Part IA Mathematics Examples paper 5 Solutions

1. a) $\frac{\partial f}{\partial x} = 2\cos x . \frac{\partial}{\partial x}(\cos x) + 0 = -2\sin x \cos x.$

$\frac{\partial f}{\partial y} = 0 + 2\sin y . \cos y$

b) $\frac{\partial f}{\partial x} = \sec^2 x \exp(-y^2)$

$\frac{\partial f}{\partial y} = \tan x . \exp(-y^2) \frac{\partial}{\partial y}(y^2) = -2y \tan x \exp(-y^2)$

(c) $\frac{\partial f}{\partial x} = \frac{1}{x^2+y^2} \frac{\partial}{\partial x}(x^2+y^2) = \frac{2x}{x^2+y^2}$

$\frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2}$ similarly

(d) $\frac{\partial f}{\partial x} = \sinh\left(\frac{x}{y}\right) \frac{\partial}{\partial x}\left(\frac{x}{y}\right) = \frac{1}{y}\sinh\left(\frac{x}{y}\right)$

$\frac{\partial f}{\partial y} = \sinh\left(\frac{x}{y}\right) \frac{\partial}{\partial y}\left(\frac{x}{y}\right) = -\frac{x}{y^2}\sinh\left(\frac{x}{y}\right).$

2. a) $\frac{\partial h}{\partial x} = \frac{10}{1000} \cos\frac{x}{1000}\left[10 - \cosh\left(\frac{y}{1000} - 1\right)\right]$

$\frac{\partial h}{\partial y} = \frac{-10}{1000} \sinh\left(\frac{y}{1000} - 1\right)\left[8 + \sin\frac{x}{1000}\right]$

So at $(0,0)$, gradients are (i) $\frac{1}{100}(10 - \cosh 1) = 0.0846 \simeq 1/11.8$

(ii) $\frac{8}{100}\sinh 1 = 0.0940 \simeq 1/10.6$

b) $\delta h \simeq \frac{\partial h}{\partial x}\delta x + \frac{\partial h}{\partial y}\delta y = 0.0846 \times 40 + 0.0940 \times 60 = 9.02$ m.

c) $h(0,0) = 50 + 80(10 - \cosh 1) = 726.55$ m

$h(40,60) = 50 + 10(8 + \sin 0.04)(10 - \cosh 0.94)$

$= 735.38$ m

So difference is $\delta h = 8.83$ m.

So b) overestimates by about 2%.

3. $a_{n+1} - a_n - 2a_{n-1} + 2a_{n-2} = 0$ Try $a_n = \lambda^n$

$\Rightarrow \lambda^3 - \lambda^2 - 2\lambda + 2 = 0 \Rightarrow (\lambda - 1)(\lambda^2 - 2) = 0 \Rightarrow \lambda = 1, \pm\sqrt{2}$

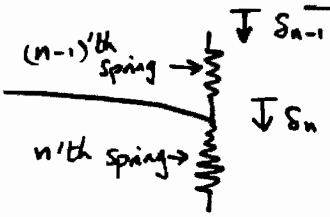
$\therefore$ General solution $\underline{a_n = A + B(\sqrt{2})^n + C(-\sqrt{2})^n}$

Initial conditions $\Rightarrow$ $A + B + C = 0$, $A + B\sqrt{2} - C\sqrt{2} = 0$, $A + 2B + 2C = 1$

Hence $A = -1$, $B = \frac{1}{2}(1 + \frac{1}{\sqrt{2}})$, $C = \frac{1}{2}(1 - 1/\sqrt{2})$

$$\Rightarrow \quad \underline{a_n = -1 + \frac{\sqrt{2}+1}{2\sqrt{2}}(\sqrt{2})^n + \frac{\sqrt{2}-1}{2\sqrt{2}}(-\sqrt{2})^n}$$

4. By putting $n = 3, \ldots$ in this expression $a_n = 0, 0, 1, 1, 3, 3, 7, 7, \ldots$



Consider n'th cantilever $(n \geqslant 2)$

Force due to $(n-1)$'th spring $= k_2(\delta_{n-1} - \delta_n)$ (+ve $\downarrow$)

Force due to n'th spring $= k_2(\delta_{n+1} - \delta_n)$ (+ve $\downarrow$)

In equilibrium force due to springs $=$ force to deflect cantilever

i.e. $k_2(\delta_{n-1} - \delta_n) + k_2(\delta_{n+1} - \delta_n) = k_1 \delta_n$

or $\underline{\delta_{n-1} - \left(2 + \frac{k_1}{k_2}\right)\delta_n + \delta_{n+1} = 0}$

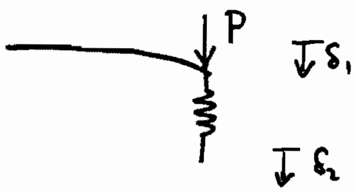
When $k_1 = k_2 = k$, Try $\delta_n = \lambda^n \Rightarrow \lambda^2 - 3\lambda + 1 = 0 \Rightarrow \lambda = \frac{3 \pm \sqrt{5}}{2}$

$\therefore$ General solution is $\underline{C_1\left(\frac{3+\sqrt{5}}{2}\right)^n + C_2\left(\frac{3-\sqrt{5}}{2}\right)^n}$

If $\delta_n \to 0$ as $n \to \infty$ then $C_1 = 0$ since $\left(\frac{3+\sqrt{5}}{2}\right)^n \to \infty$

$$\therefore \quad \underline{\delta_n = \delta_1\left(\frac{3-\sqrt{5}}{2}\right)^{n-1}}$$

For the first cantilever



$P + k(\delta_2 - \delta_1) = k\delta_1$

i.e. $P = 2k\delta_1 - k\delta_2$

$= 2k\delta_1 - k\frac{3-\sqrt{5}}{2}\delta_1$

$$\therefore \quad \underline{\frac{P}{\delta} = \frac{1+\sqrt{5}}{2}k}$$

5 (i) $\begin{vmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 2 \times \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + 0 + 1 \times \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 2 - 1 = 1$

Matrix of co-factor determinants is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}$

So Inverse $= \dfrac{1}{1}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}$

(ii) Row 3 = (row 2) − 2 × (row 1)
So matrix has zero determinant, and no inverse

(iii) $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{vmatrix} = 1 \times \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} - 2 \times \begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} + 3 \times \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix}$

$= 8$

Matrix of cofactor determinants is $\begin{bmatrix} -3 & -4 & 1 \\ -4 & -8 & -4 \\ 1 & -4 & -3 \end{bmatrix}$

So inverse is $\dfrac{1}{8}\begin{bmatrix} -3 & 4 & 1 \\ 4 & -8 & 4 \\ 1 & 4 & -3 \end{bmatrix}$

Student should be encouraged to check by multiplying by the original matrix!

6. $$\left.\begin{aligned} 3x_1 + 2x_2 &= 5y_1 - y_2 \\ 5x_1 - 4x_2 &= y_1 - 3y_2 \end{aligned}\right\} \Rightarrow \begin{bmatrix} 3 & 2 \\ 5 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

i.e. $A\underline{x} = B\underline{y} \quad \Rightarrow \quad \underline{x} = A^{-1}B\underline{y}$.

If this is $\underline{x} = C\underline{y}$ then $C = \begin{bmatrix} 3 & 2 \\ 5 & -4 \end{bmatrix}^{-1}\begin{bmatrix} 5 & -1 \\ 1 & -3 \end{bmatrix}$

$$\therefore\ C = \frac{1}{(-12-10)}\begin{bmatrix} -4 & -2 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ 1 & -3 \end{bmatrix} = -\frac{1}{22}\begin{bmatrix} -22 & 10 \\ -22 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -5/11 \\ 1 & 2/11 \end{bmatrix}$$

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7. $S = S^t$ means $S_{ij} = S_{ji}$ for each $i, j = 1, 2, 3$.
$U = -U^t$ means $U_{ij} = -U_{ji}$ for each i, j.
In particular, $U_{ii} = 0$ for each i.
Now let $T = \mathrm{Tr}(SU) = \sum_i \sum_j S_{ij} U_{ji} = \sum_i \sum_j S_{ji}(-U_{ij})$ by above

$= -\sum_i \sum_j S_{ij} U_{ji}$ renaming $i \leftrightarrow j$

So $T = -T$, so $T = 0$.

<u>Alternatively</u>, by longhand:

$\mathrm{Tr}(SU) = (SU)_{11} + (SU)_{22} + (SU)_{33}$

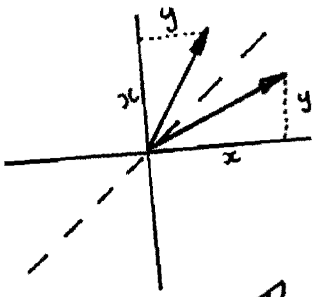
$= S_{11}U_{11} + S_{12}U_{21} + S_{13}U_{31} + S_{21}U_{12} + S_{22}U_{22} + S_{23}U_{32} + S_{31}U_{13} + S_{32}U_{32} + S_{33}U_{33}$

$= \underset{\text{using } U_{11}=0}{0} + \underbrace{S_{12}(U_{21} + U_{12})}_{\text{using } S_{12} = S_{21} \text{ etc.}} + 0 + S_{13}(U_{31} + U_{13}) + 0 + S_{23}(U_{32} + U_{23})$

$= 0$ <u>since brackets each vanish</u> using $U_{ij} = -U_{ji}$.

8(a) A Rotation by α anticlockwise $= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$\therefore$ Rotation by $90^\circ = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ Rotation by $180^\circ = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$



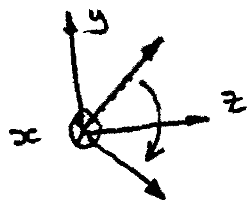
Reflection in $x=y$ has $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

ie. Reflection is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Thus (i) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

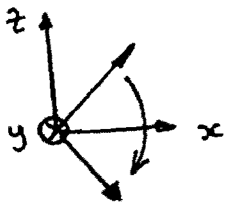
(ii) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

(b) Let $\otimes$ denote an axis <u>into</u> page



Rotation by $90^\circ \downarrow$ about x-axis given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$



Rotation by $90^\circ \downarrow$ about y-axis given by

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$\therefore$ (i) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

9 The three columns of a 3×3 orthogonal matrix whose determinant is positive must form a right-handed set of orthonormal (i.e. orthogonal unit) vectors. Thus the third column is simply the vector product of the first two

i.e. $$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{vmatrix} = \frac{-\underline{i}}{\sqrt{6}} + \frac{2\underline{j}}{\sqrt{6}} - \frac{\underline{k}}{\sqrt{6}}.$$

The three vectors made up from the rows are thus $\underline{r}_1 = \left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{6}}\right]^t$, $\underline{r}_2 = \left[\frac{1}{\sqrt{3}}, 0, \frac{2}{\sqrt{6}}\right]^t$ and $\underline{r}_3 = \left[\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{6}}\right]^t$. Each is a unit vector and they are mutually orthogonal.

10(a) $$\underline{x}' = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} - 2/\sqrt{6} \\ 1/\sqrt{3} + 4/\sqrt{6} \\ 1/\sqrt{3} - 2/\sqrt{6} \end{bmatrix}$$

$$\underline{y}' = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3} - \sqrt{2} - 1/\sqrt{6} \\ \sqrt{3} + 2/\sqrt{6} \\ \sqrt{3} + \sqrt{2} - 1/\sqrt{6} \end{bmatrix}$$

$$\underline{x}.\underline{y} = \underline{x}^t\underline{y} = 3 + 0 + 2 = 5$$

$$\underline{x}'.\underline{y}' = \underline{x}'^t\underline{y}' = \left(\frac{1}{\sqrt{3}} - \frac{2}{\sqrt{6}}\right)\left(\sqrt{3} - \sqrt{2} - \frac{1}{\sqrt{6}}\right) + \left(\frac{1}{\sqrt{3}} + \frac{4}{\sqrt{6}}\right)\left(\sqrt{3} + \frac{2}{\sqrt{6}}\right)$$
$$+ \left(\frac{1}{\sqrt{3}} - \frac{2}{\sqrt{6}}\right)\left(\sqrt{3} + \sqrt{2} - \frac{1}{\sqrt{6}}\right)$$
$$= 1 - \frac{\sqrt{2}}{\sqrt{3}} - \frac{1}{\sqrt{18}} - \frac{2\sqrt{3}}{\sqrt{6}} + \frac{2\sqrt{2}}{6} + \frac{2}{6} + 1 + \frac{2}{\sqrt{18}} + \frac{4\sqrt{3}}{6} + \frac{8}{6}$$
$$+ 1 + \frac{\sqrt{2}}{\sqrt{3}} - \frac{1}{\sqrt{18}} - \frac{2\sqrt{3}}{6} - \frac{2\sqrt{2}}{6} + \frac{2}{6} = 5$$

b) $$\underline{x}' = Q\underline{x} \Rightarrow \underline{x}'^t = (Q\underline{x})^t = \underline{x}^t Q^t$$

$$\therefore \underline{x}'.\underline{y}' = \underline{x}'^t\underline{y}' = \underline{x}^t Q^t Q \underline{y} = \underline{x}^t\underline{y} \quad \text{since } Q^tQ = I.$$