

Part IA Paper 2: Structures and Materials

MATERIALS

Solutions to Examples Paper 3 – Plasticity of Materials

*Mechanics of Plastic Deformation,
Microstructural Origin and Manipulation of Plastic Properties,
Strength-limited Design*

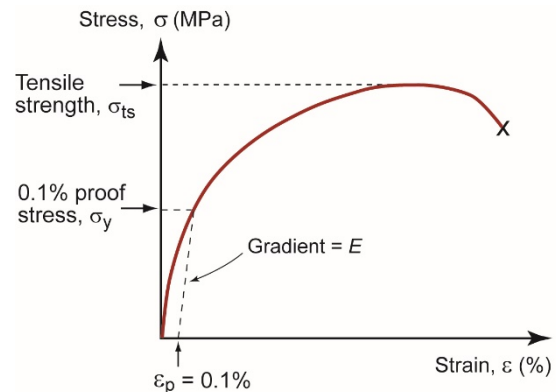
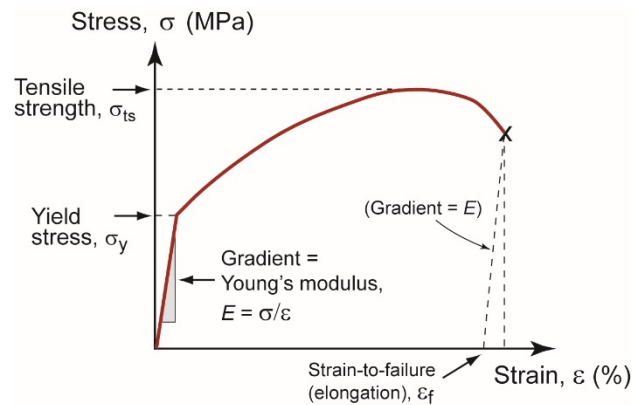
Mechanics of Plastic Deformation

1. (a) The yield stress σ_y is the stress at which yield occurs in uniaxial tension (or compression), i.e. the material begins to deform inelastically, and some of the strain is permanent (the “elastic limit”). It should strictly be called “yield strength”.

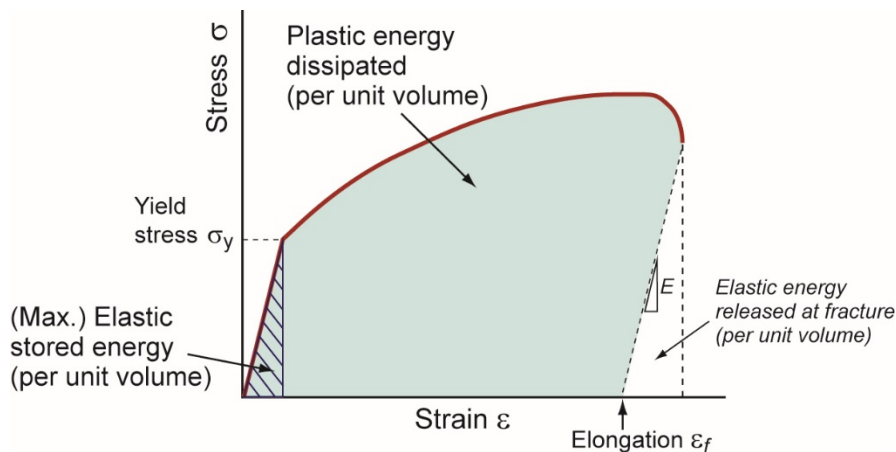
The tensile strength σ_{ts} is the maximum nominal stress in a tensile test (when the stress goes through a maximum, at the onset of necking).

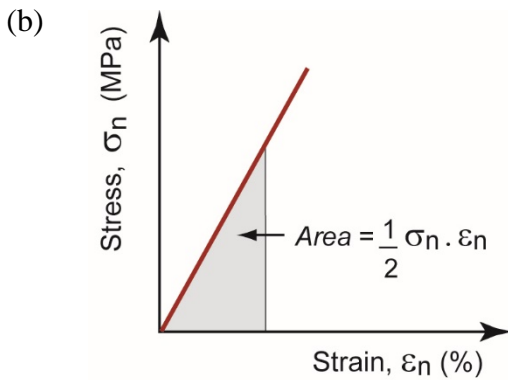
The tensile ductility (or elongation) is the nominal strain to failure in a tensile test. It is not a strict material property, since the value depends on the length and cross-section of the test sample. However, measurements taken on a standard geometry can be compared, and quoted values are based on these standards.

(b) The 0.1% proof stress $\sigma_{0.1}$ is the nominal stress corresponding to a permanent (plastic) strain of 0.1%. It is used when the onset of yielding is ill-defined.



2. (a)





Area under linear part of nominal stress-strain graph = $\frac{1}{2} \sigma_n \epsilon_n = \frac{1}{2} (F/A_o) \times (u/l_o)$

Work done = elastic stored energy = $\frac{1}{2} Fu$
(area under load-extension graph)

Volume = $A_o \times l_o$

Hence area = elastic energy stored per unit volume.

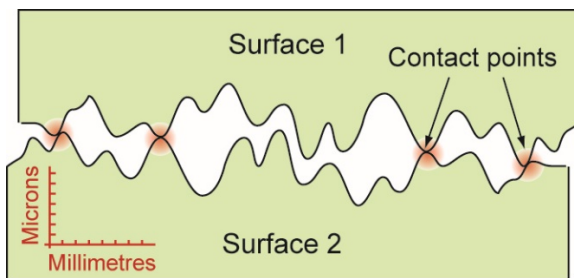
Maximum value when $\sigma_n = \sigma_y$ (elastic limit).

Substituting for $\epsilon_n = \sigma_n / E$, the maximum elastic energy stored per unit volume = $\frac{1}{2} \sigma_y^2 / E$

(c) Applications in which maximising elastic stored energy per unit volume is important: springs (vaulting poles, golf clubs, suspension parts such as leaf springs etc).

Applications in which maximising plastic energy per unit volume is important: crash resistance for cars (motorway crash barriers, crash boxes behind car bumpers etc).

3. (a) Surfaces have asperities on them. Typical mean heights and wavelengths are 0.5 and 100 μm respectively for ground surfaces. When the two metal surfaces come into contact they make contact at the asperity tops, with plastic deformation at each contact.



Note exaggerated vertical scale.
Contact only made at tips of asperities.

Nominal area: projected area of component over which contact is made.

True area: sum of area of microscopic contacts at tops of surface asperities, determined by the hardness of the softer metal.

The true area of contact is considerably less than the nominal area of contact.

True contact area: \propto load, and $\propto 1/\text{Hardness}$

At each contact, assume load/area $\approx H \approx 3\sigma_y$, so $a_{true} \approx \frac{W}{H} \approx \frac{W}{3\sigma_y}$

True contact area independent of nominal contact area (for small fractions, typical of metals).

True contact area carries both F and W .

Normal stress = $\frac{W}{a} \approx 3\sigma_y$; shear stress = $\frac{F}{a} \approx k$ (for dry sticking contact)

As $k = \sigma_y / 2$, $\frac{F}{W} = \mu \approx \frac{\sigma_y / 2}{3\sigma_y} \approx 1/6$

(b) (i) The presence of an oxide means that the shear strength is then determined by the shear strength of the oxide-oxide contacts (while the true area is still dependent on the yield stress of the underlying metal). So F and W are determined by independent strengths, and the coefficient of friction will vary.

(ii) For elastomers (like rubber), the low modulus allows the surfaces to conform elastically, so the true area of contact approaches the nominal area and the contacts do not yield (so it doesn't depend on hardness). The increase in true area of contact normally outweighs any drop in shear strength over the contact area, resulting in higher coefficient of friction.

4. (a) Using the formulae in the Materials Databook:

(i) In tension: nominal strain: $\varepsilon_n = \frac{\ell - \ell_o}{\ell_o}$, true strain: $\varepsilon_t = \ln\left(\frac{\ell}{\ell_o}\right)$

(ii) In compression (both negative): nominal strain: $\varepsilon_n = \frac{h - h_o}{h_o}$, true strain: $\varepsilon_t = \ln\left(\frac{h}{h_o}\right)$

(b) $\varepsilon_t = \ln(1 + \varepsilon_n) = \varepsilon_n + \text{higher order terms in } \varepsilon_n$. Hence $\varepsilon_t \approx \varepsilon_n$ when $\varepsilon_n \ll 1$, i.e. in most elastic problems (except for rubber, which deforms to large elastic strains).

(c) For a compression test, $\varepsilon_n < 0$ and $\sigma_n < 0$.

(i) $\sigma_t = \sigma_n (1 + \varepsilon_n)$, hence $|\sigma_t| < |\sigma_n|$ for $\varepsilon_n < 0$, i.e. true stress < nominal stress, in compression.

(ii) $\varepsilon_t = \ln(1 + \varepsilon_n)$, hence $|\varepsilon_t| > |\varepsilon_n|$ for $\varepsilon_n < 0$, i.e. true strain > nominal strain, in compression. Need to plug in a few numbers to show this – or note that as $\varepsilon_n \rightarrow -1$ (the lower limiting value in compression for ε_n), $(1 + \varepsilon_n) \rightarrow 0$, and so $\exp(\varepsilon_t) \rightarrow 0$, and thus $\varepsilon_t \rightarrow \text{large, negative value}$.

5. (a) True strains: $\varepsilon_{t1} = \ln\left(\frac{\ell_1}{\ell_o}\right)$, $\varepsilon_{t2} = \ln\left(\frac{\ell_2}{\ell_1}\right)$

Sum of true strains:

$$\varepsilon_{t1} + \varepsilon_{t2} = \ln\left(\frac{\ell_1}{\ell_o}\right) + \ln\left(\frac{\ell_2}{\ell_1}\right) = \ln\left(\frac{\ell_1}{\ell_o} \times \frac{\ell_2}{\ell_1}\right) = \ln\left(\frac{\ell_2}{\ell_o}\right) = (\varepsilon_t)_{total} \quad (\text{correct})$$

Overall nominal strain = $\frac{\ell_2 - \ell_o}{\ell_o}$

Sum of nominal strains = $\frac{\ell_1 - \ell_o}{\ell_o} + \frac{\ell_2 - \ell_1}{\ell_1}$, but this does not equal $\frac{\ell_2 - \ell_o}{\ell_o}$.

(b) The additive property for true strains (part (a)) is a main reason why they are useful. Hence in multi-pass tandem rolling, the total true strain is the sum of the true strains in each reduction.

(i) From part (a), the true strain (in compression) can be calculated directly from the inlet and outlet thickness:

$$\varepsilon_t = \ln\left(\frac{h_{out}}{h_{in}}\right) = \ln\left(\frac{3.2}{25}\right) = -2.06$$

(ii) Plastic deformation takes place at constant volume, so as the width of the strip remains constant, continuity of volume flow gives:

$$h_{in} \times v_{in} = h_{out} \times v_{out}$$

Hence: $v_{out} = \frac{25 \times 5}{3.2} = 39.1 \text{ cms}^{-1}$

(Note that continuity also means that a given length at inlet will exit with a length increased by a factor of 25/3.2. Hence the true strain can also be expressed as an extension of the material:

$$\varepsilon_t = \ln\left(\frac{25}{3.2}\right) = +2.06, \text{ i.e. numerically the same as the compressive true strain, but positive).}$$

6. (a) The surface is attached to the underlying material, and goes into tension when cooled but constrained by the substrate. As the substrate is thick compared with the surface layer, the balancing compression is negligibly small and the substrate does not strain (and bending effects can also be ignored). The surface layer has to remain the same length as the substrate and so the total strain = 0. If the thermal strain exceeds the yield strain, part of the strain due to the induced tensile stress will be elastic, and part plastic:

$$\varepsilon_{thermal} + \varepsilon_{elastic} + \varepsilon_{plastic} = 0$$

(b) The thermal strain of the surface layer is:

$$\varepsilon_{thermal} = \alpha \Delta T = 1.2 \times 10^{-5} \times (-300) = -3.6 \times 10^{-3}$$

From Ex. Paper 2 (Q5), the elastic strain under a biaxial stress σ is: $\varepsilon_{elastic} = \frac{\sigma}{E}(1-\nu)$

Assuming that the stress remains at the yield stress for plastic straining beyond the elastic limit (i.e. no work hardening), the maximum elastic strain is therefore: $\varepsilon_{elastic} = \frac{\sigma_y}{E}(1-\nu)$

For stainless steel (Databook mid-range values): $E = 200 \text{ GPa}$, $\sigma_y = 585 \text{ MPa}$

$$\text{Hence the elastic strain at yield is: } \varepsilon_{elastic} = \frac{585}{200 \times 10^3}(1-0.33) = 1.96 \times 10^{-3}$$

The plastic strain is thus:

$$\varepsilon_{plastic} = -(\varepsilon_{thermal} + \varepsilon_{elastic}) = (3.6 - 1.96) \times 10^{-3} = 1.64 \times 10^{-3}$$

Microstructural Origin and Manipulation of Plastic Properties

7. (a) The ideal strength is the notional tensile stress which would cause simultaneous rupture of the interatomic bonds across the whole sample. This occurs when the load on the atoms reaches the maximum in the atomic force-separation curve. It is of the order of $E/15$, where E is Young's modulus.

Dislocations enable incremental slip within a crystal at much lower stresses than the ideal strength. The stress for dislocation movement on a slip plane is defined as the shear stress τ_y required, i.e. parallel to the slip plane

Overall yielding in a polycrystal requires slip to occur on slip planes of many orientations, so the macroscopic shear yield stress k is somewhat higher than the shear stress to move individual dislocations on a slip plane (typically 1.5 times higher).

The tensile yield stress σ_y is the normal stress in uniaxial tension when yield occurs. The corresponding maximum shear stress lies on a plane at 45° , and at yield has the value k . The approximate relationship between the two is $k = \sigma_y/2$.

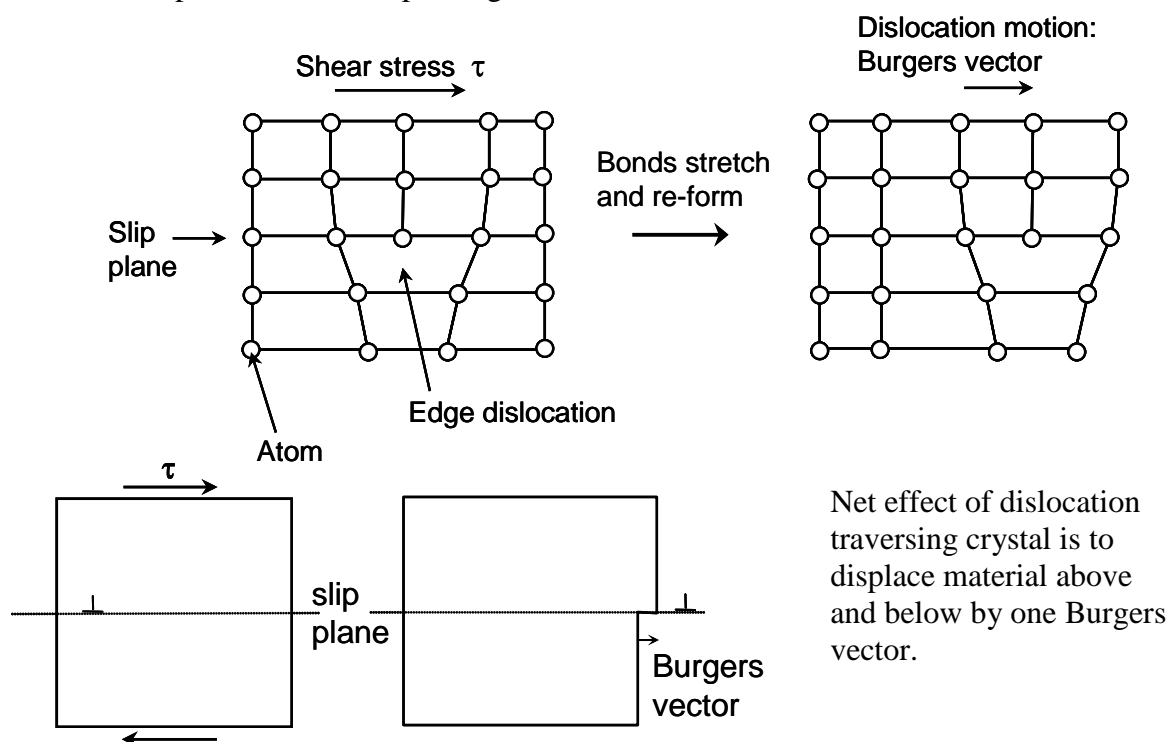
(b) A dislocation is a line defect in a crystal which can move over a slip-plane in the lattice under an applied shear stress. Dislocations concentrate the applied stress on a small proportion of the bonds, so plastic deformation takes place one row of atoms at a time at a stress which is much lower than the ideal strength. Yielding therefore occurs by very many tiny increments of shear between the blocks of crystal on either side of the slip planes. The magnitude of the incremental displacement is called the Burgers vector b .

An edge dislocation is when the dislocation line is aligned at right angles to the shear stress and slip direction, and moves in the direction of the increment of slip (the Burgers vector).

A screw dislocation is when the dislocation line is parallel to the applied stress and Burgers vector, but moves at right angles to the applied stress to achieve the same slip increment.

Dislocations are of mixed character when they are at an orientation between pure edge and pure screw (e.g. expanding loops of dislocation line).

Edge dislocations resemble the lower edge of an extra half-plane inserted into the crystal lattice – but it is important to recognise that the defect in atomic packing is local to the dislocation, and it is the distorted pattern of atomic packing that moves across the lattice, not the atoms themselves.



(c) The three dominant hardening mechanisms are: work hardening, solid solution hardening and precipitation hardening. All of them work by providing “pinning points” – obstacles which hold the dislocation back (temporarily) requiring it to bow out in a curve. This requires an increase in the imposed shear stress.

Work hardening uses plastic deformation to increase the dislocation density. Dislocations obstruct one another’s movement where they intersect, increasing the yield stress.

Solid solution hardening uses individual atoms distributed in the lattice to impede dislocation motion. Both substitutional and interstitial solute atoms perturb the atomic bonds in the lattice around them, “roughening” the slip plane, i.e. making it more difficult for the atomic bonding defect around a dislocation to move over the slip plane at this point.

Precipitation hardening involves embedding obstacles such as small crystals of compounds in the lattice. Dislocations encounter them on many slip planes, and force bowing of the dislocation between them, increasing the applied stress for yielding. Usually the precipitates are bypassed when the dislocation loop reaches a semi-circle. However, if there is some crystallographic continuity between the lattice and the precipitate, the precipitate may be sheared when the dislocation applies enough force to the precipitate – this still gives a significant increase in yield stress.

(i) Cold-rolled Al-Mn-Mg alloys are work-hardened by the rolling process, and the Mn and Mg atoms give solid solution hardening.

(ii) Brass uses strong solid-solution hardening, with the high levels of Zn added to the Cu. As the material is cast, no work hardening is used in this case (but brass is often “wrought”, i.e. shaped by deformation, and thus work hardened).

(iii) Quenching and tempering is the primary processing route to form finely dispersed precipitates of iron carbide in steels, giving very effective precipitation hardening. This dominates hardening in medium carbon steels – alloy steels also exploit solid solution hardening, and the additional effect of alloy carbide precipitates.

(d) Other important examples:

Work hardening: stainless steels, low carbon (“mild”) steels

Solid solution hardening: bronze (Cu-Sn alloys), stainless steels, most casting alloys (Mg, Zn, Al, Ti)

Precipitation hardening: alloy steels, tool steels, high temperature Ni alloys, “heat-treatable” Al alloys.

8. (a) (i) The dislocation density ρ_d is the length of dislocation per unit volume. For a 1cm cube, volume = 1000mm³. Dislocation density = 10⁸ mm⁻² (mm/mm³).

Hence total length of dislocation = 10⁸ × 1000 = 10¹¹ mm = 100,000 km!

(ii) Assume that the dislocations form a uniform square array of side d . In this case each dislocation sits at the centre of an area d^2 . For unit length of dislocation, the volume of material per dislocation is $1 \times d^2$, so the dislocation density $\rho_d = 1/d^2$.

Hence $d = 1/\sqrt{\rho_d} = 1/\sqrt{10^8} = 10^{-4}$ mm = 100 nm.

(iii) For an atomic spacing of 0.2 nm, this corresponds to 100/0.2 = 500 atoms.

(b) (i) Cross-sectional area of coiled metal:

$$\frac{\pi}{4}(d_{coil}^2 - d_{drum}^2) = \frac{\pi}{4}(1^2 - 0.15^2) = 0.768 \text{ m}^2$$

Volume of metal = 0.768 × 1 = 0.768 m³

Density of aluminium (from Databook) ≈ 2.7 Mg/m³

Hence mass of aluminium on coil ≈ 2.7 × 0.768 = 2.074 Mg = 2074 kg (approx. 2.1 tonnes).

(ii) Cross-sectional area of strip: $1 \times 0.3 \times 10^{-3} = 0.3 \times 10^{-3} \text{ m}^2$

Length of coiled strip = Volume/cross-sectional area = (0.768/0.3 × 10⁻³) = 2560 m = 2.56 km

(iii) Length of dislocation per unit volume = 10¹⁵ m/m³

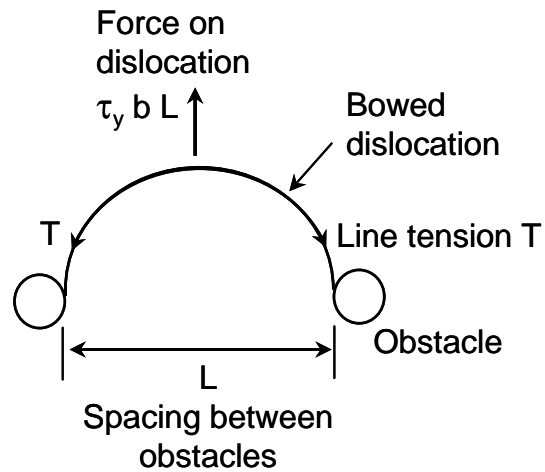
Total length of dislocation = Volume × dislocation density = 0.768 × 10¹⁵ m

(about 1/12th of a light year!).

9. (a) Balance line tension against force on dislocation:

$$\tau_y b L = 2T, \text{ where line tension} = \frac{1}{2} G b^2$$

$$\text{So } \tau_y = \frac{2T}{bL} = \frac{Gb}{L}$$



(b) For the alloy: $L = (6-1) \times 10^{-8} = 5 \times 10^{-8} \text{ m}$; $b = 2.86 \times 10^{-10} \text{ m}$; $G = 26 \text{ GPa}$.

Noting that $\sigma_y = 3 \tau_y$ for polycrystals, $\sigma_y = 446 \approx 450 \text{ MPa}$

(c) New $L = (18-3) \times 10^{-8} = 15 \times 10^{-8} \text{ m}$

Repeating the calculation, $\sigma_y = 149 \text{ MPa}$

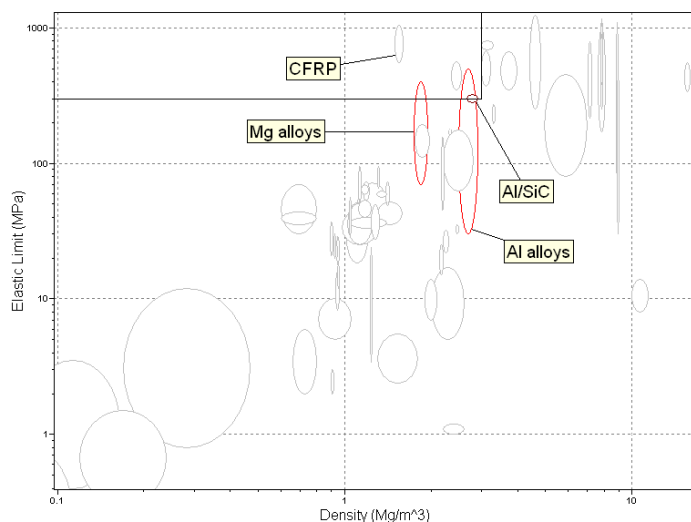
Drop in yield strength $\Delta\sigma_y = 297 \approx 300 \text{ MPa}$

10. The failure of PMMA is elastic-brittle: small defects and cracks within the material open up and lead to fracture before yielding occurs.

Polypropylene yields by slipping of the molecular chains past one another, followed by very extensive drawing. The molecules are stretched out and aligned in a stable neck of smaller cross section than the sample. The higher stress in the neck is carried by the alignment of the covalent bonds in the chains with the load. Failure will eventually occur by local necking and fracture at a stress well above the drawing stress.

Material Selection: Strength-limited Design

11. (a) In CES, elastic limit (strength)-density chart; box selection with prescribed values (or use limit stage). Ceramics excluded by a tree stage including all other classes.



Structural materials with strength $> 300 \text{ MPa}$ and density $< 3 \text{ Mg/m}^3$:

- higher strength Al and Mg alloys
- Al/SiC (just)
- CFRP: but poor elongation ($< 5\%$)

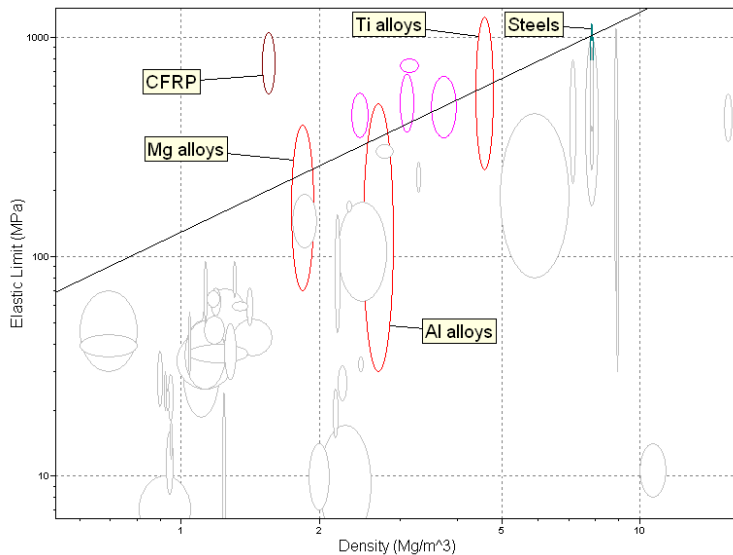
Method the same using Databook Strength-Density property chart, but low ductility of CFRP may not be expected.

Boron carbide also meets the strength and density requirements, but has very low elongation (ceramic).

Cost/kg of candidate materials:

Material	Cost/kg	Comments
Al alloys	0.75 – 1.22	The cheapest
Mg alloys	2.8 – 3.2	Both more expensive than Al alloys
Al-SiC	2.5 - 5	

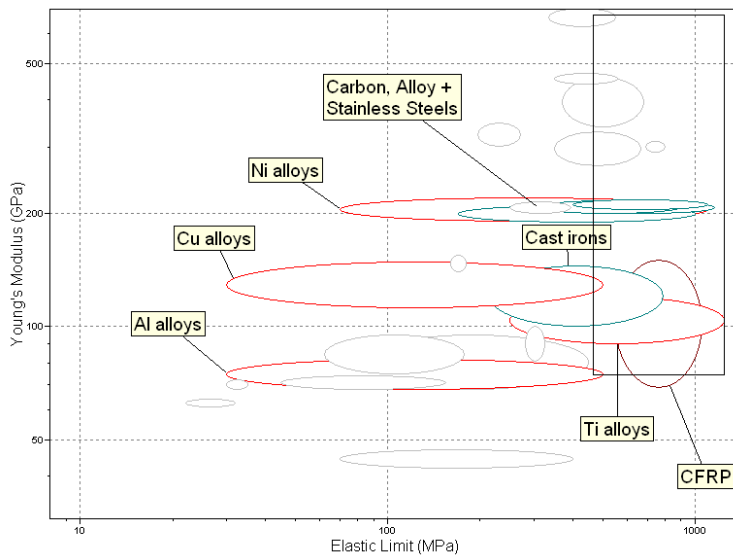
(b) In CES, elastic limit (strength)-density chart; line of slope = 1 (constant specific strength σ_f/ρ) located through top of the steels bubbles. Alternatively, use Advanced graph stage to plot σ_f/ρ for the required materials.



Specific strength of steels comparable to strongest Al alloys; Mg and Ti alloys extend to rather higher values; CFRP much higher value than all the alloys.

Method the same using Databook Strength-Density property chart.

(c) In CES, Young's modulus - elastic limit (strength) chart; box selection with bottom LH corner at high strength end of Al alloy bubble. Metals and composites selected via Tree Stage.



Metals and composites which are stiffer and stronger than strongest Al alloys:

- higher strength steels, cast irons, Ni and Ti alloys
- the very strongest Cu alloys
- CFRP

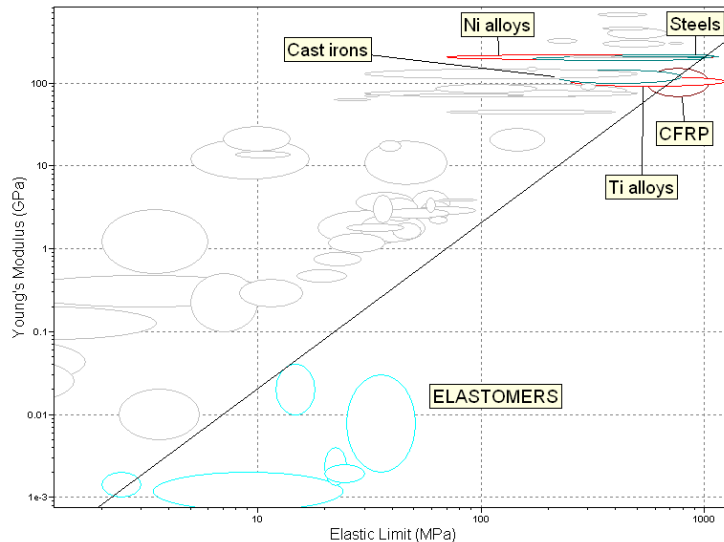
Method the same using Databook Modulus-Strength property chart.

(d) Performance index for maximum elastic stored energy per unit volume = σ_y^2 / E .

(i) In bending, the stress varies linearly from the neutral axes to the beam surfaces, so only the surface material stores the full energy associated with the maximum stress, and the stress, strain and stored energy per unit volume fall progressively to zero at the neutral axis.

(ii) Finding the stored energy over the volume of the beam in bending will involve integrating the stored energy in elements of thickness dy through the thickness. The stress in each element will be a fraction of σ_y , the strain will be a fraction of the strain σ_y/E , and so the elastic stored energy will be some fraction of σ_y^2/E . So this index still ranks the maximum elastic stored energy per unit volume, for beams in bending of constant shape.

In CES, Young's modulus – elastic limit (strength) chart: plot line of slope 2 for constant value of σ_y^2/E , move towards bottom right to optimise.



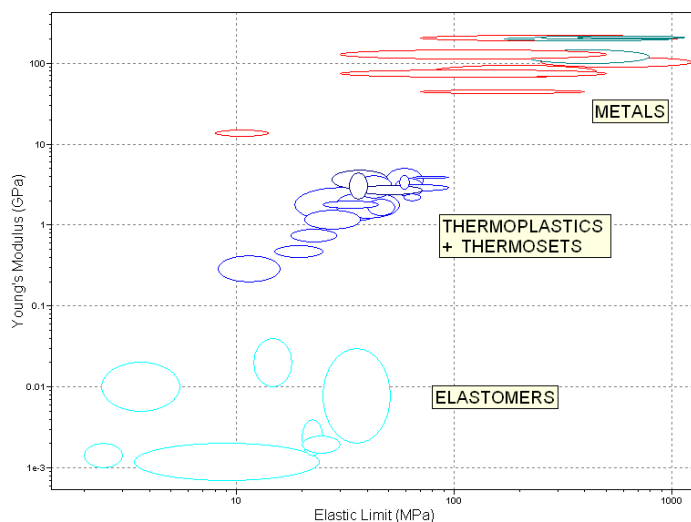
Elastomers (including rubbers) stand out as good spring materials (climbing ropes, bungee ropes)!

At the other end of the stiffness and strength spectrum, Ti alloys and CFRP are prominent (used extensively in sports products which store elastic energy).

The strongest steels (actually called “spring steels”), cast irons, and Ni alloys all join the list. Vehicle springs (coil and leaf) are mostly steel, but composites have also been trialled.

Method the same using Databook Modulus-Strength property chart.

12. Young's Modulus vs. Elastic Limit plot from CES: (chart in Databook does not show individual polymers)



The bubbles for polymers are much less elongated – a given polymer has a wider range of Young's modulus than a metal, and a more narrowly defined strength.

Variation in the microstructure of a polymer (e.g. degree of crystallinity) affects both properties more or less equally. In metals, microstructure has a strong effect on yield (e.g. precipitates, dislocation density) but little influence on stiffness.

13. Strength-limited furnace shelves

(a) Objective: mass of panel: $m = bld\rho$

Constraint: failure when maximum stress = material strength: $\sigma_f = \frac{3Wl}{2bd^2}$

Hence free variable, depth, given by: $d = \left(\frac{3Wl}{2b\sigma_f} \right)^{1/2}$

Substituting into equation for mass: $m = bl\rho\left(\frac{3Wl}{2b\sigma_f}\right)^{1/2} = \text{constant} \times \left(\frac{\rho}{\sqrt{\sigma_f}}\right)$

Hence minimise mass by maximising performance index $M = \sigma_f^{1/2} / \rho$.

Thickness and strength related by: $\sigma_f = \frac{3Wl}{2bd^2}$. Hence max. allowable thickness leads to a minimum strength (depending on values for load, length, width and max. thickness).

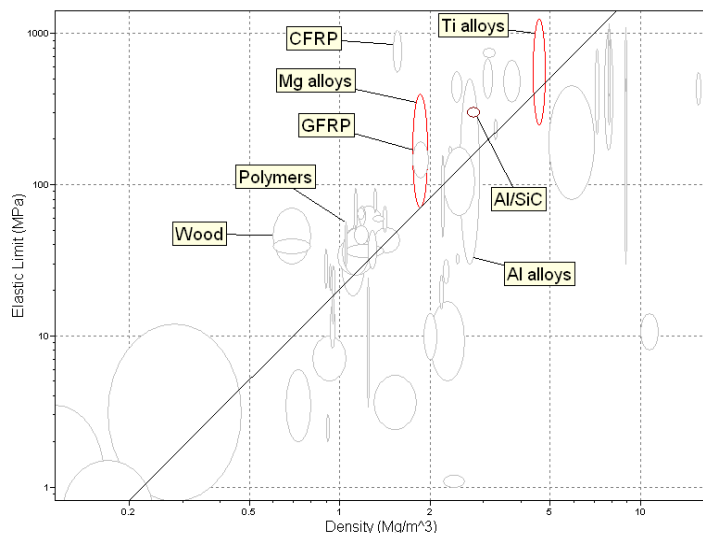
(b) In CES: plot Elastic Limit (Strength) against density, and position line of slope 2 to give a working short-list of materials.

Ceramics eliminated by selecting all other classes in a “Tree Stage”: ceramics too brittle.

Secondary constraints imposed via a limit stage:

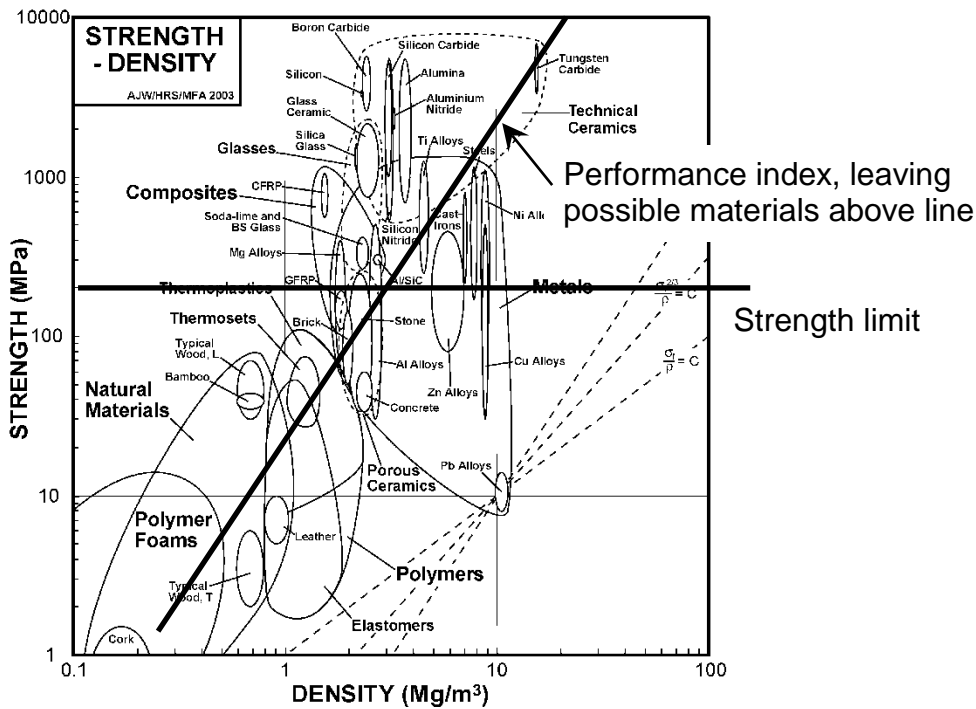
Minimum Elastic Limit = 200 MPa (for maximum allowable thickness)

Minimum value of “Maximum Service Temperature” = 250°C.



Note that ceramic data in CES is for tensile strength, and ceramics appear to compete. However, failure is by brittle fracture and ceramics do not have the required resistance to impact.

The Databook data (chart below) are for compressive strength, which is much higher than tensile strength – but the reasons for exclusion are the same.



Selection results:

Material	Comments
Ceramics (alumina etc)	Disregard – data for compressive strength, and too brittle
CFRP	Good on strength at low weight, but excluded by operating temperature
Mg alloys, Ti alloys, Al-SiC	Pass all criteria – costs compared below.
Al alloys	Excluded by operating temperature
Wood, GFRP, higher strength polymers	Below strength limit, and excluded by operating temp.

Cost/kg of candidate materials:

Material	Cost/kg	Comments
Ti alloys	15 – 40	Significantly more expensive
Mg alloys	2.8 – 3.2	Close competition on weight and cost
Al-SiC	2.5 – 5	

(b) Modify performance index, for minimum cost:

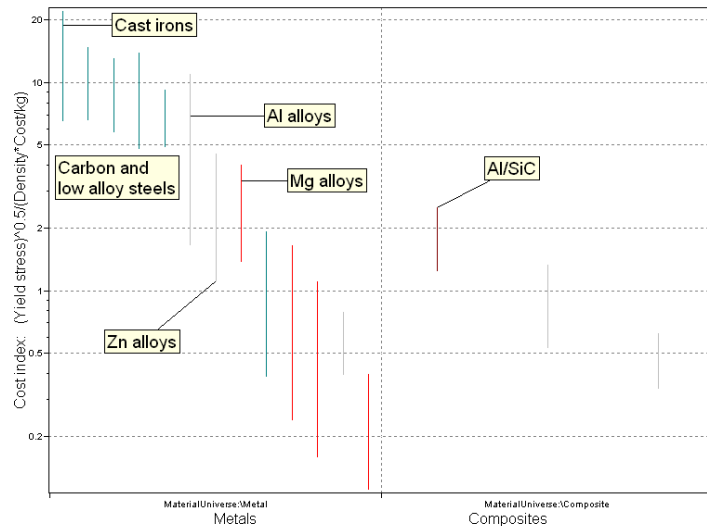
Objective: cost of panel: $C = bld \rho C_m$

Applying same constraint on failure stress and eliminating free variable d as before:

$$\text{Cost: } C = bl \rho C_m \left(\frac{3Wl}{2b\sigma_f} \right)^{1/2} = \text{constant} \times \left(\frac{\rho C_m}{\sqrt{\sigma_f}} \right)$$

Hence minimise cost by maximising performance index $M = \sigma_f^{1/2} / (\rho C_m)$.

In CES, use “Advanced” feature in graph stage to plot this index as a bar chart (for metals and composites only – other classes eliminated by operating temp. and/or strength limit).



Selection results:

Material	Comments
Cast irons, carbon & alloy steels	Wide range of ferrous alloys give strength at low cost
Al alloys	Competes on cost, excluded by operating temperature
Mg alloys, Al-SiC	Less competitive on cost
Zn alloys	Close to Mg and Al-SiC, excluded by operating temp.

14. Cable analysis (from IA Structures):

Overall equilibrium: vertical reaction = $\frac{wL}{2}$

Cut cable at centre, taking moments about cut:

$$H\delta = \frac{wL}{2} \left(\frac{L}{2} \right) - \frac{wL}{2} \left(\frac{L}{4} \right) = \frac{wL^2}{8}, \text{ so cable tension} = H = \frac{wL^2}{8\delta}$$

$$\text{Stress in cable, } \sigma = \text{tension/area} = \frac{wL^2}{8\delta A}$$

Now weight per unit length of cable, $w = \rho A g$, where ρ is the density

$$\text{Hence stress in cable} = \frac{\rho A g L^2}{8\delta A} = \frac{\rho g L^2}{8\delta}$$

Design specifies that maximum stress $< 0.8 \sigma_y$

$$\text{Hence } \frac{\rho g L^2}{8\delta} < 0.8 \sigma_y, \text{ and thus } \frac{\sigma_y}{\rho} > 0.157 \frac{g L^2}{\delta} \text{ is the lower limit for specific strength.}$$

This result is independent of the cross-sectional area – both self-weight and stress increase with cross-sectional area, so it cancels out.

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Lent Term 2019