Engineering **FIRST YEAR**

Part IA Paper 4: Mathematics

Examples paper 2

(Elementary exercises are marked †, problems of Tripos standard *)

Students are encouraged to use mathematical grammar correctly.

Revision question

For each of the following functions $f(x)$, calculate $f'(x)$:

(a) $f(x) = \sin^2\left(x^2/2\right)$ (b) $f(x) = \sin\left(\sin^2 x\right) + \sin\left(\cos^2 x\right)$

(c) $f(x) = \exp\left(\exp\left(\exp\left(x^2\right)\right)\right)$ (d) $f(x) = \ln\left\{\dfrac{\sin^2 x}{x^2}\right\}$.

Determinants

1† Evaluate the determinants of the following matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -3 \\ 3 & 0 & -4 \\ -1 & 3 & 2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & -1 & -3 \\ 2 & 0 & 1 \end{bmatrix}.$$

Verify that $\det \mathbf{AB} = \det \mathbf{A} \det \mathbf{B}$.

2 Without solving the following simultaneous equations, determine the value of s for which they have no solution when $t = 1$.

$$\begin{aligned} 2x + y + 3z &= 5 \\ 6x - 2y - z &= 3 \\ sx + z &= t \end{aligned}$$

For this value of s determine the value of t for which the equations have an infinite number of solutions. Use Python/Matplotlib to visualize the three planes for these and other values of *s* and *t*. Make sure you understand how the planes' intersections relate to the values of *s* and *t*.

See the URL

http://nbviewer.jupyter.org/github/CambridgeEngineering/Part-IA-ExamplesPapers-Python/blob/master/paper4/IA%20Paper%204%20Mathematics%2002.ipynb

You may find it easier to access this through the online version of this paper in the *IA Examples Paper repository* which can be accessed through the IA Teaching Homepage on the Department website.

3 Prove that, for any vectors **a**, **b** and **c** and scalar λ

$$\mathbf{a} \,.\, \mathbf{b} \times \mathbf{c} \;=\; (\mathbf{a} + \lambda\, \mathbf{b}\,) \,.\, \mathbf{b} \times \mathbf{c}$$

Interpret this as a rule for manipulating the rows (or columns) of 3×3 determinants.

Hence evaluate

$$\begin{vmatrix} 2 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 0 & 6 \end{vmatrix}$$ (in your head!) .

Functions and Series

4. Sketch graphs of the following functions:

(a) $(x^2 - 1)\, e^{-x}$ (b) $x - \sin x$ (c) $3x^4 - 16x^3 + 18x^2$

For what values of k does the equation $3x^4 - 16x^3 + 18x^2 = k$ have

(i) precisely two distinct real roots;

(ii) precisely three distinct real roots?

5† (i) Prove that

$$\sinh (A + B) \;=\; \sinh A \;\cosh B \;+\; \cosh A \;\sinh B$$

and find a similar expression for $\cosh (A + B)$.

(ii) Differentiate $\tanh x$.

6 (i) Express $\cosh (1 + x)$ as a power series in x .

(ii) Find the first three terms in the power series expansion of

$$\frac{1+x}{1-x^2}$$

Plot the function and the sum of the first three terms in the power series. By considering the size of the first term neglected, estimate the range of x for which you would you expect the first three terms to be a reliable approximation within 1% and 0.1% of the exact value?

7† Show that

$$\left(\frac{d}{dx}\right)^n (a + b\,x \;)^{\alpha} \;=\; \alpha \;(\,\alpha - 1)\,(\alpha - 2)...(\alpha - n + 1)\; b^n \,(\; a + b\,x \;)^{\alpha - n}.$$

Hence find the coefficient of x^7 in the power series expansion of $(2+3x)^{-1/2}$.

Limits and Approximations

8 Evaluate (a)† $\lim_{x \to 0} \frac{\sin x}{x}$

(b)† $\lim_{x \to 0} \frac{\tan x - x}{x - \sin x}$

(c) $\lim_{x \to \pi/2} \frac{\ln(x - \pi/2)}{\tan x}$

(d)* $\lim_{x \to \infty} \frac{x+1}{x^2 + 6x} \exp\left[\frac{x^2}{1+x^2}(\ln x + 2)\right]$

9 Show that if α is small, then

(a) $\frac{\sin^2 \alpha}{\alpha^2 \sqrt{1 - (\sin^2 \alpha)/3}} \approx 1 - \frac{\alpha^2}{6}$. [If you get $1 + \frac{\alpha^2}{6}$ then think again!]

(b)* $\frac{\sin^2 \alpha}{\alpha^2 \sqrt{1 - 2(\sin^2 \alpha)/3}} \approx 1 - \frac{\alpha^4}{90}$.

Suitable past Tripos questions:
02 Q2a; 03 Q2a; 04 Q2a; 05 Q1 (short); 08 Q4b; 09 Q1 (short), Q4a (long); 10 Q2 (short); 11 Q2 (short); 12 Q1 (short); 13 Q2 (short); 14 Q1 (short), Q2 (short), Q4 (long); 15 Q2(short); 16 Q1(short); 17 Q1(short) & Q4(long); 18 Q2(short).

Answers

1 det **A** = –19, det **B** = –3, det **AB** = 57

2 $s = 2$, $t = \frac{13}{5}$

3 12

4 (i) $k > 5$, $-27 < k < 0$ (ii) $k = 0$ and $k = 5$ (one root is repeated)

5 (i) $\cosh A \cosh B + \sinh A \sinh B$ (ii) $\operatorname{sech}^2 x \left[= \frac{1}{\cosh^2 x} \right]$

6 (i) $\cosh 1 + x \sinh 1 + \frac{x^2}{2!} \cosh 1 + \frac{x^3}{3!} \sinh 1$

(ii) $1 + x + x^2$ $|x| < 0.2$ (for 1%) and $|x| < 0.1$ (for 0.1%)

7 –2·531

8 (a) 1 (b) 2 (c) 0 (d) e^2

SJS/JPL October 2018