Part IA Mathematics Examples Paper 9 Solutions

1. (i) 3 choices for 1st digit then $3!$ ways of arranging others.

$\therefore$ Number of integers $= 3 \times 3! = \underline{18}$

(ii) 3 choices for 1st digit, 4 for second, third & fourth

$\therefore$ Number $= 3 \times 4^3 = \underline{192}$

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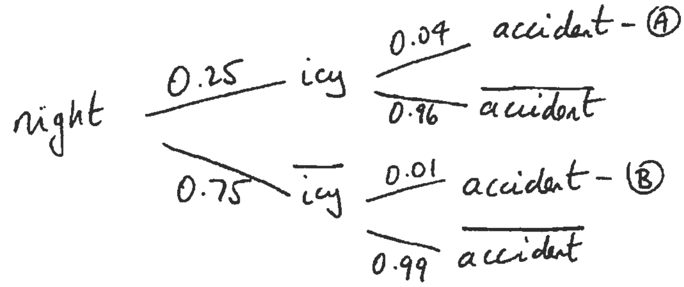
2. 3 choices for least significant digit, 5 for next, 4 for next, etc

$\therefore$ Number $= 3 \times 5! = \underline{360}$ (or ${}_5P_5$ ways of arranging)

3.



(i) $P(\text{icy} \cap \text{accident})$ (Case A)

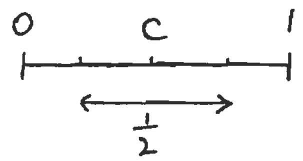
$= .25 \times .04 = 0.01$

$(= P(\text{icy})\, P(\text{accident} \mid \text{icy}))$

(ii) $P(\text{accident}) = P(\text{accident} \cap \text{icy}) + P(\text{accident} \cap \overline{\text{icy}})$ (Cases A & B)

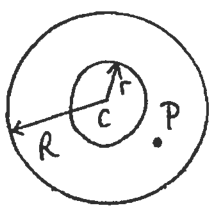
$= 0.25 \times 0.04 + 0.75 \times 0.01 = \underline{.0175}$

4.



Probability of falling in a section of line is proportional to the length.

Hence $P(\text{closer to centre}) = \underline{\frac{1}{2}}$



A point is closer to the centre $\iff r < \frac{R}{2}$

and probability of falling within a given radius is proportional to area.

$\therefore p(\text{closer to centre}) = \dfrac{\text{Area } r \le R/2}{\text{Total Area}} = \dfrac{\pi (R/2)^2}{\pi R^2} = \underline{\frac{1}{4}}$

Let C = closer to centre, $N = \bar{C}$ = not closer

The outcomes which lead to a C within the first three points picked are :—

C NC NNC

If points are chosen independently $\Rightarrow$ multiply probabilities

Thus C NC NNC

Probability $\frac{1}{4}$ $\frac{3}{4}\frac{1}{4}$ $\frac{3}{4}\frac{3}{4}\frac{1}{4}$

(a) $P(NNC) = \underline{\frac{9}{64}}$

(b) C, NC, NNC exclusive $\Rightarrow$ add probabilities

$$P(C) + P(NC) + P(NNC) = \frac{1}{4} + \frac{3}{16} + \frac{9}{64} = \underline{\frac{37}{64}}$$

k points , $P(\text{all } N) = \left(\frac{3}{4}\right)^k$

$\therefore$ $P(\text{at least one } C) = 1 - \left(\frac{3}{4}\right)^k$. This is greater than $\cdot 85$ if $\underline{k = 7}$

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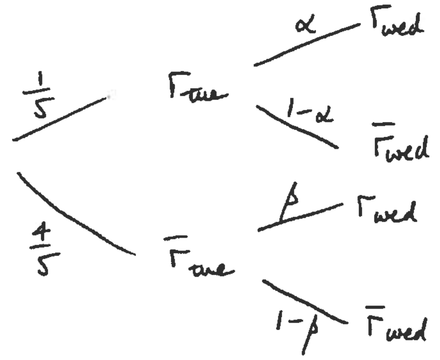
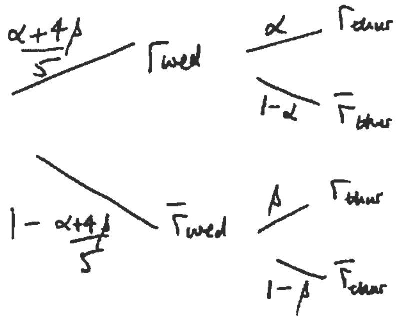
5. (a)



$$P(\Gamma_{wed}) = \underline{\alpha \cdot \frac{1}{5} + \beta \cdot \frac{4}{5}}$$

$$\left(= P(\Gamma_{wed} \mid \Gamma_{tue}) P(\Gamma_{tue}) + P(\Gamma_{wed} \mid \bar{\Gamma}_{tue}) P(\bar{\Gamma}_{tue})\right)$$

$$(\text{thus } P(\bar{\Gamma}_{wed}) = 1 - \frac{\alpha}{5} - \frac{4\beta}{5}$$



$$P(\Gamma_{thur}) = \frac{\alpha+4\beta}{5}\alpha + \left(1 - \frac{\alpha+4\beta}{5}\right)\beta$$

$$= \underline{\beta + \frac{(\alpha-\beta)(\alpha+4\beta)}{5}}$$

(b) $$P(X \mid Y) = P(X \mid \Gamma_{wed}) P(\Gamma_{wed} \mid Y) + P(X \mid \bar{\Gamma}_{wed}) P(\bar{\Gamma}_{wed} \mid Y)$$

$$= \alpha \cdot \alpha + \beta \cdot (1-\alpha)$$

$$= \alpha^2 + \beta(1-\alpha)$$

$$P(X|\bar{Y}) = P(X|r_{wed})P(r_{wed}|\bar{Y}) + P(X|\bar{r}_{wed})P(\bar{r}_{wed}|\bar{Y})$$
$$= \alpha \cdot \beta + \beta \cdot (1-\beta)$$
$$= \alpha\beta + \beta(1-\beta)$$

$\therefore\ \underline{P(X|Y) - P(X|\bar{Y})} = \alpha^2+\beta-\beta\alpha - \alpha\beta - \beta + \beta^2 = (\alpha-\beta)^2$

(c) $\alpha = \beta \Rightarrow$ P(rain| rain previous day) = P(rain | $\overline{\text{rain previous day}}$)

i.e. rain on one day is <u>independent</u> of whether it rained previous day

(& $P(X|Y) = P(X|\bar{Y}) \Rightarrow X$ independent of Y)

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6. $\boxed{52} = \overset{\text{Player A}}{\boxed{13}} + \overset{\text{Player B}}{\boxed{13}} + \overset{\text{Player C}}{\boxed{13}} + \overset{\text{Player D}}{\boxed{13}}$

Number of ways of dividing 52 into four groups of 13

$$= \underset{\text{no. ways choosing 1st 13}}{{}_{52}C_{13}} \times \underset{\text{no. ways choosing 2nd}}{{}_{39}C_{13}} \times {}_{26}C_{13} \times {}_{13}C_{13} \quad = \quad \underline{5.4\times10^{28}}$$

etc.

<u>Aliter</u>

No ways of dealing the 52 cards in which order matters $= 52!$

Player A	Player B	Player C	Player D
$\boxed{13}$	$\boxed{13}$	$\boxed{13}$	$\boxed{13}$
13! ways of arranging	13! ways of arranging	"	"

These particular four hands arise from $\dfrac{52!}{(13!)^4}$ groupings $= 5.4\times10^{28}$

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7. $52 = 39 + 13$ spades

Total no ways of drawing 13 cards from 52 $= {}_{52}C_{13}$

" " " " " " " from 39 $= {}_{39}C_{13}$

$\therefore$ Probability of having no spades $= \dfrac{{}_{39}C_{13}}{{}_{52}C_{13}} = \underline{.0128}$

$$P(\overline{\text{spades}} \cup \overline{\text{hearts}}) = P(\overline{\text{spades}}) + P(\overline{\text{hearts}}) - P(\overline{\text{spades}} \cap \overline{\text{hearts}})$$

$$P(\overline{\text{spades}}) = P(\overline{\text{hearts}}) = .0128$$

$52 = 26 + 13\text{ spades} + 13\text{ hearts}$

and $$P(\overline{\text{spades}} \cap \overline{\text{hearts}}) = \frac{{}_{26}C_{13}}{{}_{52}C_{13}}$$

$$= 1.64 \times 10^{-5} \ll 0.0128$$

$$\therefore \quad P(\overline{\text{spades}} \cup \overline{\text{hearts}}) \simeq 2 \times 0.0128$$

Similarly $$P(\overline{\text{spades}} \cup \overline{\text{hearts}} \cup \overline{\text{clubs}} \cup \overline{\text{diamonds}}) \simeq 4 \times 0.0128$$

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8. Probability of r tails from n tosses $= \binom{n}{r}\left(\frac{1}{2}\right)^r\left(\frac{1}{2}\right)^{n-r}$

(= no ways choosing which r from n × prob (r tails) × P($n-r$ heads))

$\therefore$ Probability of same no. tails

$$= \binom{5}{0}\left(\frac{1}{2}\right)^5\binom{3}{0}\left(\frac{1}{2}\right)^3 + \binom{5}{1}\left(\frac{1}{2}\right)^5\binom{3}{1}\left(\frac{1}{2}\right)^3 + \binom{5}{2}\left(\frac{1}{2}\right)^5\binom{3}{2}\left(\frac{1}{2}\right)^3$$

$$+ \binom{5}{3}\left(\frac{1}{2}\right)^5\binom{3}{3}\left(\frac{1}{2}\right)^3$$

$$= \left(\frac{1}{2}\right)^8\left[1 + 5.3 + 10.3 + 10.1\right] = \frac{56}{256} = \underline{\frac{7}{32}}$$

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9. $P(\text{correct}) = \frac{1}{3}$ $\qquad P(5 \text{ correct out of } 15) = {}_{15}C_5\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^{10}$

$$\underline{= .214}$$

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10. (i) $P(5) = {}_{10}C_5\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^5 = \underline{.246}$

(ii) No. of ways of getting 7 heads $= {}_{10}C_7 = 120$

" " " " " 8,9,10 heads $= {}_{10}C_8 + {}_{10}C_9 + {}_{10}C_{10} = 56$

$\therefore$ <u>7 heads more likely</u>

[See next sheet for solution to matlab question]

11. (i) ${}_5C_2 \left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3 = .31$

(ii) $P(4 \text{ female}) + P(5 \text{ female}) = {}_5C_4 \left(\frac{1}{2}\right)^4 \frac{1}{2} + {}_5C_5 \left(\frac{1}{2}\right)^5 = .188$

$\therefore \; P(<4) = 1 - .188 \approx \underline{.81}$

4 offspring 2 male not black

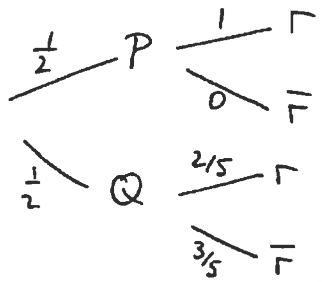
Options	4f	3f, 1m	2f, 2m	1f, 3m	4m
P(this combination)			${}_4C_2 \left(\frac{1}{2}\right)^4$	${}_4C_3 \left(\frac{1}{2}\right)^4$	$\left(\frac{1}{2}\right)^4$
P(2 non-b males / this combination)	0	0	$\left(\frac{2}{3}\right)^2$	${}_3C_2 \left(\frac{2}{3}\right)^2 \frac{1}{3}$	${}_4C_2 \left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2$

$\therefore P(2 \text{ non-black males}) = {}_4C_2 \left(\frac{1}{2}\right)^4\left(\frac{2}{3}\right)^2 + {}_4C_3\,{}_3C_2 \left(\frac{1}{2}\right)^4\left(\frac{2}{3}\right)^2\frac{1}{3} + {}_4C_2 \left(\frac{1}{2}\right)^4\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2$

$= .167 + .111 + .019 = \underline{.30}$

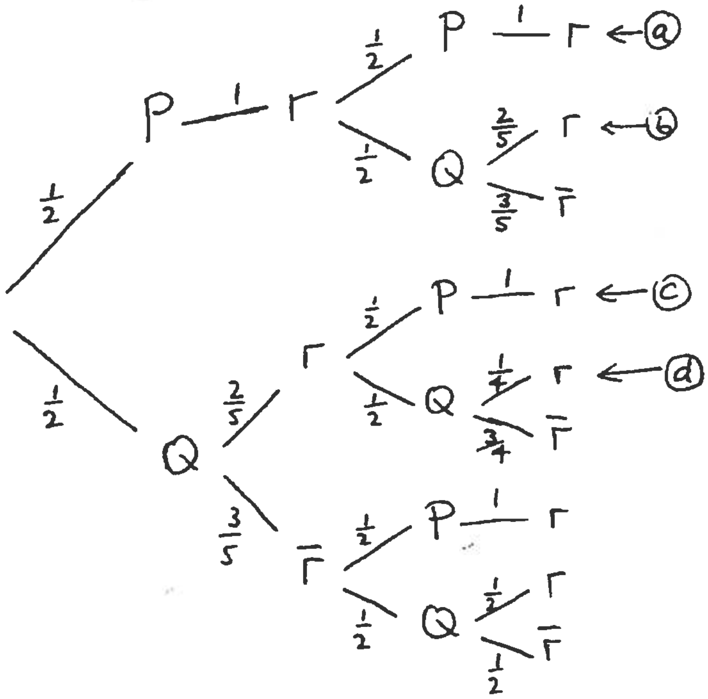
12. P: 3 red Q: 2 red 3black

(a)



P(red) $= \frac{1}{2} + \frac{1}{2}\cdot\frac{2}{5}$

$= \underline{\frac{7}{10}}$



(b) $P(\text{both red}) = P(a) + P(b) + P(c) + P(d)$

$= \frac{1}{2}\cdot 1\cdot\frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 1\cdot\frac{1}{2}\cdot\frac{2}{5}$

$+ \frac{1}{2}\cdot\frac{2}{5}\cdot\frac{1}{2}\cdot 1 + \frac{1}{2}\cdot\frac{2}{5}\cdot\frac{1}{2}\cdot\frac{1}{4}$

$= \frac{10+4+4+1}{40} = \underline{\frac{19}{40}}$

(c) P(bag P twice | 2 red)

$= \frac{P(a)}{P(a)+P(b)+P(c)+P(d)} = \frac{\frac{1}{4}}{19/40} = \underline{\frac{10}{19}}$

13.

(i) Mean $\mu = \dfrac{1+2+2+3+3+3+4+4+4+4}{10} = \underline{3}$

(ii) Standard deviation $\sigma = \sqrt{\dfrac{\Sigma x^2 - \frac{1}{n}(\Sigma x)^2}{n}} = \sqrt{\dfrac{100 - \frac{1}{10}(30)^2}{10}} = \underline{1}$

(iii) Sample mean $m = \dfrac{3+4+4+2+3+1+3+3+4+4+3+4}{12} = \underline{3.1667}$

Sample standard deviation $s = \sqrt{\dfrac{\Sigma x^2 - \frac{1}{n}(\Sigma x)^2}{n-1}}$

$= \sqrt{\dfrac{130 - \frac{1}{12}(38)^2}{11}} = \underline{0.9374}$

14.

Sample mean $= 255.4517$. Let this be called $\bar{x}$

Sample standard deviation $= \sqrt{\dfrac{6530839 - \frac{1}{100}(25545)^2}{99}} = 7.3055$

Estimate of standard deviation of the distribution of the sample mean $= \dfrac{7.3055}{\sqrt{100}} = 0.7306$. Let this be called $s(\bar{x})$

99.73% of the time $\bar{x}$ will lie within $3\,s(\bar{x})$ of the true value. This gives us a range of: $255 - 3\times0.73$ to $255 + 3\times0.73$

$\Rightarrow$ $\underline{\text{from } 253.26 \text{ up to } 257.64}$

TPH/JW/AT/RWP