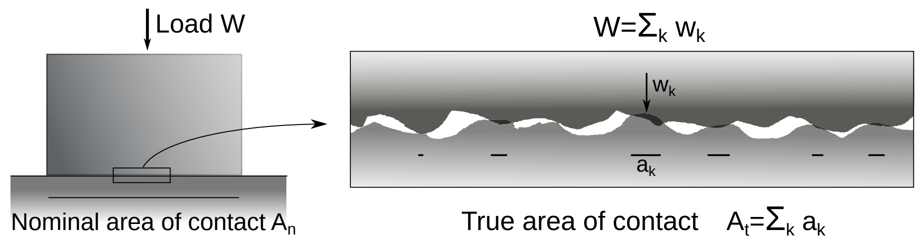
Part IA Paper 2: Materials

# Examples Paper 6 - Solutions

## Friction and wear

1. (a) Surfaces have asperities on them. Typical mean heights and wavelengths are 0.5 and 100 $\mu$m respectively for ground surfaces. When the two metal surfaces come into contact they make contact at the asperity tops, with plastic deformation at each contact. The true area of contact on the asperity tops, determined by the hardness of the softer steel (e.g. part (b)), is considerably less than the nominal area of contact.




(b) Looking up the density of medium carbon steel, the volume of the cube is 200/(7850/2)=0.0510 m$^3$. The side length is 0.371 m and nominal area of contact $A_N$ is 0.137 m$^2$. The true area of contact $A_t = W/H = W/(3\sigma_y)$, where $W$ is the weight of the cube and $H$ is the hardness of the softer steel. The lower value of $\sigma_y$ for medium carbon steel is 305 MPa (see databook). An upper limit of the true area of contact therefore is $A_t = \frac{200 \cdot 9.8}{3 \cdot 305 \cdot 10^6} \approx 2 \cdot 10^{-6}$ m$^2$.

(c) During sliding we are told that the true area is twice that of part (b). The frictional sliding force $F$ is the shear strength of the film times the true area. The friction coefficient is the shear force divided by the normal reaction $N$, which we assume is equal to the weight of the cube $W$, i.e. the minimum chute angle = $\sin^{-1}(F/W)$ = $\sin^{-1}$(0.22) = 13°.

(d) No change. According to this model for metallic contact the friction coefficient is independent of the nominal area and the weight (see the equation in (c)).

(e) The shear stresses on the two steel surfaces are the same. The softer car surface will deform more easily than the harder steel surface of the chute. Hence bits will be torn off the cube surface relatively easily, while the chute will only occasionally lose bits and so will wear much less.

(f) The true area of contact is still largely determined by the hardness of the steel, but the shear strength of the junctions is reduced to that of the additives. Hence the friction coefficient will fall.

(g) Now the true area of contact is not given by the hardness of the steel, but by the elastic deformation of the rubber. The increase in true area of contact is likely to outweigh any drop in shear stress that the junctions can stand, resulting in a higher frictional force and coefficient of friction (to be more precise, one would need to check the actual shear stress at the rubber interface).

2. (a) The wear rate $Q$ (volume of the material removed per unit sliding distance) is related to the load $W$ normal to the surfaces and the hardness $H$ of the softer surface by the following relationship, where $K$ is the wear coefficient:

$$Q = K\frac{W}{H} \, .$$

Here $W = 1$ N, $H = 240$ HV= 240kgf mm$^{-2}$ =240 $\cdot$ 9.8 $\cdot$ 10$^6$ N m$^{-2}$ = 2352 MPa.

We now need to find the wear rate $Q$. For this, we need to calculate the volume removed, and the sliding distance.

The volume removed is the volume of the mass of metal lost due to wear. Steel density is 7.9 Mg $m^{-3}$, so the volume lost is $v$ =5.1 $\cdot 10^{-11}$ $m^{-3}$.

The wear track has a diameter of 50 mm and the pin has a small cross-section. The pin rotates at 6 revolutions per second, hence 360 rpm. The sliding distance after 1 hour (60 min) is about $l$ =3400 m.

The wear coefficient is then given by $K = QH/W = \frac{vH}{lW} = 3.5 \cdot 10^{-5}$.

(b) At a load of 10 N, the wear rate Q is 1500 times higher.

This suggests that at higher load, the oxide film, which protects the asperities in mild steel to grow, can no longer sustain the load and as a result the true contact area increases resulting in severe abrasive wear.

3. (a) The material should be tested along at least three orthogonal directions. If the stiffness is the same along those directions, the stiffness is isotropic. Note that the material may still be anisotropic in terms of its microstructure and other physical quantities that were not tested here.

   (b) To test if the material is elastic, the simplest is to perform a ramp in strain and come back. If the stress follows the same path when the deformation is reversed, one can assume that the material behaves elastically (no energy dissipated). If the stress-strain curve is non-linear, then the material is non-linear elastic.

   (c) The material exhibit a time-dependent response if it is not elastic, and if the stress-strain curve depends on the rate at which deformation (or stress) is ramped up. Time dependent response could also be evidenced by a creep or relaxation test: for a given strain or stress, the state of the material appears to evolve over time.

   (d) A material could be modelled as linear visco-elastic if its mechanical response is linear with the imposed variable, either stress or strain. In practice, one can always try to fit data with models and see how the parameters vary when the stress or strain are scaled. But if the response is non-linear, it is risky to make predictions far away from the range of data measured experimentally. Note that a linear elastic material is included in linear visco-elastic material - just a single spring is required to describe it.

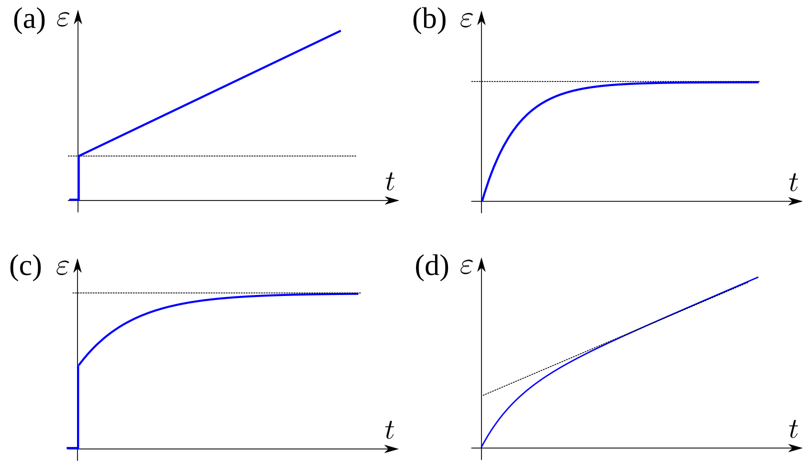
4. There are roughly 3 $\cdot 10^7$ s in a year. One can see from the figure that a stress in the order of $\sigma$ =15 MPa would produce a strain of 1% after a year, so this is the maximum allowable stress. The minimal rod cross-section is therefore $a = 500\text{N}/\sigma = 3.3 \cdot 10^{-5}$ $m^2$. The rod diameter is then found to be equal to 6.5 mm.

5. (a) The creep response of two elements in series is the sum of their creep responses. We therefore expect to have a jump at the origin, and a viscous response afterwards. At short time scales, the dashpot has no time to extend and only the spring is acting; the material is linear elastic. At long time-scales, only the dashpot contributes; the material is viscous Newtonian.

   (b) Initially, the spring is not loaded; all the load is transmitted to the dashpot, which starts to extend linearly with time. However, as it extends, an increasing part of the load is transferred to the spring, and less and less to the dashpot. Hence we expect a progressive decrease of the extension rate, until all the load is in the spring. The deformation reaches then a steady value. At short time scales, the material is dominated by the dashpot (viscous), and at long time scales, by the spring (elastic).

   (c) We expect to get the sum of the creep functions of a spring and Kelvin model (b). At short time scales, the dashpot locks the Kelvin unit, so we simply have a spring. At long time scales, the Kelvin element has reached its steady extension. We have a spring too, but with a different elastic constant.

   (d) Here as well, we need to add two contributions, the responses of the dashpot and Kelvin model (b). At short time-scales, the two dashpots control the response, making the material fluid like. At long time scales, only one plays a role; it behaves again as a fluid, but more viscous.




6. The first model was studied in the course, but not the second.

   This problem can be approached in many different ways. We could for instance compare the creep functions, relaxation functions, or ODEs of both models and find the parameters required for them to be equivalent.

   The creep response of the second model is straight-forward to determine, so we will use this approach. For a step of amplitude $\sigma_0$ in the stress, the spring $E_b$ will provide an instantaneous strain increase of amplitude $\sigma_0/E_b$. To this response, we add the response of a Kelvin element, amplitude $\sigma_0/E_a$ with a response time $\tau_a$ (expression from the lecture notes).

$$J(t) = \frac{1}{E_b} + \frac{1 - \exp(-t/\tau_a)}{E_a} \text{ for } t \geq 0$$

   By comparison with the expression obtained for the first model during the lectures:

$$J(t) = \frac{1}{E_2} - \frac{E_1}{E_2(E_1 + E_2)} e^{-t/\tau^*} \text{ for } t \geq 0$$

   we find:

$$\frac{1}{E_b} + \frac{1}{E_a} = \frac{1}{E_2}$$
$$\frac{1}{E_a} = \frac{E_1}{E_2(E_1 + E_2)}$$
$$\frac{\eta_a}{E_a} = \frac{\eta_1}{E_1}(1 + E_1/E_2)$$

   This leads to:

$$E_b = E_1 + E_2$$
$$E_a = (E_1 + E_2)\frac{E_2}{E_1}$$
$$\eta_a = \eta_1 \left(\frac{E_1 + E_2}{E_1}\right)^2$$

   **Alternative method:**

   We could also directly derive the differential equation of the second model and compare each coefficients.

Let's define $\varepsilon_a$ and $\varepsilon_b$ as the deformations on the Kelvin element $(E_a, \eta_a)$ and the spring $E_b$, respectively. The total strain is $\varepsilon = \varepsilon_a + \varepsilon_b$. The stress is the same across the Kelvin element and spring $E_b$ as they are in series. We have:

$$\sigma = E_b \varepsilon_b = E_a \varepsilon_a + \eta_a \dot{\varepsilon_a}$$

To solve this system of differential equation, we have use Laplace transform, or fiddle with the equations. Let's do the second approach here. We want to find a relationship between $\varepsilon$ and $\sigma$, using the relationship above to eliminate $\varepsilon_a$ and $\varepsilon_b$. The combination of $E_a \varepsilon_a + \eta_a \dot{\varepsilon_a}$ indicates how best start:

$$E_a \varepsilon + \eta_a \dot{\varepsilon} = E_a \varepsilon_a + \eta_a \dot{\varepsilon_a} + E_a \varepsilon_b + \eta_a \dot{\varepsilon_b}$$

Alternatively, defining $\tau_a = \eta_a / E_a$, one could write:

$$\varepsilon + \tau_a \dot{\varepsilon} = \varepsilon_a + \tau_a \dot{\varepsilon_a} + \varepsilon_b + \tau_a \dot{\varepsilon_b}$$

$$\varepsilon + \tau_a \dot{\varepsilon} = \frac{\sigma}{E_a} + \frac{\sigma}{E_b} + \tau_a \frac{\dot{\sigma}}{E_b}$$

The expression then needs to be rearranged to match the expression in the handout for the first model. Setting the coefficient in front of $\sigma$ is 1, we get a form from which the relationships between parameters can be obtained.

$$\sigma + \tau_a \frac{E_a}{E_a + E_b} \dot{\sigma} = \frac{E_a E_b}{E_a + E_b} \varepsilon + \tau_a \dot{\varepsilon}$$

7. In a linear strain ramp, the deformation $\varepsilon$ and time $t$ are related by $\varepsilon = \dot{\varepsilon} t$. So the stress strain curves are essentially stress time curves, with the $t$ axis scaled by the strain rate.

   Because a ramp is the integral of a step function, the graph we see is therefore the integral of a relaxation function. At long time scale, the slope is constant, so the relaxation function must tend to a constant value. At short time scale, the relaxation must also start high, but decrease progressively, after a certain time $\tau$, towards the constant value. The relaxation function of the standard linear solid is the only suitable candidate:

$$E(t) = E_2 + E_1 e^{-t/\tau_1}$$

   The response to a strain ramp at rate $\dot{\varepsilon}$ would be:

$$\sigma = \dot{\varepsilon} \int E_2 + E_1 e^{-t/\tau_1} = \dot{\varepsilon}(E_2 t + E_1 \tau_1 (1 - e^{-t/\tau_1}))$$

$$\sigma = E_2 \varepsilon + \dot{\varepsilon} \eta_1 (1 - e^{-\frac{\varepsilon}{\dot{\varepsilon}\tau_1}})$$

   The slope $E_2$ is well defined, about 7 MPa increase for 10% deformation, i.e. 70 MPa.

   It is clear however that the vertical shift of the curves does not scale with $\dot{\varepsilon}$ as expected from the model. The deformation at which the linear regime is reached should also be shorter for small shear rates than large shear rates, which is not visible on the curves.

   Although a good fit can be obtained, the material does not behave linearly. The whole validity of the approach should therefore be questioned.

8. (a) A string must be able to hold tension over a long period of time. The Maxwell model and the model of 5(d), which behave as fluids on long time scales, are not suitable candidates. The standard model has a relaxation behaviour but the Kelvin-Voigt model does not. So the standard model is the only plausible candidate to describe a nylon guitar string.

   (b) When the string is tuned downwards in pitch, this is equivalent to applying a negative strain step to the final steady state response to the previous problem. We are doing linear theory, so the responses for the steady state and the negative strain step can be superposed. The result for the negative strain step is the opposite of the positive step strain: an instantaneous decrease of the tension, followed by a slow increase. ”Upwards relaxation” occurs, so the standard model predicts the behaviour that guitarists observe.

TS
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