Part IA Mathematics Examples Paper 8 Solutions

1. $$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$$

where $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(t)\,dt = \frac{T}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^T \cos nt\,dt = \frac{1}{\pi}\left[\frac{\sin nt}{n}\right]_0^T = \frac{\sin nT}{n\pi}$$

$$b_n = \frac{1}{\pi}\int_0^T \sin nt\,dt = \frac{1}{\pi}\left[\frac{-\cos nt}{n}\right]_0^T = \frac{1-\cos nT}{n\pi}$$

The simplest approximation for low-pass filtering would be to truncate the series at a certain cut-off frequency. In practice, a more sophisticated approach would be needed: the amplitudes a_n, b_n would be attenuated smoothly as some decreasing function of n. Also, in practice the phase of the harmonics might be affected by transmission. This will result in the a_n and b_n being mixed together: more easily treated mathematically via the complex Fourier series.

2. The function is even: $f(\theta) = f(-\theta)$.
Cosines are all even, sines are all odd, so the Fourier series will contain only cosines.
So $$f(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta$$

where $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\,d\theta = \pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta = \underbrace{\frac{2}{\pi}}_{\times 2 \text{ because } -\pi\to 0 \text{ gives same answer}}\int_0^{\pi} \theta\cos n\theta\,d\theta$$

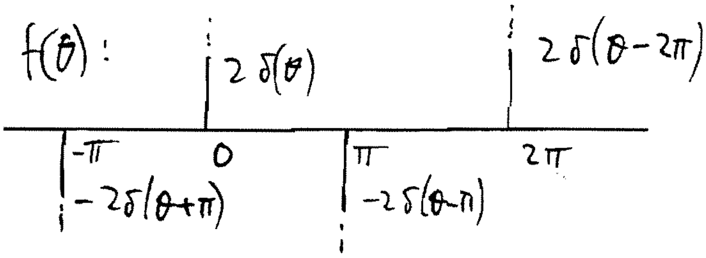
2 cont. $\therefore\ a_n = \frac{2}{\pi}\left\{\left[\theta\,\frac{\sin n\theta}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin n\theta}{n}\,d\theta\right\}$ (parts)

$$= \frac{2}{\pi}\left\{0 + \left[\frac{\cos n\theta}{n^2}\right]_0^{\pi}\right\} = \begin{cases} 0, & n \text{ even} \\ \frac{-4}{n^2\pi}, & n \text{ odd}\end{cases}$$

So $f(\theta) = \frac{\pi}{2} - \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n^2\pi}\cos n\theta$

So $\frac{df}{d\theta} = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n\pi}\sin n\theta$, as obtained for square wave in lectures.

3.



The function is even, so Fourier series needs only cosines. Convenient to choose a range for integration which doesn't end right on a delta-function — any range of 2π will do, because the function is periodic.

So write $f(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta$

where $a_0 = \frac{1}{\pi}\int_{-\pi/2}^{3\pi/2} f(\theta)\,d\theta = \frac{1}{\pi}(2-2) = 0$

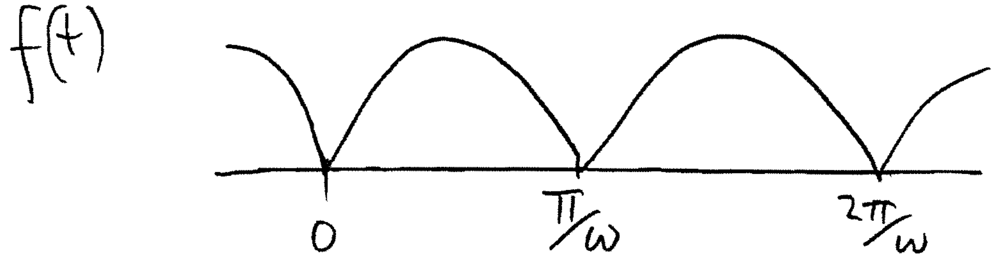
$$a_n = \frac{1}{\pi}\int_{-\pi/2}^{3\pi/2} f(\theta)\cos n\theta\,d\theta$$

$$= \frac{2}{\pi}\int_{-\pi/2}^{3\pi/2}\left(\delta(\theta) - \delta(\theta-\pi)\right)\cos n\theta\,d\theta$$

$$= \frac{2}{\pi}\left(1 - \cos n\pi\right) = \frac{2}{\pi}\left(1-(-1)^n\right) = \begin{cases} 0 & n \text{ even} \\ 4/\pi & n \text{ odd}\end{cases}$$

Integrating term by term recovers the square wave $\sum_{n=1,\, n \text{ odd}}^{\infty} \frac{4}{n\pi}\sin n\theta$

4.



f(t) has period π/ω, and is an even function. So it will have a Fourier series

$$f(t) = \frac{a_0}{2} + \sum_{1}^{\infty} a_n \cos 2n\omega t$$

Either put $\theta = 2\omega t$, which changes the period to 2π, and use that Maths Data Book formula

or use the Electrical + Information Data Book directly, to deduce

$$a_0 = \frac{2\omega}{\pi}\int_0^{\pi/\omega} \sin \omega t \, dt = \frac{2\pi}{\omega}\left[\frac{-\cos\omega t}{\omega}\right]_0^{\pi/\omega}$$

$$= \frac{4}{\pi}$$

and

$$a_n = \frac{2\omega}{\pi}\int_0^{\pi/\omega} \sin\omega t \cos 2n\omega t \, dt$$

$$= \frac{\omega}{\pi}\int_0^{\pi/\omega}\left[\sin\omega(2n+1)t - \sin\omega(2n-1)t\right]dt$$

$$= \frac{\omega}{\pi}\left[\frac{-\cos\omega(2n+1)t}{\omega(2n+1)} + \frac{\cos\omega(2n-1)t}{\omega(2n-1)}\right]_0^{\pi/\omega}$$

$$= \frac{1}{\pi}\left[\frac{1-\cos(2n+1)\pi}{2n+1} - \frac{1-\cos(2n-1)\pi}{2n-1}\right]$$

$$= \frac{2}{\pi}\left(\frac{1}{2n+1} - \frac{1}{2n-1}\right) = \frac{-4}{\pi(4n^2-1)}$$

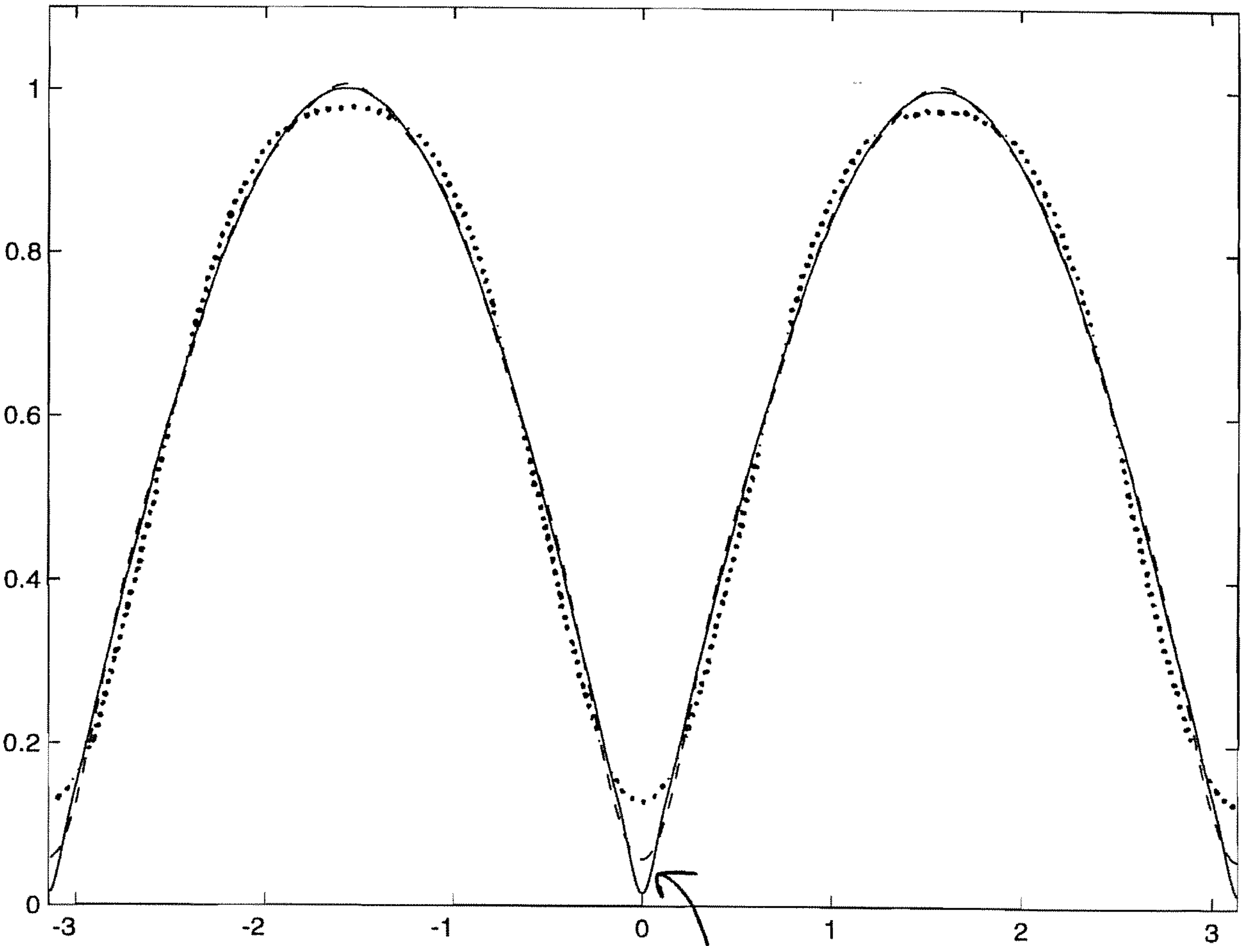
4 cont.

Computed waveforms:

... 2 terms (ie $n = 1, 2$, plus a_0 term)

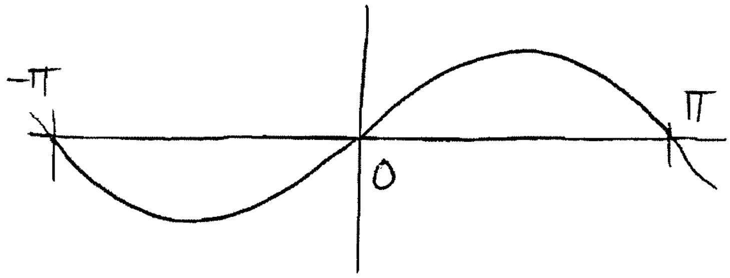
-- 5 terms

— 20 terms



Notice that we need high terms to get this sharp corner.

5. $$y = \begin{cases} x(\pi + x) & -\pi \le x \le 0 \\ x(\pi - x) & 0 \le x \le \pi \end{cases}$$



Only places where discontinuities could occur are $x = -\pi, 0, \pi$.
It is clear that y is continuous at these points.
What about $\frac{dy}{dx}$? $\left.\frac{dy}{dx}\right|_{-\pi_+} = -\pi = \left.\frac{dy}{dx}\right|_{\pi_-}$

$$\left.\frac{dy}{dx}\right|_{0-} = \pi = \left.\frac{dy}{dx}\right|_{0+}$$ so continuous

$\frac{d^2y}{dx^2}$? $$\frac{d^2y}{dx^2} = \begin{cases} 2 & -\pi \le x \le 0 \\ -2 & 0 \le x \le \pi \end{cases}$$

so has discontinuities at $0, \pi$
So y, y' are continuous but y'' is not, so expect Fourier coefficients of the order of $1/n^3$ as $n \to \infty$.

y is an odd function, so need only sines:

$$y(x) = \sum_1^\infty b_n \sin nx, \quad \pi b_n = \int_{-\pi}^{\pi} y(x) \sin nx \, dx$$

$$= 2\int_0^\pi y(x) \sin nx \, dx = 2\int_0^\pi x(\pi - x) \sin nx \, dx$$

$$= \left[-2x(\pi - x)\frac{\cos nx}{n}\right]_0^\pi + 2\int_0^\pi (\pi - 2x)\frac{\cos nx}{n} dx$$

$$= \frac{2}{n}\left[(\pi - 2x)\frac{\sin nx}{n^2}\right]_0^\pi - \frac{2}{n}\int_0^\pi (-2)\frac{\sin nx}{n} dx$$

5 cont.

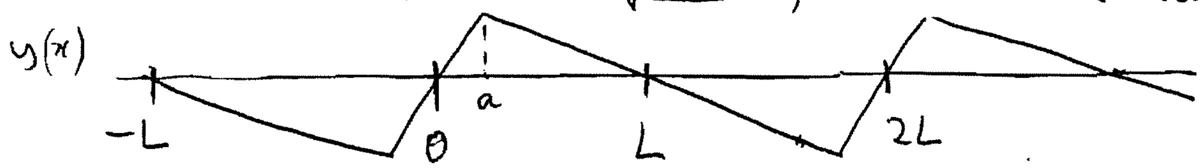
$$\therefore \quad \pi b_n = \frac{4}{n^2}\left[-\frac{\cos nx}{n}\right]_0^{\pi}$$

$$= \frac{4}{n^3}\left(1 - \cos n\pi\right) = \frac{4}{n^3}\left(1-(-1)^n\right)$$

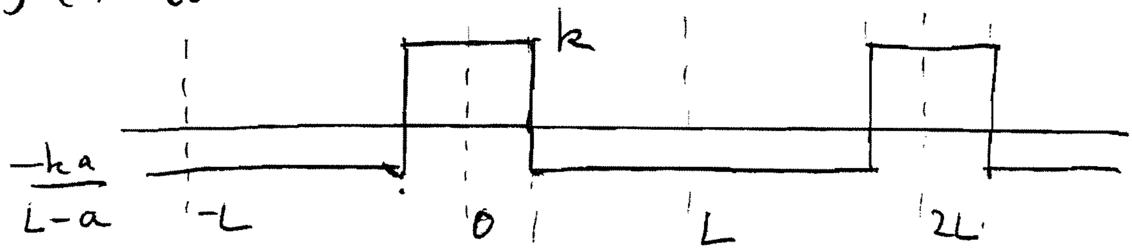
$$= \begin{cases} 8/n^3 & , \; n \text{ odd} \\ 0 & , \; n \text{ even} \end{cases}$$

So the coefficients do indeed die away like n^{-3}, as expected.

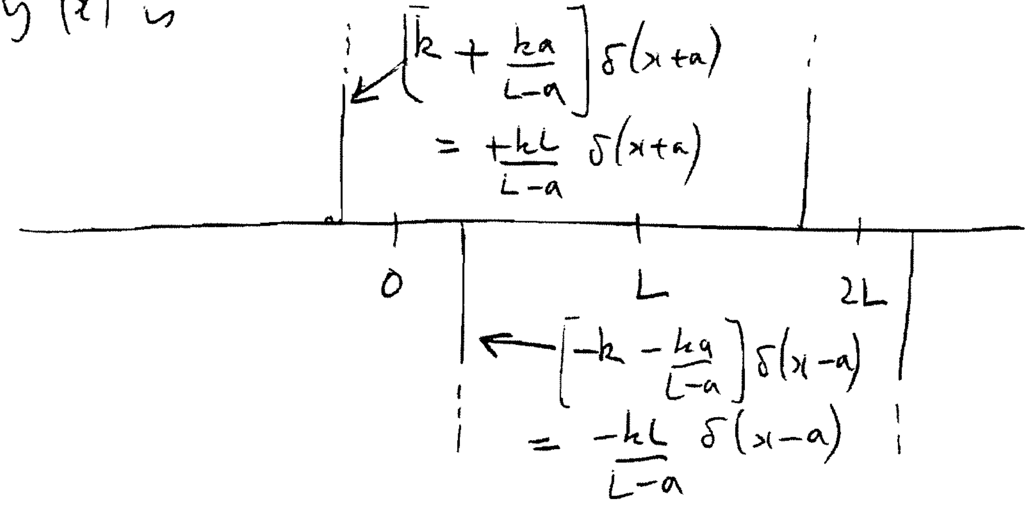
6. We have no choice how to extend the function here: the modes of the string are $\sin\frac{n\pi x}{L}$, so we use a Fourier sine series, so that the extended function is <u>odd</u>. Other extensions could represent the function, but wouldn't be modes.



To evaluate, we can either do it directly or use the trick from Q3: $y'(x)$ is:



So $y''(x)$ is

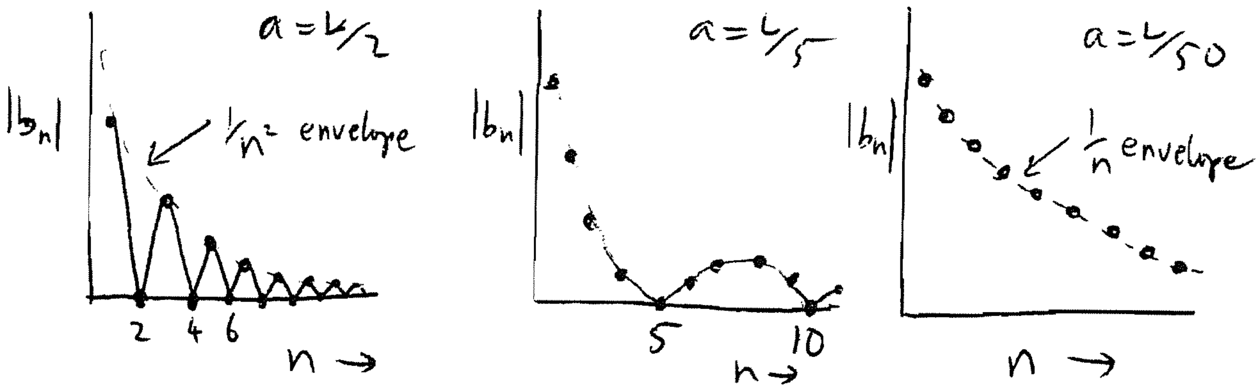


6 cont. $y''(x)$ is an odd function. Let its Fourier series be $\sum_1^\infty b_n' \sin\frac{n\pi x}{L}$ [NB. period is $2L$]

Then
$$b_n' = \frac{2}{2L}\int_{-L}^{L} y''(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L y''(x)\sin\frac{n\pi x}{L}\,dx$$
$$= \frac{2}{L}\int\left[\frac{-kL}{L-a}\,\delta(x-a)\right]\sin\frac{n\pi x}{L}\,dx$$
$$= \frac{-2k}{L-a}\sin\frac{n\pi a}{L}$$

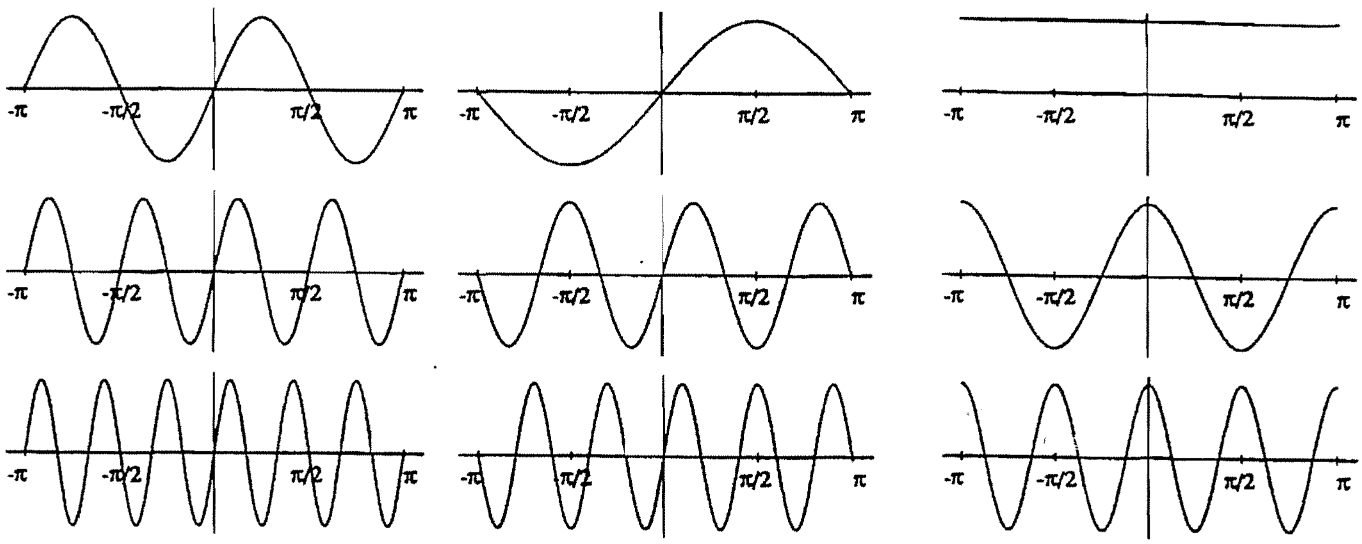
Now integrate term by term to get the Fourier series for $y(x) = \sum_1^\infty b_n \sin\frac{n\pi x}{L}$

where
$$b_n = -\frac{b_n'}{(n\pi/L)^2} = \frac{L^2}{n^2\pi^2}\cdot\frac{2k}{L-a}\sin\frac{n\pi a}{L}$$
$$\simeq \frac{aL}{n\pi}\cdot\frac{2k}{L-a} \text{ when } \frac{n\pi a}{L} \ll 1$$



$y(x)$ is continuous, but y' is not, so the Fourier coefficients die away like n^{-2} at large n. If the "corner" at the plucking point were rounded, $y'(x)$ would then be continuous, so $|b_n|$ would decay at least as fast as n^{-3}. So a rounded corner gives a less "bright" sound, because it has less high-frequency content. Contrast a guitar string plucked with the flesh of a finger, and one plucked with a narrow, hard plectrum (like a harpsichord string).

7. The key to answering this question is to consider the symmetri of the terms in the three series.



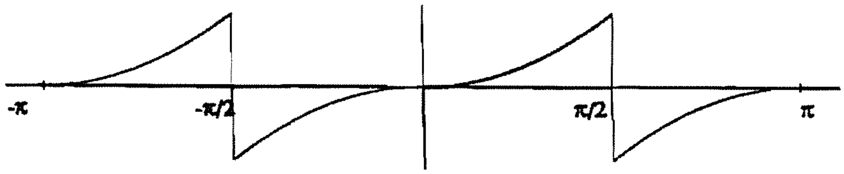
(i) Terms antisymmetric about $x = 0$ & antisymmetric about $x = \pm\pi/2$

(ii) Terms antisymmetric about $x = 0$, symmetric about $x = \pm\frac{\pi}{2}$

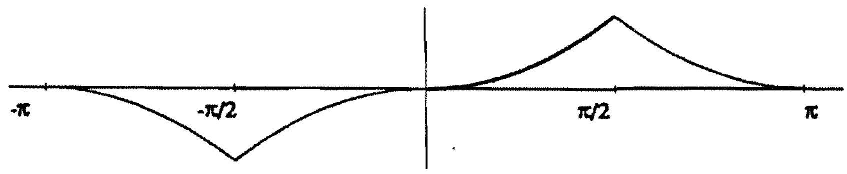
(iii) Terms symmetric about $x = 0$, symmetric about $x = \pm\pi/2$.

Functions represented by these series must have the same symmetries.

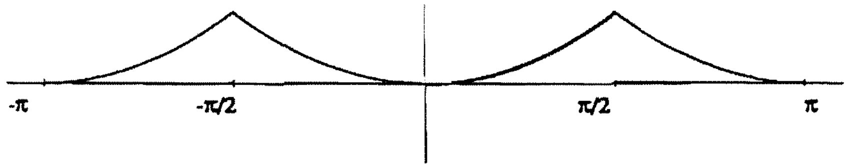
Case (i)



Case (ii)



Case (iii)



Cases (ii) & (iii) have discontinuities of slope, so $b_m, c_m = O(\frac{1}{m^2})$

Case (i) has discontinuities of value, so $a_m = O(\frac{1}{m})$ and much slower convergence.

8 $$y(t) = \begin{cases} e^{-\alpha t} & 0 < t < T/2 \\ 0 & T/2 < t < T \end{cases} = \sum_{n=-\infty}^{\infty} C_n e^{2\pi i n t/T}$$

$$\Rightarrow \quad C_n = \frac{1}{T}\int_0^T y(t)\, e^{-\frac{2\pi i n t}{T}}\, dt = \frac{1}{T}\int_0^{T/2} e^{-(\alpha + \frac{2\pi i n}{T})t}\, dt$$

$$= \frac{1}{T}\left[\frac{e^{-(\alpha+\frac{2\pi i n}{T})t}}{-(\alpha + \frac{2\pi i n}{T})}\right]_0^{T/2}$$

$$= \underline{\frac{1 - e^{-\alpha T/2 - i n\pi}}{\alpha T + 2\pi i n}}.$$

If $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi t}{T} + \sum_{n=1}^{\infty} b_n \sin\frac{2\pi n t}{T} = y(t) = \sum_{n=-\infty}^{\infty} C_n e^{2i\pi n t/T}$

then $\sum_{-\infty}^{\infty} C_n e^{2\pi i n t/T} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \frac{e^{2\pi i n t/T} + e^{-\frac{2\pi i n t}{T}}}{2} + \sum_{n=1}^{\infty} b_n \frac{e^{2\pi i n t/T} - e^{-2\pi i n t}}{2i}$

$\Rightarrow \quad a_0 = 2C_0$

and for $n \geq 1$ $\quad \frac{a_n - i b_n}{2} = C_n \quad \Rightarrow \quad a_n = 2\,\mathrm{Re}(C_n)\,; \; b_n = -2\,\mathrm{Im}(C_n)$

$\therefore \quad \underline{a_0 = \frac{2(1 - e^{-\alpha T/2})}{\alpha T}}\;; \quad a_n = 2\,\mathrm{Re}\, \frac{\alpha T - 2\pi i n}{\alpha^2 T^2 + 4n^2\pi^2}\left(1 - \cos \pi\, e^{-\alpha T/2}\right)$

$\left[e^{-in\pi} = \cos n\pi - i \sin n\pi,\ \sin n\pi = 0\right.$

ie $\quad \underline{a_n = \frac{2\alpha T(1 - e^{-\alpha T/2}\cos n\pi)}{\alpha^2 T^2 + 4n^2\pi^2}}$

and $\quad \underline{b_n = +\frac{4n\pi(1 - e^{-\alpha T/2}\cos n\pi)}{\alpha^2 T^2 + 4n^2\pi^2}}.$

JW/TPH