Part IA Mathematics Examples paper 4 Solution

1. (a) $\dfrac{d^2x}{dt^2} - 13\dfrac{dx}{dt} + 12x = 36.$ C.F. $e^{\lambda t} \Rightarrow \lambda^2 - 13\lambda + 12 = 0$

i.e. $\lambda = 1$ or 12

P.I. is $x = 3$ $\therefore$ $\underline{x = Ae^{12t} + Be^{t} + 3}$

(b) $\dfrac{d^2x}{dt^2} - 2\dfrac{dx}{dt} + 2x = e^{t}$ C.F. $e^{\lambda t} \Rightarrow \lambda^2 - 2\lambda + 2 = 0$

$\Rightarrow \lambda = 1 \pm i$

so C.F. $= e^{t}(A\sin t + B\cos t)$

P.I. Try $x = Ce^{t}$. Substituting $\Rightarrow C = 1$

$\therefore$ $\underline{x = e^{t}(A\sin t + B\cos t + 1)}$

(c) $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 2e^{3x}$ C.F. $e^{\lambda x} \Rightarrow \lambda^2 - 3\lambda + 2 = 0$

$\Rightarrow \lambda = 1$ or 2

P.I. Try $y = Ce^{3x}$. Substituting $\Rightarrow$ $C = 1$

$\therefore$ $\underline{y = Ae^{x} + Be^{2x} + e^{3x}}$

(d) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = x^2$ C.F. $e^{\lambda x} \Rightarrow \lambda^2 + 2\lambda + 1 = 0$

i.e. $\lambda = -1$ (double root)

So C.F. $= (A + Bx)e^{-x}$

P.I. Try $\alpha x^2 + \beta x + \gamma$ giving $2\alpha + 2(2\alpha x + \beta) + \alpha x^2 + \beta x + \gamma = x^2$

whence $\alpha = 1$, $\beta + 4\alpha = 0$, $2\alpha + 2\beta + \gamma = 0 \rightarrow \alpha = 1, \beta = -4, \gamma = 6$

$\therefore$ $\underline{y = (A + Bx)e^{-x} + x^2 - 4x + 6}$

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2. (a) $\dfrac{d^2x}{dt^2} + 3\dfrac{dx}{dt} = 5e^{-3t} + 2.$ C.F. $e^{\lambda t} \Rightarrow \lambda^2 + 3\lambda = 0 \Rightarrow \lambda = 0$ or -3.

P.I. has same form as C.F. $\Rightarrow$ Try $(\alpha + \beta t)e^{-3t} + \gamma t + \delta$

Substituting $\Rightarrow$ $-6\beta e^{-3t} + 9(\alpha + \beta t)e^{-3t} + 3\left[\beta e^{-3t} - 3(\alpha + \beta t)e^{-3t} + \gamma\right] = 5e^{-3t} + 2$

i.e. $-3\beta e^{-3t} + 3\gamma = 5e^{-3t} + 2$ $\Rightarrow$ $\beta = -\frac{5}{3}$, $\gamma = \frac{2}{3}$ & α & δ not necessary

$\therefore$ $\underline{x = \left(A - \frac{5t}{3}\right)e^{-3t} + \frac{2t}{3} + B}$

2(b) $\frac{d^2x}{dt^2} + 9x = \sin 3t + 2\sin 4t$ C.F. $e^{\lambda t} \Rightarrow \lambda^2 = 9 \Rightarrow \lambda = \pm 3i$

i.e. $A\cos 3t + B\sin 3t$

P.I. Try $x = t(\alpha\sin 3t + \beta\cos 3t) + \gamma\sin 4t + \delta\cos 4t$

Substituting $\Rightarrow$ $6\alpha\cos 3t - 9\alpha t\sin 3t - 6\beta\sin 3t - 9\beta t\cos 3t - 16\gamma\sin 4t - 16\delta\cos 4t$

$+ 9(\alpha t\sin 3t + \beta t\cos 3t + \gamma\sin 4t + \delta\cos 4t) = \sin 3t + 2\sin 4t$

$\rightarrow$ $6\alpha = 0$; $-6\beta = 1$; $-7\gamma = 2$; $-7\delta = 0$

$\therefore$ $x = \left(A - \frac{t}{6}\right)\cos 3t + B\sin 3t - \frac{2}{7}\sin 4t$

Aliter for P.I. Could note that R.H.S. = odd fn & differential operator in this case produces an odd function if x is odd. $\therefore$ need only try

$x = \beta t\cos 3t + \gamma\sin 4t$

3. $\frac{d}{dt}\mathrm{Re}(e^{i\omega t}) = \frac{d}{dt}\cos\omega t = -\omega\sin\omega t$

$\mathrm{Re}\frac{d}{dt}(e^{i\omega t}) = \mathrm{Re}[i\omega e^{i\omega t}] = -\omega\sin\omega t$

Writing $y = \mathrm{Re}(y_0 e^{it})$ and $\cos t = \mathrm{Re}(e^{it}) \Rightarrow (i+5)y_0 e^{it} = e^{it}$

or $y_0 = \frac{1}{5+i}$. So $y = \mathrm{Re}\left\{\frac{e^{it}}{5+i}\right\} = \mathrm{Re}\left\{\frac{5-i}{5^2+1^2}e^{it}\right\} = \frac{5\cos t + \sin t}{26}$.

4. $(D^2 + 2D + 1)y = \frac{1}{4}\cos 2x$. C.F. $e^{\lambda x} \Rightarrow \lambda^2 + 2\lambda + 1 = 0$

$\Rightarrow \lambda = -1$ twice

$\therefore$ C.F. is $(Ax + B)e^{-x}$

P.I. Try $\alpha\cos 2x + \beta\sin 2x$, substituting $\Rightarrow$

$-4\alpha\cos 2x - 4\beta\sin 2x - 4\alpha\sin 2x + 4\beta\cos 2x + \alpha\cos 2x + \beta\sin 2x = \frac{1}{4}\cos 2x$

$\Rightarrow 4\beta - 3\alpha = 1/4$ & $-3\beta - 4\alpha = 0$ $\rightarrow \beta = \frac{1}{25}$, $\alpha = -\frac{3}{100}$

Initial condition $\rightarrow B = 7/25$, $A = 1/5$

$\therefore$ $y = \frac{7 + 5x}{25}e^{-x} - (3\cos 2x - 4\sin 2x)/100$

5. $R = Cr^{\alpha} \Rightarrow r^2(\alpha-1)\alpha C r^{\alpha-2} + r\alpha C r^{\alpha-1} - m^2 C r^{\alpha} = 0$

$\therefore \quad \underline{\alpha(\alpha-1) + \alpha - m^2 = 0} \quad \Rightarrow \quad \alpha^2 - m^2 = 0 \qquad \underline{\alpha = \pm m}$

For $r^2 \dfrac{d^2R}{dr^2} + r\dfrac{dR}{dr} - 4R = r$ i.e. $m = 2$ $\quad$ C.F. $= Ar^2 + \dfrac{B}{r^2}$

P.I. Try $R = kr \Rightarrow rk - 4kr = r \Rightarrow k = -1/3$

$\therefore \quad R = Ar^2 + \dfrac{B}{r^2} - \dfrac{r}{3}$. $\quad$ B.C. $\Rightarrow B = 0$ (R finite at $r=0$)

& $A = 4/3$ ($R = 1$ at $r = 1$)

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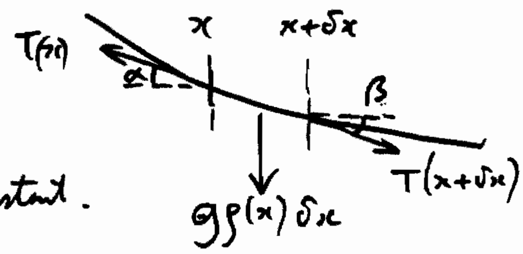
6. Resolve forces horizontally:



$T(x)\cos\alpha - T(x+\delta x)\cos\beta = 0$

But α, β both small, so $\cos\alpha \simeq 1 \simeq \cos\beta$

So $T(x) = T(x+\delta x)$, and T is constant.

Resolve vertically:

$T\sin\alpha - T\sin\beta - g\rho(x)\,\delta x = 0$

But $\sin\alpha \simeq \dfrac{dy}{dx}(x)$, and $\sin\beta \simeq \dfrac{dy}{dx}(x+\delta x) \simeq \dfrac{dy}{dx}(x) + \delta x \dfrac{d^2y}{dx^2}(x)$ by Taylor.

So $T\left(\dfrac{dy}{dx} - \dfrac{dy}{dx} - \delta x \dfrac{d^2y}{dx^2}\right) \simeq g\rho(x)\,\delta x$

$\Rightarrow \quad T\dfrac{d^2y}{dx^2} = -g\rho(x)$

When ρ is constant, general solution is $y = -\dfrac{g\rho}{T}\cdot\frac{1}{2}x^2 + Ax + B$.

But $y(0) = y(L) = 0$, so $B = 0$, $AL - \dfrac{\rho g}{2T}L^2 = 0$

So $\quad y = \dfrac{g\rho}{2T}x(L-x)$

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7. Let $x =$ no people not yet infected $(\Rightarrow x(0) = N)$

$y =$ no people currently ill $(\Rightarrow y(0) = 0)$

$z =$ no people dead $(\Rightarrow z(0) = 0)$

Then $\dfrac{dx}{dt} = -\alpha x$ So $\underline{x = Ne^{-\alpha t}}$

Also $\dfrac{dy}{dt} = \alpha x - (\beta+\gamma)y$ i.e. $\left(\dfrac{d}{dt} + \beta + \gamma\right)y = \alpha N e^{-\alpha t}$

C.F. $e^{-(\beta+\gamma)t}$ P.I. $\dfrac{\alpha N e^{-\alpha t}}{\beta+\gamma-\alpha}$

thus $y = \dfrac{\alpha N}{\beta+\gamma-\alpha}\left\{e^{-\alpha t} - e^{-(\beta+\gamma)t}\right\}$

Finally $\dfrac{dz}{dt} = \gamma y$ with $z(0) = 0$ gives

$$\underline{z = \frac{\alpha\gamma N}{\beta+\gamma-\alpha}\left\{\frac{1-e^{-\alpha t}}{\alpha} - \frac{1-e^{-(\beta+\gamma)t}}{\beta+\gamma}\right\}}$$

Let $\beta+\gamma = \alpha+\epsilon \Rightarrow z = \dfrac{\alpha\gamma N}{\epsilon}\left\{\dfrac{1-e^{-\alpha t}}{\alpha} - \dfrac{1-e^{-(\alpha+\epsilon)t}}{\alpha+\epsilon}\right\}$

$$= \frac{\alpha\gamma N}{\epsilon}\left\{\frac{1-e^{-\alpha t}}{\alpha} - \frac{1-e^{-\alpha t}(1-\epsilon t+O(\epsilon^2))}{\alpha}\left(1+\frac{\epsilon}{\alpha}\right)^{-1}\right\}$$

$$= \frac{\alpha\gamma N}{\epsilon}\left[\frac{1-e^{-\alpha t}}{\alpha} - \frac{\left(1-e^{-\alpha t}+\epsilon t e^{-\alpha t}+O(\epsilon^2)\right)}{\alpha}\left(1-\epsilon/\alpha+O(\epsilon^2)\right)\right]$$

$$= \frac{\alpha\gamma N}{\epsilon}\left[\frac{1-e^{-\alpha t}}{\alpha} - \frac{1-e^{-\alpha t}}{\alpha} - \frac{\epsilon t e^{-\alpha t}}{\alpha} + \frac{\epsilon(1-e^{-\alpha t})}{\alpha^2} + O(\epsilon^2)\right]$$

$$= \alpha\gamma N\left[-\frac{t}{\alpha}e^{-\alpha t} + \frac{1-e^{-\alpha t}}{\alpha^2} + O(\epsilon)\right] \longrightarrow \underline{\frac{\gamma N}{\alpha}\left[1-e^{-\alpha t} - \alpha t e^{-\alpha t}\right]}$$