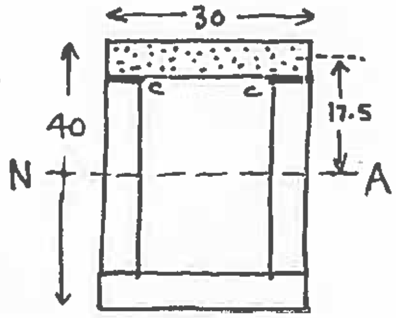
FIRST YEAR

Part IA Paper 2: Structures and Materials

STRUCTURAL MECHANICS

Solutions for Examples Paper 6

1.



$$I = \frac{1}{12}(30\times40^3 - 20\times30^3) = 11.5\times10^4\ \text{cm}^4.$$

$$M_{max} = 20\times10^3\times5 = 10^5\ \text{Nm}.$$

$$\frac{\sigma}{y} = \frac{M}{I} \Rightarrow \sigma = \frac{10^5\times0.2}{11.5\times10^{-4}}\ \frac{\text{Nm m}}{\text{m}^4}$$

$$= 17.3\times10^6\ \text{N/m}^2$$

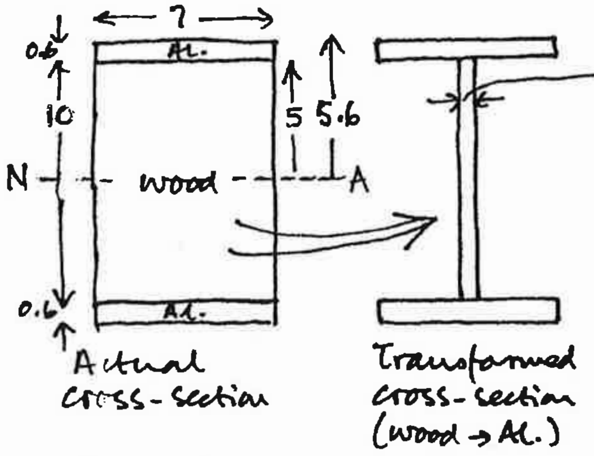
$$= \underline{17.3\ \text{N/mm}^2} = \text{greatest bending stress.}$$

Shear force per unit length of beam across the two cuts cc is equal to $\frac{SA\bar{y}}{I}$, where $A\bar{y}$ is first moment of area of shaded part of cross-section about NA.

$$\frac{SA\bar{y}}{I} = \frac{(20\times10^3)(30\times5)(17.5)}{11.5\times10^4}\ \frac{\text{N cm}^3}{\text{cm}^4} = 456\ \text{N/cm}.$$

Half of this is carried by each 'cut' c, so $\underline{228\ \text{N/cm}}$

Screws are at 7.5cm spacing, so shearing load per screw $= 228\times7.5 = \underline{1710\ \text{N}}$

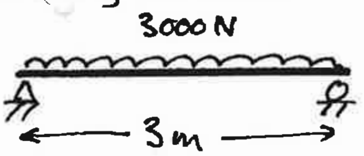
2. 

Width of equivalent Al. web $= 7 \times \dfrac{E_{wood}}{E_{Al}} = 7 \times \dfrac{9}{70}$

$= 0.9$ cm.

$$\underbrace{\begin{aligned} I &= \frac{1}{12} \times 7 \times 11.2^3 - \frac{1}{12} \times 6.1 \times 10^3 \\ &= 819.5 - 508.3 = 311.2\,\text{cm}^4 \end{aligned}}_{\text{for transformed section}}$$

Loading on beam:



$$M_{max} = \frac{3000 \times 3}{8} = 1125 \text{ Nm}$$

$$S_{max} = 1500 \text{ N}$$

Max. bending stress in Al: $\sigma = \dfrac{My}{I} = \dfrac{1125 \times 5.6 \times 10^8}{10^2 \times 311.2} \ \dfrac{\text{Nm.m}}{\text{m}^4}$

$= 20.24 \times 10^6$ N/m²

$= 20.2$ MPa.

Max bending stress in wood: $\sigma = \dfrac{My}{I_{wood}} = 1125 \times \dfrac{5.0}{10^2} \times \dfrac{10^8}{311.2} \times \dfrac{9}{70}$

$= 2.32$ MPa.

Shearing stress. First work out shearing force per unit length of beam between flange and web.

For flange about NA, $A\bar{y} = 7 \times 0.6 \times 5.3 \text{ cm}^3 = 22.3 \text{ cm}^3$

Force per unit length $= \dfrac{SA\bar{y}}{I} = \dfrac{1500 \times 22.3}{311.2} \ \dfrac{\text{N cm}^3}{\text{cm}^4} = 107.3$ N/cm

$= 10.7$ N/mm.

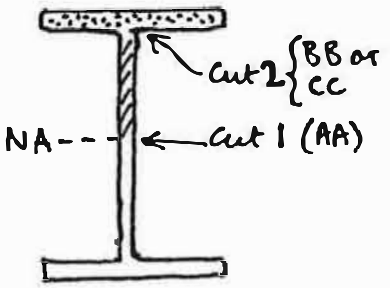
Width of glued joint $= 70$ mm.

Shearing stress in glue $= \dfrac{10.7}{70} \ \dfrac{\text{N}}{\text{mm}} \cdot \dfrac{1}{\text{mm}} = 0.153 \ \dfrac{\text{N}}{\text{mm}^2}$

$= \underline{0.153 \text{ MPa}}.$

For UB 356x171x57

3. $I_{xx} = 16040 \text{ cm}^4$.



<u>For cut 2,</u> $A\bar{y} = 1.3 \times 17.1 \times 17.2 \text{ cm}^3$
$= 382.3 \text{ cm}^3$.

This calculation applies to both cuts BB and CC: it is reasonable to neglect the small weld area, etc. in such calculations.

<u>For cut 1,</u> $A\bar{y} = 382.3 + 16.5 \times 0.8 \times \frac{16.5}{2}$
$= 382.3 + 108.9$
$= 491.2 \text{ cm}^3$.

Let us work out what shear force must be applied to the beam in order to reach the allowable shear force per unit length of beam on each of cuts AA, BB and CC separately: then the <u>smallest</u> of these is the required answer.

Allowable shear force per cm length at cuts AA or BB in the web $= 8 \times 10 \times 120 = 9{,}600$ N/cm.

web thickness, mm — 1 cm length (mm) — given allowable shear strength

Allowable shear force per cm length at cut CC is calculated as above, except that web thickness of 8 mm must be replaced by combined width of the two weld "throats" ie $\frac{8}{\sqrt{2}} \times 2$; giving 13,600 N/cm.

(a) For web at AA to govern $\frac{S A\bar{y}}{I} = 9600 \quad \therefore S = \frac{9600\, I}{A\bar{y}}$

$= \frac{9600 \times 16040}{491.2} = \underbrace{313{,}000}_{3\text{ sig. figs}} \frac{\text{N}}{\text{cm}} \times \frac{\text{cm}^4}{\text{cm}^3} = \underline{\underline{313 \text{ kN}}}$ ← smallest; governs.

(b) For web at BB to govern, $S = \frac{9600 \times 16040}{382.3} = 403$ kN

(c) For welds at CC to govern, $S = \frac{13{,}600 \times 16040}{382.3} = 571$ kN

4. from data book, $I_{xx} = 14307 \times 10^{-8}\ m^4$

$I_{yy} = 4849 \times 10^{-8}\ m^4$

$E = 210 \times 10^{9}\ N/m^2$

Euler buckling load $= \dfrac{\pi^2 E I}{l^2}$

About major axis, $P_E = \dfrac{\pi^2 \times 210 \times 10^9 \times 14307 \times 10^{-8}}{12^2}$

$= 2059\ kN$

About minor axis, $P_E = 698\ kN$

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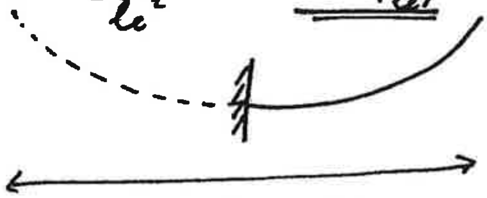
5. $P = \dfrac{\pi^2 E I}{l_e^2} = \underline{\underline{P_E \cdot \left(\dfrac{l}{l_e}\right)^2}}$

The effective length is the length between two points where the B.M. is zero (eg. free ends, curvature = 0, ..)

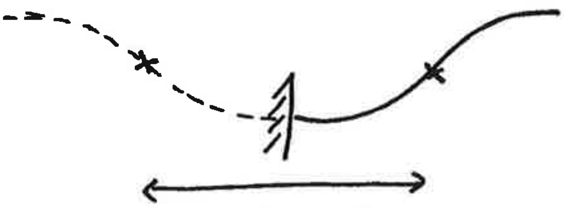
(a)



$l_e = 2l$

$\underline{\underline{P_a = \dfrac{P_E}{4}}}$

(b)



$l_e = l$

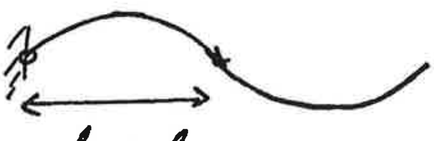
$\underline{\underline{P_b = P_E}}$

(c)

$l_e = \dfrac{l}{2}$

$\underline{\underline{P_c = 4 P_E}}$

(d)

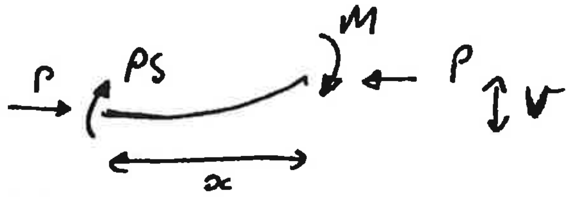


$l_e = \dfrac{l}{2}$

$\underline{\underline{P_d = 4 P_E}}$

6. Overall equilibrium

At a general position x



Equilibrium, $M = Pv - P\delta$

Elastic law, $M = EI \, . \, \Delta \kappa$

Compatibility, $\Delta \kappa = \kappa = -\dfrac{d^2 v}{dx^2}$ for small v

$$\therefore \frac{EI}{P} \cdot \frac{d^2 v}{dx^2} + v = \delta$$

Particular Integral, $v = \delta$

General solution, $v = A \sin \alpha x + B \cos \alpha x + \delta$ where $\alpha^2 = \dfrac{P}{EI}$

<u>Boundary Conditions</u>

At $x = 0$, $v = 0$ $\Rightarrow 0 = B + \delta$ $\Rightarrow B = -\delta$

At $x = 0$, $\dfrac{dv}{dx} = 0$ $\Rightarrow 0 = A\alpha$ $\Rightarrow A = 0$

At $x = l$, $v = \delta$ $\Rightarrow \delta = \delta(-\cos \alpha l + 1)$

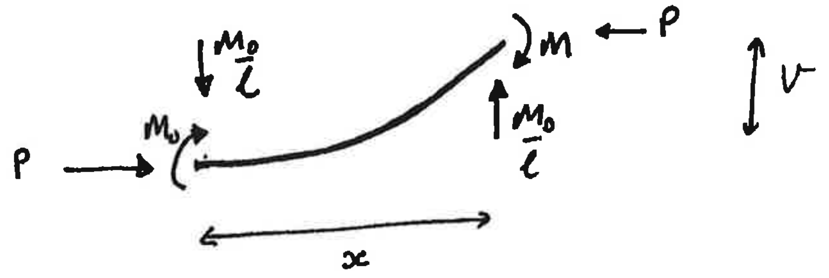
<u>Either</u> $\delta = 0$, trivial soln.

<u>or</u> $\cos \alpha l = 0$ $\Rightarrow \alpha = \dfrac{\pi}{2l}, \dfrac{3\pi}{2l} \ldots$

for $\alpha = \dfrac{\pi}{2l}$, $\underline{\underline{P = EI\alpha^2 = \dfrac{EI\pi^2}{4l^2}}}$.

7.(a) A good estimate, measured from the diagram, is $l_e = 0.7l$, which gives $P = 2.04 \frac{\pi^2 EI}{l^2}$

(b) Equilibrium at a general position x



$$M + M_0 = M_0 \frac{x}{l} + Pv$$

$$M = -EI \frac{d^2v}{dx^2}$$

$$\therefore \frac{EI}{P} \frac{d^2v}{dx^2} + v = \frac{M_0}{P}\left(1 - \frac{x}{l}\right)$$

Particular Integral, $v = \frac{M_0}{P}\left(1 - \frac{x}{l}\right)$

General solution, $v = A\sin\alpha x + B\cos\alpha x + \frac{M_0}{P}\left(1 - \frac{x}{L}\right)$ $\left[\alpha^2 = \frac{P}{EI}\right]$

Boundary conditions:

At $x = 0$, $v = 0$ $\Rightarrow 0 = B + \frac{M_0}{P}$ $\Rightarrow$ $B = -\frac{M_0}{P}$

At $x = 0$, $\frac{dv}{dx} = 0$, $\Rightarrow A\alpha - \frac{M_0}{Pl} = 0$ $\Rightarrow$ $A = \frac{M_0}{Pl\alpha}$

At $x = l$, $v = 0$ $\Rightarrow$ $0 = \frac{M_0}{Pl\alpha} \sin\alpha l - \frac{M_0}{P} \cos\alpha l$

$$\therefore 0 = \tan\alpha L - \alpha L$$

from hint, $\alpha l = 4.5$

$$\therefore P = 4.5^2 \frac{EI}{l^2} = 2.05 \frac{\pi^2 EI}{l^2}$$

8. (a) When $\eta = 0$:

<u>either</u> $\sigma_c = \sigma_y = 250 \ N/mm^2$ – general yield.

<u>or</u> $\sigma_c = \sigma_E = \dfrac{\pi^2 E}{(l/r)^2} = \dfrac{2.07 \times 10^6 \ N/mm^2}{(l/r)^2}$

– Euler buckling.

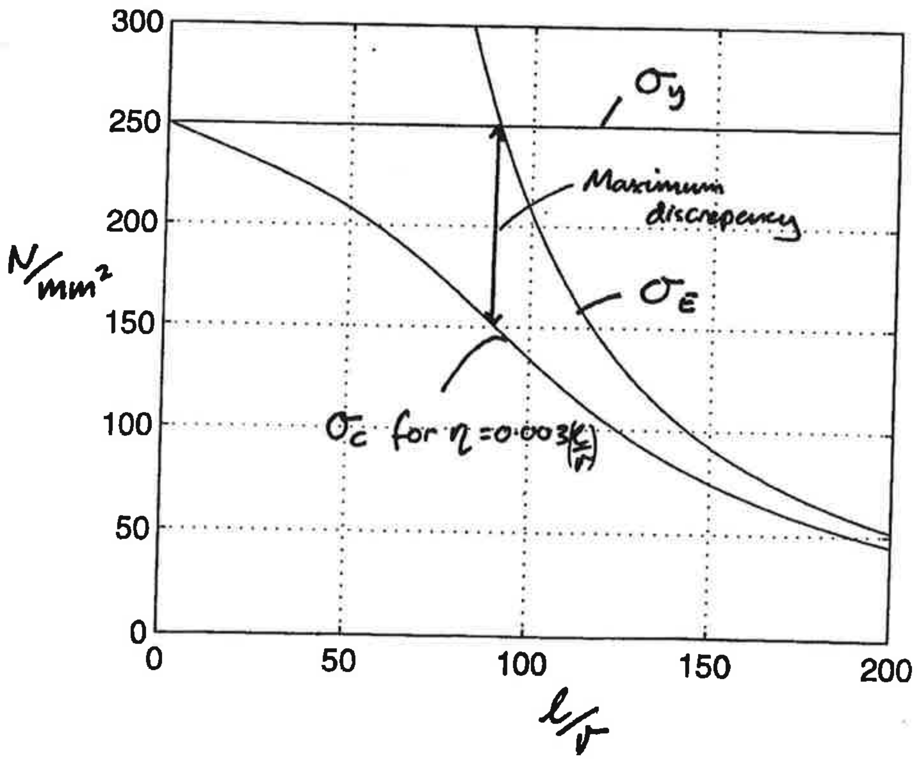
(b) For $\eta = 0.003 \ l/r$, Perry formula becomes

$$(250 - \sigma_c)\left(\frac{2.07 \times 10^6}{(l/r)^2} - \sigma_c\right) = 0.003 \times \frac{2.07 \times 10^6}{(l/r)} \sigma_c$$

$$\sigma_c^2 - \underbrace{\left(\frac{2.07 \times 10^6}{(l/r)^2} + \frac{6210}{(l/r)} + 250\right)}_{b} \sigma_c + \underbrace{\frac{517.5 \times 10^6}{(l/r)^2}}_{c} = 0$$

Critical load is then lower root, given by

$$\sigma_c = \frac{b - \sqrt{b^2 - 4c}}{2}$$



8 (b) (cont.) Maximum discrepancy at $\sigma_y = \sigma_E$

$\therefore \ l/r = 91$, $\sigma_c = 149 \ N/mm^2$

Reduction in critical load of 40%.

(c) For $\sigma_E = \sigma_y$ Perry's formula becomes

$$(\sigma_y - \sigma_c)^2 = \eta \, \sigma_y \sigma_c \quad \therefore \quad \sigma_c^2 - (2+\eta)\,\sigma_y \sigma_c + \sigma_y^2 = 0$$

$$\frac{\sigma_c}{\sigma_y} = \frac{(2+\eta) \pm \sqrt{(2+\eta)^2 - 4}}{2} \qquad \text{(choose lower root)}$$

$$= 1 + \frac{\eta}{2} - \frac{1}{2}\sqrt{\eta^2 + 4\eta} = 1 + \underset{\substack{\uparrow \\ \text{small in comparison} \\ \text{with } \sqrt{\eta}}}{\frac{\eta}{2}} - \sqrt{\eta}\,\underbrace{\sqrt{1 + \eta/4}}_{\simeq 1} =$$

$$\simeq 1 - \sqrt{\eta}$$

Struts with $\sigma_y = \sigma_E$ are very sensitive to even very small imperfections η. This was first discovered by Britvec, while a research student at CUED in the 1960's.

SDG, January 2023