Engineering **FIRST YEAR**

Part IA Paper 4: Mathematics

Examples paper 3

(Elementary exercises are marked †, problems of Tripos standard *)

Students are encouraged to use mathematical grammar correctly.

Revision Question

For each of the following functions $f(x)$, find and sketch the functions

(i) $f(x)+f(-x)$ (ii) $f(1/x)$ (iii) $f(2x)-f(x)$ (iv) $f(f(x))$

(a) $f(x) = x^2$

(b) $f(x) = \sin x$

Complex Numbers

1† (a) Use De Moivre's theorem to derive formulae for (i) $\cos 2\theta$ (ii) $\sin 3\theta$ (iii) $\sin 4\theta$ in terms of $\cos\theta$ and $\sin\theta$.

(b) From the result $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
deduce that

$$\sinh(\alpha+\beta) = \sinh\alpha\cosh\beta + \cosh\alpha\sinh\beta$$

2 Evaluate:

(a)† i^6, i^{-5}, $(3+4i)i-(5-2i)\,i^2-(6+i)$, $\dfrac{(3+2i)(2+i)}{(1-2i)(4+i)}$

$3e^{i\pi/3}2e^{i2\pi/3}$, $2e^{i\pi/3}+2e^{i2\pi/3}$

(b)* $\tan\left[\dfrac{\pi}{6}+i\dfrac{\pi}{4}\right]$, $\ln\dfrac{3-i}{3+i}$, $\cos^{-1}\dfrac{3i}{4}$, $(61.5+113.7i)^{1/3}$

3 Find all the roots of

(a) $z^4+1 = 0$

(b) $z^8-z^4+1 = 0$

and plot them on the Argand diagram.

See the URL

http://nbviewer.jupyter.org/github/CambridgeEngineering/Part-IA-ExamplesPapers-Python/blob/master/paper4/IA%20Paper%204%20Mathematics%2003.ipynb

You may find it easier to access this through the online version of this paper in the *IA Examples Paper repository* which can be accessed through the IA Teaching Homepage on the Department website.

4* If $\dfrac{z-\mathrm{i}}{z+\mathrm{i}} = 6 + 4\mathrm{i}$, find $\dfrac{\bar{z}-\mathrm{i}}{\bar{z}+\mathrm{i}}$ in the form $a + \mathrm{i}b$, where $\bar{z}$ is the complex conjugate of z .

5 (a) Find the locus of points z which satisfy $|z - \mathrm{i}| = |z - 2|$.

(b) Show that the locus of points z satisfying

$$\left| \frac{z+2}{z-1+\frac{3}{2}i} \right| = 2$$

has the equation $(x - 2)^2 + (y + 2)^2 = 5$. Sketch this curve in the Argand plane.

Find alternative expressions for this locus in the form

$|z - a - ib| = r$ and $z = z_1 + s\,\mathrm{e}^{i\theta}$ $(0 \le \theta \le 2\pi)$

where a, b , r , and s are real constants and z_1 is a complex constant.

6† The function $f(z)$ has a power series expansion

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n + \ldots$$

where the coefficients a_0 , a_1 , ... are all *real*. Show that, for any z ,

$$f(\bar{z}) = \overline{f(z)}$$

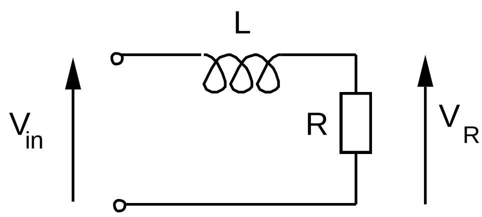
where $^-$ denotes the complex conjugate.

State whether or not this result applies to the following functions

(a) e^z , (b) e^{iz} , (c) $\mathrm{e}^{(i+1)z}$, (d) $\sin z$

[This theorem is known as the Schwarz Reflexion Principle].

7 Find the complex impedance of each of the two components of the circuit shown. Hence find the ratio of the peak value of V_R to that of V_{in} when the input voltage is sinusoidal with radian frequency ω. Find also the phase difference between V_R and V_{in}.



Differential Equations

8† Evaluate the following integrals

(a) $\displaystyle\int \frac{dx}{\sqrt{x^2+a^2}}$ (b) $\displaystyle\int \frac{dx}{x^2+a^2}$

(c) $\displaystyle\int \frac{x\,dx}{x^2+a^2}$ (d) $\displaystyle\int_{\pi/4}^{\pi/2} \frac{\cos 2t\ dt}{1+\sin 2t}$

See the URL

http://nbviewer.jupyter.org/github/CambridgeEngineering/Part-IA-ExamplesPapers-Python/blob/master/paper4/IA%20Paper%204%20Mathematics%2003.ipynb

9 Find the complete solutions of the following ordinary differential equations:

(a) $(1-x^2)\dfrac{dy}{dx} + \cot y = 0$ (b) $\sinh y\ \dfrac{dy}{dx} + \cosh^2 y\ \cos^2 x = 0$

(c) $\dfrac{dy}{dx} + \dfrac{2}{x}\,y + \dfrac{1}{1+x^3} = 0$ (d) $\dfrac{dy}{dx} + y\cot x + \cos^4 x = 0$

Suitable past Tripos questions:

02 Q2b & c ; 03 Q2b; 04 Q2b & c; 05 Q3 (short); 06 Q5 (long); 07 Q2 (short); 08 Q4a & c; 09 Q4b (long); 10 Q5 (long); 11 Q5 (long); 12 Q2 (short) & Q5 (long); 13 Q1 (short); 15 Q5 (long); 16 Q2 (short); 17 Q2 (short); 18 Q1 (short).

Answers

1 (i) $\cos^2\theta - \sin^2\theta$ (ii) $3\sin\theta\cos^2\theta - \sin^3\theta$ (iii) $4\sin\theta\cos^3\theta - 4\cos\theta\sin^3\theta$

2 (a) -1, $-i$, -5, $-\cdot 294 + \cdot 824i$, -6, $2\sqrt{3}\,i$

(b) $\cdot 288 + \cdot 765i$, $-\cdot 644i\ (+2n\pi i)$, $\pm(\pi/2 - \cdot 693i)\ \ (+2n\pi)$

$(61.5+113.7i)^{\frac{1}{3}} = 4.7351 + 1.7732i\ ,\ -3.9032 + 3.2141i\ ,\ -0.8319 - 4.9874i$

3 (a) $\exp(i\pi/4 + n\pi i/2)$, $n = 0, 1, 2, 3$ (b) $\exp(\pm i\pi/12 + n\pi i/2)$, $n = 0, 1, 2, 3$

4 $\dfrac{3}{26} + \dfrac{2}{26}\,i$

5 (a) Straight line $y = 2x - 3/2$ (b) $a = 2,\ b = -2,\ r = \sqrt{5}\ ,\ z_1 = 2 - 2i\ ,\ s = \sqrt{5}$

6 (a) True (b) False (c) False (d) True

7 $i\omega$L, R, $\dfrac{R}{\sqrt{R^2+\omega^2L^2}}$, lag of $\tan^{-1}\dfrac{\omega L}{R}$

8 (a) $\ln\left(x+\sqrt{x^2+a^2}\right)+c$ or $\sinh^{-1}\dfrac{x}{a}+c'$ (b) $\dfrac{1}{a}\tan^{-1}\dfrac{x}{a}+c$

(c) $\frac{1}{2}\ln\left(x^2+a^2\right)+c$ (d) $-\frac{1}{2}\ln 2$ (Approx = -0.3472, exact = -0.3466)

9 (a) $y = \cos^{-1}c\sqrt{\left|\dfrac{x+1}{x-1}\right|}$ (b) $y = \cosh^{-1}\left[\dfrac{4}{\sin 2x+2x+c}\right]$

(c) $y = \dfrac{c-\ln(1+x^3)}{3x^2}$ (d) $y = \dfrac{\cos^5 x+c}{5\sin x}$

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