Engineering Part IA Maths   Solutions to Examples Paper 2

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 3 & 0 & -4 \\ -1 & 3 & 2 \end{bmatrix} \Rightarrow \det A = 1(0+12) + 2(4-6) - 3(9+0)$$
$$= 12 - 4 - 27$$
$$= \underline{-19}$$

$$B = \begin{bmatrix} 0 & 1 & 2 \\ 1 & -1 & -3 \\ 2 & 0 & 1 \end{bmatrix} \Rightarrow \det B = 0(-1+0) + 1(-6-1) + 2(0+2)$$
$$= -7+4$$
$$= \underline{-3}$$

$$AB = \begin{bmatrix} 1 & 2 & -3 \\ 3 & 0 & -4 \\ -1 & 3 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 1 & -1 & -3 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -4 & -1 & -7 \\ -8 & 3 & 2 \\ 7 & -4 & -9 \end{bmatrix}$$

$$\Rightarrow \det AB = -4(-27+8) - 1(14-72) - 7(32-21) = -76 - 77 + 58 = \underline{57}$$
$$= \det A . \det B$$

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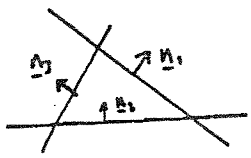
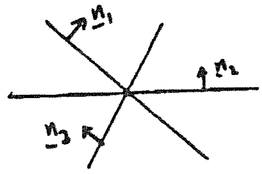
2. 3 simultaneous equations usually have no solution if left hand sides are linearly dependent i.e. $\det | \quad | = 0$. The exception is when the right hand sides have precisely the same linear dependence.

In terms of geometry: no solution if three normals lie in a plane



Exception



Line in common

No solution

N.B. This topic is covered in lectures.

Equations have no solution $\Rightarrow$
$$0 = \begin{vmatrix} 2 & 1 & 3 \\ 6 & -2 & -1 \\ s & 0 & 1 \end{vmatrix} = 2(-2+0) + 1(-s-6) + 3(0+2s)$$
$$= 5s - 10 \quad \underline{\text{i.e. } s = 2}$$

When $s = 2$:

$$2x + y + 3z = 5 \quad ①$$
$$6x - 2y - z = 3 \quad ②$$
$$2x + z = t \quad ③$$

Adding $2 \times ① + ②$
$$\Rightarrow 10x + 5z = 13 \text{ or } x + 2z = 13/5$$

This is compatible with ③ if $\underline{t = 13/5}$

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3. $(\underline{a}+\lambda\underline{b}).\underline{b}\times\underline{c} = \underline{a}.\underline{b}\times\underline{c} + \lambda\underline{b}.\underline{b}\times\underline{c} = \underline{a}.\underline{b}\times\underline{c}$

Since $\underline{a}.\underline{b}\times\underline{c} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ this means that adding a multiple of one column to another does not change the value of a determinant.

Since $\underline{a}.\underline{b}\times\underline{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ This is also true for rows.

For $\begin{vmatrix} 2 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 0 & 6 \end{vmatrix}$ Subtracting row 2 from row 1 makes the top row $1\ 0\ 0$

$\Rightarrow$ determinant is $1.2.6 = \underline{12}$

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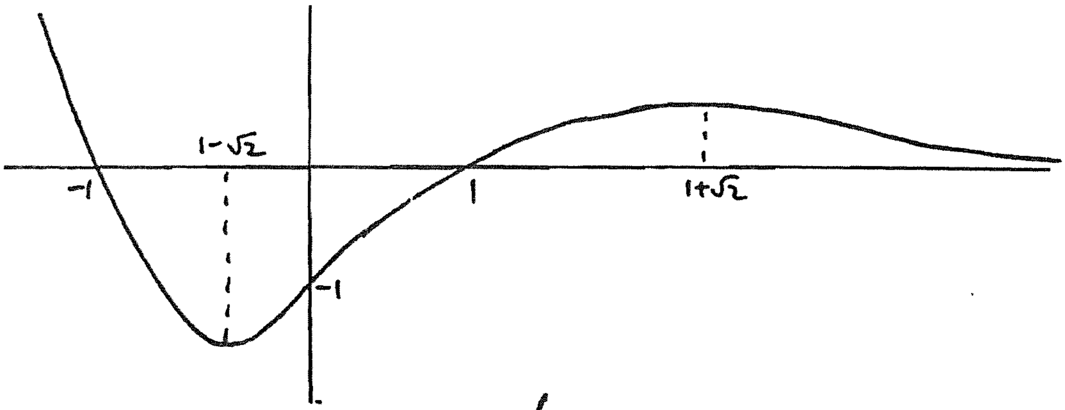
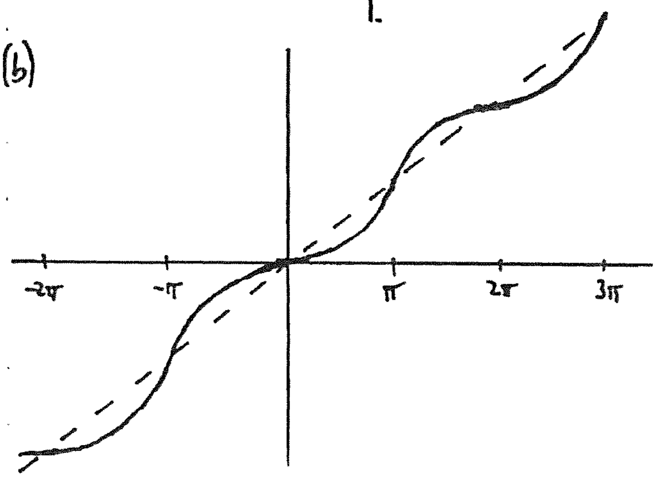
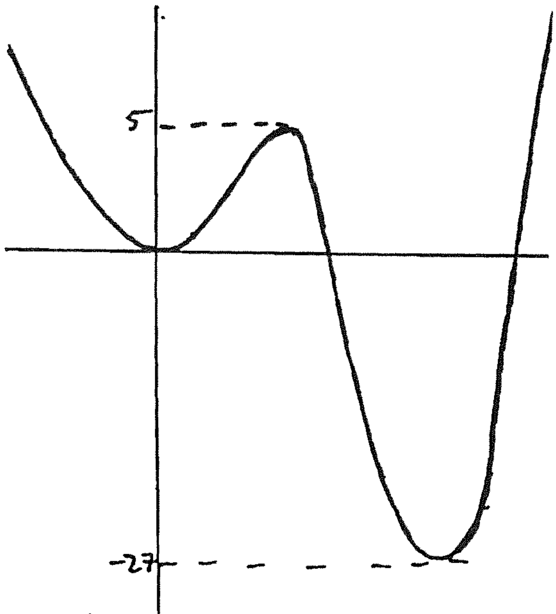
4 (a)



(b)



(c)



(d) So from graph of (c),

$k > 5$	:	2 real roots
$0 < k < 5$	:	4 real roots
$-27 < k < 0$	:	2 real roots
$k < -27$	:	no real roots

5(i) $\sinh(A+B) = \dfrac{e^{A+B} - e^{-A-B}}{2}$

$$\sinh A \cosh B + \cosh A \sinh B = \frac{e^{A}-e^{-A}}{2}\,\frac{e^{B}+e^{-B}}{2} + \frac{e^{A}+e^{-A}}{2}\,\frac{e^{B}-e^{-B}}{2}$$

$$= \frac{1}{4}\left[e^{A+B} - e^{-A+B} + e^{A-B} - e^{-A-B} + e^{A+B} + e^{-A+B} + e^{A-B} - e^{-A-B}\right]$$

$$= \frac{e^{A+B} - e^{-A-B}}{2} \qquad \underline{\text{Q.E.D.}}$$

$$\cosh(A+B) = \frac{e^{A+B} + e^{-A-B}}{2} = \frac{e^{A} \pm e^{-A}}{2}\,\frac{e^{B} \pm e^{-B}}{2} \pm \frac{e^{A} \pm e^{-A}}{2}\,\frac{e^{B} \pm e^{-B}}{2}$$

Inspection of sign of $e^{A+B} \Rightarrow (\qquad\qquad) + (\qquad\qquad)$

Inspection of sign of $e^{-A-B} \Rightarrow \quad + \quad + \quad + \quad - \quad -$

Check that with these signs e^{-A+B} & e^{A-B} terms cancel

Thus $\cosh(A+B) = \dfrac{e^{A}+e^{-A}}{2} \cdot \dfrac{e^{B}+e^{-B}}{2} + \dfrac{e^{A}-e^{-A}}{2} \cdot \dfrac{e^{B}-e^{-B}}{2}$

$$= \underline{\cosh A \cosh B + \sinh A \sinh B}$$

(ii) $\dfrac{d}{dx} \tanh x = \dfrac{d}{dx}\dfrac{\sinh x}{\cosh x} = \dfrac{\cosh x \cdot \cosh x - \sinh x \sinh x}{\cosh^2 x} = \underline{\dfrac{1}{\cosh^2 x} = \operatorname{sech}^2 x}$

6(i) $f(x) = \cosh(1+x) \qquad f'(x) = \sinh(1+x) \qquad f''(x) = \cosh(1+x) \qquad f'''(x) = \sinh(1+x)$ etc

Taylor's theorem $\Rightarrow \quad f(x) = f(0) + x f'(0) + \dfrac{x^2}{2!} f''(0) + \dfrac{x^3}{3!} f'''(0) + \dots$

$$\Rightarrow \cosh(1+x) = \cosh 1 + x \sinh 1 + \frac{x^2}{2!}\cosh 1 + \frac{x^3}{3!}\sinh 1 + \dots$$

(ii)

$$\frac{1+x}{1-x^2} = (1+x)(1+x^2+x^4+\dots) = \underline{1+x+x^2}$$

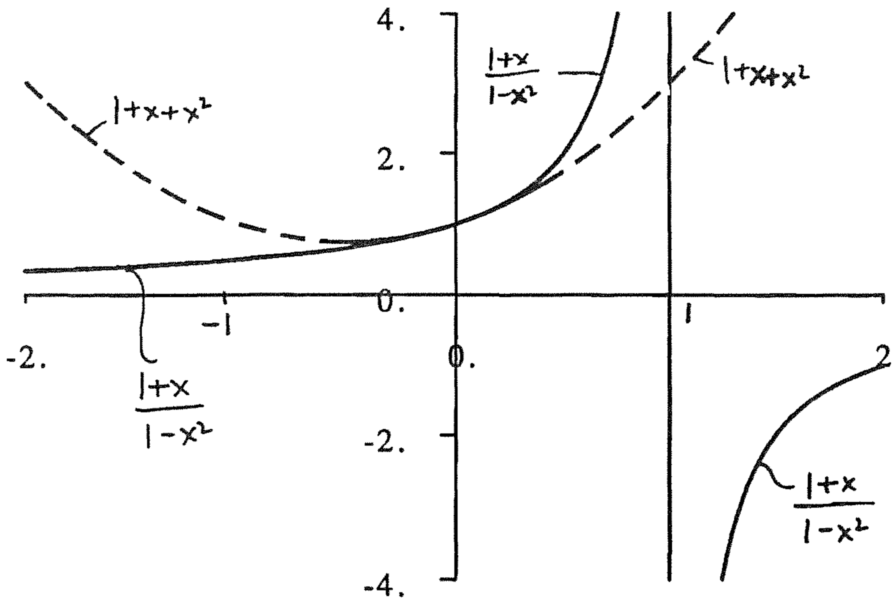
Figure overleaf shows $\dfrac{1+x}{1-x^2}$ and $1+x+x^2$

First term neglected is x^3 and typical size of the function is 1.0 (at x=0). Thus:

at 1.0%: abs(x^3)<0.01*1.0 so abs(x)<0.2

at 0.1%: abs(x^3)<0.001*1.0 so abs(x)<0.1

6.



[Seems a good approximation for about $-.3 < x < .3$]

Series probably not valid at $x=1$! Where function goes ∞.

[In fact series valid $-1 < x < 1$]

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7. $\frac{d}{dx}(a+bx)^\alpha = \alpha(a+bx)^{\alpha-1}.b \qquad \frac{d^2}{dx^2}(a+bx)^\alpha = \alpha(\alpha-1)(a+bx)^{\alpha-2}b^2$

etc $\Rightarrow \left(\frac{d}{dx}\right)^n (a+bx)^\alpha = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)\, b^n (a+bx)^{\alpha-n}$

Coefficient of x^n in expansion of $(a+bx)^\alpha = \frac{1}{n!}\left(\frac{d}{dx}\right)^n (a+bx)^\alpha \Big|_{x=0}$

$$= \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!} b^n a^{\alpha-n}.$$

$\therefore$ Coefficient of x^7 in expansion of $(2+3x)^{-\frac{1}{2}} = \frac{-\frac{1}{2}.-\frac{3}{2}.-\frac{5}{2}\ldots\ldots-\frac{13}{2}}{7!} 3^7 2^{-\frac{1}{2}-7}$

$$= -\frac{1.3.5.7.9.11.13}{7!}\frac{3^7}{2^{14}\sqrt{2}} = \underline{-2.531}$$

8a) Using Maths Data Book for power series

$\sin x = x - \frac{x^3}{6} + O(x^5) \Rightarrow \frac{\sin x}{x} = 1 - \frac{x^2}{6} + O(x^4) \to 1$ as $x \to 0$

alitter de l'Hôpital: $f(x) = \sin x,\ g(x) = x \qquad f(0) = g(0) = 0$

$f'(x) = \cos x,\ g'(x) = 1 \qquad f'(0) = g'(0) = 1$

$$\therefore \lim_{x\to 0}\frac{f(x)}{g(x)} = \frac{f'(0)}{g'(0)} = 1.$$

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b) $\tan x = x + \frac{x^3}{3} + O(x^5) \qquad \sin x = x - \frac{x^3}{6} + O(x^5)$

$$\therefore \frac{\tan x - x}{x - \sin x} = \frac{x + \frac{x^3}{3} + O(x^5) - x}{x - (x - \frac{x^3}{6} + O(x^5))} \simeq \frac{\frac{x^3}{3}}{\frac{x^3}{6}} \to 2 \text{ as } x \to 0$$

aliter de l'Hôpital:

$$f(x) = \tan x - x \qquad g(x) = x - \sin x \qquad f(0) = g(0) = 0$$

$$f'(x) = \sec^2 x - 1 \qquad g'(x) = 1 - \cos x \qquad f'(0) = g'(0) = 0$$

$$f''(x) = \frac{2\sin x}{\cos^3 x} \qquad g''(x) = \sin x \qquad f''(0) = g''(0) = 0$$

$$f'''(x) = \frac{2\cos^4 x + 6\cos^2 x \sin^2 x}{\cos^6 x} \qquad g'''(x) = \cos x \qquad f'''(0) = 2 \quad g'''(0) = 1$$

$$\therefore \lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{f'''(0)}{g'''(0)} = \underline{2}.$$

c) Put $x = \frac{\pi}{2} + \epsilon$.
$$\lim_{x\to \pi/2} \frac{\ln(x-\pi/2)}{\tan x} = \lim_{\epsilon\to 0} \frac{\ln \epsilon}{\tan(\pi/2+\epsilon)}$$
$$= \lim_{\epsilon\to 0} (-\tan\epsilon)\ln\epsilon = \lim_{\epsilon\to 0}\left[-\epsilon + O(\epsilon^3)\right]\ln\epsilon = \underline{0.}$$

$$\tan(\pi/2+\epsilon) = \frac{\sin(\pi/2+\epsilon)}{\cos(\pi/2+\epsilon)} = \frac{\cos\epsilon}{-\sin\epsilon} = -\frac{1}{\tan\epsilon}$$

d) Put $y = \frac{1}{x}$ $\Rightarrow$
$$\lim_{x\to\infty} \frac{x+1}{x^2+6x} \exp\left[\frac{x^2}{1+x^2}(\ln x + 2)\right] = \lim_{y\to 0} \frac{\frac{1}{y}+1}{\frac{1}{y^2}+\frac{6}{y}} \exp\left[\frac{\frac{1}{y^2}}{1+\frac{1}{y^2}}\left(\ln\frac{1}{y}+2\right)\right]$$

Noting that $2 = \ln e^2$ we have
$$\lim_{y\to 0} \frac{y(1+y)}{1+6y} \exp\left[\ln\frac{e^2}{y}(1+y^2)^{-1}\right] = \lim_{y\to 0} (y+\cdots)\exp\left(\ln\frac{e^2}{y}\cdots\right)$$
$$= \lim_{y\to 0} y.\frac{e^2}{y} = \underline{e^2}$$

9a)
$$\sin^2\alpha = \left\{\alpha - \frac{\alpha^3}{6} + O(\alpha^5)\right\}^2 = \alpha^2\left(1 - \frac{\alpha^2}{6} + O(\alpha^4)\right)^2 = \alpha^2\left(1-\frac{\alpha^2}{3}\right) + O(\alpha^6)$$
$$\left(1 - \tfrac{1}{3}\sin^2\alpha\right)^{-1/2} = 1 + (-\tfrac{1}{2})(-\tfrac{1}{3}\sin^2\alpha) + O(\sin^4\alpha) = 1 + \frac{\alpha^2}{6} + O(\alpha^4)$$
$$\therefore \frac{\sin^2\alpha}{\alpha^2\left[1-\frac{1}{3}\sin^2\alpha\right]^{1/2}} = \frac{\left[\alpha^2\left(1-\frac{\alpha^2}{3}\right) + O(\alpha^6)\right]\left[1 + \frac{\alpha^2}{6} + O(\alpha^4)\right]}{\alpha^2} = 1 - \frac{\alpha^2}{6} + O(\alpha^4)$$

b) Need to keep an extra power in each expansion
$$\sin^2\alpha = \left[\alpha - \frac{\alpha^3}{6} + \frac{\alpha^5}{120} + O(\alpha^7)\right]^2 = \alpha^2\left[1 - \frac{\alpha^2}{6} + \frac{\alpha^4}{120} + O(\alpha^6)\right]^2$$
$$= \alpha^2\left[1 - \frac{\alpha^2}{3} + \frac{\alpha^4}{60} + \frac{\alpha^4}{36} + O(\alpha^6)\right] = \alpha^2\left[1 - \frac{\alpha^2}{3} + \frac{2\alpha^4}{45} + O(\alpha^6)\right]$$

$$\left(1 - \tfrac{2}{3}\sin^2\alpha\right)^{-1/2} = \left[1 - \frac{2\alpha^2}{3} + \frac{2\alpha^4}{9} + O(\alpha^6)\right]^{-1/2} = 1 + (-\tfrac{1}{2})\left(-\frac{2\alpha^2}{3} + \frac{2\alpha^4}{9} + O(\alpha^6)\right)$$
$$+ \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}\left(-\frac{2\alpha^2}{3} + O(\alpha^4)\right)^2 + \cdots$$
$$= 1 + \frac{\alpha^2}{3} - \frac{\alpha^4}{9} + \frac{3}{8}.\frac{4\alpha^4}{9} + O(\alpha^6) = 1 + \frac{\alpha^2}{3} + \frac{\alpha^4}{18} + O(\alpha^6).$$

$$\therefore \frac{\sin^2\alpha}{\alpha^2\left(1-\frac{2\sin^2\alpha}{3}\right)^{1/2}} = \frac{\alpha^2}{\alpha^2}\left(1-\frac{\alpha^2}{3}+\frac{2\alpha^4}{45}+O(\alpha^6)\right)\left(1+\frac{\alpha^2}{3}+\frac{\alpha^4}{18}+O(\alpha^6)\right)$$

$$= 1+\alpha^4\left(-\frac{1}{3}\cdot\frac{1}{3}+\frac{2}{45}+\frac{1}{18}\right)+O(\alpha^6)$$

$$= 1-\frac{\alpha^4}{90}+O(\alpha^6)$$

TPH/SJS
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