Part IA Paper 4: Mathematical Methods

Solutions - Examples paper 11 (Laplace Transforms)

1 (a) $\dfrac{1}{(x-3)(x-7)} = \dfrac{A}{x-3} + \dfrac{B}{x-7}$

$\times$ by $(x-3)$, put $x=3$ $\quad A = \frac{1}{3-7} = -\frac{1}{4}$

$\times$ by $(x-7)$, put $x=7$ $\quad B = \frac{1}{7-3} = \frac{1}{4}$

$$\therefore \frac{1}{(x-7)(x-3)} = \frac{1}{4}\left\{\frac{1}{x-7} - \frac{1}{x-3}\right\}$$

(b) $\dfrac{x+7}{(x+3)(x-7)^2} = \dfrac{A}{x+3} + \dfrac{B}{x-7} + \dfrac{C}{(x-7)^2}$

$\times$ by $(x+3)$, put $x=-3$ $\quad A = \frac{4}{100} = \frac{1}{25}$

$\times$ by $(x-7)^2$ put $x=7$ $\quad C = \frac{14}{10} = \frac{35}{25}$

To find B multiply out right hand side & equate powers of x^2

ie $\quad x+7 = A(x-7)^2 + B(x+3)(x-7) + C(x+3)$

$\therefore \quad 0 = A + B \quad$ (coefficients of x^2)

$\therefore \quad B = -\frac{1}{25}$

$$\therefore \frac{x+7}{(x+3)(x-7)^2} = \frac{1}{25}\left[\frac{1}{x+3} - \frac{1}{x-7} + \frac{35}{(x-7)^2}\right]$$

check rhs $\quad = \dfrac{1}{25}\left[\dfrac{(x-7)^2 - (x+3)(x-7) + 35(x+3)}{(x+3)(x-7)^2}\right]$

$\quad = \dfrac{1}{25}\left[\dfrac{x^2-14x+49 - x^2+4x+21 + 35x+105}{(x+3)(x-7)^2}\right]$

$\quad = \dfrac{1}{25}\left[\dfrac{25x+175}{(x+3)(x-7)^2}\right]$ ✓

(c) $\dfrac{x^2+3x+4}{x(x^2+4)} = \dfrac{A}{x} + \dfrac{B+Cx}{x^2+4} = \dfrac{A(x^2+4) + Bx + Cx^2}{x(x^2+4)}$

$\times x$, put $x=0$, $\frac{4}{4} = A \Rightarrow A = 1$

Equate coefficients of x^2 $\quad 1 = A + C \quad \therefore C = 0$

Equate coefficients of x $\quad 3 = B$

$$\therefore \frac{x^2+3x+4}{x(x^2+4)} = \frac{1}{x} + \frac{3}{x^2+4}$$

2 (a) $y(t) = t$, $\quad Y(s) \equiv \int_0^\infty t e^{-st} dt$

Integrate by parts $\quad = \left[-\frac{1}{s} t e^{-st}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st} dt$

at ∞ $\; te^{-st} \to 0$, at $x=0$ $\; te^{-st} \to 0$

$$= \frac{1}{s}\left[-\frac{1}{s} e^{-st}\right]_0^\infty = \underline{\underline{\frac{1}{s^2}}}$$

(b) $\quad y(t) = e^{at} \sin \omega t$

put $y(t) = \mathcal{I} e^{(a+i\omega)t}$ $\quad \left[= \mathcal{I}(e^{at}\cos\omega t + i \sin\omega t)\right]$

for $y(t) = e^{(a+i\omega)t}$

$$Y(s) = \int_0^\infty e^{-st} e^{(a+i\omega)t} dt = \int_0^\infty e^{(a+i\omega-s)t} dt$$

$$= \left[\frac{1}{a+i\omega-s} e^{(a+i\omega-s)t}\right]_0^\infty$$

$$= -\frac{1}{a+i\omega-s} = \frac{1}{s-a-i\omega} = \frac{(s-a)+i\omega}{(s-a)^2+\omega^2}$$

Imaginary part of $Y(s) = \frac{\omega}{(s-a)^2+\omega^2}$

$\therefore$ for $y(t) = e^{at}\sin\omega t \quad \underline{\underline{\bar{y}(s) = \frac{\omega}{(s-a)^2+\omega^2}}}$

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3 If $\quad y_n(t) = t^n$, $\quad Y_n(s) = \int_0^\infty t^n e^{-st} dt$

Integrate by parts $\quad = \left[-\frac{1}{s} t^n e^{-st}\right]_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1} e^{-s}$

$[\;]_0^\infty = 0 \qquad = \frac{n}{s} Y_{n-1}(s)$

since Laplace transform of $t^{n-1} = \int_0^\infty t^{n-1} e^{-st}$

$$\therefore \quad Y_n(s) = \frac{n}{s} Y_{n-1}(s)$$

$$= \frac{n(n-1)}{s^2} \bar{y}_{n-2}(s)$$

$$= \frac{n(n-1) \cdots\cdots 2}{s^{n-1}} \, \bar{y}_1(s)$$

but $Y_1(s)$ = Laplace transform of $t = \frac{1}{s^2}$

$$\therefore \; \underline{Y_n(s) = \frac{n(n-1)\cdots 2 \cdot 1}{s^{n+1}} = \frac{n!}{s^{n+1}}}$$

4 (a) $Y(s) = \dfrac{1}{(s+1)(s+2)(s+3)} = \dfrac{A}{s+1} + \dfrac{B}{s+2} + \dfrac{C}{s+3}$ ③

To find A $\times(s+1)$ let $s \to -1$ $\quad A = \dfrac{1}{(2-1)(3-1)} = \dfrac{1}{2}$

B $\times(s+2)$ let $s \to -2$ $\quad B = \dfrac{1}{(1-2)(3-2)} = -1$

C $\times(s+3)$ let $s \to -3$ $\quad C = \dfrac{1}{(1-3)(2-3)} = \dfrac{1}{2}$

$$\therefore Y(s) = \frac{1}{2}\frac{1}{s+1} - \frac{1}{s+2} + \frac{1}{2}\frac{1}{s+3}$$

$$\underline{y(t) = \tfrac{1}{2}e^{-t} - e^{-2t} + \tfrac{1}{2}e^{-3t}}$$

(b) $Y(s) = \dfrac{1}{(s+1)(s+2)^2} = \dfrac{A}{s+1} + \dfrac{B}{s+2} + \dfrac{C}{(s+2)^2}$

To find A $\times(s+1)$ let $s \to -1$ $\quad A = \dfrac{1}{(-1+2)^2} = 1$

C $\times(s+2)^2$ let $s \to -2$ $\quad C = \dfrac{1}{(-2+1)} = -1$

B put $s=0$ solve

$$\therefore \quad \frac{1}{1 \times 2^2} = \frac{1}{1} + \frac{B}{2} - \frac{1}{2^2}$$

or $\quad \dfrac{B}{2} = \dfrac{1}{4} - 1 + \dfrac{1}{4} = -\dfrac{1}{2} \quad \therefore B = -1$

$$Y(s) = \frac{1}{s+1} - \frac{1}{s+2} - \frac{1}{(s+2)^2}$$

$$\underline{y(t) = e^{-t} - e^{-2t} - te^{-2t}}$$

(c) $Y(s) = \dfrac{1}{(s+1)(s^2+4)} = \dfrac{A}{s+1} + \dfrac{B+Cs}{s^2+4}$

To find A $\times(s+1)$ let $s \to -1$ $\quad A = \dfrac{1}{1+4} = \dfrac{1}{5}$

To find B, C $\times(s^2+4)$, put $s = 2i$ & equate real & im. pa

ie $\quad \dfrac{1}{1+2i} = B + 2iC = \dfrac{1-2i}{(1+2i)(1-2i)} = \dfrac{1-2i}{5}$

$$\therefore B = \frac{1}{5}, \quad C = -\frac{1}{5}$$

$$Y(s) = \frac{1}{5}\left[\frac{1}{s+1} + \frac{1-s}{s^2+4}\right] = \frac{1}{5}\left[\frac{1}{s+1} + \frac{1}{2}\frac{2}{s^2+4} - \frac{s}{s^2+4}\right]$$

$$\underline{y(t) = \frac{1}{5}\left[e^{-t} + \tfrac{1}{2}\sin 2t - \cos 2t\right]}$$

5 (a) $\ddot{y} + 4\dot{y} + 3y = e^{-t}$ $\quad (y(0) = \dot{y}(0) = 1)$ 4

$$\mathcal{L}\ddot{y} = s^2Y - s\,y(0) - \dot{y}(0) = s^2 Y - s - 1$$
$$\mathcal{L}\dot{y} = sY - y(0) \qquad\qquad = sY - 1$$

$\therefore$ D.E. $\rightarrow$ $s^2Y - s - 1 + 4sY - 4 + 3Y = \frac{1}{s+1}$

$$(s^2 + 4s + 3)Y = \frac{1}{s+1} + s + 5 = \frac{s^2 + 6s + 6}{s+1}$$

$$Y = \frac{s^2 + 6s + 6}{(s+1)^2(s+3)}$$

Put $\frac{s^2 + 6s + 6}{(s+1)^2(s+3)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+3}$

To find C $\quad \times (s+3)$ let $s \to -3$ $\quad C = \frac{9 - 18 + 6}{(-2)^2} = -\frac{3}{4}$

B $\quad \times (s+1)^2$ let $s \to -1$ $\quad B = \frac{1 - 6 + 6}{(-1+3)} = \frac{1}{2}$

A put $s = 0$

$$\frac{6}{1^2 \times 3} = A + B + \frac{C}{3} = A + \frac{1}{2} - \frac{1}{4}$$

$$A = 2 - \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$$

$$Y(s) = \frac{7}{4}\frac{1}{s+1} + \frac{1}{2}\frac{1}{(s+1)^2} - \frac{3}{4}\frac{1}{(s+3)}$$

$$y(t) = \frac{7}{4}e^{-t} + \frac{1}{2}te^{-t} - \frac{3}{4}e^{-3t}$$

Check $\dot{y}(t) = -\frac{7}{4}e^{-t} + \frac{1}{2}e^{-t} - \frac{1}{2}te^{-t} + \frac{9}{4}e^{-3t}$

$$\ddot{y}(t) = +\frac{7}{4}e^{-t} - \frac{1}{2}e^{-t} - \frac{1}{2}e^{-t} + \frac{1}{2}te^{-t} - \frac{27}{4}e^{-3t}$$
$$= \frac{3}{4}e^{-t} + \frac{1}{2}te^{-t} - \frac{27}{4}e^{-3t}$$
$$4\dot{y}(t) = -5e^{-t} - 2te^{-t} + 9e^{-3t}$$
$$3y(t) = \frac{21}{4}e^{-t} + \frac{3}{2}te^{-t} - \frac{9}{4}e^{-3t}$$

$\ddot{y}(t) + 4\dot{y}(t) + 3y(t) = e^{-t} \quad 0 \quad 0$ ✓

$y(0) = \frac{7}{4} - \frac{3}{4} = 1$ ✓

$\dot{y}(0) = -\frac{5}{4} + \frac{9}{4} = 1$ ✓

5 (b) $\ddot{y} - y = \sin t$ $\qquad y(0) = 1, \quad \dot{y}(0) = 0$

$$\mathcal{L}\ddot{y} = s^2 Y - s y(0) - \dot{y}(0) = s^2 Y - s$$

D.E $\quad s^2 Y - s - Y = \frac{1}{s^2+1}$

$$\text{or } (s^2-1)Y = \frac{1}{s^2+1} + s = \frac{s^3+s+1}{s^2+1}$$

$$\therefore \quad Y = \frac{s^3+s+1}{(s^2+1)(s+1)(s-1)}$$

$$= \frac{A+Bs}{(s^2+1)} + \frac{C}{s+1} + \frac{D}{s-1}$$

To find D $\times (s-1)$ let $s \to 1$ $\quad \therefore \quad D = \frac{1+1+1}{(1+1)(1+1)} = \frac{3}{4}$

C $\times (s+1)$ let $s \to -1$ $\quad C = \frac{-1-1+1}{(1+1)(-1-1)} = \frac{1}{4}$

A, B $\times (s^2+1)$ let $s \to i$

$$A + iB = \frac{i(-1+1)+1}{-2} = -\frac{1}{2}$$

$$\therefore \quad A = -\tfrac{1}{2}, \quad B = 0$$

$$\therefore \quad Y = -\frac{1}{2}\frac{1}{s^2+1} + \frac{1}{4}\frac{1}{s+1} + \frac{3}{4}\frac{1}{s-1}$$

$$y(t) = -\tfrac{1}{2}\sin t + \tfrac{1}{4}e^{-t} + \tfrac{3}{4}e^{t}$$

Check $\quad \dot{y}(t) = -\tfrac{1}{2}\cos t - \tfrac{1}{4}e^{-t} + \tfrac{3}{4}e^{t}$

$$\ddot{y}(t) = +\tfrac{1}{2}\sin t + \tfrac{1}{4}e^{-t} + \tfrac{3}{4}e^{t}$$

$$\ddot{y} - y = \tfrac{1}{2}\sin t + \tfrac{1}{4}e^{-t} + \tfrac{3}{4}e^{t} + \tfrac{1}{2}\sin t - \tfrac{1}{4}e^{-t} - \tfrac{3}{4}e^{t}$$

$$= \sin t \ \checkmark$$

$$y(0) = \tfrac{1}{4} + \tfrac{3}{4} = 1 \ \checkmark$$

$$\dot{y}(0) = -\tfrac{1}{2} - \tfrac{1}{4} + \tfrac{3}{4} = 0 \ \checkmark$$

5 (c) $\ddot{y} + y = t^3 \qquad y(0) = \dot{y}(0) = 1$

$$\mathcal{L}\ddot{y} = s^2 Y - s\,y(0) - \dot{y}(0) = s^2 Y - s - 1$$

$$\therefore \quad (s^2+1)Y - s - 1 = \frac{6}{s^4}$$

$$Y = \frac{1}{s^2+1}\left[\frac{6}{s^4} + s + 1\right]$$

$$= \frac{1}{s^2+1}\left[\frac{s^5+s^4+6}{s^4}\right]$$

$$= \frac{A}{s^4} + \frac{B}{s^3} + \frac{C}{s^2} + \frac{D}{s} + \frac{E+Fs}{s^2+1}$$

To find E, F $\times (s^2+1)$ put $s = i$

$$E + iF = \frac{i+1+6}{1} = 7 + i$$

$$E = 7, \quad F = 1.$$

To find A $\times s^4$ let $s \to 0$ $\quad A = \frac{6}{1} = 6$

To find B, C, D put $s = 1, -1, 2$ (say)

$s=1 \quad \therefore \quad \frac{1}{2}\left[\frac{1+1+6}{1}\right] = 6 + B + C + D + \frac{7+1}{2} \qquad$ (i)

$s=-1 \quad \frac{1}{2}\left[\frac{-1+1+6}{1}\right] = 6 - B + C - D + \frac{7-1}{2} \qquad$ (ii)

$s=2 \quad \frac{1}{5}\left[\frac{32+16+6}{16}\right] = \frac{6}{16} + \frac{B}{8} + \frac{C}{4} + \frac{D}{2} + \frac{7+2}{5} \qquad$ (iii)

$\therefore$ (i) $\quad B + C + D = 4 - 10 = -6 \qquad \therefore \quad 2C = -12$

(ii) $\quad -B + C - D = 3 - 9 = -6 \qquad\qquad C = -6$

$$B + D = 0$$

(iii) $\quad \frac{27}{40} = \frac{3}{8} + \frac{B}{8} - \frac{6}{4} + \frac{D}{2} + \frac{9}{5}$

$27 = 15 + 5B - 60 + 20D + 72 \qquad$ or $5B + 20D = 0$

$$\therefore \quad B = D = 0.$$

$$\therefore \quad Y = \frac{6}{s^4} - \frac{6}{s^2} + \frac{7+s}{s^2+1}$$

$$y(t) = t^3 - 6t + 7\sin t + \cos t.$$

Check $\ddot{y} = 6t \qquad - 7\sin t \;- \cos t$

$\ddot{y} + y = t^3$

$y(0) = 1$ ✓

$\dot{y}(t) = 3t^2 - 6 + 7\cos t - \sin t = 1 \qquad t = 0$ ✓

6 $$\dot{u} + av = b$$
$$\dot{v} - au = 0 \qquad u(0) = v(0) = 0$$

Take Laplace transforms

$$\begin{cases} sU(s) + aV(s) = \frac{b}{s} \\ sV(s) - aU(s) = 0 \end{cases} \rightarrow \quad U(s) = \frac{sV(s)}{a}$$

$$\therefore \quad \frac{s^2V}{a} + aV = \frac{b}{s}$$

or $V(s^2 + a^2) = \frac{ab}{s}$

$$V(s) = \frac{ab}{s} \cdot \frac{1}{s^2+a^2} = \frac{b}{a} \, a^2 \, \frac{1}{s} \cdot \frac{1}{s^2+a^2}$$

$$= \frac{b}{a}\left[\frac{1}{s} - \frac{s}{a^2+s^2}\right]$$

$$\therefore \quad V(t) = \frac{b}{a}\left[1 - \cos at\right]$$

By original equation $u = \frac{\dot{v}}{a}$

$$\therefore \quad u(t) = \frac{b}{a} \sin at$$

Check at $t = 0$ $u = 0$, $v = 0$

Sub in differential eqns

① $\underset{(\dot{u})}{b\cos at} + \underset{av}{a\frac{b}{a}(1 - \cos at)} = b$ ✓

② $+ \underset{\dot{v}}{b\sin at} - \underset{au}{b\sin at} = 0.$

7 $\quad m\ddot{x} + eB\dot{y} = eE$

$\quad m\ddot{y} - eB\dot{x} = 0$

$x = \dot{x} = y = \dot{y} = 0$ at $t = 0$. At rest at origin

Divide by m

$$\left.\begin{aligned} \ddot{x} + \frac{eB}{m}\dot{y} &= \frac{eE}{m} \\ \ddot{y} - \frac{eB}{m}\dot{x} &= 0 \end{aligned}\right] \quad (1)$$

(a) C.f. question 5. $\quad v = \dot{y}, \quad u = \dot{x},$

$a = \dfrac{eB}{m}, \quad b = \dfrac{eE}{m}$

$$\therefore \quad \dot{y} = v = \frac{b}{a}[1 - \cos at] = \frac{E}{B}\left[1 - \cos\frac{eB}{m}t\right]$$

$$\dot{x} = u = \frac{b}{a}\sin at = \frac{E}{B}\sin\frac{eB}{m}t.$$

$$x = \int_0^t \dot{x}\,dt' \qquad \text{since } x = 0 \quad t = 0$$

$$= \frac{b}{a}\left[\frac{1 - \cos at}{a}\right] = \frac{E}{B}\left[\frac{1 - \cos at}{a}\right], \quad a = \frac{eB}{m}$$

$$y = \int_0^t \dot{y}\,dt' = \frac{b}{a}\left[t - \frac{\sin at}{a}\right]$$

$$= \frac{E}{B}\left[t - \frac{\sin at}{a}\right] \quad a = \frac{eB}{m}$$

(b) Take Laplace transform of (1) with $\dfrac{eB}{m} = a$, $\dfrac{eE}{m} = b$

$$s^2X(s) + asY(s) = \frac{b}{s}$$

$$s^2Y(s) - asX(s) = 0 \quad \Rightarrow X(s) = \frac{sY(s)}{a}$$

$$\therefore \quad \frac{s^3Y}{a} + asY = \frac{b}{s}$$

$$\text{or} \quad (s^3 + a^2s)Y = \frac{ab}{s}$$

$$Y = \frac{b}{a}a^2\frac{1}{s^2}\cdot\frac{1}{s^2+a^2}$$

6(b) Cont. $Y(s) = \frac{b}{a}\left[\frac{a^2}{s^2}\cdot\frac{1}{s^2+a^2}\right] = \frac{b}{a}\left[\frac{1}{s^2} - \frac{1}{s^2+a^2}\right]$

$$= \frac{b}{a}\left[\frac{1}{s^2} - \frac{1}{a}\,\frac{a}{s^2+a^2}\right]$$

$$\therefore\ y(t) = \frac{b}{a}\left[t - \frac{\sin at}{t}\right] = \frac{E}{B}\left[t - \frac{\sin at}{t}\right].$$

$x(t)$ follows by integration.

8(a) $\int_{0^-}^{\infty} 1.\,e^{-st}\,dt = \left[-\frac{1}{s}e^{-st}\right]_{0^-}^{\infty} = \frac{1}{s} = \int_{0^+}^{\infty} 1.\,e^{-st}\,dt$

$$\int_{0^-}^{\infty} \delta(t)e^{-st}\,dt = e^{-s.0} = 1$$

$\int_{0^+}^{\infty} \delta(t)e^{-st}\,dt = 0$ since range of integration does not contain 0.

(b) Taking Laplace transforms (with 0^- lower limit):

$$sY(s) - \cancel{y(0^-)} + Y(s) = sU(s) - \cancel{u(0^-)} + 2U(s)$$

$$\Rightarrow\ Y(s) = \frac{s+2}{s+1}U(s) = \frac{s+2}{s+1}\cdot\frac{1}{s} = \frac{2}{s} - \frac{1}{s+1}$$

$$\Rightarrow\ y(t) = 2 - e^{-t} \text{ for } t \geqslant 0.$$

If the initial condition is given at $t = 0^+$ it is convenient to redefine the Laplace transform to have a lower limit of 0^+. Note that the derivative rule now becomes:

$$\mathcal{L}(\dot{y}(t)) = sY(s) - y(0^+).$$

Taking Laplace transforms gives

$$sY(s) - \cancel{y(0^+)} + Y(s) = sU(s) - u(0^+) + 2U(s)$$

$$\Rightarrow\ (s+1)Y(s) = (s+2)U(s) - 1 = (s+2)\frac{1}{s} - 1$$

8(b) cont.

$$\Rightarrow \quad Y(s) = \frac{2}{(s+1)s} = \frac{2}{s} - \frac{2}{s+1}$$

$$\Rightarrow \quad y(t) = 2 - 2e^{-t} \quad \text{for } t > 0.$$

9 (i)

$$h(t) = \int_0^t \tau e^{t-\tau} d\tau = e^t \int_0^t \tau e^{-\tau} d\tau$$

$$= e^t \left\{ \left[-\tau e^{-\tau}\right]_0^t - \int_0^t -e^{-\tau} d\tau \right\}$$

$$= e^t \left\{ te^{-t} - e^{-t} + 1 \right\}$$

$$= -t - 1 + e^t$$

From the tables: $H(s) = \frac{-1}{s^2} - \frac{1}{s} + \frac{1}{s-1} = \frac{-(s-1)(s+1)+s^2}{s^2(s-1)} = \frac{1}{s^2(s-1)}$

(ii) $\quad H(s) = F(s)\,G(s) = \frac{1}{s^2} \cdot \frac{1}{s-1}$.