Paper 1: Mechanical Engineering

Examples Paper 1

Elementary exercises are marked †, problems of Tripos standard ∗.
Answers can be found at the back of the paper.

Background

1 Draw a suitable Free Body Diagram for each of the following examples, thinking carefully about the direction of the forces and where they act:

(a) A car driving at constant speed up a hill of constant slope;

(b) A coffee cup on a seat table in an aeroplane during take-off;

(c) A cyclist travelling at constant speed around a corner, taking the cyclist and bike as one body (draw two diagrams, one from behind and one from the side);

(d) The oar of a rowing boat during a stroke;

(e) A rowing boat including the oars within the Free Body Diagram;

(f) A sailing yacht travelling at 90 degrees to the apparent wind, as illustrated in Figure 1. Note: the interaction of the wind and sails generates a force that has a *lift* component that is orthogonal to the wind, and a *drag* component that is inline with the wind. The boat itself experiences drag as it travels through the water, and there is a keel under the boat that acts in a similar way to wheels: producing very little resistance to motion longitudinally, but a large reaction force laterally.

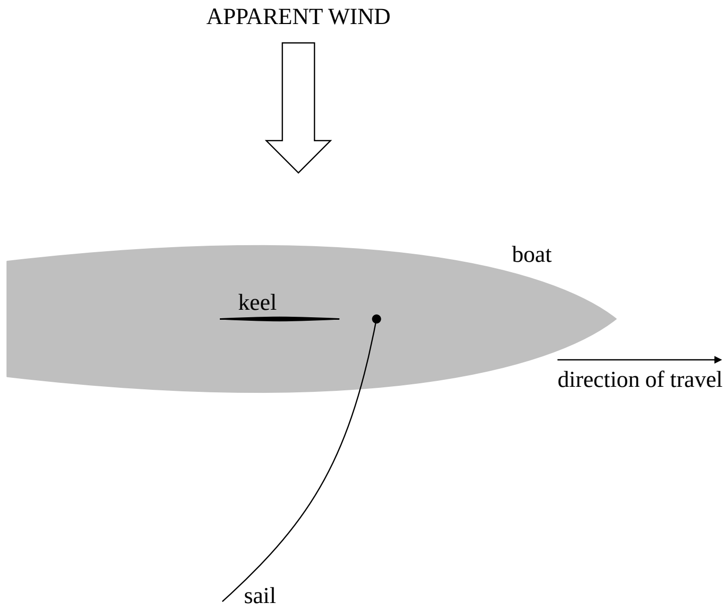

Figure 1

2 Which of the following can reasonably be approximated as inertial frames of reference:

(a) Reference frame fixed to a car during a 0-60 mph test?

(b) Reference frame fixed to a car travelling at 60 mph in a straight line on a smooth road?

(c) Reference frame fixed to a wind turbine blade rotating at constant speed with z-axis always aligned to one of the blades?

(d) Reference frame fixed to a wind turbine blade rotating at constant speed, with z-axis of reference frame always point vertically upwards?

3 What is the weight in Newtons of a person whose mass is 75 kg? If the person were to jump out of an aeroplane what would be their weight during free fall?

4 At what altitude h above the north pole is the weight of an object reduced to one half of its value on the earth's surface? Assume the earth is a sphere of radius R and express h as a fraction of R.

Kinematics of Particles

5 A particle P moves around a circle having a fixed centre C, radius R, and origin O on the circle's circumference as illustrated in Figure 2.

(a) Derive an expression for the Cartesian coordinates (x, y) of P in terms of R and ψ.

(b) Using Cartesian coordinates and associated unit vectors, find an expression for the position $\mathbf{r}$, velocity $\dot{\mathbf{r}}$ and acceleration $\ddot{\mathbf{r}}$ of the particle in terms of R, ψ and its derivatives $\dot{\psi}$ and $\ddot{\psi}$.

(c) If $\dot{\psi}$ is constant, what can you say about the speed of the particle? Show the direction of travel along the path if $\dot{\psi} > 0$.

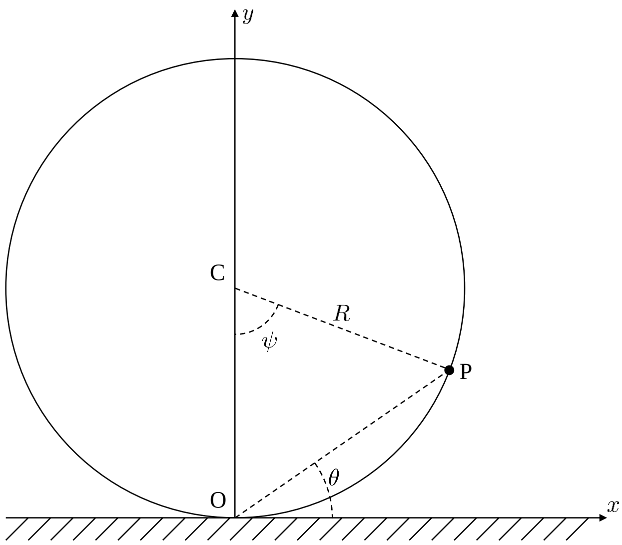

Figure 2

6 For the particle moving around the circle shown in Figure 2 find the vector expressions for the position $\mathbf{r}$, velocity $\dot{\mathbf{r}}$ and acceleration $\ddot{\mathbf{r}}$ in polar coordinates (r, θ) with unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$. Take the origin for polar coordinates at O so that r is the distance from O to P and θ is the anticlockwise angle between the x-axis and OP.

7 (a) For the particle moving around the circle shown in Figure 2 find the vector expressions for the velocity $\dot{\mathbf{r}}$ and acceleration $\ddot{\mathbf{r}}$ in intrinsic coordinates (s, ψ) with unit vectors $\mathbf{e}_t$ and $\mathbf{e}_n$.

(b) By considering the relationship between the three pairs of unit vectors ($\mathbf{e}_t$, $\mathbf{e}_n$), ($\mathbf{e}_r$, $\mathbf{e}_\theta$), and ($\mathbf{i}$, $\mathbf{j}$), express $\mathbf{e}_t$ and $\mathbf{e}_n$ in terms of:

(i) $\mathbf{i}$ and $\mathbf{j}$;

(ii) $\mathbf{e}_r$ and $\mathbf{e}_\theta$.

(c) Three coordinate systems have been used to describe the same particle's velocity and acceleration (Cartesian in Question 5, polar in Question 6 and intrinsic in this question). To demonstrate the equivalence, substitute the expressions for $\mathbf{e}_t$ obtained above into the expression for the velocity of the particle obtained in intrinsic coordinates above to confirm that you obtain the velocity expressions found in Cartesian coordinates in Question 5 and in polar coordinates in Question 6.

Note: Similar substitutions can be made into the expression for the acceleration above to demonstrate equivalence.

8 A taut string CP is unwrapped from a fixed drum, centre O and radius R, with a uniform angular velocity $\dot{\psi}$. The end of the string P is initially in contact with the drum at A, then traces out a planar curved path AB as shown in Figure 3.

(a) Find an expression for the position vector of P relative to O in terms of the intrinsic coordinate unit vectors $\mathbf{e}_t$ and $\mathbf{e}_n$ as shown (note that this is a rare case for which the position vector is readily found in intrinsic coordinates).

(b) By differentiation find the velocity $\dot{\mathbf{r}}$ and acceleration $\ddot{\mathbf{r}}$ of P in this coordinate system.

(c) From the acceleration verify that the radius of curvature of the path of P is equal to CP. Would you expect this?

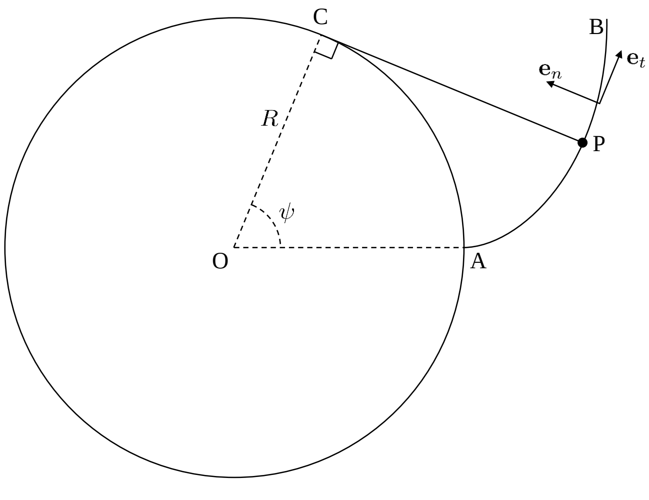

Figure 3

Note: This path is known as an 'involute' and is the most commonly used profile for gear teeth; you can learn why this is the case later in the course.

9 A tracking radar detects an aircraft due south at a range of 10 km. The radar points continually at the aircraft. It measures a range that is decreasing at a constant rate of 200 ms^{-1} while the radar is rotating clockwise (viewed from above) at a constant rate of 0.6 deg s^{-1}. Take θ to measure the angle clockwise from North.

(a) What is the course (direction) and speed of the aircraft?

(b) What is the component of the aircraft's acceleration along its path?

(c) What is the value of the instantaneous radius of curvature of the path of the aircraft?

(d) Use the Python template `p1q9_template.ipynb` to produce a plot of the path of the aircraft. It is mostly complete and only requires a few changes to make it work, and it can be run online without installing Python. See computing help at the end of this examples paper for more information

10 $*$ A crank OA is driven by a piston AB such that it rotates at a constant angular speed ω as shown in Figure 4. Point B is constrained to move horizontally.

(a) Find an expression for the position vector $\mathbf{r}_A$, $\mathbf{r}_{B/A}$, and $\mathbf{r}_B$ as a function of ω;

(b) Find the velocity of A and B;

(c) Write down an expression that would give a numerical approximation of the velocity of A and B using your analytical expression for the position vectors;

(d) Using the template Python file `p1q10_template.ipynb` numerically differentiate the position vector and plot the position and velocity of B over one complete revolution of the crank OA, and compare this with a plot of your analytic solution from (b);

(e) What factors affect the accuracy of your numerical approximation?

(f) For the ratios $a/b = 0.1, 0.5, 0.9$ identify the time in the cycle at which the maximum velocity occurs. What happens as $a \to b$?

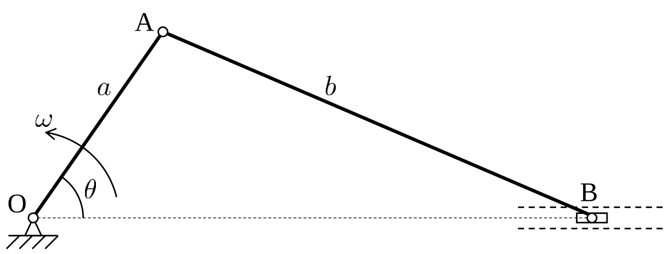

Figure 4

Computing Help

The Python examples are very easy to run online without any installation:

1. Go to: https://notebooks.azure.com/torebutlin/libraries/ia-mechanics
2. click on the relevant template file;
3. click on the 'clone' button (near top left);

4. if needed: log in to Azure using your Raven account;
5. agree to creating a clone when prompted;
6. click on the relevant template file again: this will start a working iPython Notebook that you can run and edit.

You can also run the files locally by installing Python. The most straightforward way is to download 'Anaconda' from: https://www.anaconda.com/download/. Once installed, then open the 'Jupyter Notebook' app from the start menu found inside the Anaconda folder. You can navigate to the folder where you are keeping your `*.ipynb` files and open the templates.

Suitable past Tripos questions

Background: This material is rarely examined in isolation, but forms the background to other questions.
Kinematics of Particles: IA 2017 Q7; IA 2016 Q9; IA 2015 Q10; IA 2014 Q10; IA 2013 Q10

Answers

1. For discussion with supervisors.

2. Only (b). Please discuss reasons with supervisors.

3. Weight = 736 N

4. $h/R = (\sqrt{2} - 1) \approx 0.41$

5(a). $x = R \sin \psi$, $y = R(1 - \cos \psi)$

5(b). $\mathbf{r}_P = R \sin \psi \mathbf{i} + R(1 - \cos \psi)\mathbf{j}$

5(b). $\dot{\mathbf{r}}_P = R\dot{\psi} \cos \psi \mathbf{i} + R\dot{\psi} \sin \psi \mathbf{j}$

5(b). $\ddot{\mathbf{r}}_P = (R\ddot{\psi} \cos \psi - R\dot{\psi}^2 \sin \psi)\mathbf{i} + (R\ddot{\psi} \sin \psi + R\dot{\psi}^2 \cos \psi)\mathbf{j}$

5(c). Speed is constant, $u = \dot{\psi} R$, anticlockwise

6. HINT: start by determining the directions of the unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$.

6. $\mathbf{r}_P = 2R \sin(\psi/2)\mathbf{e}_r$

6. $\dot{\mathbf{r}}_P = R\dot{\psi} \cos(\psi/2)\mathbf{e}_r + R\dot{\psi} \sin(\psi/2)\mathbf{e}_\theta$

6. $\ddot{\mathbf{r}}_P = [R\ddot{\psi} \cos(\psi/2) - R\dot{\psi}^2 \sin(\psi/2)]\mathbf{e}_r + [R\ddot{\psi} \sin(\psi/2) + R\dot{\psi}^2 \cos(\psi/2)]\mathbf{e}_\theta$

7(a). HINT: start by determining the directions of the unit vectors $\mathbf{e}_t$ and $\mathbf{e}_n$.

7(a). $\dot{\mathbf{r}}_P = R\dot{\psi}\mathbf{e}_t$

7(a). $\ddot{\mathbf{r}}_P = R\ddot{\psi}\mathbf{e}_t + R\dot{\psi}^2\mathbf{e}_n$

7(b)(i). $\mathbf{e}_t = \cos \psi \mathbf{i} + \sin \psi \mathbf{j}$, $\mathbf{e}_n = -\sin \psi \mathbf{i} + \cos \psi \mathbf{j}$

7(b)(ii). $\mathbf{e}_t = \cos(\psi/2)\mathbf{e}_r + \sin(\psi/2)\mathbf{e}_\theta$, $\mathbf{e}_n = -\sin(\psi/2)\mathbf{e}_r + \cos(\psi/2)\mathbf{e}_\theta$

8(a). $\mathbf{r}_P = R\mathbf{e}_t - R\psi\mathbf{e}_n$

8(b). HINT: determine the rate of rotation of the unit vectors $\mathbf{e}_t$ and $\mathbf{e}_n$

8(b). $\dot{\mathbf{r}}_P = R\psi\dot{\psi}\mathbf{e}_t$

8(b). $\mathbf{r}_P = R\dot{\psi}^2\mathbf{e}_t + R\psi\dot{\psi}^2\mathbf{e}_n$

9(a). Speed $v = 225.76\,\mathrm{ms}^{-1}$, heading North West $\theta = 332.5$ degrees (measured clockwise from North)

9(b). $\ddot{s} = -0.97\,\mathrm{ms}^{-2}$

9(c). $\rho = 12.1\,\mathrm{km}$

10(a). $\mathbf{r}_A = a\cos(\omega t)\mathbf{i} + a\sin(\omega t)\mathbf{j}$

10(a). $\mathbf{r}_{B/A} = \sqrt{b^2 - a^2\sin^2(\omega t)}\mathbf{i} - a\sin(\omega t)\mathbf{j}$

10(a). $\mathbf{r}_B = \left[a\cos(\omega t) + \sqrt{b^2 - a^2\sin^2(\omega t)}\right]\mathbf{i}$

10(b). $\dot{\mathbf{r}}_A = -a\omega\sin(\omega t)\mathbf{i} + a\omega\cos(\omega t)\mathbf{i}$

10(b). $\dot{\mathbf{r}}_B = \left[-a\omega\sin(\omega t) - \frac{a^2\omega\sin(\omega t)\cos(\omega t)}{\sqrt{b^2 - a^2\sin^2(\omega t)}}\right]\mathbf{i}$

10(c). $\hat{\mathbf{v}}_A(kT) \approx \frac{\mathbf{r}_A(kT) - \mathbf{r}_A([k-1]T)}{T}$, $\hat{\mathbf{v}}_B(kT) \approx \frac{\mathbf{r}_B(kT) - \mathbf{r}_B([k-1]T)}{T}$

10(f). Maximum velocity occurs at $t = 0.23, 0.19, 0.18$ for $a/b = 0.1, 0.5, 0.9$ respectively.