Part IA Paper 4: Mathematical Methods

Examples Paper 5

Elementary exercises are marked †, problems of Tripos standard ∗.
Answers can be found at the back of the paper.

Revision Question

A rectangular sheet of steel of dimensions $a \times b$ is to be made into an open-topped box by cutting a square of side h from each corner and folding the four sides up. Find the value of h that allows the maximum volume of box to be made from a given sheet. Hence show that, if the sheet is square with a side of 1 m, the maximum volume is $2/27\ \mathrm{m}^3$.

Partial Derivatives

1. † For each of the following functions $f(x, y)$, calculate $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$:

   (a) $f = \cos^2 x + \sin^2 y$ (b) $f = e^{-y^2} \tan x$

   (c) $f = \ln(x^2 + y^2)$ (d) $f = \cosh(x/y)$

2. An area of a hillside has a height in metres which is well approximated by the function

$$h(x, y) = 50 + 10 \left[8 + \sin \frac{x}{1000}\right] \left[10 - \cosh\left(\frac{y}{1000} - 1\right)\right]$$

   where x and y are distances in metres from a certain point, P, in the easterly and northerly directions respectively.

   (a) Calculate the gradients of the hillside at P experienced by walking (i) due east; and (ii) due north.

   (b) Use the partial derivative formula to estimate the height above P of the point Q, which is 40 m east and 60 m north of P.

   (c) Compare this with the exact difference in height between P and Q.

Linear Difference Equations

3. Find the general solution of the difference equation

$$a_{n+1} = a_n + 2a_{n-1} - 2a_{n-2}. \tag{1}$$

   Find the particular solution which satisfies the condition $a_0 = a_1 = 0, a_2 = 1$.

   Using this solution and Python/NumPy, evaluate $a_3, \ldots, a_{20}$. Verify that the values agree with those obtained by repeated use of eq. (1), starting from the specified values of a_0, a_1, and a_2.

Hints

Python/NumPy allows us to evaluate the solution at multiple values of `n` with ease. In this case, we are interested in the range 0 to 20, so we start by setting up a vector of these values as follows: `n = np.arange(20)`. Now we can do element-by-element arithmetic to calculate a for each value of n. For example, try `(2**(1/2))**n` and see what you get. Extend this principle to find a vector containing the solution at each value of n. Alternatively, using eq. (1) and starting with `a = [0,0,1,0,...]`, we can calculate the next value of a as follows: `a[n] = a[n-1] + 2*a[n-2] - 2*a[n-3]`. All we need to do is put this line of code inside a `for` loop that counts `n` from 3 to 20. You should find that you get the same sequence either way, though one of the methods is affected by small numerical rounding errors. Why?

A Jupyter notebook which implements this can be found at `https://github.com/CambridgeEngineering/PartIA-Paper4-Mathematics`

4. $*$ A large number of cantilevers, whose tips are joined by springs, are connected as shown. The stiffness of each cantilever (measured at its tip) is k_1, and each spring has modulus k_2, i.e. the vertical force necessary to move the end of a cantilever by an amount δ has magnitude $k_1\delta$ and the force necessary to compress, for example, the first spring has magnitude $k_2(\delta_1 - \delta_2)$.

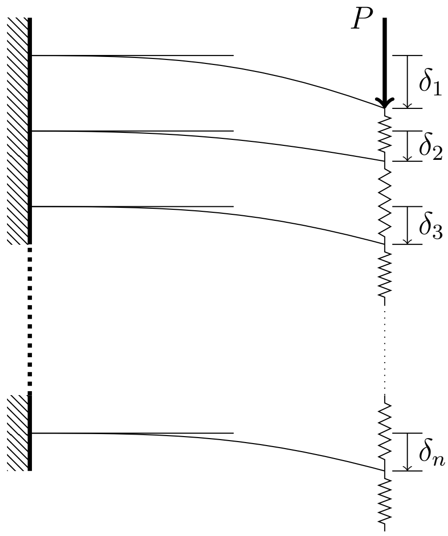
   A load P is applied to the tip of the first cantilever, producing deflections $\delta_1, \delta_2, \ldots, \delta_n, \ldots$, where δ_n is the deflection of the n-th cantilever. Show by considering equilibrium of the n-th cantilever that

   $$\delta_{n-1} - (2 + k_1/k_2)\delta_n + \delta_{n+1} = 0.$$

   Find the general solution of this equation for the case $k_1 = k_2 = k$.
   Deduce that, if $\delta_n \to 0$ as $n \to \infty$, then

   $$\delta_n = \delta_1 \left[\frac{3-\sqrt{5}}{2}\right]^{n-1}$$

   and, using this expression for δ_n, find the ratio P/δ_1.



Matrices

5. † Find the determinant and (if it exists) the inverse of each of the matrices

   (i) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 1 & 0 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}$

6. † x_1, x_2, y_1 and y_2, the components of the vectors $\underline{x}$ and $\underline{y}$, satisfy the simultaneous equations

   $$3x_1 + 2x_2 = 5y_1 - y_2$$
   $$5x_1 - 4x_2 = y_1 - 3y_2.$$

Find a matrix C such that $\underline{x} = C\underline{y}$.

7. The 3×3 matrices S and U satisfy $S = S^t$ and $U = -U^t$, where $(.)^t$ denotes the transpose. Show that
$$\mathrm{Tr}(SU) = 0,$$
where Tr(), the *trace* of a matrix, is defined as the sum of the diagonal elements, i.e.
$$\mathrm{Tr}(A) = \sum_{i=1}^{3} A_{ii}.$$

8. (a) † Find the 2×2 matrices which represent (i) an anticlockwise rotation of $90°$ followed by a reflection in the line which bisects the angle between the positive axes; and (ii) a reflection in the line which bisects the angle between the axes followed by a rotation by $180°$.

   (b) Find the 3×3 matrices which represent (i) a rotation of an object by $90°$ about the x axis followed by a rotation of $90°$ about the y axis; and (ii) a rotation of $90°$ about the y axis followed by a rotation of $90°$ about the x axis.

   [Rotations are taken as positive if they appear clockwise when viewed outwards along the positive axis in question.]

9. Find the third column which makes the matrix
$$\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & . \\ 1/\sqrt{3} & 0 & . \\ 1/\sqrt{3} & -1/\sqrt{2} & . \end{bmatrix}$$
orthogonal, with determinant $+1$. Verify that the *rows* of this matrix also form an orthonormal set, i.e. a set of mutually orthogonal unit vectors.

10. (a) † In a coordinate system C_1 two vectors are represented by $\underline{x} = [1, 0, 2]^t$ and $\underline{y} = [3, -2, 1]^t$. Calculate the representations $\underline{x}', \underline{y}'$ of the same vectors in a coordinate system C_2 which is related to C_1 by the transformation $\underline{x}' = Q\underline{x}$ (or $\underline{y}' = Q\underline{y}$), Q being the orthogonal matrix obtained as the solution to Question 9. Verify that the scalar products $\underline{x} \cdot \underline{y}$ and $\underline{x}' \cdot \underline{y}'$ are equal.

    (b) Prove that the result $\underline{x} \cdot \underline{y} = \underline{x}' \cdot \underline{y}'$ holds for any pair of vectors $\underline{x}$ and $\underline{y}$ and any orthogonal transformation matrix Q, i.e. prove that the value of a scalar product is independent of any coordinate system used to evaluate it.

Suitable past Tripos questions:
2002 Q3b, 2004 Q3b, 2005 Q5 (long), 2006 Q3 (short), 2007 Q5ab (long), 2008 Q3 (short), 2009 Q3 (short), 2010 Q1 (short), 2012 Q4a, 2015 Q1 (short).

Answers

1. (a) $-2\cos x \sin x, \quad 2\sin y \cos y$ (b) $\sec^2 x\, e^{-y^2}, \quad -2y\tan x\, e^{-y^2}$

   (c) $\dfrac{2x}{x^2+y^2}, \quad \dfrac{2y}{x^2+y^2}$ (d) $\dfrac{1}{y}\sinh\left(\dfrac{x}{y}\right), \quad -\dfrac{x}{y^2}\sinh\left(\dfrac{x}{y}\right)$

2. (a) At an angle of (i) $\tan^{-1}\dfrac{1}{11.8}$, (ii) $\tan^{-1}\dfrac{1}{10.6}$

   (b) 9.02 m (c) 8.83 m, so 2% error.

3. $A + B(\sqrt{2})^n + C(-\sqrt{2})^n; \quad -1 + \dfrac{\sqrt{2}+1}{2\sqrt{2}}(\sqrt{2})^n + \dfrac{\sqrt{2}-1}{2\sqrt{2}}(-\sqrt{2})^n;$

   $0, 0, 1, 1, 3, 3, 7, 7, \ldots$

4. $C_1\left[\dfrac{3+\sqrt{5}}{2}\right]^n + C_2\left[\dfrac{3-\sqrt{5}}{2}\right]^n; \quad \dfrac{1+\sqrt{5}}{2}k.$

5. (i) 1; $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}$ (ii) 0; no inverse (iii) 8; $\dfrac{1}{8}\begin{bmatrix} -3 & 4 & 1 \\ 4 & -8 & 4 \\ 1 & 4 & -3 \end{bmatrix}$

6. $\begin{bmatrix} 1 & -\frac{5}{11} \\ 1 & \frac{2}{11} \end{bmatrix}$

8. (a) (i) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

   (b) (i) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

9. $\begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$

10. (a) $\underline{x}' = \begin{bmatrix} 1/\sqrt{3} - 2/\sqrt{6} \\ 1/\sqrt{3} + 4/\sqrt{6} \\ 1/\sqrt{3} - 2/\sqrt{6} \end{bmatrix}, \quad \underline{y}' = \begin{bmatrix} \sqrt{3} - \sqrt{2} - 1/\sqrt{6} \\ \sqrt{3} + 2/\sqrt{6} \\ \sqrt{3} + \sqrt{2} - 1/\sqrt{6} \end{bmatrix}$

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