Engineering Part IA Mathematics Examples 1 — Solutions

1. (i) $\underline{a}.\underline{b} = 1.2 + 4.(-1) + 6.3 = \underline{16}$

Formula

$$\underline{a}\times\underline{b} = (a_y b_z - a_z b_y)\underline{i} + (a_z b_x - a_x b_z)\underline{j} + (a_x b_y - a_y b_x)\underline{k}$$
$$= (4.3 - 6.(-1))\underline{i} + (6.2 - 1.3)\underline{j} + (1.(-1) - 4.2)\underline{k}$$
$$= \underline{18\underline{i} + 9\underline{j} - 9\underline{k}}$$

or Sarrus

$$\begin{array}{cccccc} \underline{i} & \underline{j} & \underline{k} & \underline{i} & \underline{j} & \underline{k} \\ 1 & 4 & 6 & 1 & 4 & 6 \\ 2 & -1 & 3 & 2 & -1 & 3 \end{array}$$

terms +ve $\searrow$ −ve $\swarrow$

$$\underline{a}\times\underline{b} = 12\underline{i} + 12\underline{j} - \underline{k} - (-6\underline{i}) - 3\underline{j} - 8\underline{k}$$
$$= \underline{18\underline{i} + 9\underline{j} - 9\underline{k}}$$

or determinant (recommended but determinants not covered yet)

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 4 & 6 \\ 2 & -1 & 3 \end{vmatrix} = \underline{i}(4.3 - (-1).6) - \underline{j}(1.3 - 2.6) + \underline{k}(1.(-1) - 2.4)$$
$$= \underline{18\underline{i} + 9\underline{j} - 9\underline{k}}$$

(ii) $\underline{a}\times\underline{b} = 0$ if $\underline{a} \parallel \underline{b}$ $\therefore s = -3$

2. (i) Equation of line is $\underline{r} = \underline{a} + \lambda\underline{t}$ where $\underline{a}$ is pt on line & $\underline{t} \parallel$ line

In this case take $\underline{a} = (1, -5, 2)$ and $\underline{t} = (6, 3, -1) - (1, -5, 2)$

$\therefore$ line is $\underline{\underline{r} = (1, -5, 2) + \lambda(5, 8, -3)}$

(ii) Equation of plane is $\underline{r}.\underline{n} = p$ $\quad \underline{n}$ = unit normal, p = dist from O

In this case $\underline{n} = \dfrac{(1, -2, -3)}{\sqrt{14}} \Rightarrow$ plane is $x - 2y - 3z = \sqrt{14}\,p = p'$ (say)

The point $(1, 6, 2)$ lies on the plane $\Rightarrow 1 - 12 - 6 = p'$, plane is $\underline{x - 2y - 3z = -17}$

2(iii) To find a plane through points $\underline{a}, \underline{b}$ & $\underline{c}$ probably parametric form is easiest

$$\underline{r} = \underline{a} + \lambda\underline{u} + \mu\underline{v}$$ where $\underline{u}$ & $\underline{v}$ any indpt vectors in plane

In this case

$$\underline{r} = (4,0,2) + \lambda\{(2,-1,3)-(4,0,2)\} + \mu\{(3,1,0)-(4,0,2)\}$$
$$= \underline{(4,0,2) + \lambda(-3,3,0) + \mu(-1,1,2)}$$

(iv) As in part (iii), the equation of a plane through $\underline{a}, \underline{b}$ & $\underline{c}$ is

$$\underline{r} = \underline{a} + \lambda(\underline{b}-\underline{a}) + \mu(\underline{c}-\underline{a})$$
$$= \underline{a}(1-\lambda-\mu) + \lambda\underline{b} + \mu\underline{c}$$

Comparing with $\underline{r} = \alpha\underline{a} + \beta\underline{b} + \gamma\underline{c}$ gives $\alpha = 1-\lambda-\mu$, $\beta = \lambda$, $\gamma = \mu$

$$\Rightarrow \underline{\alpha + \beta + \gamma = 1}$$

<u>Aliter</u> $\underline{a}, \underline{b}$ & $\underline{c}$ lie on plane $\underline{r}.\underline{n} = p$

$$\Rightarrow \quad \underline{a}.\underline{n} = p \qquad \underline{b}.\underline{n} = p \qquad \underline{c}.\underline{n} = p$$

$\therefore$ For $\underline{r} = \alpha\underline{a} + \beta\underline{b} + \gamma\underline{c}$, $\underline{r}.\underline{n} = (\alpha\underline{a} + \beta\underline{b} + \gamma\underline{c}).\underline{n}$

$$= \alpha\underline{a}.\underline{n} + \beta\underline{b}.\underline{n} + \gamma\underline{c}.\underline{n}$$
$$= (\alpha+\beta+\gamma)p$$

$\therefore$ $\underline{r}$ lies on same plane as $\underline{a}, \underline{b}$ & $\underline{c}$ if $\underline{\alpha+\beta+\gamma = 1}$

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3. One form of the equation of a plane is $\underline{r}.\underline{n} = p$

Hence the normals are $(1,3,-5)$ and $(2,1,1)$

The line of intersection lies in both planes and so is perpendicular to both normals

i.e. parallel to $(1,3,-5) \times (2,1,1) = \underline{(-8,9,7)}$

4. The plane ABC contains $\underline{a}-\underline{b}$ & $\underline{a}-\underline{c}$. The normal $\underline{n}$ is perpendicular to both of these vectors,

i.e. $\underline{n}$ is in the direction $(\underline{a}-\underline{b})\times(\underline{a}-\underline{c}) = \overset{0}{\underline{a}\times\underline{a}} - \underline{b}\times\underline{a} - \underline{a}\times\underline{c} + \underline{b}\times\underline{c}$

$$= \underline{a}\times\underline{b} + \underline{b}\times\underline{c} + \underline{c}\times\underline{a}$$

Aliter: Let the plane through $\underline{a}$ with $\underline{n} = \underline{a}\times\underline{b} + \underline{b}\times\underline{c} + \underline{c}\times\underline{a}$ be $\underline{r}.\underline{n} = p$

Plane goes through $\underline{a}$ $\Rightarrow$ $\underline{a}.\underline{n} = p = \underline{a}.(\underline{a}\times\underline{b} + \underline{b}\times\underline{c} + \underline{c}\times\underline{a})$

$$= \underline{a}.(\underline{b}\times\underline{c}) \qquad \left[\begin{array}{l}\underline{a} \perp \underline{a}\times\underline{b} \\ \underline{a} \perp \underline{c}\times\underline{a}\end{array}\right]$$

$\underline{b}.\underline{n} = \underline{b}.(\underline{a}\times\underline{b} + \underline{b}\times\underline{c} + \underline{c}\times\underline{a})$

$= \underline{b}.\underline{c}\times\underline{a} = \underline{a}.\underline{b}\times\underline{c} = p$ ie. $\underline{b}$ lies on plane

$\underline{c}.\underline{n} = \underline{c}.(\underline{a}\times\underline{b} + \underline{b}\times\underline{c} + \underline{c}\times\underline{a})$

$= \underline{c}.\underline{a}\times\underline{b} = \underline{a}.\underline{b}\times\underline{c} = p \Rightarrow \underline{c}$ lies on plane

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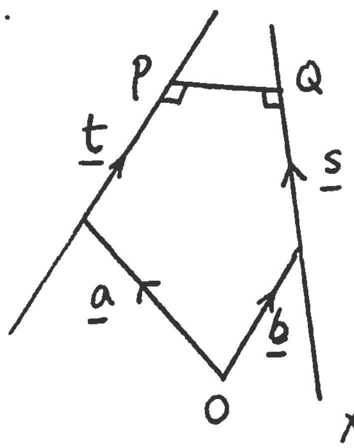
5.



The shortest distance between two lines is the length PQ of their common normal.

If the lines are $\underline{r}_1 = \underline{a} + \lambda\underline{t}$
and $\underline{r}_2 = \underline{b} + \mu\underline{s}$

then $\overrightarrow{PQ}$ is in the direction $\underline{s}\times\underline{t} = \underline{n}$ $\left(\hat{\underline{n}} = \dfrac{\underline{s}\times\underline{t}}{|\underline{s}\times\underline{t}|}\right)$

Now P lies on the first line, so its co-ordinates are of the form $\underline{a} + \lambda\underline{t}$ for some λ.

The co-ordinates of Q must, therefore, be of the form

$\underline{a} + \lambda\underline{t} + d\hat{\underline{n}}$ where $\hat{\underline{n}}$ is a unit vector & d = distance

But Q lies on the second line, hence

$$\mathbf{a} + \lambda \mathbf{t} + d\,\hat{\mathbf{n}} = \mathbf{b} + \mu \mathbf{s} \quad \text{for some value of } \mu$$

Taking the dot product with $\hat{\mathbf{n}}$ remembering that $\hat{\mathbf{n}}.\mathbf{t} = \hat{\mathbf{n}}.\mathbf{s} = 0$

gives
$$\mathbf{a}.\hat{\mathbf{n}} + d\,\underbrace{\hat{\mathbf{n}}.\hat{\mathbf{n}}}_{\overset{\shortparallel}{1}} = \mathbf{b}.\hat{\mathbf{n}}$$

i.e. $$d = (\mathbf{b}-\mathbf{a}).\hat{\mathbf{n}} = (\mathbf{b}-\mathbf{a}).\frac{\mathbf{s}\times\mathbf{t}}{|\mathbf{s}\times\mathbf{t}|}$$

{This may come out $-$ve but distance is always $+$ve}

Now $\mathbf{s}\times\mathbf{t} = (5,6,7)\times(4,5,6) = (1,-2,1) \Rightarrow \hat{\mathbf{n}} = \frac{(1,-2,1)}{\sqrt{6}}$

$$\therefore\ d = (1,1,-1).\frac{(1,-2,1)}{\sqrt{6}} = -\frac{2}{\sqrt{6}} \Rightarrow \text{distance} = \frac{2}{\sqrt{6}}.$$

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6. (i) $$(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b}) = \mathbf{a}\times\mathbf{a} + \mathbf{b}\times\mathbf{a} - \mathbf{a}\times\mathbf{b} - \mathbf{b}\times\mathbf{b}$$
$$= 0 + \mathbf{b}\times\mathbf{a} + \mathbf{b}\times\mathbf{a} + 0$$
$$= 2\,\mathbf{b}\times\mathbf{a}$$

(ii) $$\mathbf{t}\times\mathbf{r} = \mathbf{t}\times\mathbf{a} \Leftrightarrow \mathbf{t}\times(\mathbf{r}-\mathbf{a}) = 0$$
$$\Leftrightarrow \mathbf{r}-\mathbf{a} \parallel \mathbf{t} \quad \text{or} \quad \mathbf{r}-\mathbf{a} = 0 \qquad (\text{since } \mathbf{t} \neq 0)$$

i.e. $\mathbf{r}-\mathbf{a} = \lambda\mathbf{t}$ for some λ [This includes $\mathbf{r}-\mathbf{a}=0$ with $\lambda = 0$]

$\Rightarrow \mathbf{r} = \mathbf{a} + \lambda\mathbf{t}$ (line through $\mathbf{a} \parallel \mathbf{t}$)

7 a)
$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -1 & 0 & -1 \end{vmatrix} = -2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$$

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -2 & 2 \\ 0 & 1 & 2 \end{vmatrix} = -6\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$$

b) $(\underline{a}\times\underline{b})\times\underline{c} = \underline{a}.\underline{c}\,\underline{b} - \underline{b}.\underline{c}\,\underline{a} = (2+6)\begin{bmatrix}-1\\0\\-1\end{bmatrix} - (-2)\begin{bmatrix}1\\2\\3\end{bmatrix}$

$$= -6\underline{i} + 4\underline{j} - 2\underline{k}$$

8 $(\underline{a}\times\underline{b})\times\underline{c} + (\underline{b}\times\underline{c})\times\underline{a} + (\underline{c}\times\underline{a})\times\underline{b}$

$= \underline{a}.\underline{c}\,\underline{b} - \underline{b}.\underline{c}\,\underline{a} + \underline{b}.\underline{a}\,\underline{c} - \underline{c}.\underline{a}\,\underline{b} + \underline{c}.\underline{b}\,\underline{a} - \underline{a}.\underline{b}\,\underline{c}$

$= 0$

9 (a) If $\underline{b}$ and $\underline{c}$ are not parallel, then $\underline{b}, \underline{c}$ & $\underline{b}\times\underline{c}$ are three independent (i.e. non-coplanar) vectors. It follows that any vector $\underline{x}$ can be written

$$\underline{x} = \alpha\underline{b} + \beta\underline{c} + \gamma\,\underline{b}\times\underline{c}$$

(b) Substituting into (1),

$$\alpha\underline{b} + \beta\underline{c} + \gamma\underline{b}\times\underline{c} + (\alpha\underline{b}.\underline{a} + \beta\underline{c}.\underline{a} + \gamma\,\underline{b}\times\underline{c}.\underline{a})\,\underline{b} = \underline{c}$$

Since $\underline{b}, \underline{c}$ & $\underline{b}\times\underline{c}$ are independent, we can equate coefficients to give

$$\alpha(1+\underline{b}.\underline{a}) + \beta\,\underline{c}.\underline{a} + \gamma\,\underline{b}\times\underline{c}.\underline{a} = 0$$

$$\beta = 1 \qquad \gamma = 0$$

$\therefore \quad \alpha = \dfrac{-\underline{c}.\underline{a}}{1+\underline{b}.\underline{a}}$ provided $1+\underline{b}.\underline{a} \neq 0$

$$\Rightarrow \quad \underline{x} = \frac{-\underline{c}.\underline{a}}{1+\underline{b}.\underline{a}}\,\underline{b} + \underline{c}$$

(c) If $\underline{b} = \lambda\underline{c}$ then (1) $\Rightarrow$ $\underline{x}$ is also parallel to $\underline{c}$. Put $\underline{x} = \mu\underline{c}$

$\Rightarrow \mu\underline{c} + \mu\underline{c}.\underline{a}\,\underline{b} = \underline{c} \quad \Rightarrow \mu\underline{c} + \mu\underline{c}.\underline{a}\,\lambda\underline{c} = \underline{c}$

$\Rightarrow \mu = \dfrac{1}{1+\lambda\underline{c}.\underline{a}}$ and $\underline{x} = \dfrac{\underline{c}}{1+\lambda\underline{c}.\underline{a}}$

[Note answer to part (b) reduces to this when $\underline{b} = \lambda\underline{c}$]

10. $(\underline{a}\times\underline{b}).(\underline{a}\times\underline{c}) = \underline{a}.[\underline{b}\times(\underline{a}\times\underline{c})]$ swapping $\times$ & $\cdot$

$$= \underline{a}.[\underline{b}.\underline{c}\,\underline{a} - \underline{b}.\underline{a}\,\underline{c}]$$

$$= (\underline{a}.\underline{a})(\underline{b}.\underline{c}) - (\underline{a}.\underline{c})(\underline{a}.\underline{b})$$

A is the angle between two planes: one containing $\underline{a}$ & $\underline{b}$ and one containing $\underline{a}$ & $\underline{c}$.

This is also the angle between the normals to the two planes i.e. the angle between $\underline{a}\times\underline{b}$ and $\underline{a}\times\underline{c}$

Thus $\cos A = \dfrac{\underline{a}\times\underline{b}}{|\underline{a}\times\underline{b}|} . \dfrac{\underline{a}\times\underline{c}}{|\underline{a}\times\underline{c}|}$

$$= \frac{\underline{a}.\underline{a}\ \underline{b}.\underline{c} - \underline{a}.\underline{b}\ \underline{a}.\underline{c}}{|\underline{a}\times\underline{b}||\underline{a}\times\underline{c}|}$$

$\underline{a}.\underline{a} = 1$ $\quad \underline{b}.\underline{c} = |\underline{b}||\underline{c}|\cos\alpha = \cos\alpha$ since $|\underline{b}| = |\underline{c}| = 1$

Similarly $\underline{a}.\underline{b} = |\underline{a}||\underline{b}|\cos\gamma = \cos\gamma$

$\underline{a}.\underline{c} = \cos\beta$

$|\underline{a}\times\underline{b}| = |\underline{a}||\underline{b}|\sin\gamma = \sin\gamma$

$|\underline{a}\times\underline{c}| = |\underline{a}||\underline{c}|\sin\beta = \sin\beta$

$$\Rightarrow \cos A = \frac{\cos\alpha - \cos\beta\cos\gamma}{\sin\beta\sin\gamma}$$

or $\underline{\cos\alpha = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos A.}$

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