

## Part IA Paper 4: Mathematical Methods

## Examples Paper 4

Elementary exercises are marked †, problems of Tripos standard \*.

Answers can be found at the back of the paper.

## Revision Questions

Making use of expansion in partial fractions, evaluate the following integrals:

$$(a) \int \frac{dx}{(x+1)(x+2)} \quad (b) \int \frac{dx}{2x^2 - 5x + 2} \quad (c) \int \frac{dx}{a^4 - x^4} \quad (a \text{ is a constant})$$

## Differential Equations

Find complete solutions of the following ordinary differential equations:

1. † (a)  $\frac{d^2x}{dt^2} - 13\frac{dx}{dt} + 12x = 36$  (b)  $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + 2x = e^t$   
       (c)  $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2e^{3x}$  (d)  $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x^2$
2. (a)  $\frac{d^2x}{dt^2} + 3\frac{dx}{dt} = 5e^{-3t} + 2$  (b)  $\frac{d^2x}{dt^2} + 9x = \sin 3t + 2 \sin 4t$
3. Verify that  $\frac{d}{dt} \operatorname{Re}[e^{i\omega t}] = \operatorname{Re}[\frac{d}{dt} e^{i\omega t}]$ . By noting that  $\cos t = \operatorname{Re}[e^{it}]$ , find the (complex) value of  $y_0$  for which  $y = \operatorname{Re}[y_0 e^{it}]$  is a solution of

$$\frac{dy}{dt} + 5y = \cos t$$

and hence find a particular integral for the equation.

4. Find the solution of the differential equation  $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = \frac{1}{4} \cos 2x$  which satisfies the condition  $\frac{d^2y}{dx^2} = \frac{dy}{dx} = 0$  at  $x = 0$ .
5. \* Find values of  $\alpha$  for which  $R = cr^\alpha$  is a solution of the differential equation

$$r^2 \frac{d^2R}{dr^2} + r \frac{dR}{dr} - m^2 R = 0,$$

where  $c$  and  $m$  are constants. Find the solution of the differential equation

$$r^2 \frac{d^2R}{dr^2} + r \frac{dR}{dr} - 4R = r$$

which is finite at  $r = 0$ , and has value 1 at  $r = 1$ .

6. A wire is stretched with tension  $T$  between the points  $x = 0$  and  $x = L$  in a horizontal plane. A mass distribution  $\rho(x)$  per unit length is hung from the wire so that it displaces downwards by a distance  $y(x)$  *which may be assumed small*. By considering equilibrium of a short section of the wire, show (i) that the tension  $T$  is approximately constant; and (ii) that the displacement satisfies the differential equation

$$T \frac{d^2 y}{dx^2} = -g\rho(x).$$

(The wire's own mass may be ignored.) Hence find the displaced shape  $y(x)$  when a curtain of uniform mass per unit length  $\rho$  hangs from the wire, occupying its entire length.

7. \* In an epidemic, there are at any particular time  $x$  people not yet infected and  $y$  people who are ill. The rate at which people become ill is  $\alpha x$ , where  $\alpha$  is a constant. If  $x$  is initially equal to  $N$ , find an expression for  $x$  at time  $t$ . (Regard the numbers of people  $x$  and  $y$  as continuous variables.)

The rates of recovery and death of those who are ill are  $\beta y$  and  $\gamma y$ , respectively. If  $y$  is initially equal to zero, find an expression for the number of deaths up to time  $t$  from the start of the epidemic. (Assume that those who recover are immune from further infection.)

The expression for the number of deaths appears to be indeterminate if  $\alpha = \beta + \gamma$ . Find the limiting form of the expression as  $\beta + \gamma \rightarrow \alpha$ .

Not all differential equations can be solved algebraically: sometimes numerical integration is the only option. Use Jupyter/Python to solve the system of equations numerically, and compare the numerical and exact solutions.

### Hints

A Jupyter/Python notebook for numerical solution of this system of equations can be found at <https://github.com/CambridgeEngineering/PartIA-Paper4-Mathematics>.

The program splits time up into intervals `dt` and uses a simple Euler rule to update the variables, for example `x = x - dt*alpha*x`. This expression is only approximate, since it assumes a constant value of `x` across the time interval `dt`. The approximation improves as `dt` gets smaller, but we then require a greater number of time steps to cover the same time interval, and this has extra computation cost.

Experiment with different values of `dt`, and also different values of `alpha`, `beta` and `gamma`.

Suitable past Tripos questions:

2003 Q3, 2004 Q3a, 2005 Q2 (short), 2006 Q4 (long), 2007 Q3 (short), 2008 Q2 (short), 2009 Q2 (short), 2010 Q4, 2011 Q5, 2012 Q5, 2013 Q5, 2014 Q5, 2015 Q4 , 2016 Q5.

### Answers

1. (a)  $x = Ae^{12t} + Be^t + 3$  (b)  $x = e^t(A \sin t + B \cos t + 1)$   
(c)  $y = Ae^x + Be^{2x} + e^{3x}$  (d)  $y = (A + Bx)e^{-x} + x^2 - 4x + 6$
2. (a)  $x = (A - 5t/3)e^{-3t} + B + 2t/3$   
(b)  $x = (A - t/6) \cos 3t + B \sin 3t - \frac{2}{7} \sin 4t$
3.  $y_0 = \frac{1}{5+i}$  so that  $y = \frac{5}{26} \cos t + \frac{1}{26} \sin t$
4.  $y = (7 + 5x)e^{-x}/25 - (3 \cos 2x - 4 \sin 2x)/100$
5.  $\alpha = \pm m$ ,  $R = \frac{4}{3}r^2 - \frac{1}{3}r$
6.  $y(x) = \rho g x(L - x)/2T$
7.  $Ne^{-\alpha t}$ ;  $\frac{\alpha \gamma N}{\beta + \gamma - \alpha} \left[ \frac{1 - e^{-\alpha t}}{\alpha} - \frac{1 - e^{-(\beta + \gamma)t}}{\beta + \gamma} \right]$ ;  $\frac{\gamma N}{\alpha} [1 - e^{-\alpha t} - \alpha t e^{-\alpha t}]$