Part IA Mathematics Solutions To Examples 3

Q1a) i)
$$\cos 2\theta = \Re(e^{2i\theta}) = \Re((\cos\theta + i\sin\theta)^2)$$
$$= \Re(\cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta)$$
$$= \cos^2\theta - \sin^2\theta \quad = \quad 2\cos^2\theta - 1$$

ii)
$$\sin 3\theta = \Im(e^{3i\theta}) = \Im((\cos\theta + i\sin\theta)^3)$$
$$= \Im(\cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta)$$
$$= 3\cos^2\theta\sin\theta - \sin^3\theta$$

iii)
$$\sin 4\theta = \Im(e^{4i\theta}) = \Im((\cos\theta + i\sin\theta)^4)$$
$$= 4\cos\theta\sin^3\theta - 4\cos^3\theta\sin\theta$$

b)
$$\sin(a+b) = \sin a\cos b + \cos a\sin b$$
$$\sin(i\alpha + i\beta) = \sin(i\alpha)\cos(i\beta) + \cos(i\alpha)\sin(i\beta) \qquad a = i\alpha \text{ etc.}$$
$$i\,\mathrm{sh}(\alpha+\beta) = i\,\mathrm{sh}\alpha\,\mathrm{ch}\beta + i\,\mathrm{ch}\alpha\,\mathrm{sh}\beta \qquad \sin i\alpha = i\,\mathrm{sh}\alpha \text{ etc.}$$
$$\sinh(\alpha+\beta) = \sinh\alpha\cosh\beta + \cosh\alpha\sinh\beta$$

Q2a)
$$(i)^6 = (e^{i\pi/2})^6 = e^{i3\pi} = -1$$
$$(i)^{-5} = (e^{i\pi/2})^{-5} = e^{-i5\pi/2} = e^{-i\pi/2} = -i$$
$$(3+4i)i - (5-2i)i^2 - (6+i) = 3i - 4 + 5 - 2i - 6 - i = -5$$
$$\frac{(3+2i)(2+i)}{(1-2i)(4+i)} = \frac{4+7i}{6-7i} = \frac{(4+7i)(6+7i)}{(6-7i)(6+7i)} = \frac{-25+70i}{36+49} = 0.294 + 0.824i$$
$$3e^{i\pi/3}\,2e^{i2\pi/3} = 6e^{i\pi} = -6$$
$$2e^{i\pi/3} + 2e^{i2\pi/3} = 4e^{i\pi/2}\left(\frac{e^{i\pi/6} + e^{-i\pi/6}}{2}\right) = 4i\cos\frac{\pi}{6} = 2\sqrt{3}\,i$$

b)
$$\tan\left(\tfrac{\pi}{6} + i\tfrac{\pi}{4}\right) = \frac{\tan(\frac{\pi}{6}) + \tan(i\frac{\pi}{4})}{1 - \tan(\frac{\pi}{6})\tan(i\pi/4)} = \frac{\tan(\frac{\pi}{6}) + i\,\mathrm{th}(\pi/4)}{1 - i\tan(\pi/6)\,\mathrm{th}(\pi/4)}$$
$$= \frac{0.5774 + 0.6558i}{1 - i(0.5774)(0.6558)} = 0.288 + 0.765i$$

$$\ln\left(\frac{3-i}{3+i}\right) = \ln\left|\frac{3-i}{3+i}\right| + i\,\mathrm{Arg}\left(\frac{3-i}{3+i}\right) = \ln\frac{|3-i|}{|3+i|} + i\,\mathrm{Arg}(3-i) - i\,\mathrm{Arg}(3+i)$$
$$= \ln\left(\frac{1}{1}\right) - 2i\,\mathrm{Arg}(3+i) = 0 - 2i\tan^{-1}(1/3) = -0.644i\ (+2n\pi i)$$

Q2b) cont. $x + iy = \cos^{-1}\left(\frac{3i}{4}\right) \Rightarrow \frac{3i}{4} = \cos(x+iy) = \cos x \,\mathrm{ch}\, y - i \sin x \,\mathrm{sh}\, y$

$$\Rightarrow \left.\begin{array}{l} \cos x \,\mathrm{ch}\, y = 0 \\ \sin x \,\mathrm{sh}\, y = -3/4 \end{array}\right\} \quad \mathrm{ch}\, y \neq 0 \Rightarrow x = \pm\frac{\pi}{2} + 2n\pi$$

$$\Rightarrow \mathrm{sh}\, y = \mp 3/4 \quad \Rightarrow y = \mp 0.693$$

$$\Rightarrow \cos^{-1}\left(\frac{3i}{4}\right) = \pm\left(\frac{\pi}{2} - 0.693i\right) + 2n\pi$$

$$61.5 + 113.7i = 129.267\, e^{i(1.07497)}$$

$$\Rightarrow (61.5 + 113.7i)^{1/3} = (129.267)^{1/3} e^{i(1.07497/3 + 2n\pi/3)}$$

$$= 5.05626\, e^{i(0.35832 + 2n\pi/3)}$$

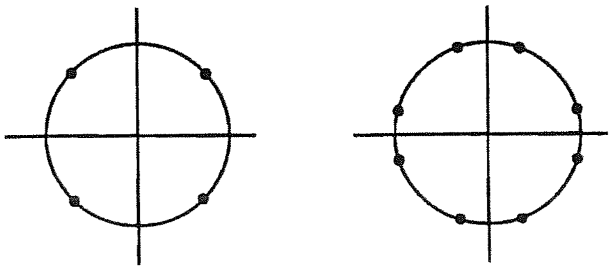
$$= 5.05626 \exp(i\, 0.35832) = 4.7351 + 1.7732i$$

$$\text{or } 5.05626 \exp(i\, 2.45272) = -3.9032 + 3.2141i$$

$$\text{or } 5.05626 \exp(i\, 4.54711) = -0.8319 - 4.9874i$$

Q3(a) $z^4 = -1 = e^{i\pi(+2n\pi i)} \Rightarrow z = e^{i\pi/4 + in\frac{\pi}{2}}$

(b) $z^4 = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{3}i}{2} = e^{\pm i\pi/3 (+2n\pi i)} \Rightarrow z = e^{\pm i\pi/12 + n\pi i/2}$



(a) (b)

Q 4. It is shown in lectures that all complex equations should be still true if you change the sign of i.

Thus $\frac{z-i}{z+i} = 6+4i \Rightarrow \frac{\bar{z}+i}{\bar{z}-i} = 6-4i$

or Taking complex conjugate

$$6+4i = \frac{z-i}{z+i} \Rightarrow 6-4i = \overline{\left(\frac{z-i}{z+i}\right)} = \frac{\overline{z-i}}{\overline{z+i}} = \frac{\bar{z}+i}{\bar{z}-i}$$

Then $\frac{\bar{z}-i}{\bar{z}+i} = \frac{1}{6-4i} = \frac{6+4i}{36+16} = \frac{3}{26} + \frac{2}{26}i$

Now $\bar{z} = \dfrac{-8+51i}{41} \Rightarrow \dfrac{\bar{z}-i}{\bar{z}+i} = \dfrac{-8+51i-41i}{-8+51i+41i} = \dfrac{-8+10i}{-8+92i} = \dfrac{-4+5i}{-4+46i}$

$= \dfrac{-4+5i}{-4+46i} \cdot \dfrac{-4-46i}{-4-46i} = \dfrac{246+164i}{2132} = \underline{\dfrac{3+2i}{26}}$

Q 5. (a) $|z-i|^2 = |z-2|^2 \Rightarrow x^2+(y-1)^2 = (x-2)^2+y^2$

$\Rightarrow x^2+y^2-2y+1 = x^2-4x+4+y^2 \Rightarrow \underline{y = 2x - 3/2}$

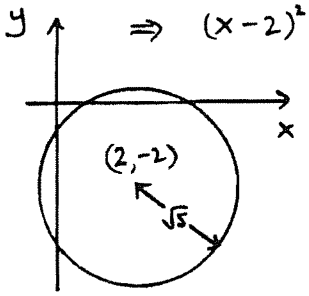
(b) $|z+2|^2 = 4|z-1+3/2\, i|^2 \Rightarrow (x+2)^2+y^2 = 4\left[(x-1)^2+(y+3/2)^2\right]$

$\Rightarrow x^2+4x+4+y^2 = 4\left[x^2-2x+1+y^2+3y+9/4\right]$

$\Rightarrow 3x^2-12x+3y^2+12y+9 = 0$

$\Rightarrow x^2-4x+y^2+4y+3 = 0$

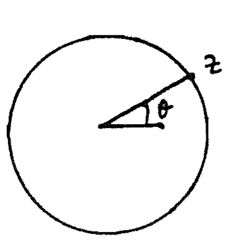
$\Rightarrow (x-2)^2-4+(y+2)^2-4+3 = 0 \Rightarrow \underline{(x-2)^2+(y+2)^2 = 5}$



This is a circle centre $(2,-2)$ radius $\sqrt{5}$

On the circle all points are $\sqrt{5}$ from centre

i.e. $|z-2+2i| = \sqrt{5}$



The point z is $2-2i+\sqrt{5}\,e^{i\theta}$

and as θ varies $0 \le \theta \le 2\pi$ z traces out the circle.

Q 6. $\overline{f(z)} = \overline{a_0+a_1z+a_2z^2+\dots} = \overline{a_0}+\overline{a_1z}+\overline{a_2z^2}+\dots$ $(\overline{z_1+z_2} = \bar{z}_1+\bar{z}_2)$

$= \bar{a}_0+\bar{a}_1\bar{z}+\bar{a}_2\bar{z}^2+\dots$ since $\overline{z_1z_2} = \bar{z}_1\bar{z}_2$

and a_n's real $\Rightarrow \underline{\overline{f(z)}} = a_0+a_1\bar{z}+a_2\bar{z}^2+\dots = f(\bar{z})$

(a) $e^z = 1+z+z^2/2!+\dots$ real coefficients $\Rightarrow \overline{e^z} = e^{\bar{z}}$

(b) $e^{iz} = 1+iz-z^2/2!+\dots$ complex " $\Rightarrow \overline{e^{iz}} \ne e^{i\bar{z}}$ (infact $\overline{e^{iz}} = e^{-i\bar{z}}$)

(c) $e^{(i+1)z} = 1+(i+1)z\dots$ " " $\Rightarrow \overline{e^{(i+1)z}} \ne e^{(i+1)\bar{z}}$ (" " $= e^{(-i+1)\bar{z}}$)

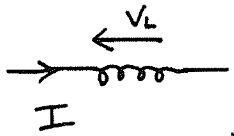
(d) $\sin z = z - z^3/3! + \dots$ real " $\Rightarrow \overline{\sin z} = \sin\bar{z}$

Q 7. Denoting complex voltages and currents by $\tilde{V}, \tilde{I}$ etc

i.e. $I = \mathrm{Re}\,\tilde{I}$, $V = \mathrm{Re}\,\tilde{V}$

and for sinusoidal voltages and currents $\tilde{I} = I_0 e^{i\omega t}$ etc

For an inductor



$$V_L = L\frac{dI}{dt} \rightarrow \tilde{V}_L = L\frac{d\tilde{I}}{dt} = Li\omega\tilde{I}$$

$$\therefore \text{complex impedance} = \frac{\tilde{V}_L}{\tilde{I}} = \underline{i\omega L}$$

For resistor $\tilde{V}_R = R\tilde{I} \Rightarrow$ impedance $= \underline{R}$

For the circuit

$$\tilde{V}_{in} = \tilde{I}(i\omega L + R) \quad \text{and} \quad \tilde{V}_R = \tilde{I}R$$

$$\text{dividing} \Rightarrow \tilde{V} = \tilde{V}_{in}\frac{R}{R+i\omega L} = \frac{R}{\sqrt{R^2+\omega^2L^2}}e^{-i\varphi}\tilde{V}_{in}$$

$$\text{where} \quad \cos\varphi = \frac{R}{\sqrt{R^2+\omega^2L^2}} \qquad \sin\varphi = \frac{\omega L}{\sqrt{R^2+\omega^2L^2}}$$

$\therefore$ Ratio of peak value of V_R to that of V_{in} $= \underline{\dfrac{R}{\sqrt{R^2+\omega^2L^2}}}$

Phase difference $= \underline{-\tan^{-1}\dfrac{\omega L}{R}}$

Q 8 a) Put $x = a\sinh\theta$, so $dx = a\cosh\theta\,d\theta$

$$\text{Then} \int\frac{dx}{\sqrt{x^2+a^2}} = \int\frac{a\cosh\theta}{a\cosh\theta}d\theta = \int d\theta = \theta + c' = \underline{\sinh^{-1}\frac{x}{a} + c'}$$

Alternative form: since $\dfrac{x}{a} = \sinh\theta = \dfrac{e^{\theta}-e^{-\theta}}{2} \Rightarrow \sqrt{\dfrac{x^2+a^2}{a^2}} = \cosh\theta = \dfrac{e^{\theta}+e^{-\theta}}{2}$

we have $e^{\theta} = \cosh\theta + \sinh\theta = \dfrac{x}{a} + \dfrac{\sqrt{x^2+a^2}}{a}$

So $\theta = \ln\left[x + \sqrt{x^2+a^2}\right] + \text{const}$ & integral $= \underline{\ln(x+\sqrt{x^2+a^2}) + c}$

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b) Put $x = a\tan\theta$, $dx = a\sec^2\theta\,d\theta = a(1+\tan^2\theta)\,d\theta = a(1+x^2/a^2)\,d\theta = (x^2+a^2)\,d\theta/a$

$$\therefore \int\frac{dx}{x^2+a^2} = \int\frac{d\theta}{a} = \frac{\theta}{a} + c = \underline{\frac{1}{a}\tan^{-1}\frac{x}{a} + c}$$

(c) $\displaystyle\int \frac{dx}{x^2+a^2} = \frac{1}{2}\int \frac{\frac{d}{dx}(x^2+a^2)\,dx}{x^2+a^2} = \underline{\frac{1}{2}\ln(x^2+a^2)+c}$

(d) Similarly $\displaystyle\int_{\pi/4}^{\pi/2} \frac{\cos 2t\,dt}{1+\sin 2t} = \frac{1}{2}\Big[\ln(1+\sin 2t)\Big]_{\pi/4}^{\pi/2} = \frac{1}{2}\Big[\ln 1 - \ln 2\Big]$

$$= \underline{-\frac{1}{2}\ln 2}$$

Q 9. (a) $\displaystyle(1-x^2)\frac{dy}{dx} + \cot y = 0 \Rightarrow \int \tan y\,dy = \int \frac{dx}{x^2-1}$

i.e. $\displaystyle\int \frac{\sin y}{\cos y}\,dy = -\ln(\cos y) = \frac{1}{2}\left(\int\frac{dx}{x-1} - \int\frac{dx}{x+1}\right) = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + A$

$\displaystyle\Rightarrow \cos y = c\sqrt{\left|\frac{x-1}{x+1}\right|}$ or $\underline{y = \cos^{-1} c\sqrt{\left|\frac{x-1}{x+1}\right|}}$.

(b)

$\displaystyle\sinh y\frac{dy}{dx} + \cosh^2 y\cos^2 x = 0 \Rightarrow -\int\frac{\sinh y}{\cosh^2 y}\,dy = \int \cos^2 x\,dx$

$\displaystyle\Rightarrow \frac{1}{\cosh y} = \int \frac{\cos 2x + 1}{2}\,dx = \frac{\sin 2x}{4} + \frac{x}{2} + c \Rightarrow \underline{y = \cosh^{-1}\frac{4}{\sin 2x + 2x + c}}$

(c) $\displaystyle\frac{dy}{dx} + \frac{2}{x}y + \frac{1}{1+x^3} = 0$. Integrating factor $e^{\int\frac{2}{x}dx} = e^{2\ln x} = x^2$

Multiplying by this $\displaystyle\Rightarrow x^2\frac{dy}{dx} + 2xy = \frac{d}{dx}(x^2y) = -\frac{x^2}{1+x^3}$

Integrating gives $\displaystyle x^2y = -\frac{1}{3}\ln(1+x^3) + A$ i.e. $\displaystyle\underline{y = \frac{c - \ln(1+x^3)}{3x^2}}$

(d) $\displaystyle\frac{dy}{dx} + y\cot x + \cos^4 x = 0$. Integrating factor $e^{\int\cot x\,dx} = e^{\int\frac{\cos x}{\sin x}dx}$

$$= e^{\ln\sin x} = \sin x$$

Multiplying by int. factor

$\displaystyle\Rightarrow \sin x\frac{dy}{dx} + \cos x\, y = \frac{d}{dx}(y\sin x) = -\cos^4 x\sin x$

Integrating $\displaystyle y\sin x = \frac{\cos^5 x}{5} + A$ or $\displaystyle\underline{y = \frac{\cos^5 x + c}{5\sin x}}$

TPH/SJS

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