Part IA Paper 2: Structures and Materials

MATERIALS

Solutions to Examples Paper 5 – Fracture (incl. Fatigue) and Weibull Statistics

1. (a) (i) The stress concentration factor (SCF) tells us the magnification factor above the remote load at a hole or change in cross section.

(ii) The stress intensity factor K characterises the loading at the tip of a sharp crack. Assuming that the material is ideally elastic, the stresses perpendicular to the crack, ahead of the crack, are given by $\sigma = K/\sqrt{2\pi x}$. A stress concentration factor is for a blunt feature, a stress intensity factor is for a sharp crack. Note K has dimensions stress×sqrt(length) while the stress concentration factor is dimensionless.

(iii) G is an alternative way of characterising the loading at the tip of a sharp crack, being the strain energy available for crack propagation. G and K are both **equivalent** measures of the loading at the crack tip, related by $K = \sqrt{EG}$. They only differ in that G is in terms of energy, K is in terms of stresses. K and G do have a meaning at sub-critical loads before the crack is propagating.

(iv) K_{IC} for a given material is the value of the stress intensity factor at a crack tip needed for a crack to propagate. K_{IC} is a materials parameter, while K is a loading parameter.
Only K_{IC} is a materials parameter, the others are geometry/loading parameters.

(b) Ahead of the crack tip ductile metals have a good deal of plastic flow. Eventually voids form and join up, allowing crack advance. This plasticity dissipates a lot of energy, as compared with the limited damage before bonds break just ahead of the crack tip in brittle materials like ceramics. This greater energy dissipation requires a greater strain energy release rate to drive it, hence a higher value of G_{IC}.

2. Maximum bending moment = $FL/4$ at the mid-span of the beam. Using $\frac{\sigma}{y} = \frac{M}{I}$ for the extreme fibre stress with $y = t/2$ and $I = bt^3/12$ gives the maximum tensile stress at the surface of a beam as

$$\sigma = \frac{3FL}{2bt^2}$$

The configuration shown is like a centre notched crack where $K = \sigma\sqrt{\pi a}$ if a is taken as the half-length of the crack (given in the notes). By comparison we should take a as the **radius** here.
Fracture occurs when

$$\sigma\sqrt{\pi a} = K$$

i.e. when

$$\frac{3FL}{2bt^2}\sqrt{\pi a} = K$$

Hence, the maximum load which can be sustained by the adhesive joint is

$$F = \frac{2bt^2 K}{3L\sqrt{\pi a}}$$

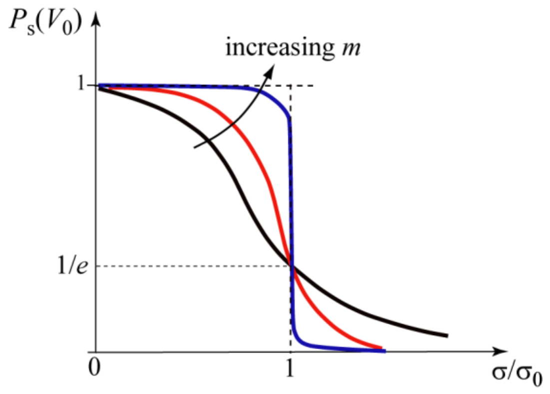
Taking K_{IC} as 1.3 MPa $m^{1/2}$ for epoxy (mid-range from the Data Book, p13), for the joint shown

$$F = \frac{2\times(0.1)^3\times 1.3\times 10^6}{3\times 2\sqrt{\pi\times 0.001}}$$

$$= \underline{7.7\,kN}$$

[In fact the factor Y in the stress intensity factor calibration is $2/\pi$ for a penny shaped crack, radius a, rather than the assumed value of 1.]

3. (a)

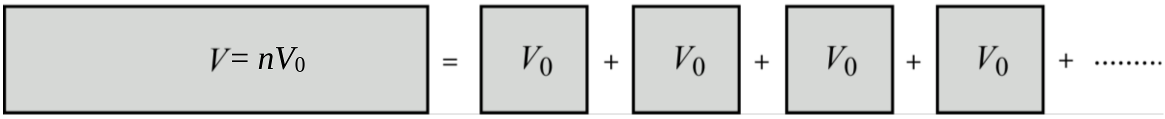


$\sigma = 0$; $P_s = 1$
$\sigma = \sigma_0$; $P_s = 1/e$
$\sigma \to \infty$; $P_s = 0$

$$\frac{dP_s}{d\sigma/\sigma_0} = -m\left(\frac{\sigma}{\sigma_0}\right)^{m-1}\exp\left(-\left(\frac{\sigma}{\sigma_0}\right)^m\right)$$

$$\sigma = \sigma_0; \quad \frac{dP_s}{d\sigma/\sigma_0} = -\frac{m}{e}$$

(b)



If we 'link together' n samples of volume V_0 each with probability of survival $P_s(V_0)$ then the weakest link theory requires that

$$P_s(V = nV_0) = \{P_s(V_0)\}\times\{P_s(V_0)\}\ldots(n \text{ times}) = \{P_s(V_0)\}^n = \{P_s(V_0)\}^{V/V_0}$$

The Weibull model has:

$$P_s(V) = \exp\left(-\frac{V}{V_0}\left(\frac{\sigma}{\sigma_0}\right)^m\right) \quad \text{and} \quad P_s(V_0) = \exp\left(-\left(\frac{\sigma}{\sigma_0}\right)^m\right)$$

This does satisfy $P_s(V) = \{P_s(V_0)\}^{V/V_0}$ as $y = \{\exp x\}^n \quad \Rightarrow \quad \ln y = nx \quad \Rightarrow \quad y = \exp(nx)$

(c) Batch 1 = small specimens; batch 2 = large specimens

$$\frac{P_{s2}(V_2)}{P_{s1}(V_1)} = \exp\left(-\frac{V_2}{V_0}\left(\frac{\sigma_2}{\sigma_0}\right)^m\right) \bigg/ \exp\left(-\frac{V_1}{V_0}\left(\frac{\sigma_1}{\sigma_0}\right)^m\right)$$

$$\frac{\ln(P_{s2}(V_2))}{\ln(P_{s1}(V_1))} = \frac{\ln(P_{s2}(V_2))}{\ln(0.5)} = \frac{V_2\,\sigma_2^m}{V_1\,\sigma_1^m} = \frac{50\times 11^2}{25\times 5^2}\left(\frac{40}{120}\right)^5 \qquad \Rightarrow P_{s2} = 0.9728$$

Note the advantage of taking ratios and substituting in values at the end.

4. (a) Weight of material below section at x = $\rho g V = \rho g \times \left(\frac{1}{3}\pi(\alpha x)^2 x\right)$

Cross sectional area = $\pi(\alpha x)^2$ Stress = Force/Area = $\frac{1}{3}\rho g x$

(b) Integrate over the volume using discs of thickness $\mathrm{d}x$ with corresponding volume $\mathrm{d}V = \pi(\alpha x)^2 \, dx$

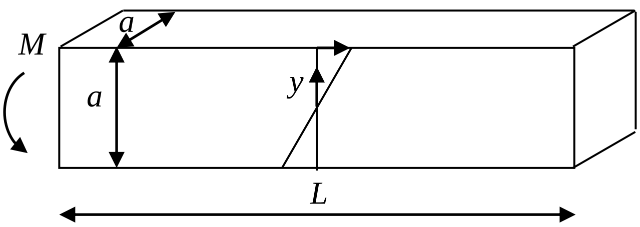
$$P_\mathrm{s}(L) = \exp\left\{\int_V -\left(\frac{\sigma}{\sigma_0}\right)^m \frac{\mathrm{d}V}{V_0}\right\} = \exp\left\{\int_0^L -\left(\frac{\rho g x}{3\sigma_0}\right)^m \frac{\pi(\alpha x)^2 \,\mathrm{d}x}{V_0}\right\}$$

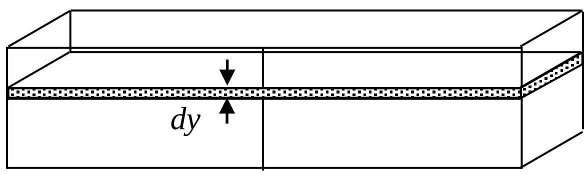
$$P_\mathrm{s}(L) = \exp\left(-\left(\frac{\rho g}{3\sigma_0}\right)^m \frac{\pi\alpha^2 L^{m+3}}{(m+3)V_0}\right)$$

The probability of survival falls with increasing α because, although the stresses are the same, the amount of material which is stressed increases with α, and hence the chances of meeting a critical flaw increase.

(c) Flaws induced during sample preparation & gripping. Use bend test instead.

5.





For the specimen in bending consider only tensile failure as the compressive strength would be very much larger. Integrate by considering slices of length L, breadth a and height $\mathrm{d}y$ a distance y from the neutral axis, to give:

$$P_\mathrm{sb} = \exp\left\{\int_V -\left(\frac{\sigma}{\sigma_0}\right)^m \frac{\mathrm{d}V}{V_0}\right\} = \exp\left\{\frac{-1}{V_0\sigma_0^m}\int_0^{a/2} La\sigma^m \mathrm{d}y\right\}$$

Putting $\dfrac{\sigma}{\sigma_\mathrm{b}} = \dfrac{y}{a/2}$ gives $-\ln(P_\mathrm{sb}) = \dfrac{aL\sigma_\mathrm{b}^m}{V_0\sigma_0^m}\displaystyle\int_0^{a/2}\left(\frac{y}{a/2}\right)^m dy = \frac{a^2L\sigma_\mathrm{b}^m}{2V_0\sigma_0^m(m+1)}$ (1)

For the specimen in uniform tension σ_t:

$$P_\mathrm{st} = \exp\left(-\frac{a^2L}{V_0}\left(\frac{\sigma_\mathrm{t}}{\sigma_0}\right)^m\right) \Rightarrow -\ln(P_\mathrm{st}) = \frac{a^2L}{V_0}\left(\frac{\sigma_\mathrm{t}}{\sigma_0}\right)^m \qquad (2)$$

Combining (1) and (2), and noting that P_s is the same in both cases:

$$\frac{-\ln(P_{sb})}{-\ln(P_{st})} = \frac{a^2L\sigma_\mathrm{b}^m}{2V_0\sigma_0^m}\frac{1}{(m+1)}\frac{V_0}{a^2L}\frac{\sigma_0^m}{\sigma_\mathrm{t}^m} \Rightarrow \sigma_\mathrm{t} = \sigma_\mathrm{b}/(2(m+1))^{1/m}$$

σ_t = 367 MPa, putting σ_b = 500 MPa and m = 10

Note again the advantage of leaving substitution of variables until the end.

6. (a) Reading directly from the graph, 2×10^6 cycles.

(b) Use Goodman's rule, page 7 of the data book to convert from the stress range $\Delta\sigma$ with mean stress σ_m to an equivalent stress range $\Delta\sigma_0$ for zero mean stress.

$$\Delta\sigma = \Delta\sigma_0\left(1-\frac{\sigma_m}{\sigma_{ts}}\right) \Rightarrow \Delta\sigma_0 = \Delta\sigma\left(1-\frac{\sigma_m}{\sigma_{ts}}\right)^{-1} = 1200\times\left(1-\frac{100}{1100}\right)^{-1} \approx 1300\text{ MPa}$$

Now use the S-N curve to find a lifetime of about 2×10^5 with this stress range of 1300 MPa. The percentage reduction in lifetime is about 90%, i.e. a reduction of lifetime by a factor of 10 - an enormous effect.

(c) We can either change the design to reduce the stresses – mountain bikes have larger fork tubes with a large second moment of area to reduce the stresses, or change the material to one with a higher fatigue threshold.

[There has recently been a spate of fork fatigue failures reported due to substandard-metal forks; better materials can also allow us to reduce the weight, though stiffness may be a factor here too.]

7. (a) $\Delta\sigma\left(N_f\right)^{\alpha} = C$

$$280\left(10^5\right)^{\alpha} = 200\left(10^7\right)^{\alpha} = C$$

$$\frac{280}{200} = 1.4 = \left(\frac{10^7}{10^5}\right)^{\alpha} = \left(10^2\right)^{a}$$

$$\log_{10}1.4 = 2\alpha; \quad \alpha = 0.07306$$

$$C = 280\left(10^5\right)^{0.07306} = 649.3\text{ MPa}$$

At 150 MPa, $N_f = \dfrac{C^{1/\alpha}}{\Delta\sigma^{1/\alpha}} = \left(\dfrac{649.3}{150}\right)^{1/0.07306} = 5.13\times10^8$ cycles

(b) The amount of life used up in the first 4×10^8 cycles is, $\dfrac{N}{N_f} = \dfrac{4\times10^8}{5.13\times10^8} = 0.780$

Miner's rule gives: $\dfrac{N}{N_f} + \dfrac{N^1}{N_f^1} = 1$

Hence the amount of life that is available for the twilight years of the aircraft is

$$\frac{N^1}{N_f^1} = 1 - 0.780 = 0.220$$

$$N_f^1 = \frac{N^1}{0.220} = \frac{4\times10^8}{0.220} = 1.82\times10^9\text{ cycles}$$

For this $\Delta\sigma = \dfrac{C}{\left(N_f^1\right)^{0.07306}} = \dfrac{649.3}{\left(1.82\times10^9\right)^{0.07306}} = 136.7$ MPa

The **decrease** in stress range = 150 – 136.7 = 13.3 ≈ 13 MPa

8. (a) In pressure vessels, the main concern is to avoid catastrophic rupture, particularly when the pressurised fluid is a gas - in which case an explosion is likely to result. Such rupture most commonly occurs in the form of fast crack propagation. The "leak-before-break" criterion states that if the critical crack size for fast fracture is greater than the wall thickness of the vessel, then gas will leak out through the crack, causing a drop in pressure. This will reduce the stress levels, so the system should be failsafe and an explosion impossible.

A proof test involves pressurizing the vessel to a level above the planned working pressure σ_p. If it survives, then we have a fix on the **maximum** crack length a_0 which can be present in the vessel from $K_{IC} = Y\sigma_p\sqrt{\pi a_0}$. We can then calculate the **minimum** number of cycles at the working stress range which would propagate cracks of this length to the level which would cause fast fracture or leakage.

(b) $$\sigma_{\text{working}} = \frac{pr}{t} = \frac{5.1\times(7.5/2)}{0.04} = \underline{478\ \text{MPa}}$$

$$K_{IC} = \sigma_{\text{working}}\sqrt{\pi a}\ \text{at fracture.}$$

$$a = \frac{1}{\pi}\left\{\frac{200\ \text{MPa m}^{1/2}}{478\ \text{MPa}}\right\}^2 = \underline{5.6\times10^{-2}\ \text{m}}$$

This critical depth for fast fracture is greater than the wall thickness of 40 mm. The vessel will fail by leaking before the crack length becomes critical and it fails by fast fracture.

[In practice we should allow for the complicated geometry of the crack, by looking up the geometry calibration factor Y in a stress intensity factor handbook. This will be particularly important as the crack approaches the free outside of the wall (*not examinable*)]

(c) Rearranging the crack growth equation

$$\frac{da}{dN} = 2.44\times10^{-14}\{\text{MPa}\}^{-4}\,\text{m}^{-1}(\Delta K)^4$$

$$= 2.44\times10^{-14}\{\text{MPa}\}^{-4}\,\text{m}^{-1}\cdot(\Delta\sigma)^4\,\pi^2 a^2$$

We can then integrate this, from the initial condition after the proof test (assuming that a crack of length a_0 is present), to the required end point where after 3000 cycles the crack has grown out to the wall, where failure would occur by leakage.

$$2.44\times10^{-14}\{\text{MPa}\}^{-4}\,\text{m}^{-1}(478)^4\{\text{MPa}\}^4\,\pi^2\int_0^{N_f} dN = \int_{a_0}^{4.0\times10^{-2}\,\text{m}}\frac{da}{a^2}$$

$$1.257\times10^{-2}\,\text{m}^{-1}N_f = \left[-\frac{1}{a}\right]_{a_0}^{4.0\times10^{-2}\ \text{m}}$$

$a_0 = 0.016$ m

This is the initial flaw size that will penetrate the wall after 3000 loading cycles.

The proof stress P_{Proof} must be sufficient to cause flaws of this size to propagate by fast fracture.

$$K = \sigma_{\text{Proof}}\sqrt{\pi a_0} \geq K_{\text{IC}}$$

where $\sigma_{\text{Proof}} = \dfrac{P_{\text{Proof}}\, r}{t}$

Hence $P_{\text{Proof}} \geq \dfrac{tK_{\text{IC}}}{r\sqrt{\pi a_0}} = \dfrac{0.04 \times 200}{7.5/2\sqrt{\pi\ 0.016}} = 9.5$ MPa

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