Mechanical Vibrations

EXAMPLES PAPER 3 SOLUTIONS

Q1. a) Double Pendulum:

One coordinate (e.g. angle to vertical) needed to specify position of each pendulum, total = 2

b) Manometer:

Only one coordinate needed to specify liquid position, (assuming no large air bubbles in the liquid)

c) Point mass:

3 coordinates (e.g. x, y, z)

(For point mass, rotations are irrelevant)

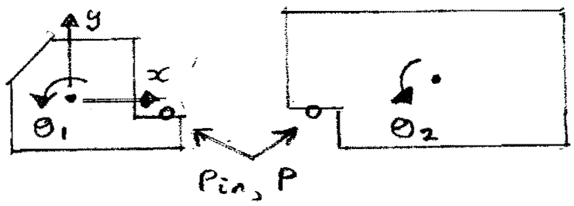
d) Engine block:

6 coordinates required to specify the configuration of any rigid body in space

e) Articulated Vehicle model:

For left hand mass, 3 coordinates

e.g. Horizontal translation x
Vertical translation y
Pitch rotation θ_1



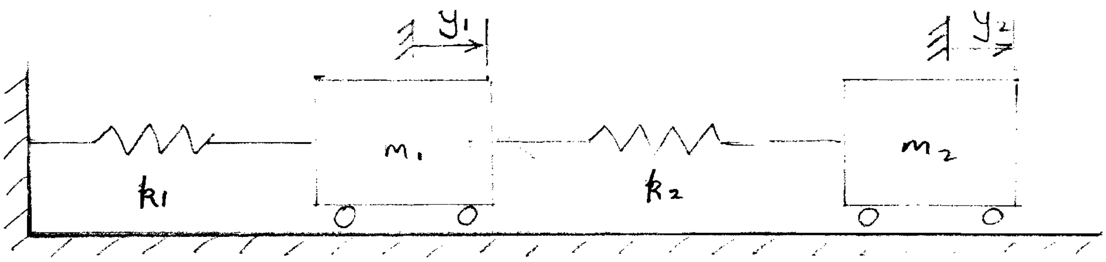
These 3 coordinates specify the position of the pin P which joins the 2 masses. So the right hand mass has only one independent degree of freedom – θ_2 say.

$\therefore$ Total = 4

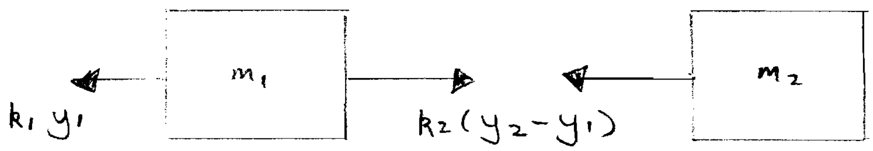
f) Tacoma Narrows bridge:

This is a continuous system made up of flexible beams and cables $\therefore$ ∞ coordinates required.

Q2



Forces



Mass x Accel^n

$R(\rightarrow)$ $\quad m_1\ddot{y}_1 = k_2(y_2-y_1) - k_1 y_1$

$\quad m_2\ddot{y}_2 = -k_2(y_2-y_1)$

$$\therefore \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or $\quad [M]\,\underline{\ddot{y}} + [K]\,\underline{y} = 0.$

For a normal mode $\underline{y}_j = \underline{Y}_j \cos\omega_j t \quad \therefore \quad \underline{\ddot{y}}_j = -\omega_j^2 \underline{Y} \cos\omega_j t$

$\therefore \{[K] - \omega_j^2[M]\}\underline{Y}_j \cos\omega t = 0 \qquad \therefore \quad \underline{\{[K] - \omega_j^2[M]\}\underline{Y}_j = 0}$

To find ω_j, put $|\{[K] - \omega_j^2[M]\}| = 0$, with $k_2 = k_1$ and $m_2 = m_1$

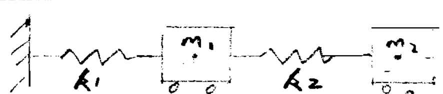
$$\begin{vmatrix} 2k_1 - \omega_j^2 m_1 & -k_1 \\ -k_1 & k_1 - \omega_j^2 m_1 \end{vmatrix} = m_1^2\omega_j^4 - 3k_1 m_1 \omega_j^2 + k_1^2 = 0$$

$$\therefore \omega_j^2 = \frac{3k_1 m_1 \pm \sqrt{9k_1^2 m_1^2 - 4k_1^2 m_1^2}}{2m_1^2} = \frac{k_1}{m_1}\left(\frac{3 \pm \sqrt{5}}{2}\right)$$

$$\therefore \underline{\omega_1^2 = 0.382(k_1/m_1)} \quad \text{and} \quad \underline{\omega_2^2 = 2.618\,(k_1/m_1)}$$

From 1st equation of motion;-

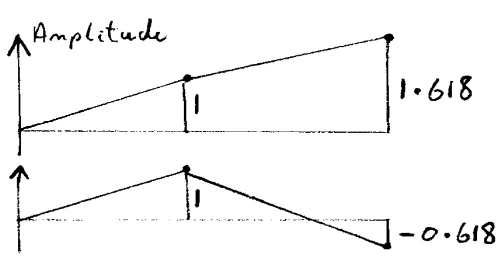
$[-m_1\omega_n^2 Y_1 + (k_1+k_2)Y_1 - k_2 Y_2]\cos\omega_n t = 0.$



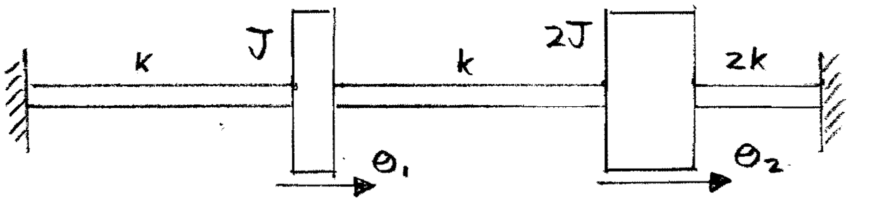
$\therefore \dfrac{Y_1}{Y_2} = \dfrac{k_2}{k_1+k_2 - m_1\omega_n^2}$. Putting $k_2 = k_1$, $m_2 = m_1$

At ω_1, $\dfrac{Y_1}{Y_2} = \dfrac{k_1}{2k_1 - m_1\omega_1^2} = \dfrac{1}{2-0.382} = \dfrac{1}{1.618}$

At ω_2, $\dfrac{Y_1}{Y_2} = \dfrac{k_1}{2k_1 - m_1\omega_2^2} = \dfrac{1}{2-2.618} = \dfrac{-1}{0.618}$



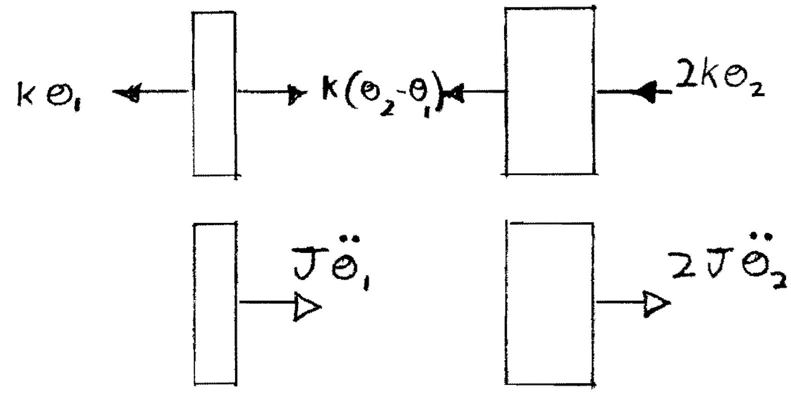
Q3



<u>Torques on discs</u>

=

<u>Moments of Inertia × Angular accel^ns</u>



$$M(\rightarrow)\quad J\ddot{\theta}_1 = k(\theta_2-\theta_1) - k\theta_1$$

$$2J\ddot{\theta}_2 = -k(\theta_2-\theta_1) - 2k\theta_2$$

$$\therefore \begin{bmatrix} J & 0 \\ 0 & 2J \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 3k \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = 0$$

As in question 2, find ω_n by putting $\left|\{[K]-\omega_n^2[J]\}\right| = 0$

$$\begin{vmatrix} 2k-\omega^2 J & -k \\ -k & 3k-\omega^2 2J \end{vmatrix} = 2J^2\omega^2 - 7Jk\omega^2 + 5k^2$$

$$= (J\omega^2 - k)(2J\omega^2 - 5k) = 0$$

$\therefore$ $\underline{\omega_1^2 = k/J}$ and $\underline{\omega_2^2 = 5k/2J}$

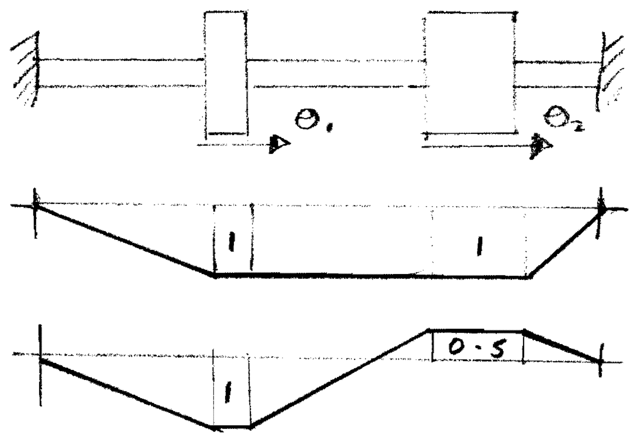
From 1st equation of motion; putting $\underline{\theta} = \underline{\Theta}\cos\omega_n t$

$$-J\omega_n^2\Theta_1 + 2k\Theta_1 - k\Theta_2 = 0$$

$$\therefore \frac{\Theta_1}{\Theta_2} = \frac{k}{2k-\omega_n^2 J}$$

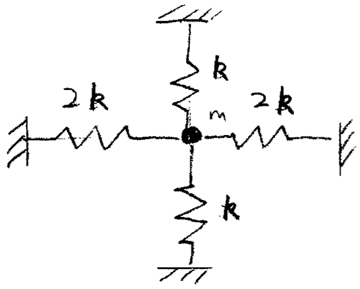
At ω_1, $\underline{\dfrac{\Theta_1}{\Theta_2} = \dfrac{k}{2k-k} = \dfrac{1}{1}}$

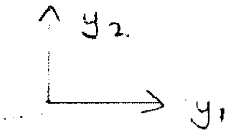
At ω_2 $\underline{\dfrac{\Theta_1}{\Theta_2} = \dfrac{k}{2k-\frac{5k}{2}} = \dfrac{1}{-0.5}}$



For Matlab results, see Supplement on p12

Q4 a)



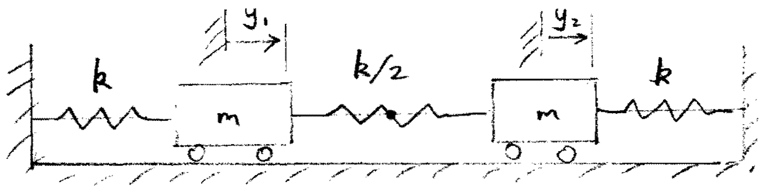


Modes are;-

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \omega_1^2 = \frac{2k+2k}{m} = \frac{4k}{m}$$

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \omega_2^2 = \frac{k+k}{m} = \frac{2k}{m}$$

b)

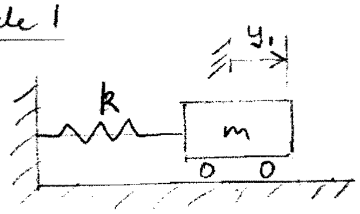


Modes are;

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ 2 masses move in phase. No force in central spring

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$ 2 masses 180° out of phase. Node at centre of central spring.

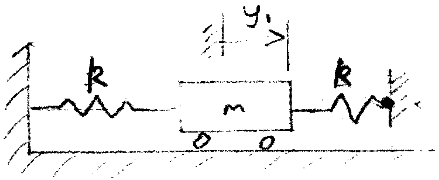
Mode 1



Since no force in central spring;-

$$\underline{\omega_1^2 = k/m}$$

Mode 2



Since $y_2 = -y_1$ in this mode, the central spring of stiffness $\frac{k}{2}$ is compressed thro' a distance $2y_1$ and so experiences a compressive force of magnitude $\frac{k}{2} \times 2y_1 = ky_1$. Hence its apparent stiffness is k. [Alternatively, since there must be a node at the centre of the spring, its stiffness is doubled, stiffness $= 2 \times \frac{k}{2} = k$] $\therefore$ $\underline{\omega_2^2 = \frac{k+k}{m} = \frac{2k}{m}}$

Q4(b) Cont.

(i) Initial Condition $\underset{\sim}{y} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$, $\dot{\underset{\sim}{y}} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ excites only the first mode

$$\therefore \quad y_1 = y_2 = \cos \omega_1 t$$

(ii) Initial condition $\underset{\sim}{y} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$, $\dot{\underset{\sim}{y}} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ excites both modes, and by inspection an equal amount of the $\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$ and $\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$ modes satisfy the initial conditions

$$\therefore \quad y_1 = \frac{1}{2}\left(\cos \omega_1 t + \cos \omega_2 t\right)$$
$$y_2 = \frac{1}{2}\left(\cos \omega_1 t - \cos \omega_2 t\right)$$

(iii) Initial condition $\underset{\sim}{y} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$, $\dot{y} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$ is less obvious.

Free vibration in mode 1 gives, for any initial condition

$$y_1 = y_2 = A \cos \omega_1 t + B \sin \omega_1 t$$

and for mode 2,

$$y_1 = -y_2 = C \cos \omega_2 t + D \sin \omega_2 t$$

Any general motion is the sum of the normal modes

$$\therefore \quad \begin{aligned} y_1 &= A \cos \omega_1 t + B \sin \omega_1 t + C \cos \omega_2 t + D \sin \omega_2 t \\ y_2 &= A \cos \omega_1 t + B \sin \omega_1 t - C \cos \omega_2 t - D \sin \omega_2 t \end{aligned}$$

and at $t = 0$, $\left.\begin{aligned} A + C &= y_1 = 0 \\ A - C &= y_2 = 0 \end{aligned}\right\}$ $A = C = 0$

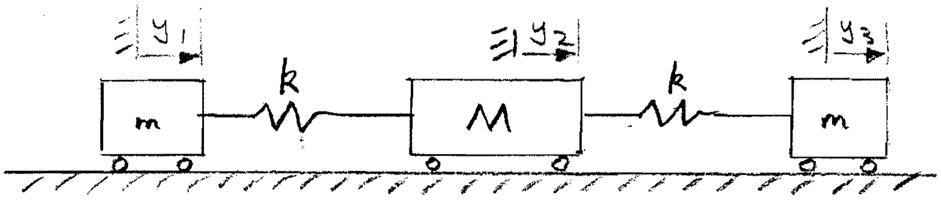
Differentiate for velocities

$$\begin{aligned} \dot{y}_1 &= B\omega_1 \cos \omega_1 t + D\omega_2 \cos \omega_2 t \\ \dot{y}_2 &= B\omega_1 \cos \omega_1 t - D\omega_2 \cos \omega_2 t \end{aligned}$$

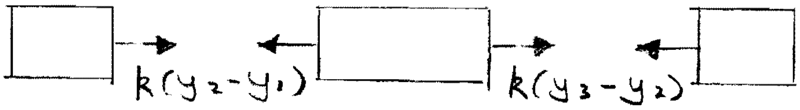
and at $t = 0$, $\left.\begin{aligned} B\omega_1 + D\omega_2 &= \dot{y}_1 = 1 \\ B\omega_1 - D\omega_2 &= \dot{y}_2 = 0 \end{aligned}\right\}$ $B = \frac{1}{2\omega_1}$, $D = \frac{1}{2\omega_2}$

$$\therefore \quad \begin{aligned} y_1 &= \frac{1}{2}\left(\frac{1}{\omega_1} \sin \omega_1 t + \frac{1}{\omega_2} \cos \omega_2 t\right) \\ y_2 &= \frac{1}{2}\left(\frac{1}{\omega_1} \sin \omega_1 t - \frac{1}{\omega_2} \cos \omega_2 t\right) \end{aligned}$$

Q5



Forces



Mass x Accel^{n}s

$R(\rightarrow)$
$$m\ddot{y}_1 = k(y_2 - y_1)$$
$$M\ddot{y}_2 = k(y_3 - y_2) - k(y_2 - y_1)$$
$$m\ddot{y}_3 = -k(y_3 - y_2)$$

$$\therefore \begin{bmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{bmatrix}\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_3 \end{bmatrix} + \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad - (1)$$

For normal modes, $\underline{y}_j = \underline{Y}_j \cos \omega_j t \;\therefore \{[K] - \omega_j^2[M]\}\underline{Y}_j = 0 \quad (2)$

Natural frequencies ω_n found from $|\{[K] - \omega_j^2[M]\}| = 0$

$$\therefore \begin{vmatrix} k-\omega_j^2 m & -k & 0 \\ -k & 2k-\omega_j^2 M & -k \\ 0 & -k & k-\omega_j^2 m \end{vmatrix} = 0$$

$$\therefore (k-\omega_j^2 m)\left[(2k-\omega_j^2 M)(k-\omega_j^2 m) - k^2\right] + k(-k(k-\omega_j^2 m)) = 0$$

$$\therefore (k-\omega_j^2 m)\left[2k^2 - \omega_j^2 k(M+2m) + \omega_j^4 Mm - k^2 - k^2\right] = 0$$

$$\therefore -\omega_j^2 (k-\omega_j^2 m)\left[k(M+2m) - \omega_j^2 Mm\right] = 0.$$

So natural frequencies are:-

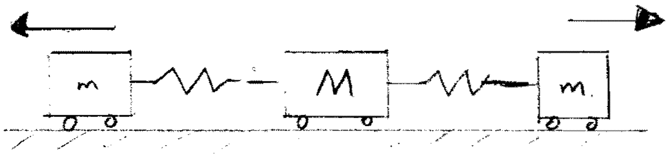
$$\omega_1^2 = 0, \quad \omega_2^2 = k/m, \quad \omega_3^2 = \frac{k(M+2m)}{Mm}$$

Q5 (cont) Normal modes

$\omega_1^2 = 0$: (2) $\Rightarrow$ $Y_1 = Y_2 = Y_3$ i.e. Rigid Body motion $\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\omega_2^2 = \frac{k}{m}$: (2) $\Rightarrow \begin{bmatrix} 0 & -k & 0 \\ -k & 2k - \frac{kM}{m} & -k \\ 0 & -k & 0 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow Y_2 = 0 \Rightarrow Y_1 = -Y_3$

i.e. $\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$



$\omega_3^2 = \frac{k(M+2m)}{Mm}$

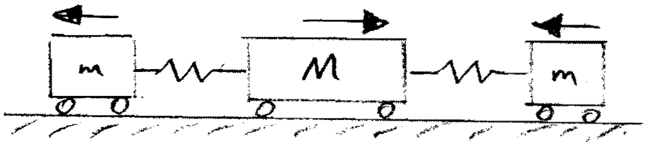
(2) $\Rightarrow k \begin{bmatrix} 1-(M+2m)/M & -1 & 0 \\ -1 & 2-(M+2m)/m & -1 \\ 0 & -1 & 1-(M+2m)/M \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Rows 1 & 3 $\Rightarrow$ $Y_1 = Y_3$

Row 2 $\Rightarrow$ $-2Y_1 + (2 - (M+2m)/m)Y_2 = 0$

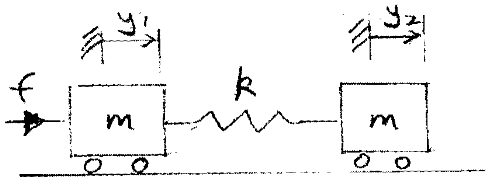
$\therefore \frac{Y_1}{Y_2} = 1 - \frac{M+2m}{2m} = \frac{2m - M - 2m}{2m} = \frac{-M}{2m}$

$\therefore \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2m/M \\ 1 \end{bmatrix}$

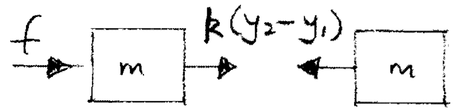


For Matlab results, see Supplement on p12

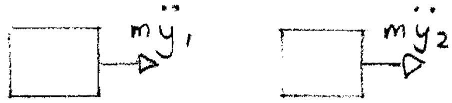
Q6



Forces



Mass x Accel^n



R(→). $m\ddot{y}_1 = k(y_2-y_1)+f$

$m\ddot{y}_2 = -k(y_2-y_1)$

$\therefore \underline{[M]\underline{\ddot{y}} + [K]\underline{y} = \underline{f}}$ where $[M]=\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$, $[K]=\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$ and $\underline{f}=\begin{bmatrix} f \\ 0 \end{bmatrix}$

a) Free motion, $\underline{f}=0$. To find ω_j, put $|[K]-\omega_j^2[M]|=0$ as in question 2.

$\therefore \Delta = (k-\omega_j^2 m)^2 - k^2 = m\omega_j^2(m\omega_j^2-2k)=0 \quad \therefore \underline{\omega_1^2=0,\ \omega_2^2=\frac{2k}{m}}$

From $\{[K]-\omega_j^2[M]\}\underline{Y}_j=0$. for $\omega_1^2=0$ $\underline{\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}=\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$, Rigid body motion

and for $\omega_2^2=2k/m$, $\begin{bmatrix} -k & -k \\ -k & -k \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \therefore \underline{\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}=\begin{bmatrix} 1 \\ -1 \end{bmatrix}}$

b) For $\underline{f}=\begin{bmatrix} F \\ 0 \end{bmatrix}\cos\omega t$, harmonic response in $\underline{y}=\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}\cos\omega t$, $\underline{\ddot{y}}=-\omega^2\underline{y}$

$\therefore \{[K]-\omega^2[M]\}\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}\cos\omega t=\begin{bmatrix} F \\ 0 \end{bmatrix}\cos\omega t \quad \therefore \underline{\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}=\{[K]-\omega^2[M]\}^{-1}\begin{bmatrix} F \\ 0 \end{bmatrix}}$

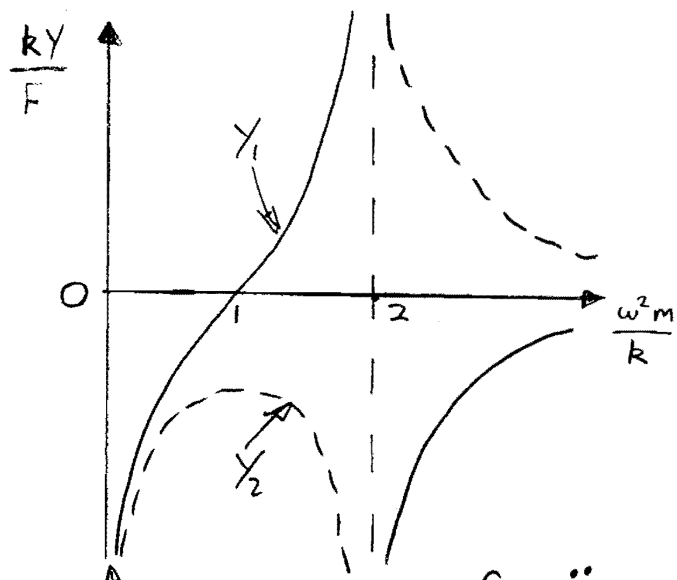
$\therefore \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}=\frac{-1}{m\omega^2(2k-m\omega^2)}\begin{bmatrix} k-\omega^2 m & k \\ k & k-\omega^2 m \end{bmatrix}\begin{bmatrix} F \\ 0 \end{bmatrix}$

$\therefore Y_1 = \frac{-(k-\omega^2 m)F}{m\omega^2(2k-m\omega^2)}$

$\therefore \underline{Y_1 = \frac{-\left(1-\frac{\omega^2 m}{k}\right)}{\left(\frac{\omega^2 m}{k}\right)\left(2-\frac{\omega^2 m}{k}\right)}\cdot\frac{F}{k}}$

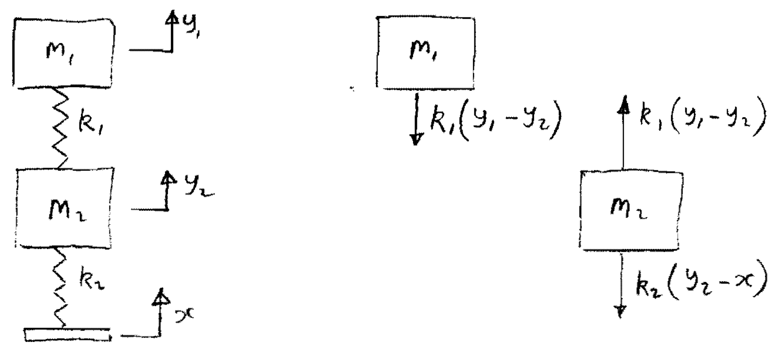
and $Y_2 = \frac{-kF}{m\omega^2(2k-m\omega^2)}$

$\therefore \underline{Y_2 = \frac{-1}{\left(\frac{\omega^2 m}{k}\right)\left(2-\frac{\omega^2 m}{k}\right)}\cdot\frac{F}{k}}$



↑ Negative because $f=m\ddot{Y} = -\omega^2 mY$

Q7.



Sum of forces = mass x accel $\left\{ \begin{array}{ll} \text{Mass } ① \therefore & m_1\ddot{y}_1 + k_1(y_1-y_2) = 0 \\ \text{Mass } ② \therefore & M_2\ddot{y}_2 + k_2(y_2-x) - k_1(y_1-y_2) = 0 \end{array} \right.$

put into matrix form : $\underline{\underline{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} + \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1+k_2 \end{bmatrix}\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ k_2 x \end{Bmatrix}}}$

Substitute $\underset{\sim}{x} = \underset{\sim}{X}\cos\omega t$ and $\underset{\sim}{y} = \underset{\sim}{Y}\cos\omega t$, $\ddot{\underset{\sim}{y}} = -\omega^2\underset{\sim}{Y}\cos\omega t$

$$\therefore \left[-\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\omega^2 + \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1+k_2 \end{bmatrix}\right]\begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ k_2 X \end{Bmatrix}$$

Now $m_1 = 500\text{ kg}$ $\quad m_2 = 40\text{ kg}$
$k_1 = 20\text{ kN/m}$ $\quad k_2 = 160\text{ kN/m}$
and $\omega = \dfrac{2\pi V}{L} = \dfrac{2\pi \times 50}{1{\cdot}25 \times 3{\cdot}6} = 69{\cdot}8\text{ rad/s}$ $\quad (11{\cdot}1\text{ Hz})$

$$\therefore \begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} = \begin{bmatrix} 20{,}000 - 500\omega^2 & -20{,}000 \\ -20{,}000 & 180{,}000 - 40\omega^2 \end{bmatrix}^{-1}\begin{Bmatrix} 0 \\ 160{,}000 \times 25 \end{Bmatrix} \text{ mm}$$

$$= \begin{bmatrix} -2417 & -20 \\ -20 & -15{\cdot}0 \end{bmatrix}^{-1}\begin{Bmatrix} 0 \\ 4000 \end{Bmatrix} \text{ mm}$$

$$\Delta = 2417 \times 15{\cdot}0 - 20^2 = 35855$$

$$\therefore \begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} = \frac{1}{35855}\begin{bmatrix} -15{\cdot}0 & 20 \\ 20 & -2417 \end{bmatrix}\begin{Bmatrix} 0 \\ 4000 \end{Bmatrix}$$

$$= \begin{Bmatrix} 20 \times 4000 / 35855 \\ -2417 \times 4000 / 35855 \end{Bmatrix} = \begin{Bmatrix} 2{\cdot}2 \\ -270 \end{Bmatrix} \text{ mm}$$

<u>Caravan amplitude = 2·2 mm</u> ; <u>Axle amplitude = 270 mm</u>
(The two are out-of-phase)

For natural frequencies (with $x = 0$) use the method of Q2

$$\begin{vmatrix} 20 - 0.5\omega^2 & -20 \\ -20 & 180 - 0.04\omega^2 \end{vmatrix} = 0$$

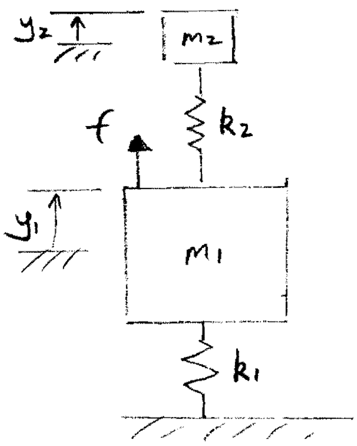
Excitation very close to resonance. Damping provided by shock absorbers prevents excessive amplitude

$\therefore (20 - 0.5\omega^2)(180 - 0.04\omega^2) - 400 = 0$

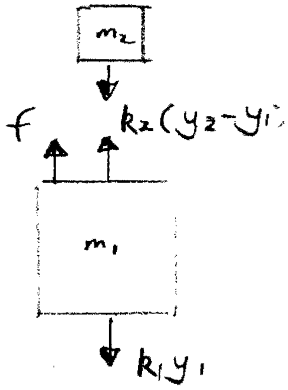
$\therefore 0.02\omega^4 - 90{\cdot}8\omega^2 + 3600 - 400 = 0$

$\therefore \omega^2 = \dfrac{90{\cdot}8 \pm \sqrt{7989}}{0.04}$ $\quad \therefore \omega_1, \omega_2 = 5{\cdot}96,\ 67{\cdot}1\text{ rad/s}$

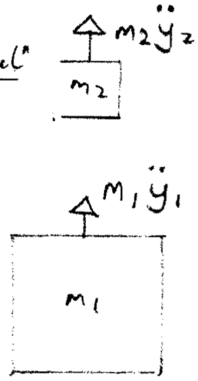
Q8 $\Omega = (s_1/m_1)^{1/2} = (2.7\times10^6/10^4)^{1/2} = (270)^{1/2} = \underline{16.4 \text{ rad/s}}$ $\therefore f = 2.62$ Hz



Forces



Mass x Accel^n



R(↑) $m_1\ddot{y}_1 = f + k_2(y_2-y_1) - k_1 y_1$

$m_2\ddot{y}_2 = -k_2(y_2-y_1)$

$$\therefore \underline{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}}\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} + \underline{\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}}$$

If $f = F\cos\omega t$, then harmonic response is $\underline{y} = \underline{Y}\cos\omega t$,

since $\ddot{\underline{y}} = -\omega^2\underline{y}$:- $\begin{bmatrix} k_1+k_2-\omega^2 m_1 & -k_2 \\ -k_2 & k_2-\omega^2 m_2 \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}$

$$\therefore \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} k_2-\omega^2 m_2 & k_2 \\ k_2 & k_1+k_2-\omega^2 m_1 \end{bmatrix}\begin{bmatrix} F \\ 0 \end{bmatrix}$$

where $\Delta = (k_1+k_2-\omega^2 m_1)(k_2-\omega^2 m_2) - k_2^2$

$= m_1 m_2\omega^4 - [m_2(k_1+k_2) + m_1 k_2]\omega^2 + k_1 k_2$

$\therefore \underline{Y_1 = (k_2-\omega^2 m_2)F/\Delta}$

The condition for $Y_1 = 0$ when excited at $\omega = \Omega$ is thus

$k_2 - \Omega^2 m_2 = 0$ ie $\underline{k_2/m_2 = \Omega^2}$.

For the case $m_2 = 1000$ kg $k_2 = 270$ kN/m, by inspection

$\frac{k_2}{m_2} = \frac{k_1}{m_1} = \Omega^2$ and this condition is met, ie the

absorber is "tuned".

Q8 (cont). Natural frequencies ω_n are found from solution of $\{[K] - \omega_j^2[M]\}\underline{Y}_j = 0$

Solution is $\Delta = 0$

ie $m_1 m_2 \omega_j^4 - [m_2(k_1+k_2) + m_1 k_2]\omega_j^2 + k_1 k_2 = 0$

For the case $m_1 = 10\,000$ kg $\quad m_2 = 1000$ kg
$k_1 = 2.7$ MN/m $\quad k_2 = 270$ kN/m $\Big\} \frac{m_1}{m_2} = \frac{k_1}{k_2} = 10$

$$\therefore \frac{m_1^2}{10}\omega_j^4 - \left[\frac{m_1}{10}\left(\frac{11}{10}\right)k_1 + \frac{m_1 k_1}{10}\right]\omega_n^2 + \frac{k_1^2}{10} = 0$$

$$\therefore \Delta = \frac{m_1^2}{10}\left[\omega_j^4 - 2.1\left(\frac{k_1}{m_1}\right)\omega_j^2 + \left(\frac{k_1}{m_1}\right)^2\right] = 0$$

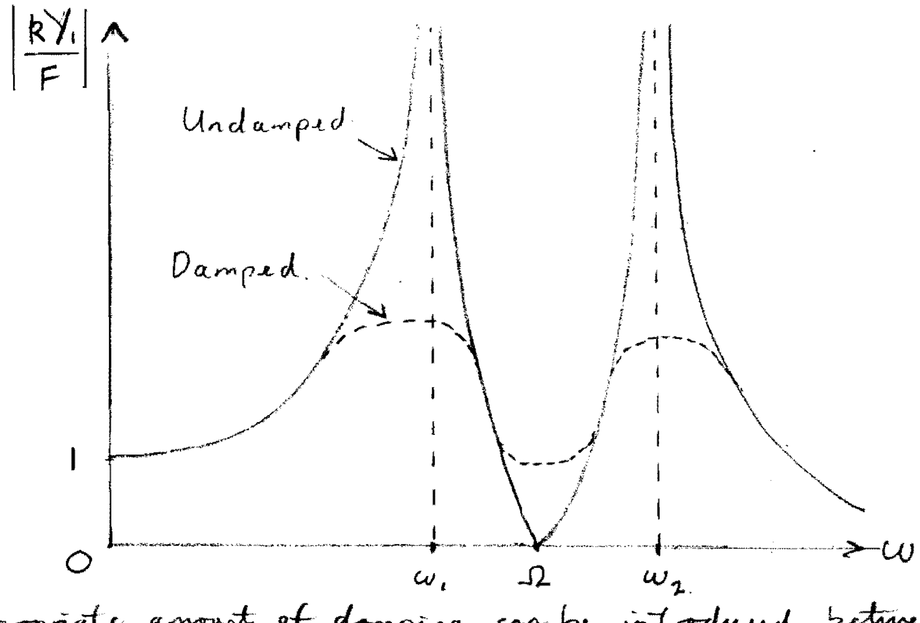
$$\therefore \omega_j^2 = \left(\frac{k_1}{m_1}\right)^2\left[\frac{2.1 \pm \sqrt{2.1^2 - 4}}{2}\right]$$

$$= \Omega^2\left[1.05 \pm 0.32\right] = \Omega^2\left[0.73,\ 1.37\right]$$

$$\therefore \omega_1 = \Omega \times 0.854 = 14.0 \text{ rad/s} \quad (\equiv 2.23 \text{ Hz})$$
$$\omega_2 = \Omega \times 1.17 = 19.2 \text{ rad/s} \quad (\equiv 3.06 \text{ Hz})$$

$\therefore$ We can write $\quad \Delta = \dfrac{m_1^2}{10}(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)$

and hence $\quad Y_1 = \dfrac{(k_2 - \omega^2 m_2)F}{\Delta} = \dfrac{(\Omega^2 - \omega^2)\Omega^2}{(\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)}\dfrac{F}{s_1}$



An appropriate amount of damping can be introduced between the two masses to limit the amplitudes of the resonant peaks

Q3 supplement: The given Matlab script:

```
%  Check IA Vibrations EP3 Q3
close all
clear all
K=[2 -1;-1 3];
M=[1 0;0 2];
[V,D]=eig(K,M);
for n=1:2
  mode=V(:,n);
  mode=mode/mode(1); % Normalise to unit amplitude on mass 1
  freq2=D(n,n);
  disp(sprintf('Mode %i has squared frequency %g and mode [%g,
%g]',n,freq2,mode))
end
```

produces this output:

```
Mode 1 has squared frequency 1 and mode [1, 1]
Mode 2 has squared frequency 2.5 and mode [1, -0.5]
```

Q5 supplement: The Matlab script:

```
%  Check IA Vibrations EP3 Q5
close all
clear all
m1=1;
m2=1;
K=[1 -1 0;-1 2 -1;0 -1 1];
M=[m1 0 0;0 m2 -0;0 0 m1];
[V,D]=eig(K,M);
for n=1:3
  mode=V(:,n);
  mode=mode/mode(1);
  freq2=D(n,n);
  disp(sprintf('Mode %i has squared frequency %g and mode [%g, %g,
%g]',n,freq2,mode))
end
```

produces this output:

```
Mode 1 has squared frequency 9.99658e-17 and mode [1, 1, 1]
Mode 2 has squared frequency 1 and mode [1, -1.37383e-16, -1]
Mode 3 has squared frequency 3 and mode [1, -2, 1]
```

Notice that the zero frequency of mode 1 and the zero motion of the middle mass in mode 2 both come out as very small but non-zero numbers, because of rounding errors.

To explore other mass ratios, change the values of m1 and m2 in the code. For example, m1=1, m2=2 gives

```
Mode 1 has squared frequency 4.94396e-17 and mode [1, 1, 1]
Mode 2 has squared frequency 1 and mode [1, -5.55112e-17, -1]
Mode 3 has squared frequency 2 and mode [1, -1, 1]
```

Note that only the 3rd mode has changed, as expected. A few cases like this will verify the theoretical expression involving the mass ratio.

J Woodhouse

Easter 2014