

**Part IA Paper 2: Structures and Materials  
MATERIALS**

**Solutions to Examples Paper 2 – Elasticity of Materials**

*Mechanics of Elastic Deformation,  
Microstructural Origin and Manipulation of Elastic Properties,  
Stiffness-limited Design*

Mechanics of Elastic Deformation

1. (a) For hydrostatic pressure  $p$ ,  $\sigma_1 = \sigma_2 = \sigma_3 = -p$ , Hooke's Law in 3D gives:

$$\varepsilon_1 = \frac{1}{E}(-p + \nu p + \nu p) = -\frac{p}{E}(1 - 2\nu)$$

From symmetry,  $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = -\frac{p}{E}(1 - 2\nu)$

Hence volumetric strain  $\Delta = (\varepsilon_1 + \varepsilon_2 + \varepsilon_3) = -\frac{3p}{E}(1 - 2\nu)$

(b) (i) Bulk modulus is:  $K = \frac{p}{\Delta} = \frac{E}{3(1 - 2\nu)}$

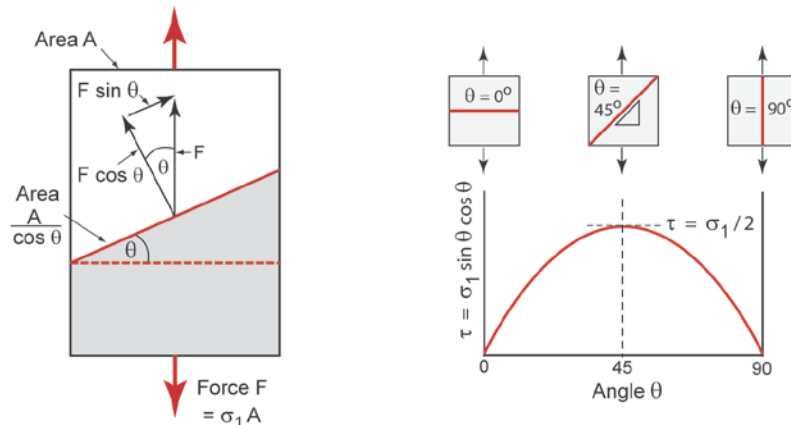
(ii) For metals ( $\nu \approx 0.3$ ),  $K \approx E/1.2 \approx E$

(iii) For rubber ( $\nu \approx 0.5$ ),  $\Delta \approx 0$ ,  $K \rightarrow \infty$ , i.e. the material is effectively incompressible (and thus has a very high stiffness under hydrostatic loading – an unexpected response from the solids with the lowest Young's modulus).

2. (a) The vertical force from the applied stress is:  $F = \sigma_1 A$ , equal to the net force acting on the inclined plane, shown in the LH figure resolved normal and parallel to the plane.

The inclined area =  $A/\cos \theta$ , so the corresponding stresses on the inclined plane are:

$$\sigma = F \cos \theta / (A/\cos \theta) = \sigma_1 \cos^2 \theta \quad \text{and} \quad \tau = F \sin \theta / (A/\cos \theta) = \sigma_1 \sin \theta \cos \theta$$



(b) To find the direction of maximum shear:

$$\tau = (\sigma_1/2) \sin 2\theta, \quad d\tau/d\theta = \sigma_1 \cos 2\theta = 0 \text{ at maximum, hence } \theta = 45^\circ \text{ when } \tau = \sigma_1/2$$

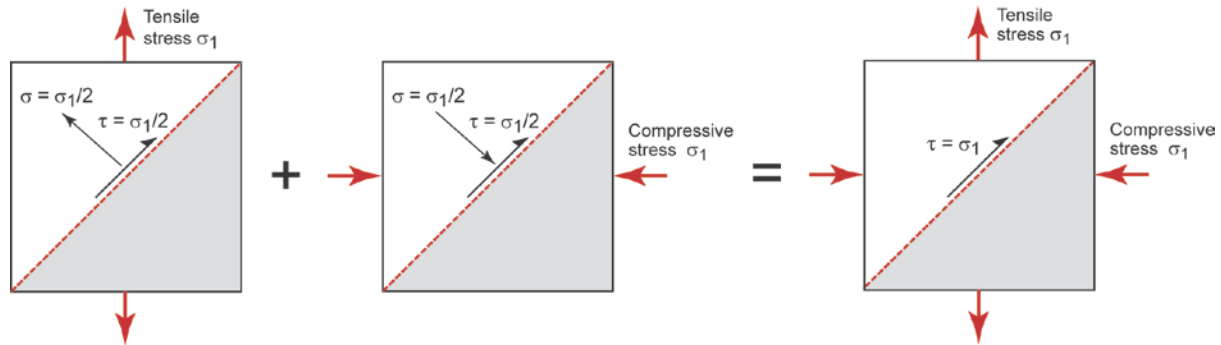
The variation of the shear stress  $\tau$  is plotted in the RH figure.

(c) See figure below.

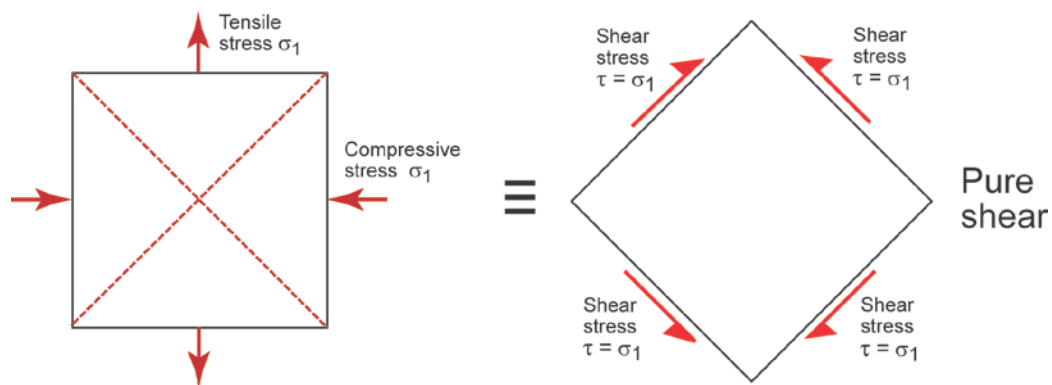
For the tensile stress, the stresses on a plane at  $45^\circ$  are:  $\sigma = \tau = \sigma_1/2$

For the perpendicular compressive stress, the stresses on the same plane are:  $-\sigma = \tau = \sigma_1/2$

Using superposition, the normal stresses are equal and opposite, so cancel to a net stress of zero; the shear stresses are in the same direction and superpose to give  $\tau = \sigma_1$ . The same applies on both  $45^\circ$  diagonal planes.



So if a material element at  $45^\circ$  is considered in a biaxial equal tension/compression stress state, the element is loaded in pure shear (see figure).



3. From the lecture notes, the effective modulus for constraint in one transverse direction is:

$$\frac{\sigma_1}{\varepsilon_1} = \frac{E}{(1 - \nu^2)}$$

From the Materials Databook, mid-range value for butyl rubber:  $E = 0.0015 \text{ GPa}$ ; and  $\nu = 0.5$ .

Applied stress:  $\sigma_1 = \frac{50}{40 \times 40} = 0.313 \text{ MPa}$

Strain in 1 direction:  $\varepsilon_1 = \frac{\sigma_1}{E} (1 - \nu^2) = \frac{0.313}{0.0015 \times 10^3} (1 - 0.5^2) = 0.0156$

Deflection in 1 direction:  $\delta = \varepsilon_1 \times l_o = 0.0156 \times 40 = 0.624 \text{ mm}$

Hence stiffness =  $(F / \delta) = 50 / 0.624 = 80 \text{ N/mm}$

The stiffness ratio is given directly by the ratio of effective modulus to  $E$ :  $\frac{1}{(1 - \nu^2)}$

For rubber,  $\nu = 0.5$ , this stiffness ratio is 1.33, i.e. 33% stiffer with constraint in one transverse direction.

4. (a) For full transverse constraint:  $\varepsilon_2 = \varepsilon_3 = 0$

Hence from Hooke's Law in 3D:  $\varepsilon_2 = 0 = \frac{1}{E}(-\nu\sigma_1 + \sigma_2 - \nu\sigma_3)$

$$\varepsilon_3 = 0 = \frac{1}{E}(-\nu\sigma_1 - \nu\sigma_2 + \sigma_3)$$

So  $\sigma_2 - \nu\sigma_3 = -\nu\sigma_2 + \sigma_3$ , i.e.  $\sigma_2 = \sigma_3$  (as it must be, from symmetry)

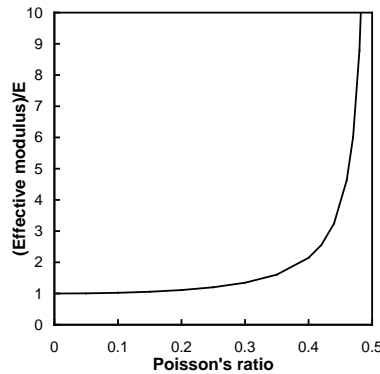
Thus for the 2 direction:  $\varepsilon_2 = 0 = \frac{1}{E}(-\nu\sigma_1 + \sigma_2 - \nu\sigma_2) \Rightarrow \sigma_2 = \frac{\nu\sigma_1}{(1-\nu)}$

and from symmetry, for the 3 direction:  $\sigma_3 = \frac{\nu\sigma_1}{(1-\nu)}$

Hence in the 1 direction:  $\varepsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2 - \nu\sigma_3) = \frac{\sigma_1}{E}\left(1 - \frac{2\nu^2}{1-\nu}\right) = \frac{\sigma_1}{E}\left(\frac{1-\nu-2\nu^2}{1-\nu}\right)$

Hence the “effective modulus” in the 1 direction is given by:  $\frac{\sigma_1}{\varepsilon_1} = \frac{E(1-\nu)}{(1-\nu-2\nu^2)}$

(b)



For  $\nu = 0$  (foam),  $(\sigma_1/\varepsilon_1) = E$  (i.e. constraint has no effect, as no lateral expansion).

For  $\nu \rightarrow 0.5$  (rubber),  $(\sigma_1/\varepsilon_1) \rightarrow \infty$  (i.e. incompressible, but lateral expansion prevented). So rubber, which has the lowest Young's modulus of all solid materials, has a very stiff response when constrained.

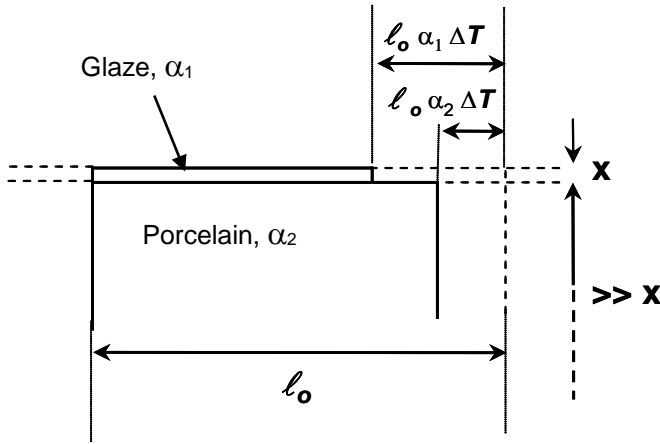
(c) A cylindrical hole full of rubber is constrained fully in all transverse directions, i.e. as in (a). This is not a good design, as the stiffness is very high, providing little elastic cushioning.

(d) Running shoe treads provide grip (by allowing the shoe sole to deform around surface irregularities, giving higher friction) but also provide elastic cushioning to reduce impact forces (and avoid injury). Solid rubber would be uncomfortable – the sole is thin relative to its length and width and would be unable to strain laterally under the foot (giving the high constraint effect above). Foaming the rubber reduces Poisson's ratio, treads allow local transverse strain of the rubber, while air and gel pockets are compressible – all giving a much more elastic response (and a great deal of freedom to customise the design – it's not all marketing hype).

5. (a) Provided the interface between the glaze and the porcelain is strong enough, the glaze and porcelain are constrained by one another to contract by the same amount  $\Delta l$ . If the glaze has a larger expansion coefficient than the porcelain, it wishes to contract more than  $\Delta l$  and goes into tension; if smaller, it goes into compression.

To find the stress in the glaze, use superposition: first allow the glaze and porcelain to contract independently without constraint, then apply the induced stress in the glaze to bring the final lengths into coincidence. The stress in the porcelain is negligible (as the glaze is thin).

A square patch of glaze contracts equally in both dimensions on cooling. There is no Poisson interaction between two perpendicular directions for purely thermal contraction, as there is no stress. Hence the thermal strain is simply found in one dimension by considering an initial length  $l_o$  of glazed porcelain at the initial high temperature, cooled by a temperature drop  $\Delta T = (T_1 - T_o)$ . Assume initially that  $\alpha_1$  (glaze)  $>$   $\alpha_2$  (porcelain).



Unconstrained thermal contraction

$$\text{Cooled length of glaze} = l_o (1 - \alpha_1 \Delta T)$$

$$\text{Extension required to match cooled length of porcelain} = l_o (\alpha_1 - \alpha_2) \Delta T$$

$$\text{Required strain} = \frac{l_o (\alpha_1 - \alpha_2) \Delta T}{l_o (1 - \alpha_1 \Delta T)}$$

Since  $\alpha_1 \Delta T \ll 1$ , required strain =

$$\frac{l_o (\alpha_1 - \alpha_2) \Delta T}{l_o} = (\alpha_1 - \alpha_2) \Delta T$$

This thermal strain is resisted in both in-plane directions. The induced thermal stress is therefore biaxial, with equal stress in any two perpendicular directions. The strain produced by a biaxial stress  $\sigma$  is given by Hooke's Law:  $\epsilon_1 = \epsilon_2 = \frac{1}{E}(\sigma - \nu \sigma)$

Hence the tensile stress that balances the thermal contraction is:

$$\sigma = \frac{E}{(1-\nu)} (\alpha_1 - \alpha_2) \Delta T = \frac{E}{(1-\nu)} (\alpha_1 - \alpha_2) (T_1 - T_o)$$

The sign of the surface stress is determined purely by the difference in thermal expansion coefficients:  $\alpha_1 > \alpha_2$  gives tension,  $\alpha_1 < \alpha_2$  gives compression.

(b)	$\alpha \times 10^{-6} \text{ (K}^{-1}\text{)}$	$\sigma \text{ (MPa)}$
Porcelain	2.2	$\approx 0$
Glaze A	1.5	-53.5
Glaze B	2.3	7.6
Glaze C	2.5	23.0

For a decorative tile, choose Glaze C: first because the stress is tensile, but second because the stress is sufficient to cause cracking (as  $\alpha_{\text{glaze}} > \alpha_{\text{porcelain}}$ )

For a strong tile with compression in the glaze, choose Glaze A ( $\alpha_{\text{glaze}} < \alpha_{\text{porcelain}}$ ).

6. (a) Theoretical density at room temperature (Examples Paper 1, Exercise B6a):  $7873 \text{ kg m}^{-3}$

At  $910^\circ\text{C}$ , fractional increase in unit cell size =  $\Delta T \times \alpha = 890 \times 12 \times 10^{-6} = 10.7 \times 10^{-3} \text{ (1.07\%)}$ .

New volume  $V' = \text{old volume} \times (1.0107)^3 \approx 1.032 V \text{ (3.2\% decrease in density)}$

[Alternatively, recall that for uniform expansion with a *small* strain  $\epsilon$ , the dilatation =  $3\epsilon$ . Hence volumetric strain =  $3 \times (10.7 \times 10^{-3}) = 0.032$ ].

Mass unchanged, so density decreases by this factor to  $\approx 7629 \text{ kg m}^{-3} \text{ (}\alpha \text{ iron at } 910^\circ\text{C)}$ .

(b) From Examples Paper 1, the ratio of the atomic packing factors for FCC and BCC unit cells =  $0.74/0.68 = 1.088$  (8.8% *increase* in density).

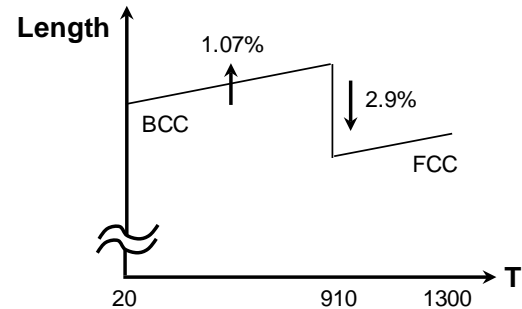
The density of BCC  $\alpha$ -Fe (ferrite) at  $910^\circ\text{C} = 7629 \text{ kg m}^{-3}$  (from above).

Hence, density of FCC  $\gamma$ -Fe (austenite) at  $910^\circ\text{C} = 7629 \times 1.088 = 8300 \text{ kg m}^{-3}$

Thermal expansion from  $20^\circ\text{C}$  to  $910^\circ\text{C}$  causes a density drop of 3.2%; the phase transformation from BCC to FCC causes a density rise of 8.8%.

The % length change  $\approx 1/3$  of % density change (with increase in length  $\equiv$  decrease in density).  
Corresponding length change when BCC changes to FCC therefore:  $8.8/3 = 2.9\%$  (decrease).

Above  $910^\circ\text{C}$ , gradient of length vs. temperature is the same as the gradient below (i.e. constant thermal expansion coefficient).



### Microstructural Origin and Manipulation of Elastic Properties

7. (a) Approximate ranges of E (from property charts):

Metals: 12 – 220 GPa

Ceramics: 150 – 800 GPa

Polymers: <0.001 – 6 GPa

(Thermoplastics: 0.2 – 6 GPa; Thermosets: 2 – 5 GPa; Elastomers: <0.001 – 0.05 GPa)

The elastic properties of crystalline materials (metals/ceramics) and polymers are determined by primary and secondary bonds, respectively. Metals and ceramics undergo very little elastic deformation due to the short range nature of stiff metallic, ionic or covalent bonds. The weak van der Waals bonds between the polymer chains give rise to a low stiffness (polymer molecules can slide over each other easily under an applied stress).

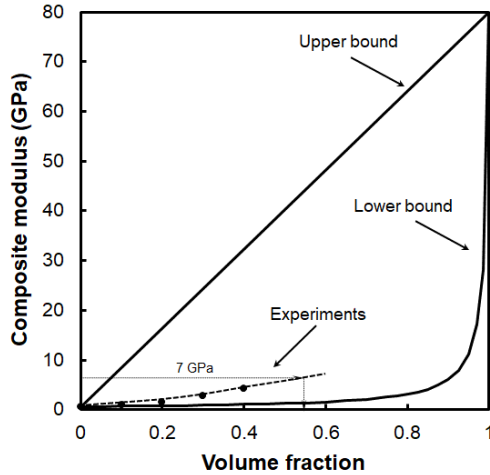
(b) The secondary van der Waals bonds between polymer chains become ineffective through the glass transition temperature,  $T_g$  (of the order  $-100$  to  $200^\circ\text{C}$ ). Below  $T_g$  the modulus is already low ( $\approx 1$ -2GPa), but the behaviour above  $T_g$  depends on the polymer class.

*Amorphous thermoplastics* lose all bonding between chains but retain a very low modulus (of order 1 MPa) due to chain entanglements. In *semi-crystalline thermoplastics*, crystalline regions of organised molecular packing form – in these regions the bonding between chains is retained until the polymer melting point (typically  $T_g \approx 2/3 T_m$ , in K), giving a “rubbery” region. The modulus above  $T_g$  is sensitive to the degree of crystallinity.

*Elastomers* have low  $T_g$  (all below room temperature), and have low moduli (of order 1 MPa) above  $T_g$ : the chains are only bonded by occasional cross-links (strong covalent bonds between chains). These allow large recoverable strains at low loads. *Thermosets* have extensive cross-linking, such that the glass transition has little effect on stiffness (modulus of order 2 GPa).



9. PE-hydroxyapatite composite: bounds for Young's modulus plotted below.



Experimental data closer to the lower bound, but still underestimates the modulus by a factor of 5 (by  $V_f = 0.4$ ). Use the *shape* of the lower bound curve to extrapolate the experimental data to a modulus of 7 GPa – the required volume fraction is around 55%. This is impractical, as mixing ceramic volume fractions  $>50\%$  into a polymer is difficult – probably why the experiments stop at 40% – the particles contact one another, preventing infill with the polymer matrix. 40% HA gives 4.3 GPa, much closer to the value of 7 GPa for bone (compared to steel, at 210 GPa).

10. (a) The density of a composite is derived directly from mass conservation:

$$\text{Composite mass } m_c = \text{Reinforcement mass } m_f + \text{Matrix mass } m_m$$

Replacing each mass by (density  $\times$  volume), and dividing through by total volume:

$$\rho_c = \rho_f V_f + (1 - V_f) \rho_m, \quad \text{i.e. simple rule of mixtures.}$$

Result does not depend on the form of the reinforcement.

(b) The average stress is given from the weighted mean of the stress in the fibre and matrix:

$$\sigma = \sigma_f V_f + (1 - V_f) \sigma_m$$

The strains in both fibres and matrix are equal along the fibre direction so:

$$\sigma = E_f \varepsilon V_f + (1 - V_f) E_m \varepsilon$$

$$\text{Thus: } E_c = \frac{\sigma}{\varepsilon} = E_f V_f + (1 - V_f) E_m$$

(c) Density and Young's moduli of the fibre composites are both derived using the rule of mixtures:

$$\text{Carbon-fibre epoxy: } \rho_c = (0.5 \times 1.90) + (0.5 \times 1.15) = \underline{1.53 \text{ Mg/m}^3}$$

$$E_c = (0.5 \times 390) + (0.5 \times 3) = \underline{197 \text{ GPa}}$$

$$\text{Glass-fibre polyester: } \rho_c = (0.5 \times 2.55) + (0.5 \times 1.15) = \underline{1.85 \text{ Mg/m}^3}$$

$$E_c = (0.5 \times 72) + (0.5 \times 3) = \underline{37.5 \text{ GPa}}$$

$$(d) \text{ The lower bound composite modulus is given by: } E_c = \left( \frac{V_f}{E_f} + \frac{1 - V_f}{E_m} \right)^{-1}$$

$$\text{Density follows rule of mixtures, as in (a): } \rho_c = \rho_f V_f + (1 - V_f) \rho_m$$

From Materials Databook, approx. mid-range data for Young's modulus and density:

$$\text{Mg alloys: } E = 44.5 \text{ GPa, } \rho = 1.85 \text{ Mg m}^{-3}$$

$$\text{SiC: } E = 380 \text{ GPa, } \rho = 3.1 \text{ Mg m}^{-3}$$

Hence for  $V_f = 0.2$ :

$$E_c = 1/[(0.2/380) + (0.8/44.5)] = \underline{54.0 \text{ GPa}}$$

$$\rho_c = (0.2 \times 3.1) + (0.8 \times 1.85) = \underline{2.1 \text{ Mg m}^{-3}}$$

(e)

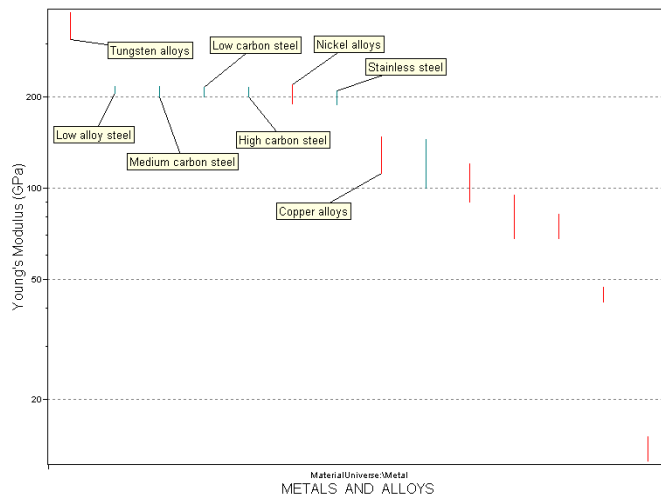
Composite	$E_c / \rho$ (GPa Mg <sup>-1</sup> m <sup>3</sup> )
Carbon-fibre/epoxy resin	129
Glass-fibre/polyester resin	20
Mg/SiC particulate	26
Steels	28

Carbon-fibre/epoxy resin has much higher specific stiffness than steels, while the MMC is comparable to steels, and the glass-fibre polyester has a lower value.

[Footnote: specific stiffness is a good indicator for tensile loading, but in the next part of the course it will be shown that other combinations of E and  $\rho$  are usually appropriate in bending – e.g.  $E^{1/3} / \rho$  (see Examples paper 3). In this case it is readily shown that all of the composites perform better than steels. But there will be other important factors to consider in using one of the composites instead of steel: geometry and shape limits, cost, strength, durability, ability to manufacture and join etc. ]

### Material Selection: Stiffness-limited Design

11. (a) In CES: plot bar chart for Young's modulus, restricted to metals.



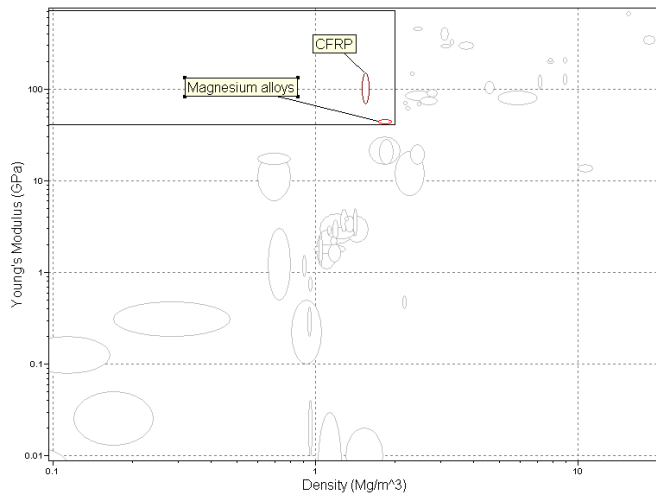
Alloys with higher Young's modulus than Cu alloys:

- tungsten alloys
- steels (low alloy, carbon and stainless)
- nickel alloys

In Databook, use table for Young's modulus, or refer to Modulus-Density property chart.

(b) In CES: plot Young's modulus - density chart and use a selection box. As only 2 materials pass this stage it is not worth plotting a second chart to consider cost – just browse the two records for the price/kg (or use the Databook).





Materials with  $E > 40$  GPa and

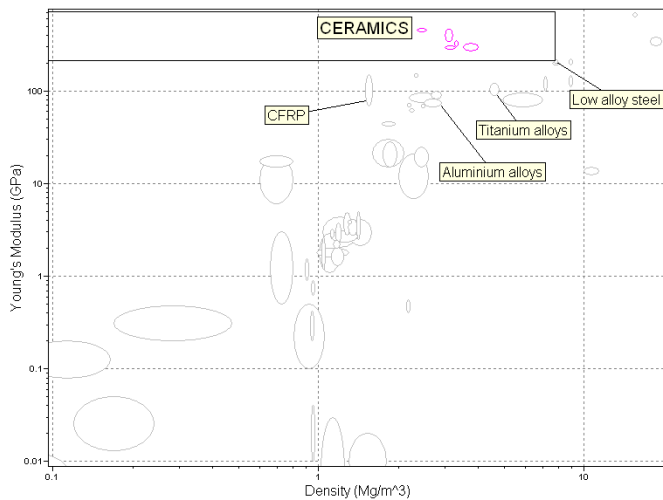
$\rho < 2$  Mg/m<sup>3</sup> are:

- CFRP
- Mg alloys

Mg is the cheaper material.

*In Databook*, use Modulus-Density property chart, exactly as in CES. For more than one property limit, it is tedious to look data up in the tables.

(c) *In CES*: plot Young's modulus – density chart and use a selection box with the bottom RH corner located on the material of interest (example shown is for steel).

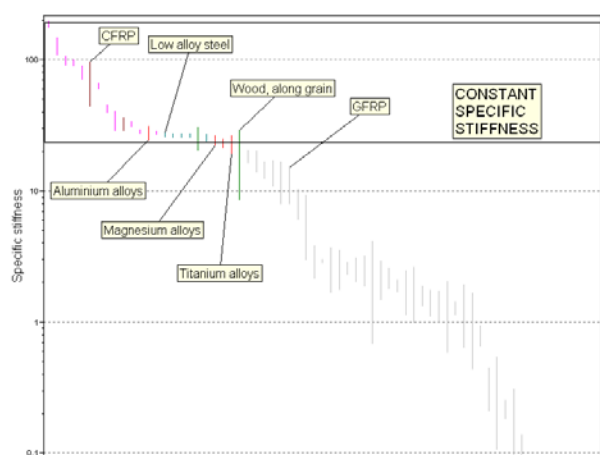
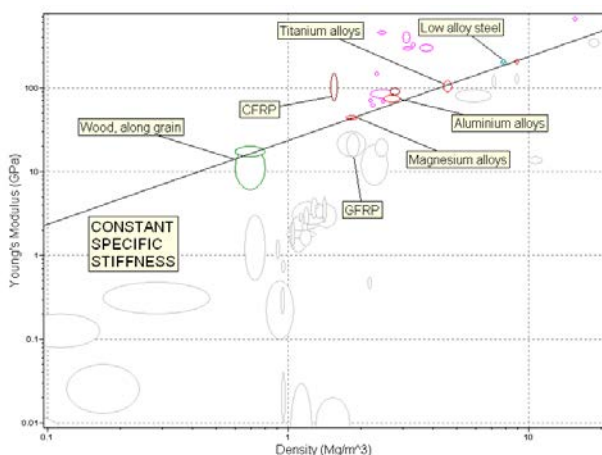


Metals and composites with higher  $E$  and lower  $\rho$  are:

- steels: there are none (only ceramics, which are already excluded: see figure)
- Ti, Al alloys: only CFRP

*In Databook*, use Modulus-Density property chart, exactly as in CES.

(d) *In CES*: either use line of slope unity on  $E$ - $\rho$  plot, or bar chart of  $E/\rho$ .

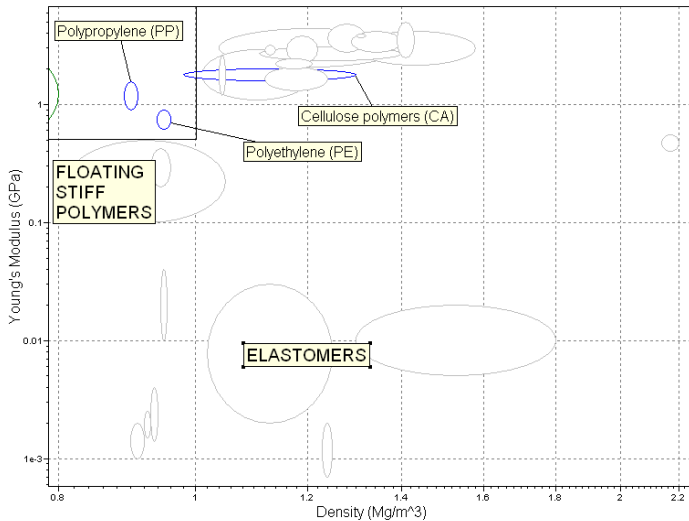


Both charts show that  $E/\rho$  is remarkably similar for steels, Ti, Al, and Mg alloys, and wood (along the grain). CFRP has a higher specific stiffness; GFRP a little lower.

*In Databook*, use  $E$ - $\rho$  chart in exactly the same way (using guideline for constant  $E/\rho$ ).

(e) The approach in (c) is weak, since there is no sense of a trade-off between stiffness and density. Nothing is stiffer and lighter than steels, but clearly the “light alloys” (Mg, Ti, Al) and CFRP compete strongly for light, stiff design. The performance index approach in (d) captures this competition, revealing that higher density is compensated in proportion with increased stiffness for a wide range of materials. Only CFRP emerges as superior to all other materials – on the basis of specific stiffness.

12. (a) *In CES*: Plot a modulus - density (restricted to Polymers), with a box selection set to a maximum of 1 Mg/m<sup>3</sup> (density of water), and  $E > 0.5$  GPa.



Polymers which float ( $\rho < 1 \text{ Mg/m}^3$ ) and  $E > 0.5 \text{ GPa}$  are:  
 - polypropylene (PP) and polyethylene (PE)  
 (- also cellulose polymers (CA), but only at the very lowest densities, most would not pass this selection stage).

*In Databook*, E- $\rho$  chart only reveals that some thermoplastics are possible – so have to use the data tables to find which ones (easy enough, by seeking  $\rho < 1 \text{ Mg/m}^3$  first).

(b) Environmental resistance to Fresh Water and Salt Water are “Very Good” for both, but “Poor” resistance to Sunlight (UV radiation) may give problems, leading to a need for frequent replacement.

Results (both stages):

[Cellulose polymers (CA)]  
 Polyethylene (PE)  
 Polypropylene (PP)

(c) A CES search for “ropes” indicates the polymers PA, PP, PE (as well as bamboo). Nylon (PA) doesn’t float, but PP and PE are both used (as also indicated by the applications tabulated in the Databook).

13. (a) For the stiffness constraint: the panel deflection  $\delta = \frac{W L^3}{48 E I}$ .

Substituting for  $I = \frac{b d^3}{12}$ :  $\delta = \frac{12 W L^3}{48 E b d^3} \Rightarrow E d^3 = \frac{12 (W / \delta) L^3}{48 b} = \text{constant}$ , since all the parameters on the RHS are specified.

Since an upper limit on  $d$  is specified, this translates into a minimum  $E$  to satisfy this relationship.

(b) Objective: minimum mass. Mass =  $m = \rho b d L$

From constraint equation, free variable  $d = \left( \frac{12 (W / \delta) L^3}{48 b E} \right)^{1/3}$

Substituting for the free variable  $d$  in the objective equation:  $m = \rho b L \left( \frac{12 (W/\delta) L^3}{48 b E} \right)^{1/3}$

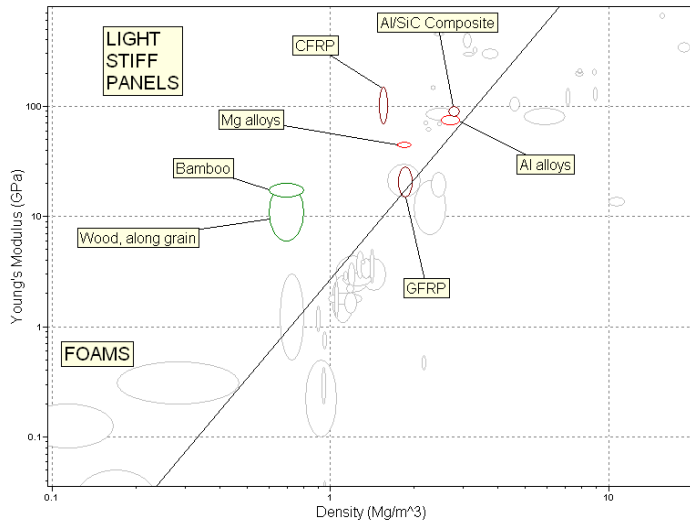
$$\text{Hence } m = \left( \frac{\rho}{E^{1/3}} \right) \left[ b L \left( \frac{(W/\delta) L^3}{4 b} \right)^{1/3} \right] = \left( \frac{\rho}{E^{1/3}} \right) \times \text{constant}$$

So material performance index to maximise for minimum mass is  $M_1 = \left( \frac{E^{1/3}}{\rho} \right)$ .

(c) On an E- $\rho$  chart, constant values of the performance index have slope = 3, since:

$$\log(M_1) = \frac{1}{3} \log E - \log \rho \Rightarrow \log E = 3 \log \rho + \log(M_1)$$

*In CES:* plot E- $\rho$  chart with line of slope 3 positioned to leave a reasonable shortlist. Ceramics and glasses are excluded because of their brittleness. Further constraints imposed in limit stage to see which materials fail on the other criteria:  $E > 5$  GPa; environmental resistance in salt water = Good or Very Good.



Material	Comments
<b>CFRP</b>	The best.
Bamboo	Not available as flat panels, and inadequate sea water resistance
Wood	Inadequate sea water resistance
Foams	Modulus too low – would have to be too thick
Mg alloys	Good, but inadequate sea water resistance
<b>Al alloys, Al/SiC composite, GFRP</b>	Close competition, and all cheaper options than CFRP.

*In Databook:* use E- $\rho$  chart as above (using guideline for appropriate performance index), also imposing horizontal line at  $E = 5$  GPa. Check environmental resistance from table.

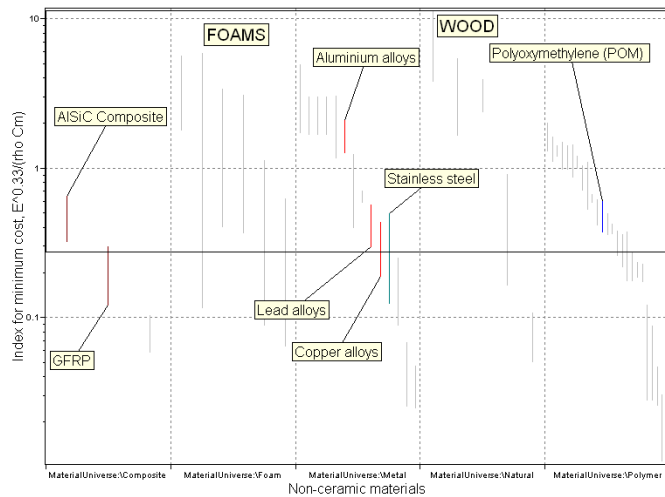
(d) Material cost = mass  $\times C_m$  where  $C_m$  is the material cost/kg. Since, everything else is constant, this can be incorporated directly into the performance index:

$$\text{Material cost} = m \times C_m = \left( \frac{\rho C_m}{E^{1/3}} \right) \times \text{constant}$$

Hence for minimum material cost, maximise the index  $M_2 = \left( \frac{E^{1/3}}{\rho C_m} \right)$ .

This is difficult and slow using the Databook (substituting in data for all non-ceramic materials with  $E > 5$  GPa, and good resistance to sea water) – i.e. not just those that performed well on the previous index.

*In CES,* either plot  $E - (\rho C_m)$  on a chart with a line of slope 3, or plot a bar chart showing the index (as below).



Material	Comments
<b>Al alloys</b>	The best.
<b>Al/SiC composite, GFRP, Pb alloys, Cu alloys, Stainless steel, POM</b>	Possible alternatives, but 5-10 times more expensive. Panel weight may be unacceptable (e.g. Pb)
Wood	Very cheap, but inadequate sea water resistance
Foams	Very cheap, but modulus too low – would have to be too thick

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Lent Term 2019