Part IA Mathematics Examples paper 7 Solutions

(1) $F\delta(t) = 0$ except when $t = 0$, and $\int_{-\infty}^{\infty} F\delta(t)\,dt = F$

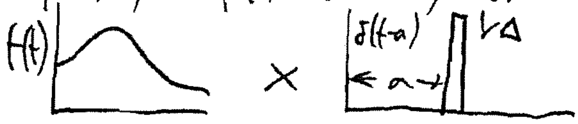
So F is a "very large force for a very short time", in other words an impulse of magnitude F at time $t=0$

$P\delta(x-a)$ is zero except at $x-a=0$, i.e. $x=a$. The integrated value is P. So this represents a point load P at the point $x=a$

2. $\delta(t)$ is a unit impulse at $t=0$, the limit as $\Delta \to 0$ of the function:

$\delta(t-a)$ is a unit impulse occurring where $t-a=0$ i.e. at $t=a$.

For a continuous function $f(t)$, $f(t)\,\delta(t-a)$ is approximately the product



As $\Delta \to 0$, the product is only non-zero very near $t=a$

So $\int \delta(t-a)\,f(t)\,dt \simeq \int \delta(t-a)\,f(a)\,dt = f(a)$

(provided $t=a$ lies within the range of integration)

(i) $\int_0^5 \delta(t-3)\,dt = 1$ since $t=3$ lies between 0 and 3

(ii) $\int_{-1}^{1} \delta(x)\sin(x)\,dx = \sin 0 = 0$

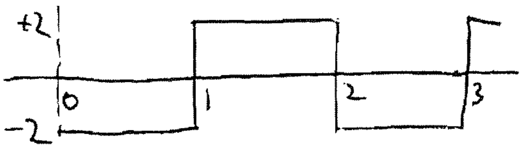
(iii) $\int_0^1 \delta(x-\pi/4)\exp(\cos(1/x))\,dx = \exp(\cos(1/(\pi/4)))$

because $0 < \pi/4 < 1$

$= \exp(\cos(4/\pi)) = 1.3407$

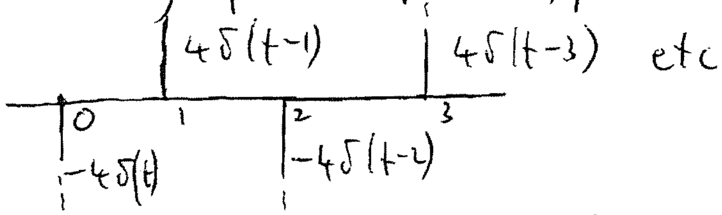
(v) $\int_0^1 \delta(x+\pi/4)\cos x\,dx = 0$ since $-\frac{\pi}{4}$ is not in the range $0 \to 1$

3 (i) $\frac{df}{dt}$ is the square wave



All jumps of magnitude 4.

$\frac{d^2f}{dt^2}$ consists entirely of delta functions, from these jumps

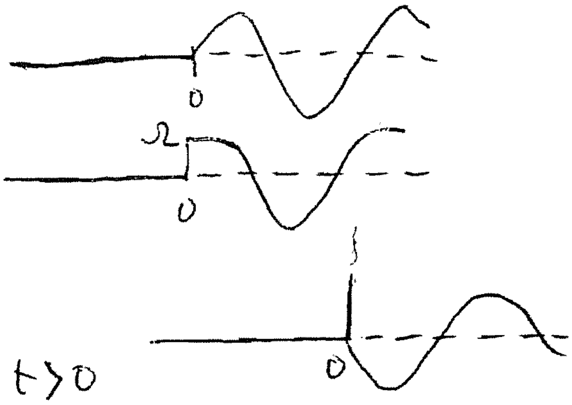


(ii) $f(t) = \begin{cases} 0 & t<0 \\ \sin \Omega t & t>0 \end{cases}$

$\frac{df}{dt} = \begin{cases} 0 & t<0 \\ \Omega \cos \Omega t & t>0 \end{cases}$

Note jump at $t=0$

$\frac{d^2f}{dt^2} = \begin{cases} 0 & t<0 \\ \Omega\delta(t) - \Omega^2 \sin \Omega t, & t>0 \end{cases}$



4(i) For $t<0$ $f=0$, and $y=0$ is a solution

For $t>0$ $f=1$ and the equation is

$$\frac{dy}{dt} + 3y = 1 \qquad (1)$$

Complementary function: solve $\dot{y} + 3y = 0$ to get $y = Ae^{-3t}$

Particular integral: try $y = \alpha$ (constant)

Then from (1) $3\alpha = 1$, so $\alpha = 1/3$

So general solution is CF + PI, $y = Ae^{-3t} + 1/3$

Now at $t=0$ y must be <u>continuous</u>: if y had a jump, $\dot{y}$ would have a delta function, and nothing else in (1) could balance this.

So at $t=0$ we have $y=0$, so $A + 1/3 = 0$

$\therefore$ step response is $y = \begin{cases} 0, & t<0 \\ \frac{1}{3}(1-e^{-3t}) & t>0 \end{cases}$

Impulse response $= \frac{d}{dt}(\text{step response}) = \begin{cases} 0 & t<0 \\ e^{-3t} & t>0 \end{cases}$

4 (ii) Same method as (i). For $t<0$, $y=0$.

For $t>0$ we have $\ddot{y}+4y=1$

C.F. : $\ddot{y}+4y=0 \rightarrow y = A\cos 2t + B\sin 2t$

P.I. Try $y=\alpha$, then need $4\alpha=1$, so $\alpha = 1/4$

So general solution is $y = 1/4 + A\cos 2t + B\sin 2t$

At $t=0$: if y had a jump, $\dot{y}$ would have a delta function and $\ddot{y}$ would have $\frac{d}{dt}(\delta(t))$. This could not satisfy the equation, so y is continuous.

Similarly, if $\dot{y}$ had a jump, $\ddot{y}$ would have $\delta(t)$, and so $\dot{y}$ must be continuous.

So at $t=0$ $\begin{cases} y=0 \rightarrow 1/4 + A = 0 \rightarrow A = -1/4 \\ \dot{y}=0 \rightarrow B=0 \end{cases}$

So step response is $y = \begin{cases} 0 & t<0 \\ \frac{1}{4}(1-\cos 2t) & t>0 \end{cases}$

Impulse response $= \frac{d}{dt}$ (step response) $= \begin{cases} 0 & t<0 \\ \frac{1}{2}\sin 2t & t>0 \end{cases}$

5. (i) For $t<0$ $f=0$, and $y=0$

For $t>0$ $f=1$ and equation is $\dot{y}+\alpha y = 1$

C.F. is $y = Ae^{-\alpha t}$

P.I. : try $y=B \rightarrow \alpha B = 1 \rightarrow B = 1/\alpha$

So general solution is $y = 1/\alpha + Ae^{-\alpha t}$

At $t=0$ $y=0$, so $1/\alpha + A = 0 \rightarrow A = -1/\alpha$

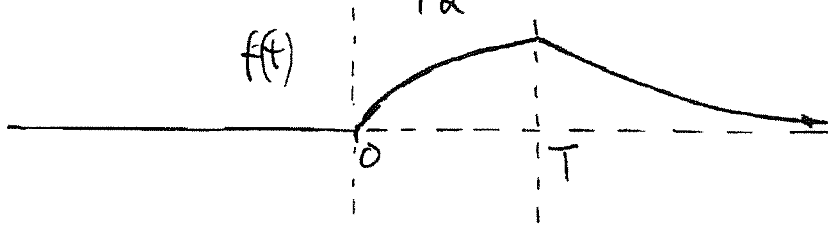
So step response is $y = \frac{1}{\alpha}\left(1-e^{-\alpha t}\right)$ for $t>0$

(ii) If the unit step function is $H(t)$, the required input is $f(t) = \frac{1}{T}\left(H(t) - H(t-T)\right)$

The output thus a superposition of two step responses, as found in (i)

5 cont. Output $y = \begin{cases} 0 & t < 0 \\ \frac{1}{T\alpha}\left(1 - e^{-\alpha t}\right) & 0 \leq t \leq T \\ \frac{1}{T\alpha}\left[\left(1 - e^{-\alpha t}\right) - \left(1 - e^{-\alpha(t-T)}\right)\right] & t \geq T \end{cases}$

The last part simplifies to $\frac{1}{T\alpha} e^{-\alpha t}\left(e^{\alpha T} - 1\right)$



As $T \to 0$, $e^{\alpha T} \simeq 1 + \alpha T + \frac{(\alpha T)^2}{2!} + \dots$

So $\frac{1}{\alpha T} e^{-\alpha t}\left(e^{\alpha T} - 1\right) \simeq e^{-\alpha t} \frac{\alpha T + \alpha^2 T^2/2 \dots}{\alpha T}$

$\simeq e^{-\alpha t}$

So response tends to $y = \begin{cases} 0 & t < 0 \\ e^{-\alpha t} & t > 0 \end{cases}$

which is indeed the impulse response.

6 Impulse response from 4(i) was $g(t) = \begin{cases} 0 & t < 0 \\ e^{-3t} & t > 0 \end{cases}$

(i) Input $f(t) = \begin{cases} 0 & t < 0 \\ e^{-2t} & t > 0 \end{cases}$

Output is $y(t) = \int_{-\infty}^{t} g(t-\tau) f(\tau)\, d\tau$ for $t > 0$

$= \int_0^t e^{-3(t-\tau)} e^{-2\tau}\, d\tau = e^{-3t}\left[e^{\tau}\right]_0^t$

$= e^{-2t} - e^{-3t}$

Check: $\dot{y} = -2e^{-2t} + 3e^{-3t}$

So $\dot{y} + 3y = e^{-2t} = f(t)$ for $t > 0$ ✓

6 (ii) Input $f(t) = \begin{cases} 0 & t<0 \\ \sin t & t>0 \end{cases}$

Output $y(t) = \int_{-\infty}^{t} g(t-\tau)\, f(\tau)\, d\tau$

$$= \int_0^t e^{-3(t-\tau)} \sin\tau \, d\tau$$

$$= e^{-3t} \int_0^t e^{3\tau}\left(\frac{e^{i\tau}-e^{-i\tau}}{2i}\right) d\tau$$

$$= \frac{e^{-3t}}{2i}\left[\frac{e^{(3+i)\tau}}{3+i} - \frac{e^{(3-i)\tau}}{3-i}\right]_0^t$$

$$= \frac{e^{-3t}}{2i}\left[e^{3t}\left(\frac{e^{it}}{3+i} - \frac{e^{-it}}{3-i}\right) - \frac{1}{3+i} + \frac{1}{3-i}\right]$$

$$= \frac{(3-i)e^{it} - (3+i)e^{-it}}{2i(3^2+1^2)} - \frac{e^{-3t}}{2i}\left(\frac{3-i-3-i}{3^2+1^2}\right)$$

$$= \frac{1}{10}\left(3\sin t - \cos t + e^{-3t}\right) \quad \text{for } t>0$$

Check: $\dot{y} = \left(3\cos t + \sin t - 3e^{-3t}\right)/10$ for $t>0$

$\therefore\ \dot{y} + 3y = \sin t$ ✓

7 (i) $\frac{1}{6}\ddot{y} + \frac{11}{12}\dot{y} + y = 1$ for $t>0$

C.F.: Try $y = e^{\lambda t} \Rightarrow \frac{1}{6}\lambda^2 + \frac{11}{12}\lambda + 1 = 0 \Rightarrow \lambda = -4$ or $-\frac{3}{2}$

so C.F. is $Ae^{-\frac{3}{2}t} + Be^{-4t}$

P.I: Try $y = \alpha \Rightarrow \alpha = 1$

so general solution is $y = 1 + Ae^{-\frac{3}{2}t} + Be^{-4t}$

At $t=0$, $y = \dot{y} = 0 \Rightarrow 1 + A + B = 0$ and $\frac{-3A}{2} - 4B = 0$

Thus $A = -\frac{8}{5}$ and $B = \frac{3}{5}$

So the step response is

$$y = 1 - \frac{8e^{-\frac{3}{2}t}}{5} + \frac{3e^{-4t}}{5} \quad \text{for } t>0.$$

7 (ii) Impulse response $= \dfrac{d}{dt}$ (step response)

$$\therefore\ g(t) = \frac{12e^{-3t/2}}{5} - \frac{12e^{-4t}}{5} \quad \text{for } t>0$$

(iii) Now impose input $f(t) = \begin{cases} 0 & t<0 \\ e^{-t} & t>0 \end{cases}$

Output is $y(t) = \displaystyle\int_{-\infty}^{t} g(t-\tau)\, f(\tau)\, d\tau$

$$= 12\int_0^t \left\{ \frac{e^{-3(t-\tau)/2}}{5} - \frac{e^{-4(t-\tau)}}{5} \right\} e^{-\tau}\, d\tau$$

$$= \frac{e^{-3t/2}}{5} 12 \left[2e^{\tau/2} \right]_0^t - \frac{e^{-4t}}{5} 12 \left[\frac{e^{3\tau}}{3} \right]_0^t$$

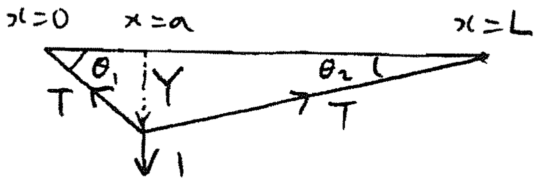
$$= \left(\frac{2e^{-t}}{5} - \frac{2e^{-3t/2}}{5} - \frac{e^{-t}}{15} + \frac{e^{-4t}}{15} \right) 12$$

$$= \frac{12}{15} \left\{ e^{-4t} - 6e^{-3t/2} + 5e^{-t} \right\}$$

[For 7(iv) see next sheet]

changed to "rubber band" argument requires flexibility

8 (i) Stretched curtain wire in response to a point load moves to a displaced shape consisting of two straight lines.



Exaggerated vertical scale

θ_1, θ_2 are assumed small, so that

$$\begin{cases} \sin\theta_1 \simeq \theta_1 \simeq \tan\theta_1 = \dfrac{Y}{a}, & \cos\theta_1 \simeq 1 \\ \sin\theta_2 \simeq \theta_2 \simeq \tan\theta_2 = \dfrac{Y}{L-a}, & \cos\theta_2 \simeq 1 \end{cases}$$

A unit load is applied, so force balance requires

$$1 = T\sin\theta_1 + T\sin\theta_2$$

8 cont.

i.e. $TY\left[\frac{1}{L-a} + \frac{1}{a}\right] \simeq 1$

$\therefore \quad \frac{TYL}{a(L-a)} = 1$, so $Y = \frac{a(L-a)}{LT}$

So $g(x,a) = \begin{cases} \frac{(L-a)x}{LT} & x < a \\ \frac{a(L-x)}{LT} & x > a \end{cases}$

(ii) For a continuous load $f(x)$, we can approximate it by a row of small point loads (think of the curtain as a bead curtain!)
Each point load produces a suitably scaled version of $g(x,a)$ as found in (i). By superposition, total displacement is the sum of these. In the limit, this sum becomes an integral, just as for the time-varying case from lectures.
So $y(x) = \int_0^a g(x,a)\, f(a)\, da$

(iii) For $0 \leq x \leq L/2$:

$$y(x) = \underbrace{\int_0^x \frac{Fa(L-x)}{LT}\,da}_{\text{In here, } a \leq x \text{ or } x \geq a} + \underbrace{\int_x^{L/2} \frac{F(L-a)x}{LT}\,da}_{\text{In here, } a \geq x \text{ so } x \leq a}$$

$$= \frac{F(L-x)}{LT}\frac{x^2}{2} + \frac{Fx}{LT}\left[La - \frac{a^2}{2}\right]_x^{L/2}$$

$$= \frac{F}{2LT}x^2(L-x) + \frac{Fx}{LT}\left[\frac{L^2}{2} - \frac{L^2}{8} - Lx + \frac{x^2}{2}\right]$$

$$= \frac{F}{8LT}\left\{4x^2L - 4x^3 + 3L^2x - 8Lx^2 + 4x^3\right\}$$

$$= \frac{Fx}{8T}(3L - 4x)$$

8 cont.

For $x > L/2$: $$y(x) = \int_0^{L/2} \frac{Fa(L-x)}{LT}\, da$$

$\uparrow$

$x > a$ for whole range this time

$$= \frac{F(L-x)}{LT} \cdot \frac{L^2}{8} = \frac{FL}{8T}(L-x)$$

Check: At $x = L/2$, wire should be not broken.

From first solution, $y = \frac{FL}{16T}(3L - 2L) = \frac{FL^2}{16T}$

From second solution, $y = \frac{FL^2}{16T}$ ✓

Note that this problem involving convolution in <u>space</u> is a bit more tricky than problems involving <u>time</u>. The reason is that an impulse response in time is always zero for $t < 0$, but a spatial "impulse response" like $g(x,a)$ here has non-zero values on both sides of the delta function. This means that more care is needed to get the convolution integral right.

TPH/JW