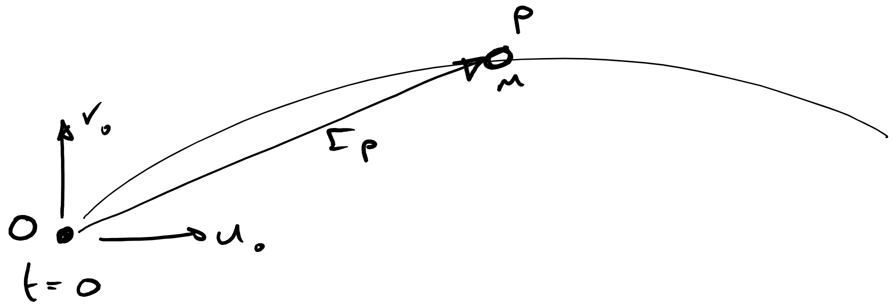
Paper 1: Mechanical Engineering

## Examples Paper 2

Question 1

1 † A ball of mass $m$ is thrown from the ground with initial horizontal velocity $u_0$, and initial vertical velocity $v_0$ at time $t = 0$.

(a) If there is no resistance to motion, what is the vector equation of motion of the ball after it has been released?




FBD:


equation of motion: $\underline{F} = m\underline{a}$

$$-mg\underline{j} = m\ddot{\underline{r}}_P$$

$$m\ddot{\underline{r}}_P + mg\underline{j} = 0$$

(b) Show that the height $y$ of the ball above the ground at time $t$ is given by:

$$y = v_0 t - \frac{gt^2}{2}$$

equation of motion: $m\ddot{\underline{r}}_P + mg\underline{j} = 0$.

$\underline{i}$-direction: $ma_x = 0$ —— not relevant.

$\underline{j}$-direction: $ma_y + mg = 0$.

$$a_y = -g.$$

$$v_y = -gt + v_0$$

$v_0$ — initial velocity

$$y = -gt^2/2 + v_0 t + 0$$

$0$ — initial height.

so $y = v_0 t - gt^2/2$ as required.

Question 1 (continued)

(c) Find the position vector of the ball as a function of time $t$;

x-direction: $ma_x = 0$.

$v_x = u_0$ — initial velocity

$x = u_0 t$.

so $\underline{r}_p = u_0 t\, \underline{i} + \left(v_0 t - g t^2/2\right) \underline{j}$ //.

(d) If $v_0/g = 0.8\,\text{s}$, find the time $t$ at which the ball first strikes the ground;

strikes ground when $y = 0$

ie $v_0 t - g t^2/2 = 0$.

$t\left(v_0 - g t/2\right) = 0$.

so $t = 0$ : initial condition

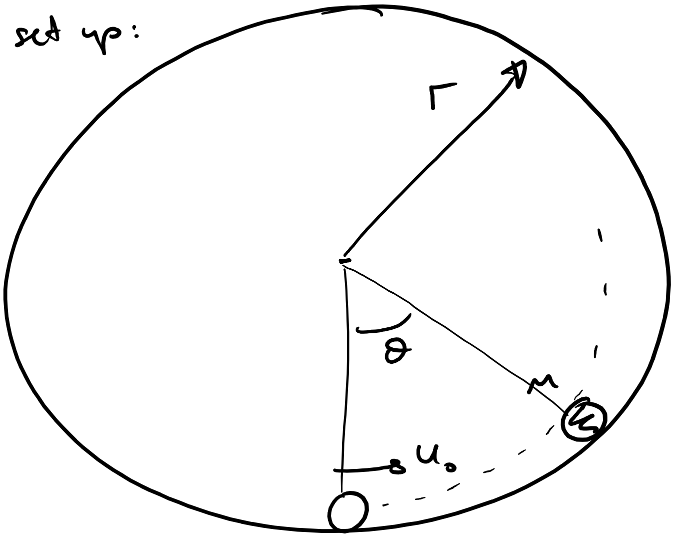
$t = 2v_0/g = 1.6\,\text{s}$ // solution.

Question 2

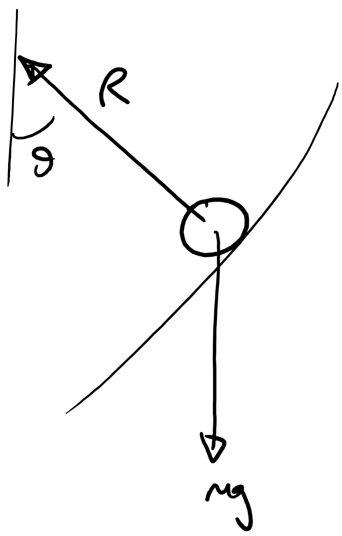
2 A particle of mass $m$ is given an initial speed $u_0$ at the bottom of a frictionless loop-the-loop of radius $r$. The loop-the-loop is circular and is in a vertical plane.

(a) Draw a Free Body Diagram of the particle at an arbitrary angular position $\theta$ around the loop;

Problem set up:



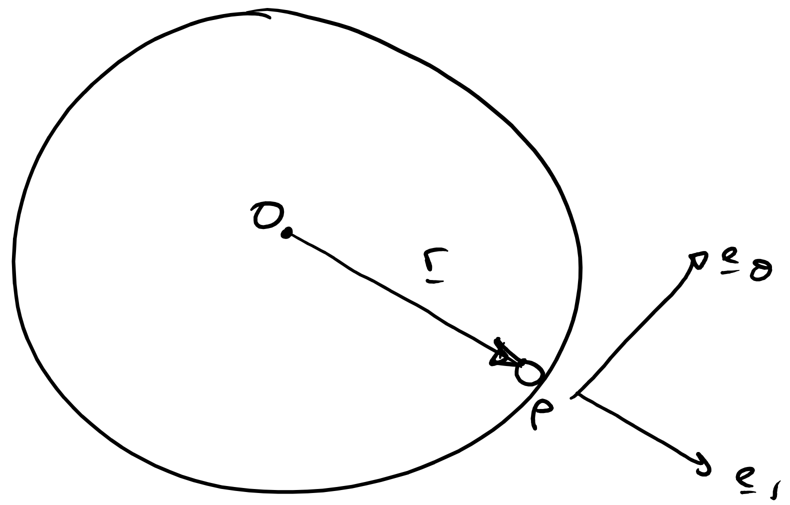

FBD:



Question 2 (continued)

(b) By differentiating the position vector in polar coordinates (taking the origin to be the centre of the loop), derive a vector expression for the acceleration of the particle and verify that your answer agrees with the Mechanics Databook;

Define origin at circle centre:




$$\underline{r}_P = r\,\underline{e}_r$$

$$\dot{\underline{r}}_P = \dot{r}\,\underline{e}_r + r\,\dot{\underline{e}}_r \qquad (\text{n/b } \dot{\underline{e}}_r = \dot{\theta}\,\underline{e}_\theta.)$$

$$= \underbrace{\dot{r}}_{=0}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta .$$

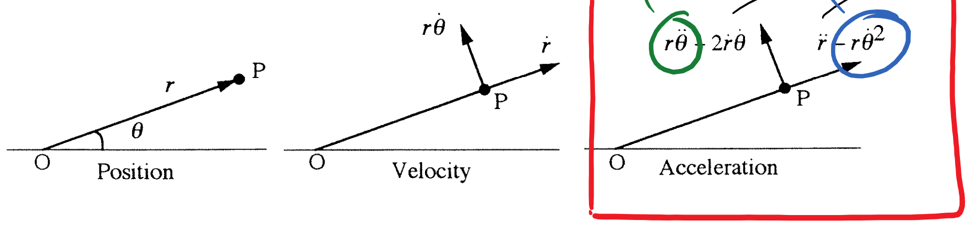
$$= r\dot{\theta}\,\underline{e}_\theta .$$

$$\ddot{\underline{r}}_P = \underbrace{\dot{r}}_{=0}\dot{\theta}\,\underline{e}_\theta + r\ddot{\theta}\,\underline{e}_\theta + r\dot{\theta}\,\dot{\underline{e}}_\theta \qquad (\text{n/b } \dot{\underline{e}}_\theta = -\dot{\theta}\,\underline{e}_r)$$

$$= r\ddot{\theta}\,\underline{e}_\theta - r\dot{\theta}^2\,\underline{e}_r$$

0 as $\dot{r} = \ddot{r} = 0$

**1.1: Velocity and acceleration in polar coordinates**

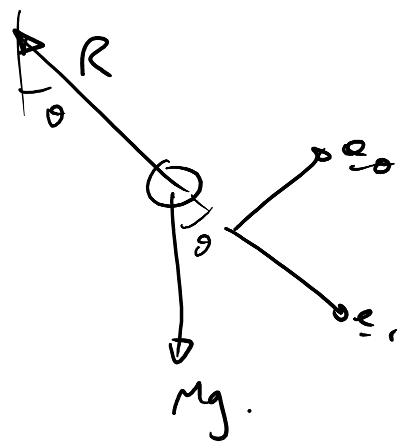



verified.

(c) Find an expression for the normal reaction force acting on the particle;

$$\ddot{\underline{r}}_p = r\ddot{\theta}\,\underline{e}_\theta - r\dot{\theta}^2\,\underline{e}_r .$$

from FBD :




in radial direction :

$$(-R + mg\cos\theta)\,\underline{e}_r = -mr\dot{\theta}^2\,\underline{e}_r$$

$$R = (mg\cos\theta + mr\dot{\theta}^2) \quad \text{(+ve if in contact)}$$

(d) Derive the equation of motion of the particle;

$$m\ddot{\underline{r}}_p = mg\cos\theta\,\underline{e}_r - mg\sin\theta\,\underline{e}_\theta - R\,\underline{e}_r$$

$$= \cancel{mg\cos\theta\,\underline{e}_r} - mg\sin\theta\,\underline{e}_\theta - (\cancel{mg\cos\theta} + mr\dot{\theta}^2)\,\underline{e}_r$$

$$m(r\ddot{\theta}\,\underline{e}_\theta - r\dot{\theta}^2\,\underline{e}_r) = -mg\sin\theta\,\underline{e}_\theta - mr\dot{\theta}^2\,\underline{e}_r$$

$\underline{e}_r$ terms cancel, leaving equation of motion in $\underline{e}_\theta$: $\ddot{\theta} = -g/r\,\sin\theta$

(e) Use Python to compute a numerical solution to the equation of motion. A template solution is available on Moodle.

Two possible schemes:

Euler:

$$\begin{cases} \ddot{\theta}_{k-1} = -g/r\,\sin\theta_{k-1} & \text{— from EOM} \\ \dot{\theta}_k = \dot{\theta}_{k-1} + \ddot{\theta}_{k-1}\,T & \\ \theta_k = \theta_{k-1} + \dot{\theta}_{k-1}\,T & \end{cases} \quad \text{integration (last two lines)}$$

Semi-implicit: same but last line:

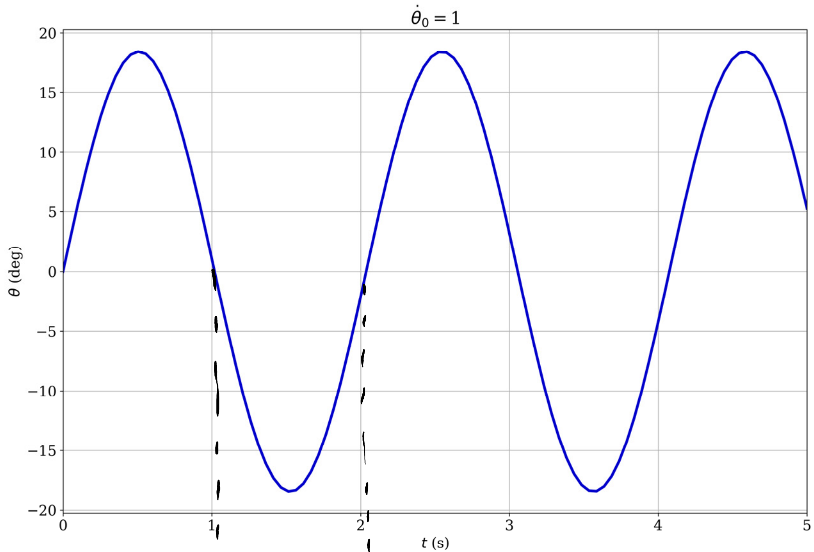
$$\theta_k = \theta_{k-1} + \dot{\theta}_k T \quad \text{more robust}$$

Try both: first is unstable, second is stable.

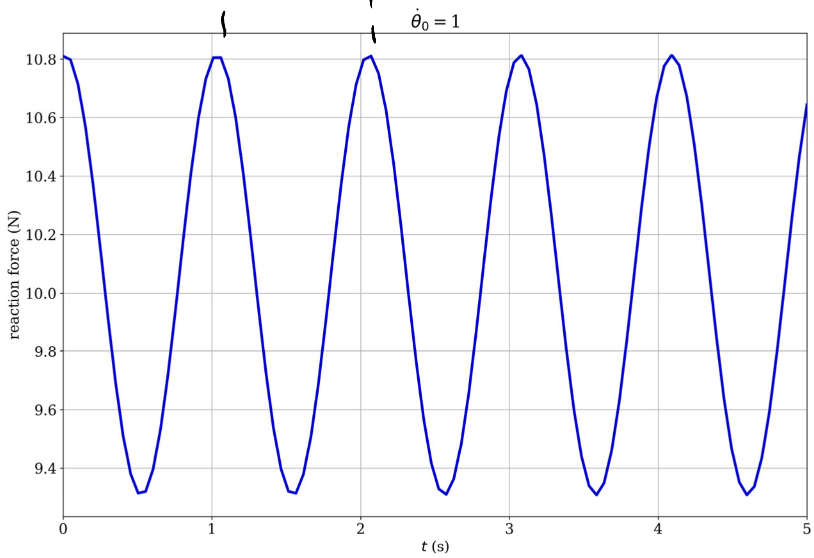
(e) Use the Python template `p2q2_template.ipynb` to compute a numerical solution to the equation of motion. See computing help at the end of the examples paper for more information.

$m=1$, $r=1$, $g=9.81$

Small amplitude oscillation: just rolling back & forth at bottom.




Note max reaction at $\theta = 0$ as expected.




reaction force frequency is twice particle frequency as peak at every zero crossing.

Use script to explore different cases.

Question 2 (continued)

(f) How could you check if your numerical simulation is giving a trustworthy solution?

Several easy checks are possible:

- speed at bottom after completing loop is $u_0$ (no energy loss)
- speed at top satisfies energy conservation:
  $$mg(2r) + \frac{1}{2} m v_{top}^2 = \frac{1}{2} m u_0^2$$
- reaction force skims zero at critical speed
  $$u_0 = \sqrt{5gr} \, / r$$

3 A proposed design for a bistable spring is shown in Figure 1: the aim is to design a mechanical device that can be at rest in one of two positions. It consists of a light inextensible rod AB connected to a pivot A via a torsional spring. The spring applies a torque $T$ proportional to the angle of rotation such that $T = K\theta$, where $K$ is the torsional stiffness of the spring. The other end of the rod B is connected via a linear spring to point O, such that the tension force in the spring is $F = kx$, where $k$ is the linear spring stiffness and $x$ is the extension of the spring OB (assuming its natural length is zero).

(a) Find an expression that relates $x$ and $\theta$;

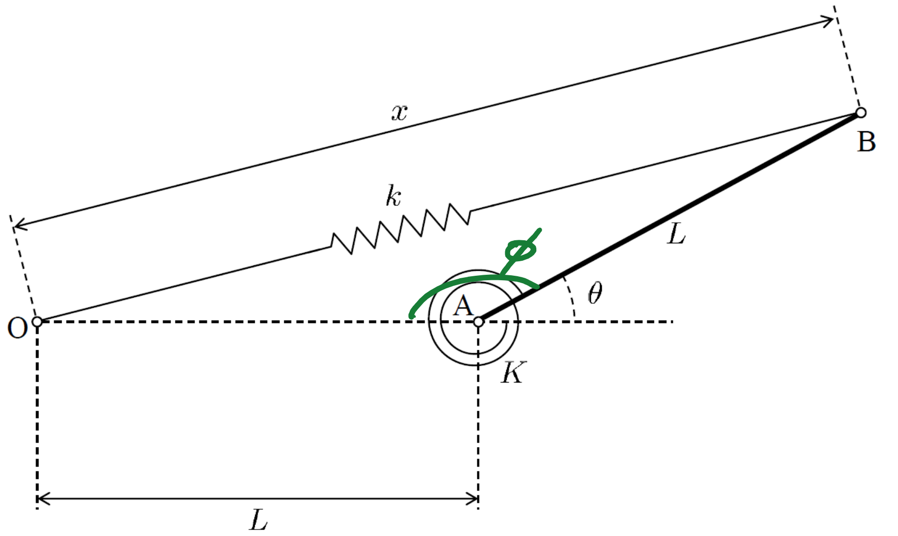


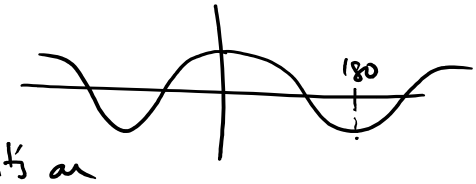

Figure 1

cosine rule: $x^2 = L^2 + L^2 - 2L^2 \cos\phi$

now $\phi = 180 - \theta$

so $\cos\phi = \cos(180-\theta)$

$= \cos(\theta - 180)$ as it's an even function




$= -\cos\theta$

hence: $x^2 = 2L^2(1 + \cos\theta)$

(b) Find an expression for the potential energy of the system as a function of $\theta$;

$V = \frac{1}{2}K\theta^2 + \frac{1}{2}kx^2$

$= \frac{1}{2}K\theta^2 + \frac{1}{2}k \cdot 2L^2(1+\cos\theta)$

$= \frac{1}{2}K\theta^2 + kL^2(1+\cos\theta)$ //

(c) How many equilibria are there and what does your answer depend on?

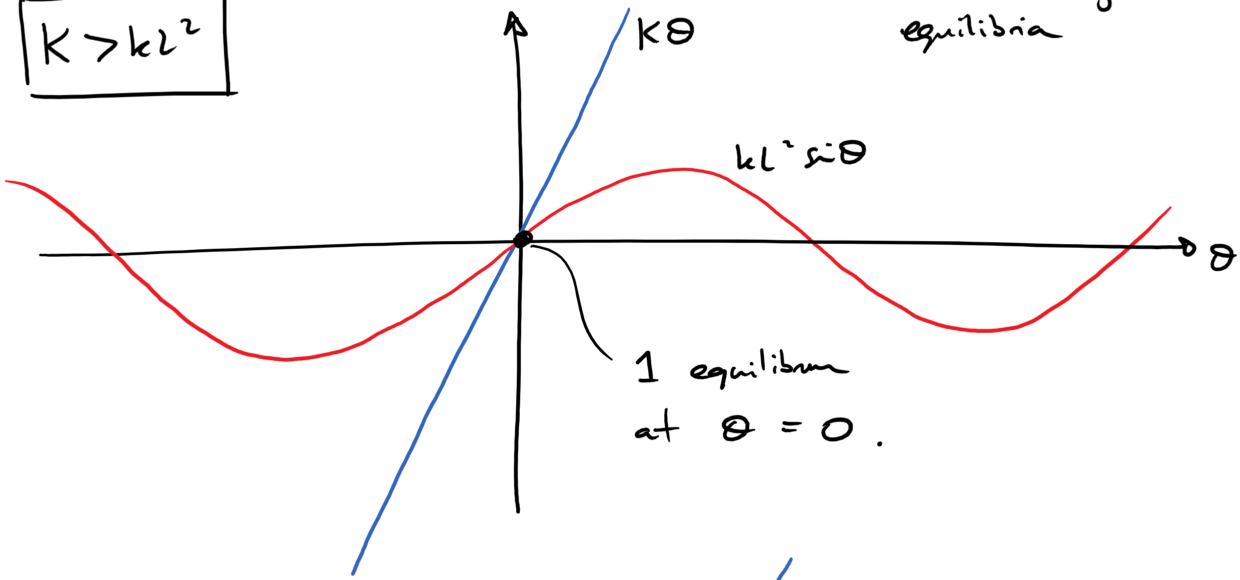
$$V = \frac{1}{2} K\theta^2 + k l^2 (1 + \cos\theta)$$

at equilibrium $V' = 0$

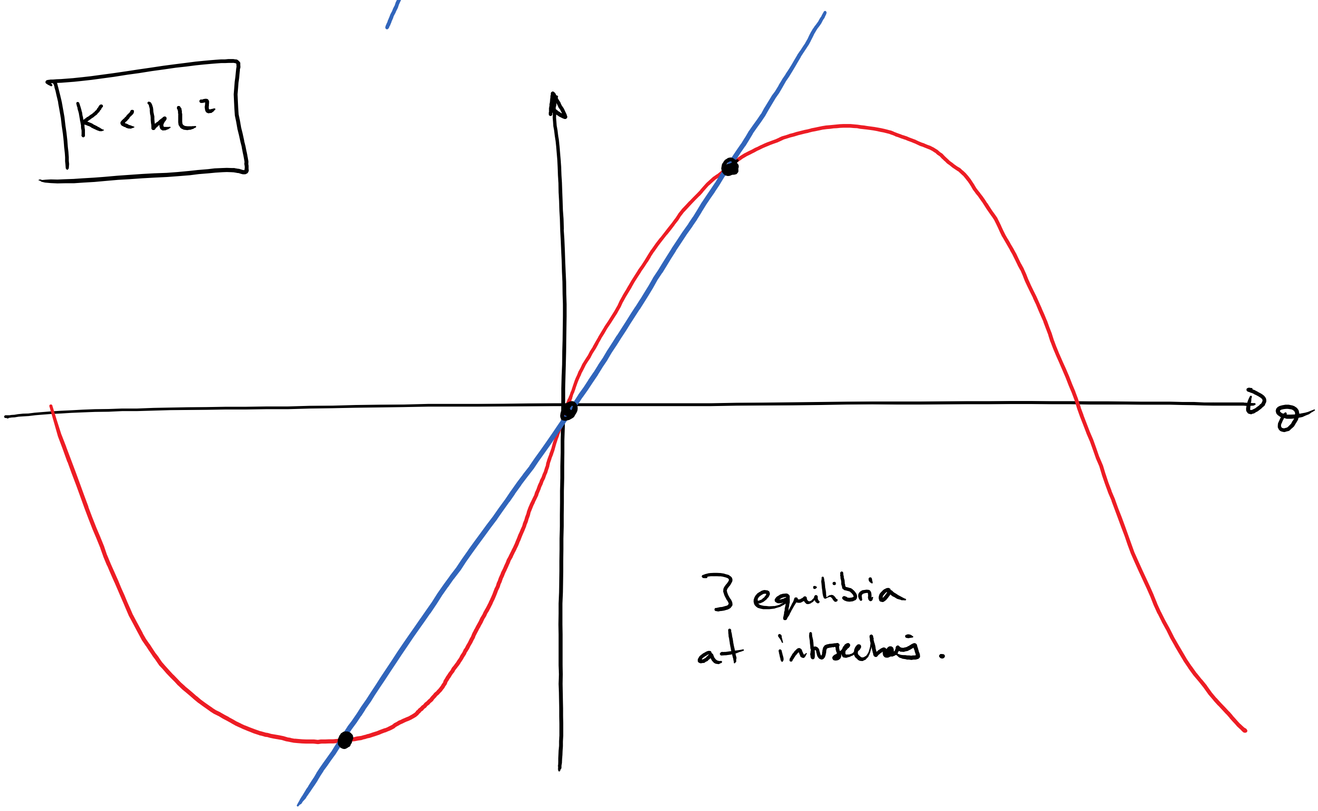
$$V' = K\theta - k l^2 \sin\theta = 0 .$$

$$K\theta = k l^2 \sin\theta$$ — intersection of two lines gives equilibria

$\boxed{K > kl^2}$




$\boxed{K < kl^2}$




note: $\frac{d}{d\theta}(K\theta) = K$, $\frac{d}{d\theta}(kl^2 \sin\theta) = kl^2 \cos\theta \Big|_{\text{eval } @\ \theta = 0} = kl^2$, hence $kl^2$ & $K$ determine number of intersections.

Question 3 (continued)

(d) Under what conditions is the switch unstable at $\theta = 0$?

$$V = \frac{1}{2} K\theta^2 + kl^2(1+\cos\theta)$$

$$V' = K\theta - kl^2 \sin\theta$$

$$V'' = K - kl^2 \cos\theta \, .$$

equilibrium @ $\theta = 0$ gives $V'' = K - kl^2$

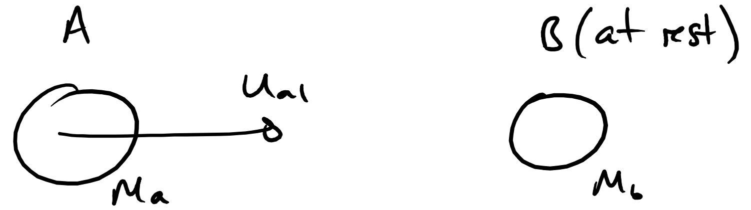
unstable if $kl^2 > K$ //

Question 4

4 A particle of mass $m_a$ and initial velocity $u_{a1}$ collides with a second particle $m_b$ that is initially at rest.

(a) Write down an expression for the velocities of the two particles after the collision. Why can the individual velocities not be determined from momentum considerations alone?

BEFORE




AFTER: conservation of linear momentum:


$$m_a u_a = m_a v_a + m_b v_b$$

One equation, two unknowns so cannot determine $v_a$ & $v_b$ : need more information to provide second equation.

(b) Is it possible for the first particle to have a larger velocity after the collision than before? Why or why not?

Nothing in momentum equation prevents $v_a > u_a$, so yes it is possible.

But note energy expression:

BEFORE : $KE_1 = \frac{1}{2} m_a u_a^2$.

AFTER : $KE_2 = \frac{1}{2} m_a v_a^2 + \frac{1}{2} m_b v_b^2$

If $v_a > u_a$, then $KE_2$ is <u>greater</u> than $KE_1$. So it is possible if energy added at impact (eg spring-loaded mechanism released at impact), but not possible if <u>passive</u> collision.

(c) If the collision is perfectly elastic (i.e. the coefficient of restitution $e = 1$), what ratio of masses $m_b/m_a$ results in the second particle having a large velocity and what is the limiting velocity? [Hint: recall the velocity result for elastic 1D collisions]

MOMENTUM: $m_a u_a = m_a v_a + m_b v_b$ (1)

ENERGY (elastic 1D): $u_a + v_a = 0 + v_b$ (2)

eliminate $v_a$ (2): $v_a = v_b - u_a$.

$\longrightarrow$ (1): $m_a u_a = m_a(v_b - u_a) + m_b v_b$

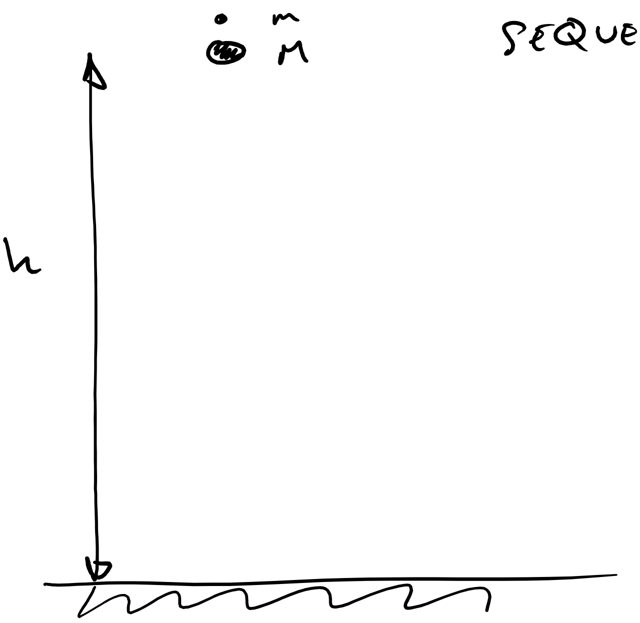
$$v_b(m_a + m_b) = 2 m_a u_a$$

$$v_b = \frac{2 m_a u_a}{m_a + m_b} = \frac{2 u_a}{1 + m_b/m_a}$$

$v_b$ large when $1 + m_b/m_a$ as small as possible, ie when $m_b \ll m_a$, then $v_b \to 2u_a$.

5 Two balls are released from rest at the same time from height $h$: the upper ball has mass $m$ while the lower ball has mass $M$: the upper ball is much lighter than the lower ball, i.e. $m \ll M$, and there is a small gap between the balls at the time of release. Assume all collisions are elastic (with coefficient of restitution $e = 1$). Find an approximate expression for the maximum height reached by the lighter ball (treating the balls as particles of negligible size compared to $h$).

INITIALLY




SEQUENCE : (1) release
(2) just before first ground impact
(3) lower ball impact
(4) two balls impact
(5) upper ball reaches max height.

DROP

For lower mass $M$:

$$PE_1 = Mgh.$$

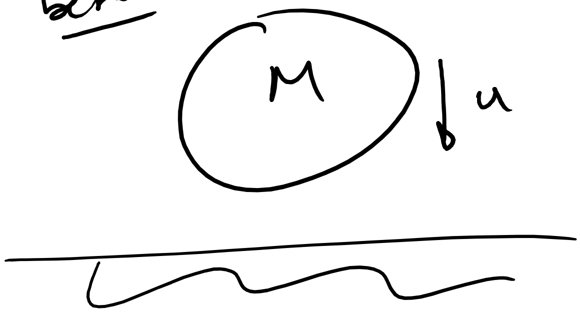
$$KE_2 = \tfrac{1}{2}Mu^2$$

$KE_2 = PE_1$ so $\tfrac{1}{2}\cancel{M}u^2 = \cancel{M}gh$

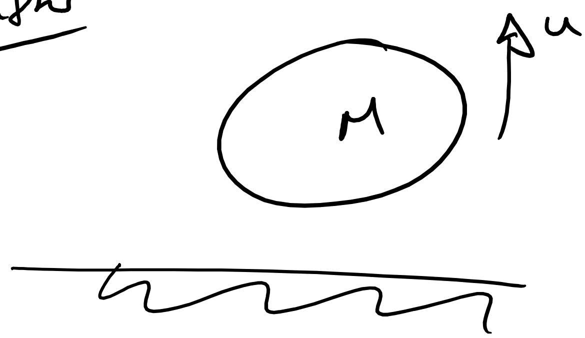
$u^2 = 2gh$. just before ground impact.

GROUND COLLISION

before




after



BALL COLLISION


For an elastic 1D collision: $\underbrace{v_2 + u}_{\text{sum before \& after velocities of } M} = \underbrace{v_1 - u}_{\text{sum before \& after velocities of } m}$

so Momentum: $Mu - mu = Mv_2 + mv_1$ ①

Energy (using above): $v_2 + u = v_1 - u$ ②

eliminate $v_2$ : ② : $v_2 = v_1 - 2u$.

$\rightarrow$ ① : $Mu - mu = M(v_1 - 2u) + mv_1$

& solve: $v_1(M+m) = Mu - mu + 2Mu$

$$v_1 = \frac{(3M - m)u}{(M+m)}$$

when $M \gg m$ $v_1 \rightarrow 3u$

upper: $KE = \frac{1}{2} m v_1^2 = \frac{1}{2} m \, 9u^2$

$\longrightarrow PE = mgH$

$mgH = \frac{1}{2} m \cdot 9\underbrace{u^2}_{(2gh)}$.

$\cancel{m}gH = 9\cancel{m}gh$

$H = 9h$ //

max height that can be reached by upper ball.

Question 6

6 The Space Shuttle has a mass of 2000 tonnes on take-off. It ejects fuel at a constant rate of 9000 $kgs^{-1}$ for the first 30 s of its flight. The initial acceleration of the Shuttle at take-off is $1.2\,g$, where $g = 9.81\ ms^{-1}$.

(a) What is the speed relative to the Shuttle that the fuel is being ejected?

$$M = 2 \times 10^6 \text{ kg.}$$

$$\frac{dM}{dt} = -9000 \text{ kgs}^{-1} \qquad (\text{for } 0 < t < 30\text{ s})$$

$$a_0 = 1.2g$$

$$F = \frac{dM}{dt} u$$

relative speed.

at $t=0$, $F - Mg = M \cdot 1.2g$.

up thrust

down weight

up acc.

so $F = 2.2Mg = \dot{M}u$

$$u = \frac{2.2\,Mg}{\dot{M}} = \frac{2.2 \cdot 2\times10^6 g}{9000}$$

$$= 4796\ ms^{-1}$$ relative velocity.

Question 6 (continued)

(b) Assuming the mass of the Shuttle remains constant, what is its acceleration and speed after 30 s?

if $M$ constant: acceleration constant, $a = 1 \cdot 2g$

$$\frac{dv}{dt} = a \quad , \quad \int_0^v dv = \int_0^t a \, dt$$

$$v = 1 \cdot 2g \,.\, 30 = 353 \; ms^{-1}$$

(c) ∗ Calculate the acceleration and speed of the Shuttle after 30 s, assuming the mass does not remain constant.

$$F_t = \frac{dM}{dt} \cdot u$$

$$F = -\frac{dM}{dt} u - Mg = Ma$$

[n/b $\frac{dM}{dt}$ = rate of increase of mass.]

$$-\dot{M} u - Mg = M \dot{v}$$

@ $t = 30s$: $M_{30} = 2 \times 10^6 - 9000 \times 30 = 1 \cdot 73 \times 10^6$ kg

$$a = \frac{-\dot{M} u - Mg}{M} = \frac{9000 \,.\, 353 - 1 \cdot 73 \times 10^6 \,.\, 9 \cdot 81}{1 \cdot 73 \times 10^6 \,.\, 9 \cdot 81}$$

$$= 15 \cdot 14 \; ms^{-2}$$

$$-\dot{M} u - Mg = M \dot{v}$$

$$\times \frac{dt}{M} \Rightarrow \quad -\frac{dM}{M} u - g dt = dv$$

& integrate ... $$-\int_{2\times 10^6}^{1 \cdot 73 \times 10^6} \frac{u}{M} dM - \int_0^{30} g \, dt = \int_0^v dv$$

$$-\left[u \ln M\right]_m^m - \left[gt\right]_0^{30} = \left[v\right]_0^v$$

$$-\int_{2\times10^6}^{1.73\times10^6} \frac{u}{M}\, dM - \int_0^{30} g\, dt = \int_0^v dv$$

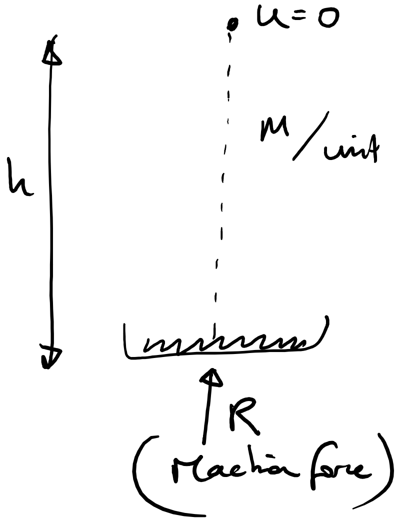
$$-\left[u \ln M\right]_m^m - \left[gt\right]_0^{30} = \left[v\right]_0^v$$

$$+\, 4796.\ \ln \frac{2}{1.73} - 30g = v = 401\ \text{ms}^{-1}$$

swap fraction
& sign

Question 7

7 $*$ Coffee beans are being dropped from rest onto the scale pan of an electronic balance from fixed height $h$ and at a constant rate $\dot{m}$. They do not bounce. The flow of beans is cut off at source when the balance reads the exact weight required. Show that the weight of beans in the air at that instant will exactly compensate for the false reading of the balance caused by the change in momentum of the falling beans.




$M$ / unit time. $\equiv \dot{M}$

$$R = Mg + \dot{M}v$$

└── final velocity.

energy of bean: $\cancel{m_b} gh = \frac{1}{2} \cancel{m_b} v^2$

$$v = \sqrt{2gh}.$$

$$R = Mg + \dot{M}\sqrt{2gh}$$

Mass in air $= \dot{M}T$ with $T =$ fall time.

final velocity = acc. $T$ $\Rightarrow$ $\sqrt{2gh} = gT$

$$T = \sqrt{2h/g}.$$

$$M_{air} = \dot{M}\sqrt{2h/g}.$$

want to show $\dot{M}\sqrt{2gh} = M_{air}\, g = \dot{M}\sqrt{\frac{2h}{g}} \cdot g$

$= \dot{M}\sqrt{2hg}$ // as required.

Question 8

8 Figure 2 shows a frictionless horizontal plane upon which a particle A of mass $m$ slides. This particle is attached to a similar particle B by a light inextensible string, which passes through a small frictionless hole. The string is of sufficient length that particle B always remains clear of the plane. Particle B moves in a vertical line at all times.

Particle A is a distance $r$ from the hole when it is projected with a speed $V_1$ perpendicular to the string. Some time later the particle has reached a distance of $2r$ from the hole and its speed is now $V_2$, again perpendicular to the string.

(a) † Determine an expression in terms of $V_1$ for the speed $V_2$ of the particle in the new position;

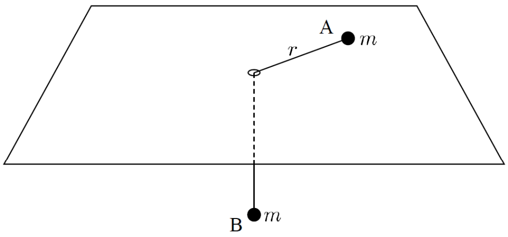
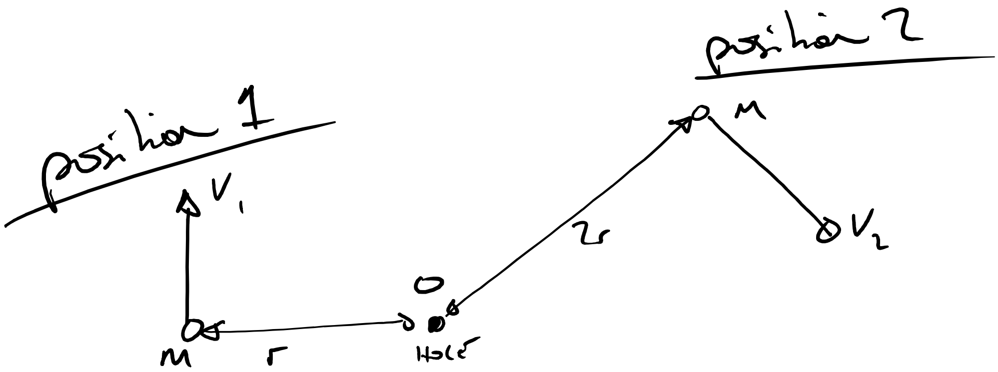



Figure 2




String tension always towards hole, so moment of momentum conserved about O.

$$\cancel{m}V_1\cancel{r} = \cancel{m}V_2 2\cancel{r} \quad \Rightarrow \quad V_2 = V_1/2$$

(b) Find an expression for $V_1$ in terms of the gravitational acceleration $g$ and $r$;

no friction, so energy conserved.
Don't forget suspended mass!

$$\frac{1}{2}mV_1^2 = \frac{1}{2}m\underbrace{V_2^2}_{(V_1/2)} + mg\underbrace{r}_{\text{change in height of second mass.}}$$

$$\frac{1}{2}mV_1^2 = \frac{1}{2}m\left(\frac{V_1}{2}\right)^2 + mgr$$

$$\frac{1}{2}\cancel{m}\frac{3}{4}V_1^2 = \cancel{m}gr$$

$$V_1 = \sqrt{\frac{8gr}{3}}$$

(c) For these two positions do you expect the string tension to be equal to $mg$?

$V_1$ & $V_2$ perpendicular to string so instantaneously velocity of suspended mass zero, but not acceleration so not in equilibrium & tension not $mg$.

Question 9

9 (a) In the motion of a particle, under what circumstances is:

(i) total mechanical energy conserved?

(ii) moment of momentum conserved about an axis?

Note that moments taken about an axis and about a point are different: see the lecture notes, or discuss with your supervisor.

i) if KE + PE = constant, ie no non-conservative forces act on particle.

ii) if all forces acting on particle pass through a single axis

(b) A smooth conical vessel with a cone angle of $\theta = 70°$ and a height of $h_2 = 2\,\mathrm{m}$ is fixed with its vertex downwards and its axis vertical, as shown in Figure 3. A particle is projected with horizontal velocity $v_1$ on the inner surface at a height $h_1 = 1\,\mathrm{m}$ measured vertically above the vertex. Show that if $v_1$ is approximately equal to $5.1\,\mathrm{ms}^{-1}$ the particle will *just* remain inside the vessel. Will $v_1$ be different if a cone with a different cone angle is used?

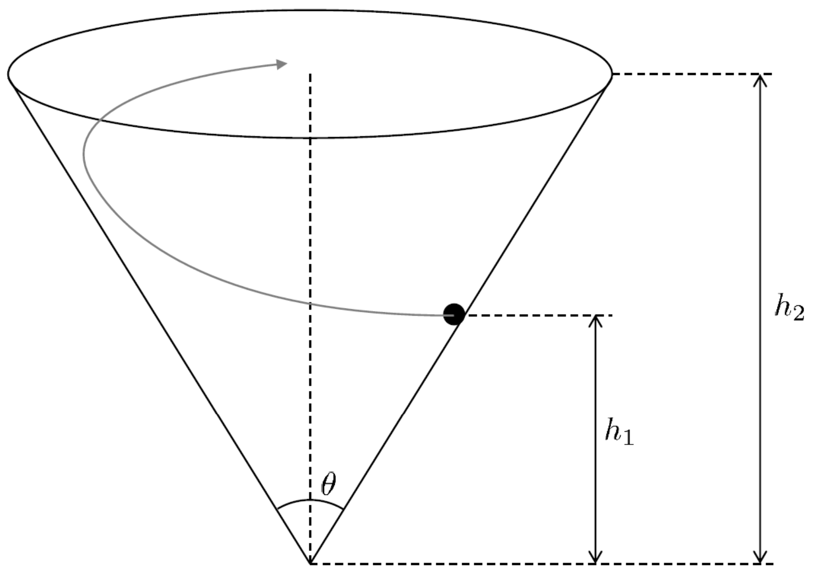
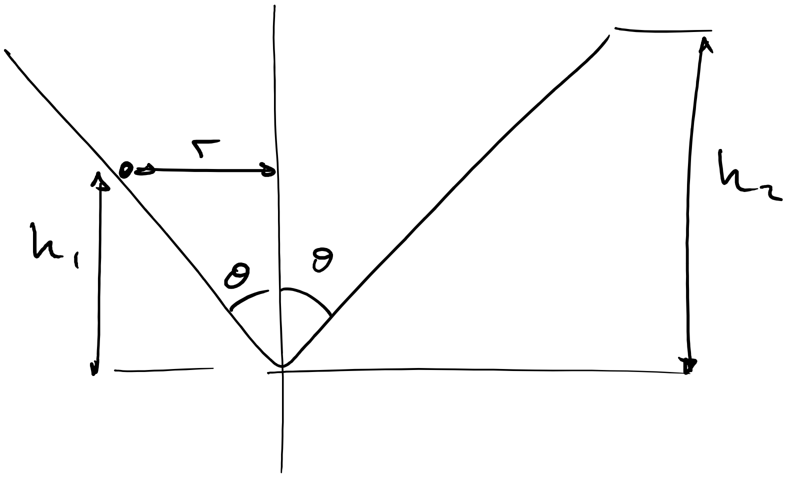



Figure 3




moment of momentum conserved about cone axis as reaction force intersects this always.

$$MoM \Rightarrow \quad \cancel{m} v_1 r_1 = \cancel{m} v_2 r_2 \qquad (\& \ v_2 \text{ horizontal})$$

$$r_2 = 2 r_1 \quad (\text{geometry}).$$

$$v_1 \cancel{r_1} = v_2 \cdot 2 \cancel{r_1} \qquad v_2 = v_1/2$$

$$\text{energy} \Rightarrow \quad \frac{1}{2}\cancel{m} v_1^2 = \frac{1}{2}\cancel{m}\left(\frac{v_1}{2}\right)^2 + \cancel{m} g (h_2 - h_1)$$

$$\frac{1}{2}\frac{3}{4} v_1^2 = g(h_2 - h_1), \qquad v_1 = \sqrt{\frac{8}{3} g} = 5.11\ ms^{-1}$$

[answer independent of cone angle]

Question 10

10 A satellite of mass 2000 kg is in elliptical orbit about the earth. At its perigee (point of closest approach) it has an altitude of 1100 km and a speed of 7900 $\mathrm{ms}^{-1}$. The earth's radius is 6400 km and $g$ at the earth's surface is $g = 9.81\,\mathrm{ms}^{-2}$.

(a) What is the energy of the satellite? How much work must be done on the satellite to put it into orbit?

<u>satellite.</u>

$m = 2000\text{kg}.$

$r = 1100\text{km}.$

$v = 7900\text{ms}^{-1}$

<u>earth</u>

$R = 6400\text{km}.$

$g = 9.81\text{ms}^{-2}$ @ surface.

$$\frac{GM\cancel{m}}{R^2} = \cancel{m}g$$

$$GM = gR^2 = 9.81 \times (6400)^2 = 401.8 \times 10^3$$

a) $E = T + V = \text{const}$

$$E_0 = \frac{1}{2}mv^2 - \frac{GMm}{R+r} = \text{const.} = -4.47 \times 10^{10}\ \text{J}$$

$$E_{surface} = -\frac{GMm}{R} = -1.26 \times 10^{11}\ \text{J}$$

$$WD = E_0 - E_s = 8.08 \times 10^{10}\ \text{J}$$

(b) If the 'burn' of the launching rocket is 10 minutes, at what rate (i.e. power) does the satellite gain energy?

$$P = \frac{8.08 \times 10^{10}}{10 \times 60} = 135\ \text{MW}$$

(c) What is the altitude and speed of the satellite at its apogee (the point on its orbit furthest from the earth)?

energy: $\frac{1}{2}mv_P^2 - \frac{GMm}{r_P} = \frac{1}{2}mv_A^2 - \frac{GMm}{r_A}$

MoM: $mv_P r_P = mv_A r_A \Rightarrow \frac{v_P}{v_A} = \frac{r_A}{r_P} \equiv \frac{1}{f}$

energy: $\frac{1}{2}m\left(v_P^2 - v_A^2\right) = GMm\left(\frac{1}{r_P} - \frac{1}{r_A}\right)$

$\frac{1}{2}mv_P^2\left(1 - \left(\frac{v_A}{v_P}\right)^2\right) = \frac{GMm}{r_P}\left(1 - \frac{r_P}{r_A}\right)$

$\frac{1}{2}mv_P^2\left(1 - f^2\right) = \frac{GMm}{r_P}\left(1 - f\right)$

[$v_P$ & $r_P$ known].

$1 - f^2 = (1-f)(1+f)$ so...

$\frac{1}{2}\cancel{m}v_P^2(1+f) = \frac{GM\cancel{m}}{r_P}$

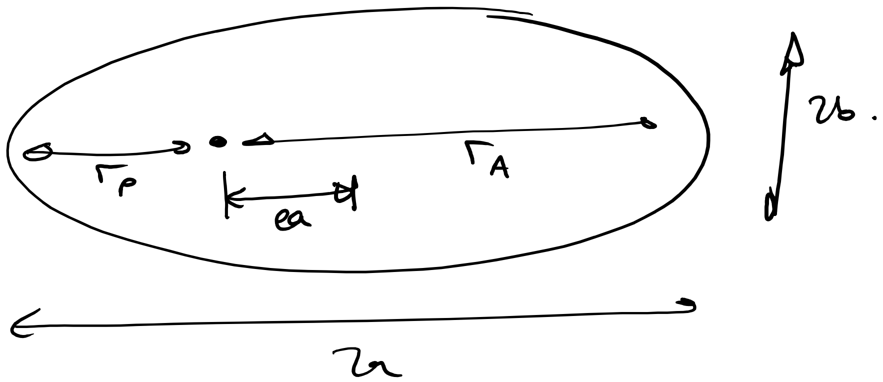
$f = \frac{2GM}{r_P v_P^2} - 1 = 0.717 = \frac{v_A}{v_P} = \frac{r_P}{r_A}$.

hence $v_A = f v_P = 5663 \text{ ms}^{-1}$ //

$r_A = r_P/f = 10.46 \times 10^3 \text{ km}$

$\text{altitude}_A = r_A - R = 4062 \text{ km}$ //

(d) What is the eccentricity of its orbit and the length of the minor axis of its elliptical path?




$$\frac{r_P + r_A}{2} = a.$$

$$ea = r_A - a$$

$$e = \frac{r_A}{a} - 1$$

$$r_A = a(1+e)$$

$$r_P = a(1-e)$$

So
$$\frac{r_P}{r_A} = \frac{1-e}{1+e}$$

$$(1+e)\, r_P = (1-e)\, r_A$$

$$e(r_P + r_A) = (r_A - r_P)$$

$$e = \frac{r_A - r_P}{r_A + r_P} = 0.165$$

$$\frac{b}{a} = \sqrt{1-e^2}, \quad b = a\sqrt{1-e^2} = \frac{r_P + r_A}{2}\sqrt{1-e^2}$$

$$\approx 8.9 \times 10^3 \text{ km}$$

$$2b = 17.7 \times 10^3 \text{ km}$$

Question 11

11 An artificial satellite of mass $m$ is transferred from one circular orbit at speed $v_P$ to another at speed $v_P/7$ by exerting a short-duration thrust (impulse $I_P$) tangentially at P, as illustrated in Figure 4 (not to scale).

(a) Using the equations of motion for the satellite in its two circular orbits, determine the ratio of the radii $r_2/r_1$ of the two orbits;

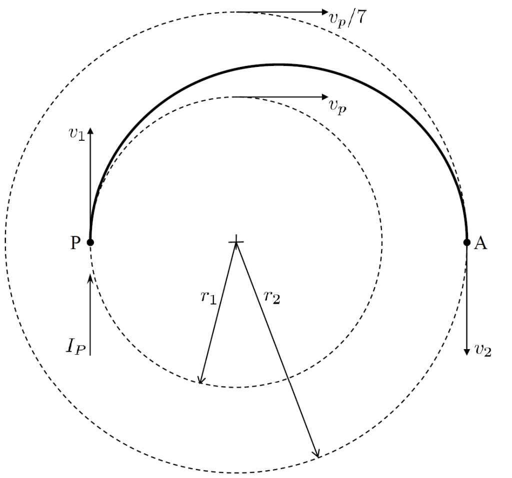



Figure 4

Eq. of motion: Force = mass × acceleration

Orbit P: $\frac{GMm}{r_1^{\cancel{2}}} = \frac{m v_P^2}{\cancel{r_1}}$ — (P)

Orbit A: $\frac{GMm}{r_2^{\cancel{2}}} = \frac{m v_A^2}{\cancel{r_2}}$ — (A)

$\frac{(P)}{(A)} \Rightarrow \frac{GMm/r_1}{GMm/r_2} = \frac{\cancel{m} v_P^2}{\cancel{m} v_A^2}$

so $\frac{r_2}{r_1} = \frac{v_P^2}{v_A^2} = \frac{1}{(1/7)^2} = 49$ ✓

(b) The impulse $I_P$ at P increases the speed of the satellite from $v_p$ to $v_1$. By considering the motion from P to A, determine the speed of arrival $v_2$ at A in terms of $v_p$;

Motion after $I_p$ : Perigee → Apogee

MoM $\quad m v_1 r_1 = m v_2 r_2 \quad \Rightarrow \quad \frac{v_1}{v_2} = \frac{r_2}{r_1} = 49.$

energy: $\quad \frac{1}{2}\cancel{m} v_1^2 - \frac{GM\cancel{m}}{r_1} = \frac{1}{2}\cancel{m} v_2^2 - \frac{GM\cancel{m}}{r_2}$

$$\frac{1}{2} v_1^2\left(1 - \left(\frac{v_2}{v_1}\right)^2\right) = \frac{GM}{r_1}\left(1 - \frac{r_1}{r_2}\right)$$

$$\frac{1}{2} v_1^2\left(1 + \tfrac{1}{49}\right) = \frac{GM}{r_1}\cancel{\left(1 - \tfrac{1}{49}\right)}$$

$$v_1 = \sqrt{\frac{2GM}{r_1} \Big/ \left(1 + \tfrac{1}{49}\right)} \quad , \quad v_p = \sqrt{\frac{GM}{r_1}}$$

So $\quad v_1 = v_p\sqrt{\frac{2}{50/49}} = 7\, v_p \cdot \sqrt{1/25}$

$$= \frac{7}{5} v_p$$

$$\frac{v_2}{v_1} = \frac{1}{49} \quad , \quad v_2 = \frac{7}{5} v_p \cdot \frac{1}{49} = v_p/35$$

(c) Determine the magnitude of the impulse $I_P$ in terms of $m$ and $v_p$.

$$I_p = \Delta mv = mv_1 - mv_p$$

$$= m\left(\frac{7}{5} - 1\right) v_p = 2mv_p/5$$

(d) Another short duration thrust (impulse $I_A$) is required at A to enter a circular orbit of radius $r_2$. Determine the magnitude and direction of this impulse in terms of $m$ and $v_p$.

$$I_A = \Delta mv = mv_A - mv_2 = \frac{m v_p}{7} - \frac{m v_p}{35}$$

$$= + 4mv_p/35 \quad \text{in direction } v_2$$

(e) What is the shape of the path traced from P to A? What happens to the satellite if no impulse is delivered at A?

- ellipse
- no impulse @ A would mean continued motion of ellipse with perigee P and apogee A.