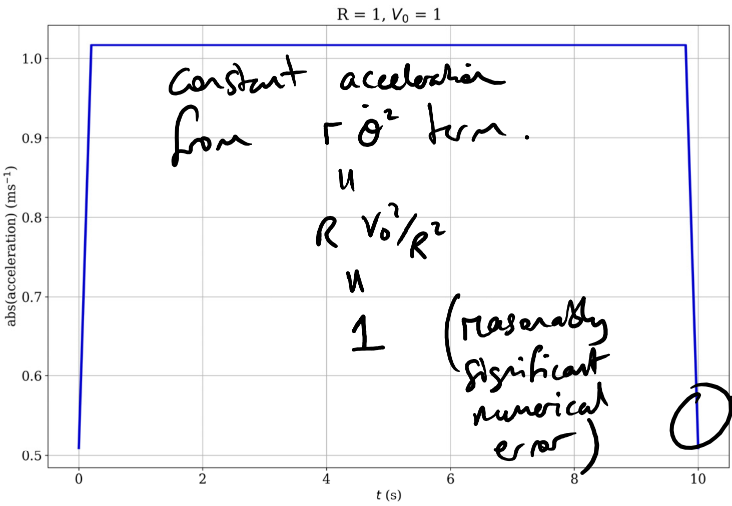
plot (t, velocity)



velocity nearly zero (missed by chosen time step)

(iii)

plot (t, acceleration)



edge effects of gradient function

Question 7

7 Point A has coordinates (1,1,0) m, and point B has coordinates (2,0,1) m. They are fixed within a 3D rigid body and have absolute velocities $\dot{\mathbf{r}}_A = (4\mathbf{i} + 2\mathbf{k})\,\mathrm{ms}^{-1}$ and $\dot{\mathbf{r}}_B = (9\mathbf{i} - \mathbf{j} - 4\mathbf{k})\,\mathrm{ms}^{-1}$.

(a) Check that the given velocities are consistent with rigid body motion;

$\underline{r}_A = \underline{i} + \underline{j}$, $\underline{r}_B = 2\underline{i} + \underline{k}$

$\underline{V}_A = 4\underline{i} + 2\underline{k}$, $\underline{V}_B = 9\underline{i} - \underline{j} - 4\underline{k}$

vector along length : $\underline{r}_B - \underline{r}_A = \underline{i} - \underline{j} + \underline{k} = \underline{e}$ doesn't need to be unit.

rigid body $\Rightarrow$ $\underline{V}_A \cdot \underline{e} = \underline{V}_B \cdot \underline{e}$

$LHS = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 6$, $RHS = \begin{pmatrix} 9 \\ -1 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 6$ // ✓

(b) Find the angular velocity $\boldsymbol{\omega} = \omega_x \mathbf{i} + \omega_y \mathbf{j} + \omega_z \mathbf{k}$ of the body when the component ω_z in the $\mathbf{k}$ direction is:

(i) $\omega_z = 0\,\mathrm{rads}^{-1}$;

(ii) $\omega_z = 1\,\mathrm{rads}^{-1}$;

i) $\underline{V}_B = \underline{V}_A + \underline{\omega} \times \underline{r}_{B/A}$

$\begin{pmatrix} 9 \\ -1 \\ -4 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} + \underline{\omega} \times \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ ie $\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \omega_x & \omega_y & 0 \\ 1 & -1 & 1 \end{vmatrix} = \begin{pmatrix} 5 \\ -1 \\ -6 \end{pmatrix}$

$\Rightarrow \omega_y = 5$, $-\omega_x = -1$ so $\omega_x = 1$

ie $\underline{\omega} = \underline{i} + 5\underline{j}$ //

ii) $\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \omega_x & \omega_y & 1 \\ 1 & -1 & 1 \end{vmatrix} = \begin{pmatrix} 5 \\ -1 \\ 6 \end{pmatrix}$ $\Rightarrow$ $\omega_y + 1 = 5$, $\omega_y = 4$.
$1 - \omega_x = -1$, $\omega_x = 2$
ie $\underline{\omega} = 2\underline{i} + 4\underline{j} + \underline{k}$ //

This leads to an apparent contradiction because we have found two *different* angular velocities giving the same relative velocity $\mathbf{v}_{B/A}$ for two points fixed in a rigid body.

(c) Subtract the two angular velocity vectors, and show that this difference gives an angular velocity vector that is parallel to $\mathbf{r}_{B/A}$. Use this to explain the apparent contradiction.

$$\underline{\omega}_{ii} - \underline{\omega}_{i} = \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = -(\underline{r}_B - \underline{r}_A)$$

So difference in angular velocity is parallel to the vector from A to B.

ie this is the component of spin about axis AB, which doesn't affect their velocities.

Question 8

8 A coriolis flow meter measures the flow rate of a fluid through a pipe by using the coriolis effect. Figure 4 shows the path of a pipe in a basic design. The pipe is nominally vertical with fluid flowing downwards at a speed V_0. It then enters a semi-circular section of pipe that follows a path of radius R. The whole pipe is set in rotation about the z-axis at a constant angular velocity $\omega = \Omega\mathbf{k}$. The unit vectors in Figure 4 are defined within this rotating reference frame.

(a) Derive an expression for the acceleration of a fluid particle P at an arbitrary position θ along the semi-circular section of pipe;

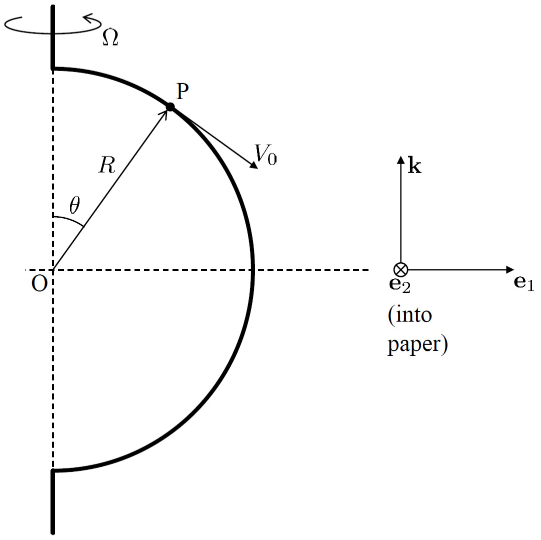


Figure 4

$$\underline{\omega} = \Omega \underline{k}$$

$$\dot{\underline{e}}_1 = \underline{\omega} \times \underline{e}_1 = \Omega \underline{e}_2$$

$$\dot{\underline{e}}_2 = \underline{\omega} \times \underline{e}_2 = -\Omega \underline{e}_1$$

$$\underline{r}_p = R\sin\theta\,\underline{e}_1 + R\cos\theta\,\underline{k}$$

$$\dot{\underline{r}}_p = R\dot{\theta}\cos\theta\,\underline{e}_1 + R\sin\theta\,\dot{\underline{e}}_1 - R\dot{\theta}\sin\theta\,\underline{k}$$

$$= R\dot{\theta}\cos\theta\,\underline{e}_1 + R\Omega\sin\theta\,\underline{e}_2 - R\dot{\theta}\sin\theta\,\underline{k}$$

$$\ddot{\underline{r}}_p = R\ddot{\theta}\cos\theta\,\underline{e}_1 - R\dot{\theta}^2\sin\theta\,\underline{e}_1 + R\dot{\theta}\Omega\cos\theta\,\underline{e}_2$$

$$+ R\Omega\dot{\theta}\cos\theta\,\underline{e}_2 - R\Omega^2\sin\theta\,\underline{e}_1$$

$$- R\ddot{\theta}\sin\theta\,\underline{k} - R\dot{\theta}^2\cos\theta\,\underline{k}$$

$$= \left[R\ddot{\theta}\cos\theta - R(\dot{\theta}^2+\Omega^2)\sin\theta\right]\underline{e}_1$$

$$+ 2R\Omega\dot{\theta}\cos\theta\,\underline{e}_2$$

$$-\left[R\ddot{\theta}\sin\theta + R\dot{\theta}^2\cos\theta\right]\underline{k}$$

(b) Identify the coriolis acceleration and show that it is proportional to the flow rate;

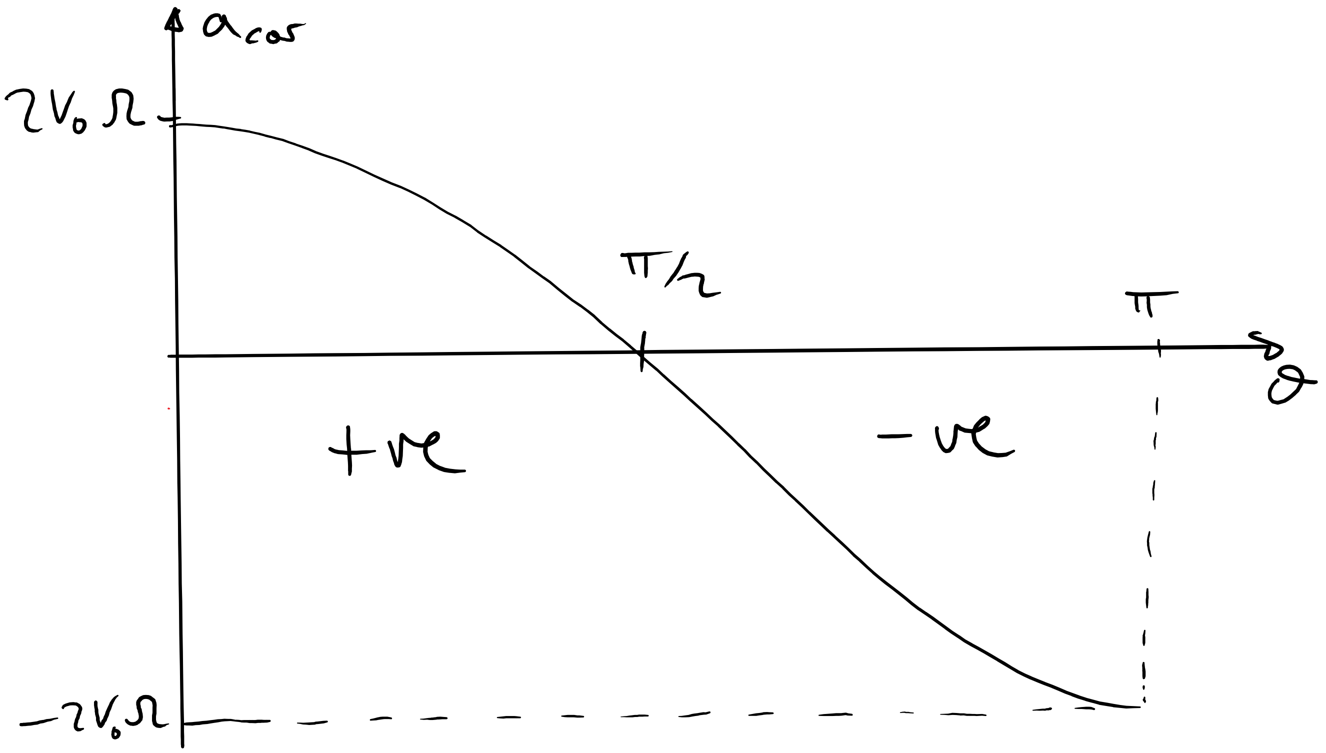
coriolis component is in $\underline{e}_2$ direction:

$$a_{cor} = \ddot{\underline{r}}_p \cdot \underline{e}_2 = 2R\Omega\dot{\theta}\cos\theta$$

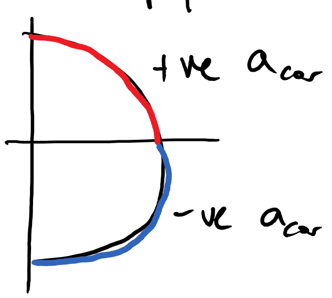
flow rate is $V_0 = \dot{\theta}R$,

so $a_{cor} = 2V_0\Omega\cos\theta$ ie proportional to flow rate

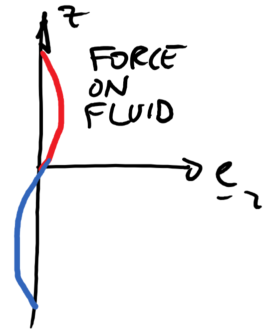
(c) Sketch the magnitude of the coriolis acceleration for $0 \leq \theta \leq \pi$ and use this to explain how the flow rate can be measured [hint: consider the force required to produce this component of acceleration and hence how the pipe may deform slightly].

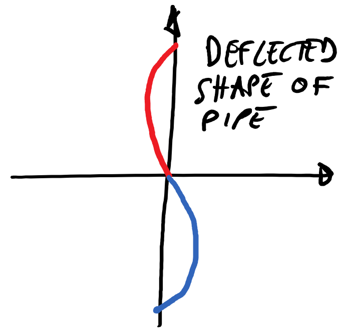


Acceleration is +ve for first half, & −ve for second half. This must be caused by a force, which means the pipe measurably twists to provide this.



OR EDGEWAYS:

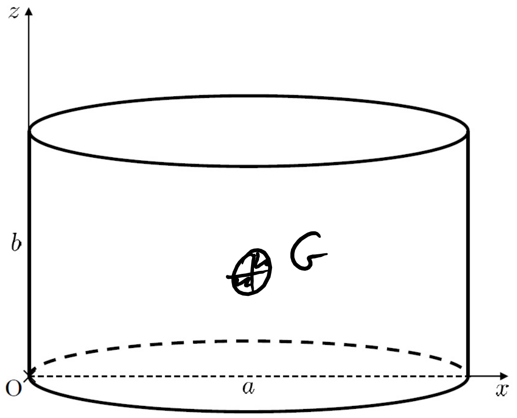




Question 9

9 Find the centre of mass as a vector $\mathbf{r}_{G/O}$ for the objects shown in Figure 5:

(a) a solid cylinder;



(a)

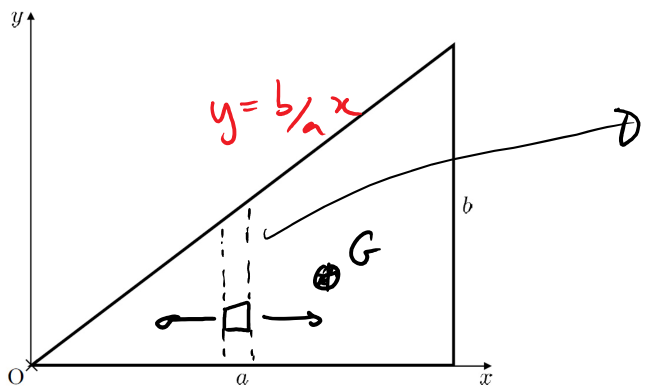
$$\underline{r}_G = \frac{1}{M}\int \underline{r}\, dM$$

But symmetry allows us to find C.o.M by inspection.

$$\underline{r}_G = \frac{a}{2}\underline{i} + \frac{b}{2}\underline{k}$$

Question 9 (continued)

(b) a triangular lamina;



(b)

1. Small area $dA = dx\,dy$
2. Integrate over y to make strip.
3. Integrate over area

Definition: $\underline{r}_{G/O} = \frac{1}{M}\int \underline{r}\, dm$

$$\underline{r} = x\underline{i} + y\underline{j}$$

$$\underbrace{dm}_{\text{small lump of mass}} = \underbrace{\frac{M}{A}}_{\frac{\text{mass}}{\text{area}}}\underbrace{dx\,dy}_{\text{small area.}}$$

So $$\underline{r}_{G/O} = \frac{1}{A}\int_{x=0}^{x=a}\int_{y=0}^{y=bx/a} x\underline{i} + y\underline{j}\ \underbrace{dy\,dx}_{dA}$$

goes together

goes together

$$\underline{r}_{G/O} = \frac{1}{A}\int_{x=0}^{x=a}\int_{y=0}^{y=bx/a} x\underline{i} + y\underline{j}\;\underbrace{dy\,dx}_{dA}$$

$$= \frac{2}{ab}\cdot\int_{x=0}^{x=a}\left[xy\underline{i} + \frac{y^2}{2}\underline{j}\right]_{y=0}^{y=bx/a} dx$$

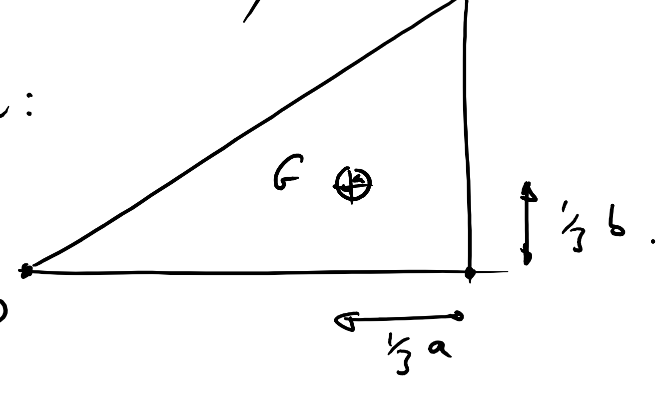
$$= \frac{2}{ab}\int_{x=0}^{x=a}\left[x.\frac{bx}{a}\underline{i} + \frac{b^2x^2}{2a^2}\underline{j}\right] dx$$

$$= \frac{2}{ab}\left[\frac{b}{a}\frac{x^3}{3}\underline{i} + \frac{b^2}{2a^2}\frac{x^3}{3}\underline{j}\right]_{x=0}^{x=a}$$

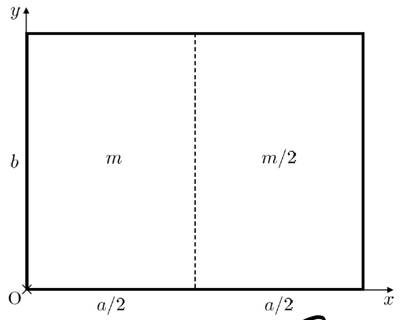
$$= \frac{2}{ab}\left[\frac{b}{a}\frac{a^3}{3}\underline{i} + \frac{b^2}{2a^2}\frac{a^3}{3}\underline{j}\right]$$

$$= \frac{2a}{3}\underline{i} + \frac{b}{3}\underline{j} \quad //$$

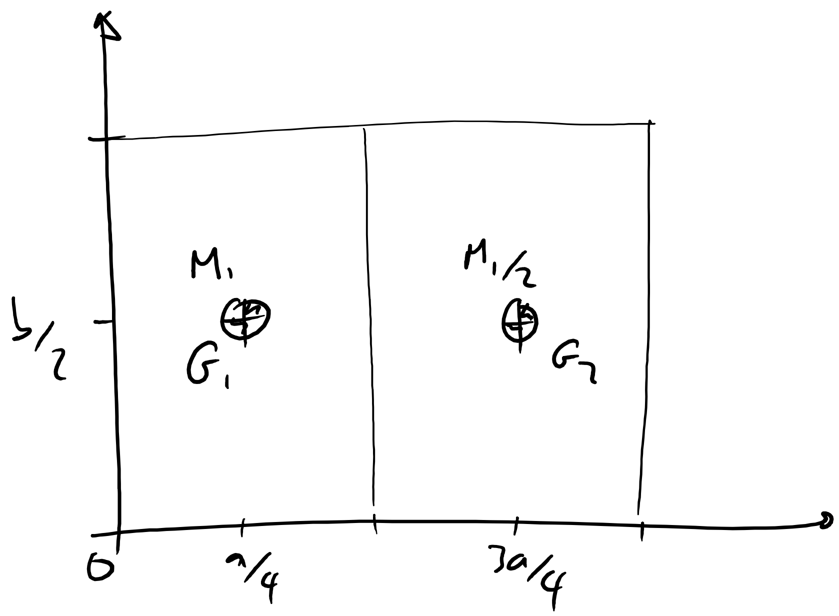
ie:



(c) a rectangular lamina made up of two sections of equal dimensions: the right side having half the mass of the left side;



(c) Centre of mass of each half by inspection:



Now discrete formula for C.o.M. gives:

$$\underline{r}_{G/O} = \frac{1}{M}\sum_{\text{all } i} m_i \, \underline{r}_{i/O}$$

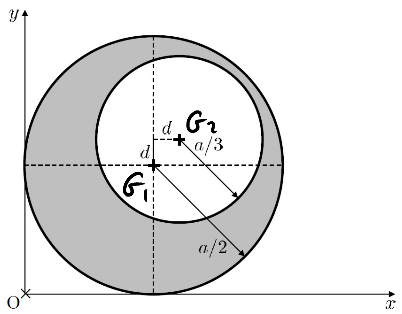
but can see that CoM must be on line joining G_1 & G_2 so only need to calculate $\underline{i}$ component:

$$\underline{r}_G \cdot \underline{i} = \frac{1}{\cancel{M_1} + \cancel{M_1}/2} \cdot \left(\cancel{M_1}\, a/4 + \frac{\cancel{M_1}}{2}\,\frac{3a}{4} \right)$$

$$= \frac{2}{3}\left(\frac{5a}{8}\right) = \frac{5a}{12}$$

$$\underline{r}_G = \frac{5a}{12}\,\underline{i} + \frac{b}{2}\,\underline{j}$$

(d) A circular disc with a circular hole offset from the centre (assume $d < a/6$).



(c)

Don't try to integrate: assemble from two pieces & consider hole as negative mass:

$$\underline{r}_{G1} = \frac{a}{2}\left(\underline{i} + \underline{j}\right) \quad \text{(centre of disc as if no hole)}$$

$$\underline{r}_{G2} = \left(\frac{a}{2} + d\right)\left(\underline{i} + \underline{j}\right) \quad \text{(centre of hole)}$$

$$A_1 = \pi\left(a/2\right)^2$$

$$A_2 = \pi\left(a/3\right)^2$$

$$\underline{r}_G = \frac{1}{A_{total}}\left(A_1 \underline{r}_{G1} - A_2 \underline{r}_{G2}\right)$$

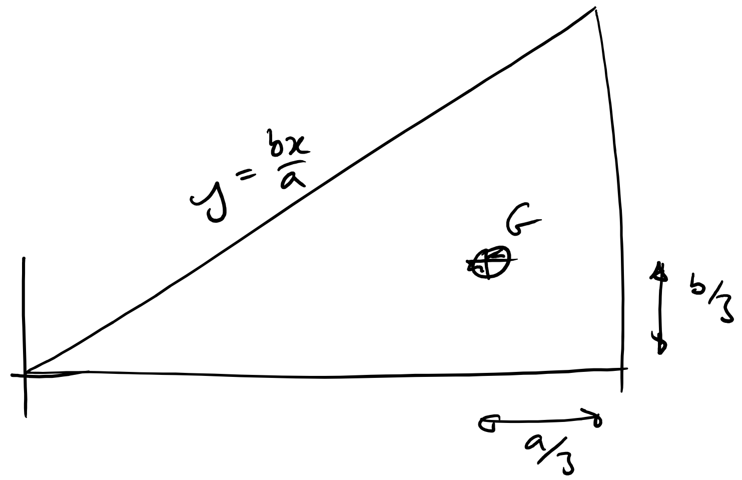
−ve, as hole is taking mass away.

$$\underline{r}_G = \frac{1}{\pi\left(\left(\frac{a}{2}\right)^2 - \left(\frac{a}{3}\right)^2\right)}\left[\pi\left(\frac{a}{2}\right)^2 \frac{a}{2}\left(\underline{i} + \underline{j}\right) - \pi\left(\frac{a}{3}\right)^2\left(\frac{a}{2} + d\right)\left(\underline{i} + \underline{j}\right)\right]$$

$$= \frac{36}{5\pi a^2} \cdot \pi a^3 \left[\frac{1}{8}\left(\underline{i} + \underline{j}\right) - \frac{1}{18}\left(\underline{i} + \underline{j}\right) - \frac{d}{9a}\left(\underline{i} + \underline{j}\right)\right]$$

$$= \left(\frac{a}{2} - \frac{4d}{5}\right)\left(\underline{i} + \underline{j}\right)$$

10 (a) Find from first principles the Mass Moment of Inertia I_{zz} of the triangular lamina in Question 9(b), about an axis passing through the origin O and parallel to the z-axis;



$$I_{zz} = \int r^2 \, dm = \int (x^2+y^2) \cdot \overbrace{\frac{M}{A}\, dA}^{\text{mass of a small area}}$$

$\frac{M}{A}$ ← mass per unit area; dA ← small area

$$= \frac{M}{A} \int_{x=0}^{x=a} \int_{y=0}^{y=bx/a} (x^2+y^2)\, dy\, dx$$

$$= \frac{M}{A} \int_{x=0}^{x=a} \left[x^2 y + y^3/3 \right]_0^{bx/a} dx$$

$$= \frac{M}{A} \int_{x=0}^{x=a} \left[x^2 . bx/a + b^3 x^3 / 3a^3 \right] dx$$

$$= \frac{M}{A} \left[\frac{b}{a} \cdot \frac{x^4}{4} + \frac{b^3}{3a^3} \cdot \frac{x^4}{4} \right]_0^a = \frac{2M}{ab} \left[\frac{a^4 b}{4a} + \frac{a^4 b^3}{12 a^3} \right]$$

$$\left(A = \frac{ab}{2} \right)$$

$$= \frac{2M}{ab} \cdot \left(\frac{a^3 b}{4} + \frac{a b^3}{12} \right)$$

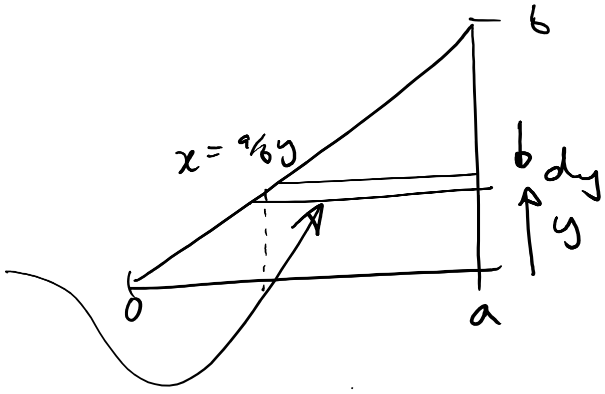
$$= M \left(\frac{a^2}{2} + \frac{b^2}{6} \right)$$

Question 10 (continued)

(b) For the triangular lamina find I_{xx} and I_{yy}, the mass moments of inertia about the x-axis and y-axis;

$$I_{xx} = \int y^2 \, dm$$

mass of a small strip at height y



$$= \frac{M}{A} \int_0^b y^2 \cdot \left(a - \frac{a}{b} y\right) \cdot dy$$

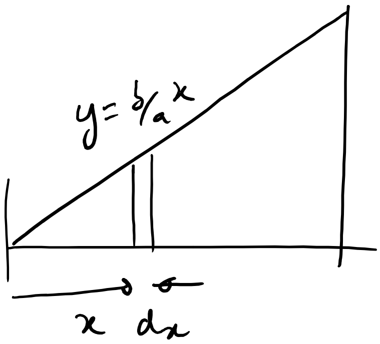
$$= \frac{2M}{ab} \left[\frac{ay^3}{3} - \frac{a}{b} \frac{y^4}{4} \right]_0^b = \frac{2M}{ab} \left(\frac{ab^3}{3} - \frac{ab^4}{4b} \right)$$

$$= 2M \left(\frac{b^2}{3} - \frac{b^2}{4} \right)$$

$$= \frac{Mb^2}{6}$$

$$I_{yy} = \int x^2 \, dm$$

$$\frac{M}{A} \cdot \frac{b}{a} x \, dx$$



$$= \frac{M}{A} \int_0^a x^2 \cdot \frac{b}{a} x \, dx = \frac{2M}{ab} \left[\frac{b}{a} \frac{x^4}{4} \right]_0^a = \frac{2M}{ab} \cdot \frac{ba^4}{4a} = \frac{Ma^2}{2}$$

(c) Use the perpendicular axis theorem to confirm the result obtained in (a);

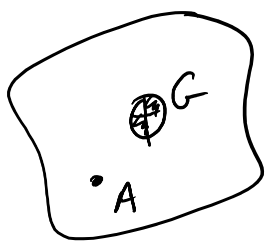
$\perp$ axis theorem : $I_{zz} = I_{xx} + I_{yy}$

a) $I_{zz} = M\left(\frac{a^2}{2} + \frac{b^2}{6}\right)$

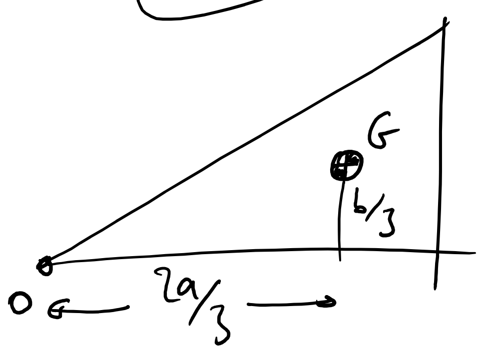
b) $I_{xx} + I_{yy} = \frac{Mb^2}{6} + \frac{Ma^2}{2} = I_{zz}$ ✓.

(d) Use the parallel axis theorem to find the moment of inertia I_G of the triangular lamina about an axis passing through its centre of mass G and parallel to the z-axis;

// axis theorem: $I_A = I_G + M|\underline{r}_{AB}|^2$



so $I_G = I_O - M\left[\left(\frac{2a}{3}\right)^2 + \left(\frac{b}{3}\right)^2\right]$



$= M\left(\frac{a^2}{2} + \frac{b^2}{6}\right) - M\left[\frac{4a^2}{9} + \frac{b^2}{9}\right]$

$= M\left(a^2/18 + b^2/18\right)$ //

(e) To which objects in Question 9 can the parallel and perpendicular axis theorems be applied?

// axis theorem valid for general bodies, so all (a,b,c,d)

$\perp$ axis theorem valid for lamina only, so (b,c,d)