ENGINEERING **FIRST YEAR**

Part IA Paper 1: Mechanical Engineering

MECHANICAL VIBRATIONS

Examples paper 1

Straightforward questions are marked with a †
Tripos-standard questions are marked * .

First-order systems: Transient response

†1. Fig. 1 shows a spring of stiffness k in series with a viscous dashpot of rate λ . Show that the displacement y is related to the input displacement x by

$$T\frac{dy}{dt} + y = x \qquad (1)$$

where $T = \lambda / k$.

By writing down a complementary function and particular integral, find an expression for the response of y if the system is initially at rest and receives a step input given by

$x = 0$ $\quad t < 0$
$x = x_0$ (constant) $\quad t \geq 0$

and sketch the response.

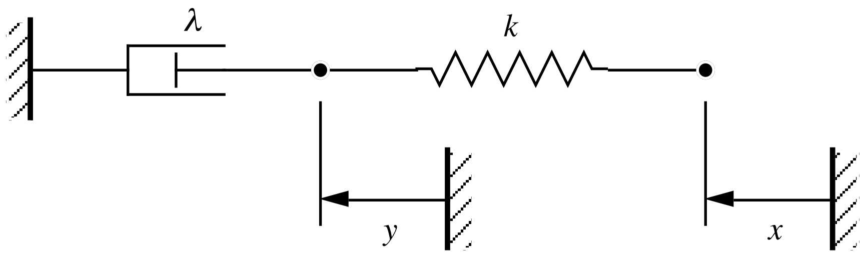

Fig. 1

†2. Fig. 2 shows a parallel RC circuit fed by a variable current source. No current is drawn at the output terminals. Show that the output voltage v is related to the current i by

$$RC\frac{dv}{dt} + v = iR$$

Initially $i = v = 0$. If at $t = 0$ the current is increased such that $i = i_0$ (constant) for $t \geq 0$, find an expression for the output voltage v as a function of time.

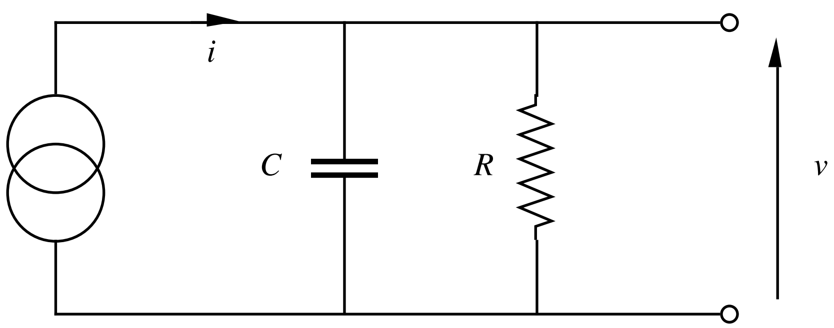

Fig. 2

3. Fig. 3 shows a model of a shock absorber from a car suspension comprising a spring of stiffness k in parallel with a viscous dashpot of rate λ. Derive a differential equation which relates the displacement y and its derivatives to the input force f.

By comparing this equation term by term with that given in Q1, write down an expression for the response in y caused by a step input force of magnitude f_0, and sketch this response.

Hence deduce the impulse response in y when the absorber receives an impulse of magnitude I, and sketch this response. (Hint: note that a unit impulse is the time derivative of a unit step input).

Now suppose that the step input is applied to the displacement y: what is the response of the force f?

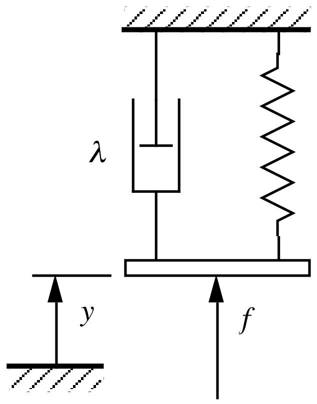

Fig. 3

4. †(i) A thermometer is in thermal equilibrium with a bath of water at 20°C. It is then quickly transferred to another bath at 35°C. The variation with time of its reading is shown in Fig. 4. Assuming the thermometer reading θ is related to the bath temperature θ_i by the differential equation

$T \dfrac{d\theta}{dt} + \theta = \theta_i$, estimate the time constant T.

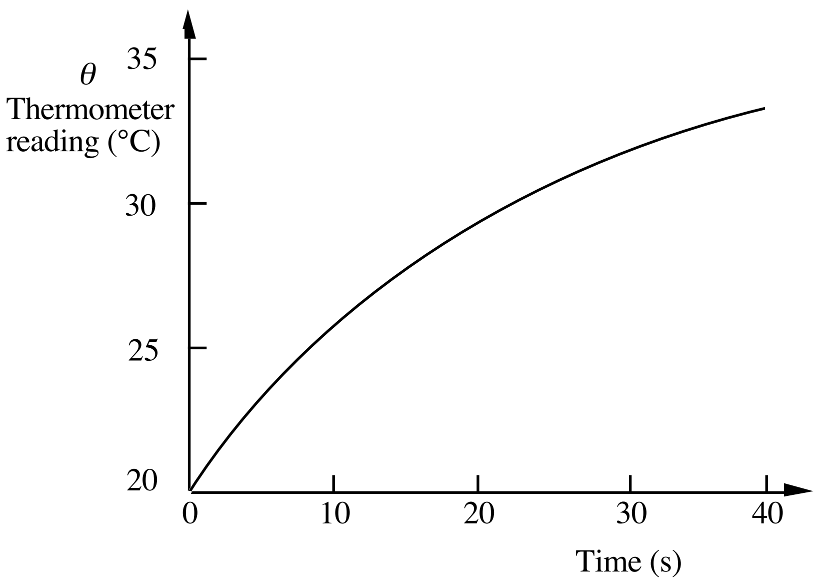

Fig. 4

*(ii) The same thermometer is again initially in thermal equilibrium with the bath of water at 20°C. The temperature of the bath is then raised at a uniform rate of 9°C/min. Deduce a solution to the differential equation to obtain an expression for θ as a function of time. Show that, after some time, θ lags θ_i by 3°C and sketch this response.

*5. A train of mass m moving at velocity v_0 strikes a viscous buffer whose damping rate is λ. After impact, the train does not lose contact with the buffer. Show that the subsequent velocity v of the train is governed by a first-order differential equation. (Hint: derive the d.e. in terms of the train velocity v and acceleration $\dot{v}$). What is the time constant?

How far does the buffer compress before the train comes to rest? (remember that $\dot{v} = v \dfrac{dv}{dx}$)

How long does this take?

First-order systems: Harmonic response

†6. The system shown in Fig. 1 is subjected to a harmonic input given by

$$x = \mathrm{Re}\{X\, e^{i\omega t}\} = X \cos \omega t$$

where X is taken to be real. By considering the particular integral of equation (1), find the steady-state harmonic response y given by

$$y = \mathrm{Re}\{Y\, e^{i\omega t}\} = |Y| \cos(\omega t - \phi)$$

and deduce expressions for the amplitude $|Y|$ and phase ϕ of the response in terms of X, ω, and the time constant T. Sketch a phasor diagram to represent each of the terms in equation (1).

Find the amplitudes of the response as $\omega \to 0$ and as $\omega \to \infty$, and give a physical explanation of the results.

7. Derive a differential equation relating the displacements x and y shown in Fig. 5. If x is forced to vary harmonically at 31.8 Hz with an amplitude of 25 mm, find the amplitude of y and its phase relative to x assuming $k = 100$ N/m and $\lambda = 0.5$ Ns/m . Sketch a phasor diagram.

Find the amplitude of the horizontal force which must be applied at A to cause the displacement x and find its phase relative to x. Indicate this force on your phasor diagram.

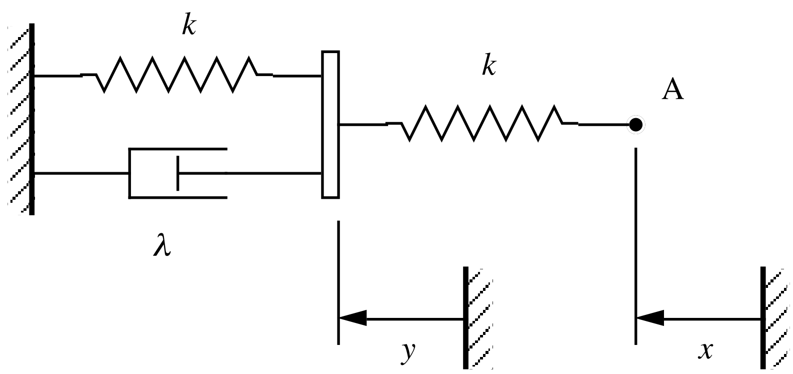

Fig. 5

Second-order systems: Transient response

(You are encouraged to use the information on page 6 of the 2000 Mechanics Data Book for Q's 8–10)

8. Fig. 6 shows a rotor with moment of inertia J mounted at one end of a light elastic shaft of torsional stiffness k. The angle of rotation of the rotor from its equilibrium position is θ. Derive an equation of motion for the rotor in terms of θ and its derivatives for free torsional oscillation of the system. If $\theta = 0$ and $\dot{\theta} = 50$ rad/s at $t = 0$ find the amplitude of the subsequent motion assuming $J = 0.2$ kg m^2 and $k = 1500$ Nm/rad.

In a real system, it is observed that after 10 cycles the amplitude of the motion has decreased by 10%. What is the logarithmic decrement δ for the damped system? Estimate how long it takes for the amplitude to decrease to 0.2% of its initial value. Find the value of the damping factor ζ and the quality factor Q.

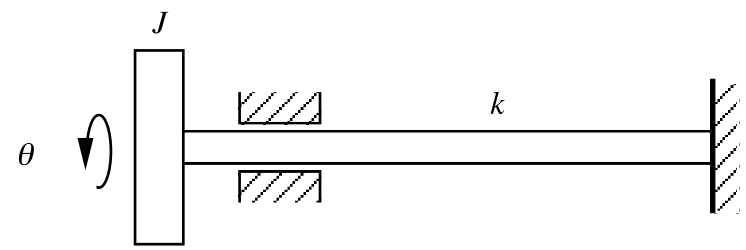

Fig. 6

9. A door is closed under the control of a spring and dashpot. The spring exerts a torque of 12.5 Nm when the door is closed and has a stiffness of 50 Nm/rad. The damping torque from the dashpot is 200 Nms/rad. The moment of inertia of the door about its hinges is 90 kg m^2.

The door is opened 90° and released. Show that the equation of the subsequent motion is

$$90\,\ddot{\theta} + 200\,\dot{\theta} + 50\,\theta = 50\left[\frac{\pi}{2} + \frac{1}{4}\right]$$

where θ is the rotation of the door from the open position.

Estimate the time of closure and the angular velocity at closure.

Explain the meaning of the term *critical damping*.

The system of the door and its closer is *over-damped*. How might you adjust the moment of inertia of the door to cause the system to be *under-damped*?

*10. For the circuit shown in Fig. 7, show that

$$LC\frac{R_1}{R_2}\frac{\mathrm{d}^2 v}{\mathrm{d}t^2} + \left(CR_1 + \frac{L}{R_2}\right)\frac{\mathrm{d}v}{\mathrm{d}t} + \left(1 + \frac{R_1}{R_2}\right)v = e.$$

(hint: let v_1 be the voltage at A and consider the sum of currents at A; then use current flow through L to find an expression for v_1 in terms of v.)

Find expressions for the undamped natural frequency ω_n and damping factor ζ of the circuit.

If the time constants L/R_2 and CR_1 are both equal to 1/3 ms and $R_1 = 3\,R_2$, sketch the response in v to a step change in e of 10 V.

What is the maximum value of v during the transient?

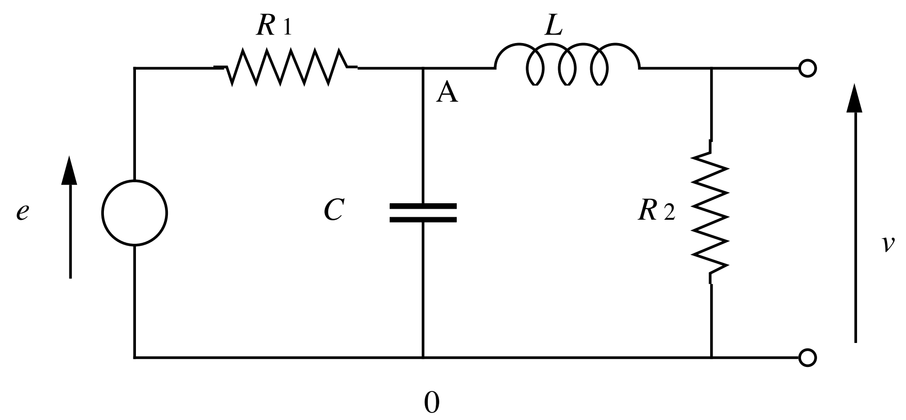

Fig. 7

No current is drawn at the output terminals.

Answers

1. $y = x_o (1 - e^{-t/T})$

2. $v = i_o R (1 - e^{-t/T})$ $T = RC$

3. $T\dfrac{dy}{dt} + y = \dfrac{f}{k}$

$y = \dfrac{f_o}{k}(1 - e^{-t/T})$

$y = \dfrac{I}{kT} e^{-t/T}$

$f = kT\delta(t) + k$ for $t \geq 0$

4. 20 s
$17 + 0.15\,t + 3\,e^{-t/20}$

5. $\dfrac{m}{\lambda}$

$\dfrac{v_o m}{\lambda}$

6. $|Y| = X / \sqrt{(1 + \omega^2 T^2)}$ $T = \lambda/k$
$\phi = \tan^{-1} \omega T$
X
0

7. $T\dfrac{dy}{dt} + y = \dfrac{x}{2}$ $T = \lambda/2k$
11 mm 26.6° lag 1.58 N 18.4° lead

8. $\dfrac{\ddot{\theta}}{\omega_n^2} + \theta = 0$ $\omega_n = \sqrt{(k/J)}$
0.577 rad 0.0105 43 s 0.00168 298

9. 7.1 s
0.075 rad/s
Increase it by more than 110 kgm^2

10. $\omega_n = \sqrt{\dfrac{1}{LC}\left(1 + \dfrac{R_2}{R_1}\right)}$

$\zeta = \dfrac{1}{2}\dfrac{R_2}{R_1}\dfrac{CR_1 + \dfrac{L}{R_2}}{\sqrt{LC\left(1 + \dfrac{R_2}{R_1}\right)}}$

2.9 V

For further practice, the following Tripos questions from Paper 1 are suitable:

2006 Q10; 2007 Q10; 2009 Q10; 2012 Q10; 2013 Q11

DC/JW