Engineering Tripos **FIRST YEAR**

Part IA Paper 2: MATERIALS

Examples Paper 1 – TEACH YOURSELF PROPERTIES AND MICROSTRUCTURE

Solutions to Questions: Part A – Mechanical Properties

A1 Let the cross section area of the chimney be A and its height be h.
The weight of the chimney is $W = \rho A h g$, where g is the acceleration due to gravity.
The axial compressive stress at the base is: $\sigma = -W/A = -\rho g h$ (note that the area A has cancelled out – the stress is independent of the shape or size of the cross-section). This is the same formula as for the hydrostatic pressure at depth h in a fluid of density ρ, but note that stress also has direction, while pressure is a scalar quantity (acting equally in all directions).
For $g = 9.81$ m/s^2, $\sigma = -1800 \times 9.81 \times 50 = -9 \times 10^5$ N/m^2 = – 0.88 MPa.

A2 The stress $\sigma = F/A = (250 \times 9.81) / 80 =$ 30.7 MPa (i.e. N/mm^2). The strain in the steel cable will be $\varepsilon =$ 30.7/200,000 = 0.00030, hence it will stretch by $\delta = 0.0003 \times 3{,}000$ mm = 0.46 mm. The stretch is inversely proportional to modulus, so the PP cable will stretch by $0.46 \times (200/1.2) = 77$ mm.

A3 Reading (approximately) from Fig. 2.1 in the Materials Databook, the yield properties are:

Alloy	Yield stress (MPa)	Tensile strength (MPa)	Ductility
Annealed copper	60	250	0.35 (35%)
Drawn 70/30 brass	280	400	0.24 (24%)
Mild steel	350	540	0.57 (57%)

(Remember to allow for elastic unloading after failure, when estimating the ductility)

In general, ductility falls as tensile strength rises, within each alloy class (copper alloys, steels etc).

A4 The applied stress is 30.7 MPa (Question A2). The safety factors between operation and yield are: 300/30.7 = 9.8 (for steel); 32/30.7 = 1.04 (for PP). The polypropylene is close to failure; the steel is much stronger than it needs to be for this load.

A5 (a) In tension: stress $\sigma = F/A_o = \sigma_y$. Hence $F = \sigma_y \times A_o = 300 \times 2 \times 25 =$ 15,000 N = 15 kN.

The strain at first yield $\varepsilon_y = \sigma_y/E = 300/210{,}000 = 0.00143$

Hence the extension $\delta = \varepsilon_y L_o =$ 0.00143 × 250mm = 0.36mm.

(b) In bending: $\sigma_{max} = \sigma_y = 300 = \frac{6 \times F \times 250}{25 \times 2^2}$. Hence $F = 20$ N.

The deflection is then: $\delta = \frac{4 \times 20 \times 250^3}{210{,}000 \times 25 \times 2^3} = 30\text{mm}$

The stiffness F/δ in tension is 15,000/0.36 ≈ 41,700 N/mm; that in bending is 20/30 = 0.67 N/mm, i.e. over 60,000 times smaller. Failure in bending depends on the applied moment, which varies with the loading configuration – but it is clear that the forces involved to reach failure in bending are also smaller by several orders of magnitude.

A6 (a) Stress: $\frac{\sigma}{\varepsilon} = E$; $\sigma = \frac{F}{A}$; $\varepsilon = \frac{\delta}{L}$. So: $\frac{F/A}{\delta/L} = E$; giving $\delta = \frac{FL}{AE}$

(b) $A = \pi R^2 = \frac{F L}{\delta E}$; giving $R = \sqrt{\frac{F L}{\pi \delta E}}$ (1)

(c) Use values in the table below with $F = 1000$ N, $\delta = 0.1$mm and $L = 0.3$ m; e.g. for PEEK, this gives:
$R = \sqrt{\frac{1000 \times 0.3}{\pi \left(0.1 \times 10^{-3}\right)\left(3.8 \times 10^{9}\right)}} = 15.9$ mm. Similarly for the other 3 materials – values in table below.

(d) $m = \rho A L = \rho \pi R^2 L$. Hence for PEEK, $m = 1300 \pi \left(15.9 \times 10^{-3}\right)^2 (0.3) = 0.31$ kg (for other materials – see table below).

(e) $m = \rho \pi R^2 L$. Using eqn. (1) this gives $m = \rho \pi \left[\frac{F L}{\pi \delta E} \right] L = \frac{F L^2}{\delta} \left(\frac{\rho}{E} \right)$. F, L and δ are fixed values, whichever material is used. Hence the material index to be minimized is ρ/E.

(f) The material index (ρ/E) is proportional to the mass of the tie, and consequently the index ranks the materials in the same order, e.g. compare copper and titanium in the table: $m_{Cu}/m_{Ti} = \left(\frac{\rho}{E} \right)_{Cu} \Big/ \left(\frac{\rho}{E} \right)_{Ti} \approx 1.8$ etc.

Material	Young's Modulus E (GPa)	Density ρ (kg/m^3)	R (mm)	m (kg)	ρ/E
PEEK	3.8	1300	15.9	0.31	342
Butyl rubber	0.0015	2400	798	1440	1.6×10^6
Titanium alloy	110	4600	2.9	0.038	41.8
Copper alloy	120	8900	2.8	0.067	74.2

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Lent Term 2019

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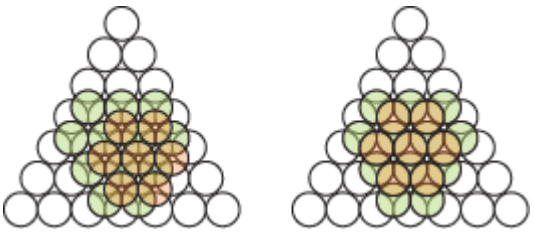
Part IA Paper 2: MATERIALS

Examples Paper 1 – TEACH YOURSELF PROPERTIES AND MICROSTRUCTURE

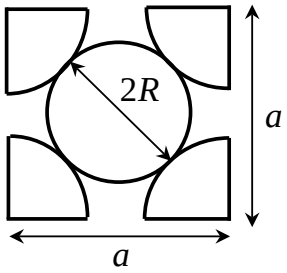
Solutions to Questions: Part B - Microstructure

B1.



FCC (ABCABC) and CPH (ABAB) stacking of close-packed layers.

B2. FCC (one face – atoms shown full size)



Let the lattice constant = a, and the atomic radius = R.

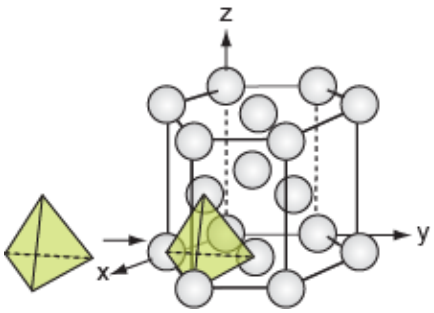
From the Figure: $a^2 + a^2 = (4R)^2$

Hence $a = 2\sqrt{2}\,R$.

B3. (a) CPH

The figure shows a regular tetrahedron between 4 atom centres: 3 atoms in the base-plane of the unit cell, and the atom from the middle layer that sits on top of them.

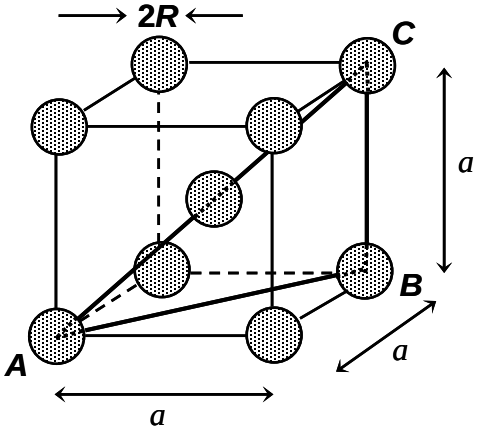
The height of the unit cell c is twice the height of the tetrahedron, and the side-length of the hexagonal base, $a = 2R$.



It is readily shown using Pythagoras' theorem that the height of a regular tetrahedron of side-length a is $a\sqrt{2}/\sqrt{3}$. Hence $c{:}a = 2\sqrt{2}/\sqrt{3} = 1.633$.

(b) For Mg, $R = 0.1605$nm, hence $a = 0.321$nm, $c = 0.524$nm.

B4. BCC



Atom radius = R (shown reduced in size for clarity). Central atom and atoms at A and C are in contact, so unit cell diagonal has length $4R$.

$$(AB)^2 = 2a^2$$

$$(AC)^2 = (AB)^2 + (BC)^2 = 2a^2 + a^2 = (4R)^2$$

Hence $a = 4R/\sqrt{3}$

B5. (a) CPH, lattice constants a and c

Base area of hexagonal unit cell: $a^2\,3\sqrt{3}/2$; height $c = a\,2\sqrt{2}/\sqrt{3}$ (from Question B3).

Volume of unit cell = base × height = $a^2\,3\sqrt{3}/2 \times c = a^2\,3\sqrt{3}/2 \times a\,2\sqrt{2}/\sqrt{3} = a^3\,3\sqrt{2} = R^3\,24\sqrt{2}$

Corner atoms shared between 6 cells, hexagonal face atoms between 2, plus 3 complete internal atoms.

Hence number of atoms per unit cell: $12 \times 1/6 + 2 \times 1/2 + 3 \times 1 = 6$

(corners) (faces) (internal)

Atomic packing fraction = $\dfrac{6 \times \frac{4}{3}\pi R^3}{R^3\,24\sqrt{2}} = 0.74$ (74%), i.e. same as FCC (both close-packed).

(b) BCC, lattice constant a

From Question B4, $a = 4R/\sqrt{3}$

Volume of unit cell = $\left(\dfrac{4R}{\sqrt{3}}\right)^3 = \dfrac{64R^3}{3\sqrt{3}}$

Number of atoms per unit cell = $8 \times 1/8 + 1 \times 1 = 2$

(corners) (internal)

Hence atomic packing fraction = $\dfrac{2 \times \frac{4}{3}\pi R^3}{64R^3/3\sqrt{3}} = 0.68$ (74%)

BCC is not close-packed: compared with the packing fraction in close-packed FCC and CPH (= 0.74), close packing is 8.8% more dense than BCC.

B6. Theoretical density $\rho = \dfrac{nA}{V_c N_A}$; data for lattice constant a and relative atomic weight A from Materials Databook.

(a) For BCC: $n = 2$, $V_c = a^3$. For Fe, $a = 2.8663$ Å, $A = 55.847$.

$$\rho = \frac{2 \times 55.847}{\left(2.8663\times10^{-10}\right)^3 \times 6.02\times10^{23}} = 7.878\ \text{Mg m}^{-3}$$

(b) For FCC: $n = 4$, $V_c = a^3$. For Au, $a = 4.0786$ Å, $A = 196.967$.

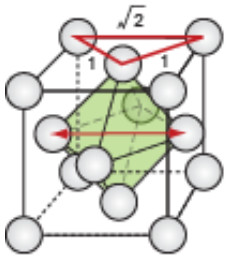
$$\rho = \frac{4 \times 196.967}{\left(4.0786\times10^{-10}\right)^3 \times 6.02\times10^{23}} = 19.29\ \text{Mg m}^{-3}$$

For CPH: $n = 6$, $V_c = a^3\,3\sqrt{2}$. For Mg, $a = 3.2094$ Å, $A = 24.312$.

$$\rho = \frac{2 \times 24.312}{3\sqrt{2} \times \left(3.2094\times10^{-10}\right)^3 \times 6.02\times10^{23}} = 1.728\ \text{Mg m}^{-3}$$

(c) In all three cases, the values agree very well with the measured densities (better than 1%). This supports the validity of the hard sphere approximation. Crystal defects such as grain boundaries make a very small difference, and the distortion of the outer electron shells is not taken into account in the hard sphere model.

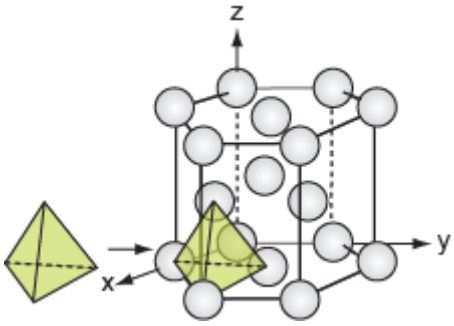
B7. The Figure shows how to calculate the octahedral hole size in the FCC structure. The face diagonals are close-packed directions, so the atom spacing along the diagonal is 1 unit. The cell edge thus has length $\sqrt{2} = 1.414$ units. That is also the separation of the centers of the atoms at opposite corners of the octahedral hole. The atoms occupy 1 unit of this, leaving a hole that will just contain a sphere of diameter 0.414 units without distortion.



The calculation of the octahedral interstitial hole size in the FCC lattice

B8. The figure shows a tetrahedral hole of the CPH structure.

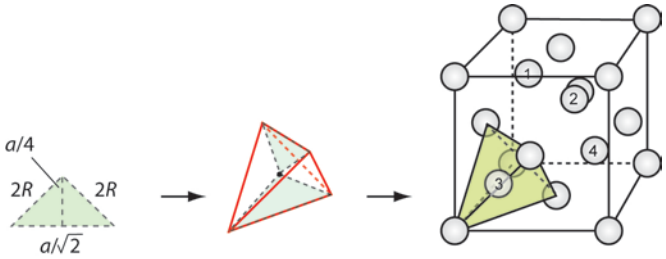
(The geometry of this tetrahedron was considered in evaluating the height of the unit cell).



A tetrahedral hole of the CPH structure.

B9. In diamond cubic there are 8 atoms per unit cell: 8×1/8 (corners) + 6×1/2 (faces) + 4 (internal).

Referring to the DC unit cell in the figure, atom number 3 sits in the middle of a regular tetrahedron. The atom spacing along the horizontal edges of the tetrahedron is equal to half the length of the diagonal of one face of the cube, i.e. $a/\sqrt{2}$, where a is the lattice constant. The shaded triangles, height $a/4$, link the centre of the internal atom number 3 to the face/corner atoms that it touches.



The geometry of the tetrahedral packing in DC.

From Pythagoras' theorem: $(2R)^2 = (a/4)^2 + (a/2\sqrt{2})^2$, from which $R = \sqrt{3}\,a/8$, or $a = 8R/\sqrt{3} = 4D/\sqrt{3} = 2.309D$. Hence the atomic packing fraction $= \dfrac{8 \times \frac{4}{3}\pi R^3}{8^3 R^3 / 3\sqrt{3}} = 0.34$ (34%). This is remarkably low compared to close-packing (74%).

B10. For DC silicon carbide, there are 4 Si atoms and 4 C atoms in the unit cell. The mass of the unit cell is therefore $[(4 \times 28.09) + (4 \times 12.01)] / 6.022 \times 10^{26} = 26.636 \times 10^{-26}$ kg. The volume of the SiC unit cell $= (0.436 \times 10^{-9})^3 = 8.288 \times 10^{-29}$ m^3.
Hence the theoretical density $= 26.636 \times 10^{-26}$ kg $/ 8.288 \times 10^{-29}$ m^3 ≈ 3.21 Mg/m^3.

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