Part IA Mathematics Examples paper 6 Solutions

i) $\begin{vmatrix} \lambda-1 & -2 \\ -2 & \lambda+2 \end{vmatrix} = (\lambda-1)(\lambda+2) - 4 = 0 \Rightarrow \lambda^2+\lambda-6=0$

$\Rightarrow \underline{\lambda = 2 \text{ or } -3}$

$\lambda = 2$: e-vector given by $\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ ie. $x_1 + 2x_2 = 2x_1$

$\Rightarrow x_1 = 2x_2$. Take $x_1 = 2$, $x_2 = 1$ or in normalised form

$$\underline{\underline{x} = \left[\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right]^t}$$

$\lambda = -3$: $\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -3\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow x_1 + 2x_2 = -3x_1$

$\Rightarrow x_2 = -2x_1$

$\therefore$ Take $\underline{\underline{x} = \left[\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right]^t}$.

<u>Aliter</u> Second e-vector is $\perp$ to 1st and is unit magnitude

$\Rightarrow \underline{x} = \left[\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right]$.

(ii) $\begin{vmatrix} \lambda-3 & -1 \\ -1 & \lambda-4 \end{vmatrix} = (\lambda-3)(\lambda-4) - 1 = 0 \Rightarrow \lambda^2 - 7\lambda + 11 = 0$

$\lambda = \dfrac{7 \pm \sqrt{49-44}}{2} = \underline{4.618 \text{ or } 2.382}$

<u>e-vectors</u> $\lambda = 4.618$ $\quad 3x_1 + x_2 = 4.618x_1 \Rightarrow x_2 = 1.618x_1$

$\therefore$ Take $\underline{x} = \dfrac{(1, 1.618)}{\sqrt{1^2+1.618^2}} = [.526, .851]^t$

$\lambda = 2.382$ $\quad 3x_1 + x_2 = 2.382x_1 \Rightarrow x_2 = -.618x_1$

$\therefore$ Take $\underline{x} = \dfrac{(1, -.618)}{\sqrt{1^2+.618^2}} = [.851, -.526]^t$

(iii) $\begin{vmatrix} \lambda-2 & 0 \\ 0 & \lambda-3 \end{vmatrix} = 0 \Rightarrow (\lambda-2)(\lambda-3) = 0 \Rightarrow \underline{\lambda = 2 \text{ or } 3}$

e-vectors $\lambda=2$; $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow$ $2x_1 = 2x_1$; $3x_2 = 2x_2$

Take $x_1 = 1$ $x_2 = 0$

Similarly for $\lambda = 3$ $\quad x_1 = 0$, $x_2 = 1$ $\quad \therefore$ <u>e-vectors $[1,0]^t$ & $[0,1]^t$</u>

(iv) $\begin{vmatrix} \lambda-2 & -1 \\ 0 & \lambda-3 \end{vmatrix} = 0 \Rightarrow \underline{\lambda = 2 \text{ or } 3}$ (again)

e-vectors $\lambda = 2$; $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\Rightarrow \underline{\underline{[1,0]^t}}$ works again

$\lambda = 3$; $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow x_1 = x_2$

$\Rightarrow$ 2nd eigenvector is $\left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]^t$

(which is NOT orthogonal to $[1,0]^t$. Why?)

(v) $\begin{vmatrix} \lambda-1 & -1 \\ 0 & \lambda-1 \end{vmatrix} = 0 \Rightarrow (\lambda-1)^2 = 0 \Rightarrow \lambda = 1, 1$

e-vectors $\lambda = 1$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$\Rightarrow x_2 = 0$, x_1 arbitrary. So <u>only</u> eigenvector is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

NOTE: Most matrices with repeated eigenvalues still have a full set of eigenvectors. This is an example of a <u>defective</u> matrix.

(vi) $\begin{vmatrix} \lambda-3 & 4 & -1 \\ 4 & \lambda-8 & 4 \\ -1 & 4 & \lambda-3 \end{vmatrix} = 0 \quad \Rightarrow (\lambda-3)\left[(\lambda-8)(\lambda-3)-16\right]$

$+4\left[-4-4(\lambda-3)\right] - 1\left[16+\lambda-8\right] = 0$

i.e. $\lambda^3 - 14\lambda^2 + 24\lambda = 0 \quad \Rightarrow \lambda = 0$ or $\lambda^2 - 14\lambda + 24 = 0$

$\Rightarrow (\lambda-2)(\lambda-12) = 0$

$\therefore \quad \underline{\lambda = 0, 2 \text{ or } 12}$

<u>e-vectors</u>

$\lambda = 0 \qquad \begin{bmatrix} 3 & -4 & 1 \\ -4 & 8 & -4 \\ 1 & -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\Rightarrow \quad 3x_1 - 4x_2 + x_3 = 0$

$-4x_1 + 8x_2 - 4x_3 = 0 \qquad \Rightarrow x_1 = x_2$ and $x_1 = x_3$

$\therefore$ Take $\underline{x} = \left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]^t$

$\lambda = 2 \quad \Rightarrow \quad 3x_1 - 4x_2 + x_3 = 2x_1$

$-4x_1 + 8x_2 - 4x_3 = 2x_2 \qquad \Rightarrow x_3 = -x_1$ & $x_2 = 0$

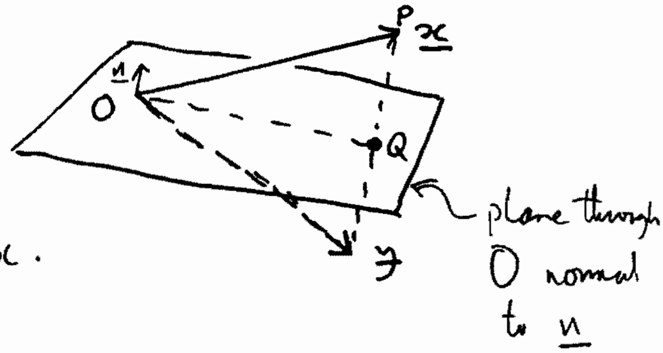
$\therefore$ Take $\underline{x} = \left[\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right]^t$

$\lambda = 12 \quad \Rightarrow \quad 3x_1 - 4x_2 + x_3 = 12x_1$

$-4x_1 + 8x_2 - 4x_3 = 12x_2 \qquad \Rightarrow \begin{array}{l} x_2 = -2x_1 \\ x_3 = x_1 \end{array}$

$\therefore$ Take $\underline{x} = \left[-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right]^t$

2. The vector $\overline{QP}$ is $(\underline{x}\cdot\underline{n})\underline{n}$, the projection of $\underline{x}$ onto the normal.



To obtain Q, we must negate this component of x.

$$\overline{OQ} = \underline{x} - (\underline{x}\cdot\underline{n})\underline{n}$$

so
$$\underline{y} = \underline{x} - 2(\underline{x}\cdot\underline{n})\,\underline{n}$$
$$= (I - 2\,\underline{n}\,\underline{n}^t)\,\underline{x} \qquad (\underline{n}^t\underline{x} = \underline{n}\cdot\underline{x})$$

So $R = I - 2\underline{n}\,\underline{n}^t$.

We expect (i) $\underline{n}$ to be an eigenvector with eigenvalue -1

(ii) any vector in the plane to be an eigenvector with eigenvalue $+1$.

Both are readily verified: (i)
$$R\underline{n} = (I - 2\underline{n}\,\underline{n}^t)\,\underline{n}$$
$$= \underline{n} - 2\underline{n}\,|\underline{n}|^2$$
$$= -\underline{n}$$

(ii) Let $\underline{a}$ satisfy $\underline{a}\cdot\underline{n} = 0$: then
$$R\underline{a} = I\underline{a} - \underline{n}(\underline{n}\cdot\underline{a})$$
$$= \underline{a} \qquad //.$$

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3. The matrix of normalised eigenvectors is

$$U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & -1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \end{bmatrix}$$

From lectures, $A = U\Lambda U^t$, $\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

So
$$A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & -1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & -1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \end{bmatrix}^t$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & -1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 3/\sqrt{2} & 3/\sqrt{2} & 0 \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 1/2\sqrt{6} & -1/2\sqrt{6} & -1/\sqrt{6} \end{bmatrix}$$

$$= \frac{1}{12}\begin{bmatrix} 23 & 13 & 2 \\ 13 & 23 & -2 \\ 2 & -2 & 8 \end{bmatrix}$$

3 contd. Similarly
$$B = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & -1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{6} & -2/\sqrt{6} \end{bmatrix}$$
$$= \frac{1}{6}\begin{bmatrix} 5 & -3 & -2 \\ -3 & 5 & 2 \\ -2 & 2 & 10 \end{bmatrix}$$

By direct multiplication, $AB = I$

This is because A^{-1} has the same eigenvectors as A, but eigenvalues $1/\lambda$:
$$A\underline{u} = \lambda\underline{u} \Rightarrow 1/\lambda \, A^{-1}A\underline{u} = A^{-1}\underline{u}$$
$$\text{ie } A^{-1}\underline{u} = \lambda^{-1}\underline{u}$$

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4. The eigenvalue equation $\begin{vmatrix} A_{11}-\lambda & A_{12} \\ A_{21} & A_{22}-\lambda \end{vmatrix} = 0$ gives

$$(A_{11}-\lambda)(A_{22}-\lambda) - A_{21}A_{12} = 0 \quad \text{ie} \quad \lambda^2 - \lambda(A_{11}+A_{22}) + A_{11}A_{22} - A_{21}A_{12} = 0$$

Comparing this with the polynomial equation $(\lambda-\lambda_1)(\lambda-\lambda_2) = 0$ which has the eigenvalues as roots

ie. $\lambda^2 - (\lambda_1+\lambda_2)\lambda + \lambda_1\lambda_2 = 0$,

we have $\lambda_1 + \lambda_2 = A_{11} + A_{22}$ and $\lambda_1\lambda_2 = A_{11}A_{22} - A_{21}A_{12}$

ie $\sum_{i=1}^{2} A_{ii} = \sum_{i=1}^{2} \lambda_i$ and $|A| = \lambda_1\lambda_2$

Supervisors may wish to show how the method can be generalised to the $n \times n$ case. We wish now to multiply out

$$\begin{vmatrix} A_{11}-\lambda & A_{12} & A_{13} & \cdots & A_{1n} \\ A_{21} & A_{22}-\lambda & A_{31} & \cdots & \\ \vdots & & & & \\ A_{n1} & \cdots & & & A_{nn}-\lambda \end{vmatrix} = 0 \qquad (1)$$

The expanded determinant consists of a sum of terms and each of these terms is in turn a product. The product is formed by taking one element from each row such that there is also only one element from each <u>column</u>. Consider for

4 contd. example a product containing the element A_{12}. This product cannot contain the element $A_{11}-\lambda$ (same row as A_{12}) or the element $A_{22}-\lambda$ (same column as A_{12}). Any term containing A_{12} (and in fact any off diagonal element) can only lead to powers of λ up to λ^{n-2}.

The product of elements which leads to terms in λ^n and λ^{n-1} is simply the product of all the diagonal ones. $(A_{11}-\lambda)(A_{22}-\lambda)\dots(A_{nn}-\lambda)$

Thus the coefficient of λ^n in the expanded polynomial $=(-1)^n$

" " " λ^{n-1} " " " " $=\sum A_{ii}(-1)^{n-1}$

Finally, the term indept of λ is the determinant of A (since this is the expansion of the determinant when $\lambda = 0$).

Equation (1) is thus

$$(-\lambda)^n + \left(\sum_{i=1}^n A_{ii}\right)(-\lambda)^{n-1} + \underbrace{\dots\dots}_{\text{These terms are very difficult to work out.}} + |A| = 0$$

The polynomial with $\lambda_1, \lambda_2, \dots \lambda_n$ as roots is

$$(\lambda_1-\lambda)(\lambda_2-\lambda)\dots(\lambda_n-\lambda)$$

which expands to

$$(-\lambda)^n + \left(\sum_i \lambda_i\right)(-\lambda)^{n-1} + \dots\dots + \lambda_1\lambda_2\dots\lambda_n$$

Comparing coefficients gives

$$\sum_{i=1}^n A_{ii} = \sum_{i=1}^n \lambda_i \quad \text{and} \quad |A| = \lambda_1\lambda_2\dots\lambda_n$$

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5. $A = \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}$. E-values satisfy $\begin{vmatrix} 4-\lambda & -2 \\ -2 & 4-\lambda \end{vmatrix} = 0$

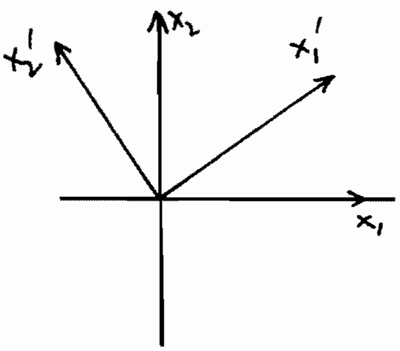
i.e. $(\lambda-4)^2 = 4 \Rightarrow \lambda - 4 = \pm 2$ or $\lambda = 2$ or 6.

The corresponding normalised eigenvectors satisfy $4x_1 - 2x_2 = 2x_1$

$\Rightarrow \underline{x} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ and $4x_1 - 2x_2 = 6x_1 \Rightarrow \underline{x} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

5 contd The required matrix which gives the new coordinates in terms of the old is, as given in lectures, formed by taking the normalised eigenvectors as rows i.e.

$$R = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \qquad \text{so that } \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = R \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



The new version of A is, again as given in lectures,

$$A' = RAR^t = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{2}} & -\frac{6}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} & \frac{6}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$$

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6. We have $A\underline{y}_1 = \lambda_1 \underline{y}_1$, $A\underline{y}_2 = \lambda_2 \underline{y}_2$, $A\underline{y}_3 = \lambda_3 \underline{y}_3$

The vectors $\underline{y}_1, \underline{y}_2, \underline{y}_3$ form an orthogonal triad since A is symmetric, so we can use them as a coordinate basis set, and express any vector as a linear combination $\underline{x} = \alpha \underline{y}_1 + \beta \underline{y}_2 + \gamma \underline{y}_3$

(where $\alpha = \underline{x}\cdot\underline{y}_1$, $\beta = \underline{x}\cdot\underline{y}_2$, $\gamma = \underline{x}\cdot\underline{y}_3$ if the $\underline{y}$'s are normalised to be unit vectors).

So (b) $\underline{x}^t A \underline{x} = (\alpha \underline{y}_1^t + \beta \underline{y}_2^t + \gamma \underline{y}_3^t)(\alpha\lambda_1 \underline{y}_1 + \beta\lambda_2 \underline{y}_2 + \gamma\lambda_3 \underline{y}_3)$

$= \alpha^2 \lambda_1 |\underline{y}_1|^2 + \beta^2\lambda_2 |\underline{y}_2|^2 + \gamma^2 \lambda_3 |\underline{y}_3|^2$

(since $\underline{y}_1^t \underline{y}_2 = \underline{y}_1\cdot\underline{y}_2 = 0$, etc.)

$= \alpha^2\lambda_1 + \beta^2\lambda_2 + \gamma^2\lambda_3$ as $|\underline{y}|^2 = 1$.

(a) $\underline{x}^t \underline{x} = (\alpha\underline{y}_1^t + \beta\underline{y}_2^t + \gamma\underline{y}_3^t)(\alpha\underline{y}_1 + \beta\underline{y}_2 + \gamma\underline{y}_3)$

$= \alpha^2 + \beta^2 + \gamma^2$ by a similar argument.

So $$\frac{\underline{x}^t A \underline{x}}{\underline{x}^t \underline{x}} = \frac{\alpha^2\lambda_1 + \beta^2\lambda_2 + \gamma^2\lambda_3}{\alpha^2 + \beta^2 + \gamma^2}$$

But $\lambda_1 < \lambda_2, \lambda_3$, so $\alpha^2\lambda_1 + \beta^2\lambda_2 + \gamma^2\lambda_3 \geq \lambda_1(\alpha^2 + \beta^2 + \gamma^2)$

and $\lambda_3 > \lambda_1, \lambda_2$, so $\alpha^2\lambda_1 + \beta^2\lambda_2 + \gamma^2\lambda_3 \leq \lambda_3(\alpha^2+\beta^2+\gamma^2)$

So $$\lambda_1 \leq \frac{\underline{x}^t A \underline{x}}{\underline{x}^t \underline{x}} \leq \lambda_3$$ //.

Equality is only achieved if $\beta = \gamma = 0$ (for the first) or $\alpha = \beta = 0$ (for the second) ie if $\underline{x} = \alpha\underline{y}_1$ or $\underline{x} = \gamma\underline{y}_3$ respectively

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7. If $\underline{x} = \alpha \underline{u}_1 + \beta \underline{u}_2 + \gamma \underline{u}_3$,

$$A\underline{x} = \alpha \lambda_1 \underline{u}_1 + \beta \lambda_2 \underline{u}_2 + \gamma \lambda_3 \underline{u}_3$$

$$\therefore A^2 \underline{x} = \alpha \lambda_1^2 \underline{u}_1 + \beta \lambda_2^2 \underline{u}_2 + \gamma \lambda_3^2 \underline{u}_3$$

$$\cdots\cdots$$

$$\therefore A^n \underline{x} = \alpha \lambda_1^n \underline{u}_1 + \beta \lambda_2^n \underline{u}_2 + \gamma \lambda_3^n \underline{u}_3$$

This will be progressively dominated by whichever eigenvalue has the largest absolute value $|\lambda_i|$. So $A^n \underline{x}$ tends towards $\lambda_i^n \underline{u}_i$ (times a constant), where $|\lambda_i| > |\lambda_j|, j \neq i$.
A^{-1} has the same eigenvectors $\underline{u}_i$, but corresponding eigenvalues $1/\lambda_i$ (see q. 2).
Thus $(A^{-1})\underline{x} = \alpha \lambda_1^{-1} \underline{u}_1 + \beta \lambda_2^{-1} \underline{u}_i + \gamma \lambda_3^{-1} \underline{u}_3$

$$\cdots$$

$$\therefore (A^{-1})^n \underline{x} = \alpha \lambda_1^{-n} \underline{u}_1 + \beta \lambda_2^{-n} \underline{u}_2 + \gamma \lambda_3^{-n} \underline{u}_3$$

So $(A^{-1})^n \underline{x}$ tends towards a multiple of $\lambda_k^{-n} \underline{u}_k$, where $|\lambda_k| < |\lambda_p|$, $p \neq k$ (ie the smallest absolute value).
Performing the matrix product in the usual way gives

$$A\underline{x} = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}, \quad A^2\underline{x} = \begin{bmatrix} 26 \\ -48 \\ 22 \end{bmatrix}, \quad A^3\underline{x} = \begin{bmatrix} 292 \\ -576 \\ 284 \end{bmatrix}, \quad A^4\underline{x} = \begin{bmatrix} 3464 \\ -6912 \\ 3448 \end{bmatrix}$$

Taking ratios of the last two, component by component, gives $\frac{3464}{292} = 11 \cdot 86$, $\frac{6912}{576} = 12 \cdot 00$, $\frac{3448}{284} = 12 \cdot 14$.

The average ratio gives a reasonable estimate for the eigenvalue, $12 \cdot 00$. Normalising the last vector gives $\underline{u} \simeq [0 \cdot 4092, -0 \cdot 8165, 0 \cdot 4073]^t$.

The exact answer (from sheet 4) is $\lambda = 12$,

$$\underline{u} = [1/\sqrt{6}, -2/\sqrt{6}, 1/\sqrt{6}]^t = [0 \cdot 4082, -0 \cdot 8165, 0 \cdot 4082]$$

Convergence is quite rapid, as the other eigenvalues are 0, 2