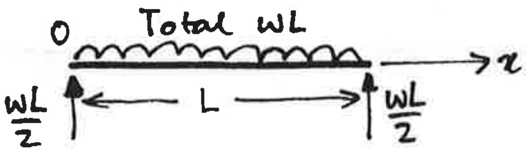
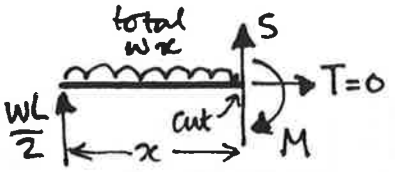
Part IA Engineering Tripos

STRUCTURAL MECHANICS

Solutions for Examples Paper 4

1.





$$S = wx - \frac{wL}{2}$$

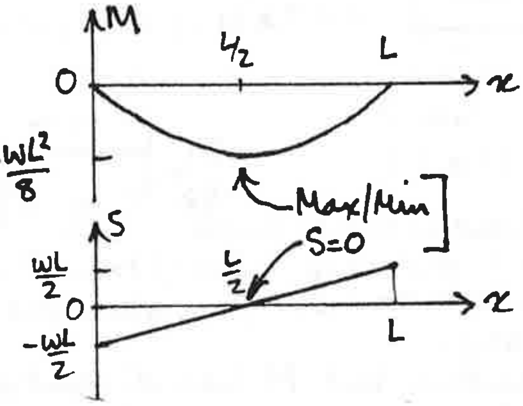
$$M = \frac{wx^2}{2} - \frac{wLx}{2} = \frac{wx(x-L)}{2}$$

Notes for supervisors:

(i) Need only one free body.

W = total load = wL

$\frac{WL}{8} = -\frac{wL^2}{8}$



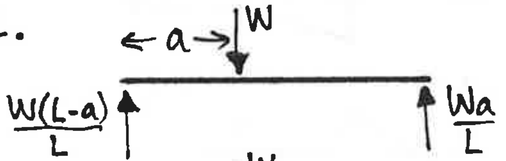
(ii) M is parabolic, with |M| maximum at $x = L/2$.

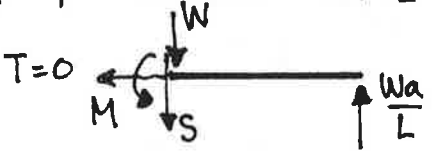
(iii) S is linear, with $S = 0$ at $x = L/2$

(iv) |M| is max. where $S = 0$. This is a useful general result!

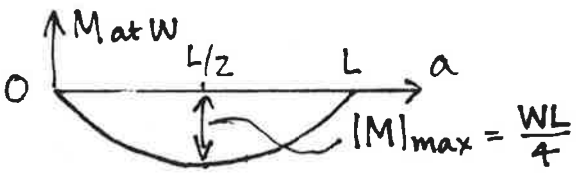
$$M_{centre} = -\frac{w}{2}\cdot\frac{L}{2}\cdot\frac{L}{2} = -\frac{wL^2}{8} = -\frac{WL}{8} \quad \text{c.f.} \; -\frac{WL}{4} \text{ for concentrated load}$$

2.



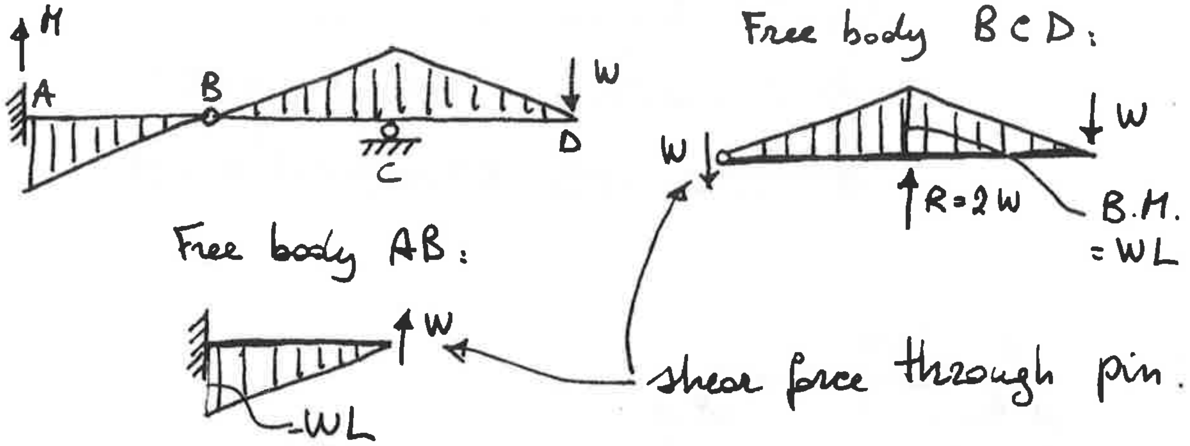


$$M = -\frac{Wa}{L}(L-a)$$

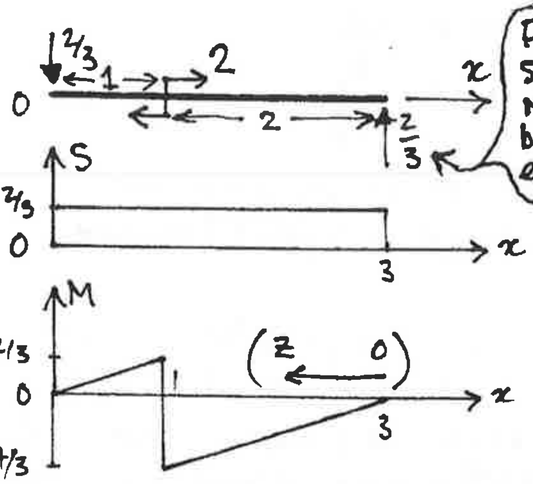


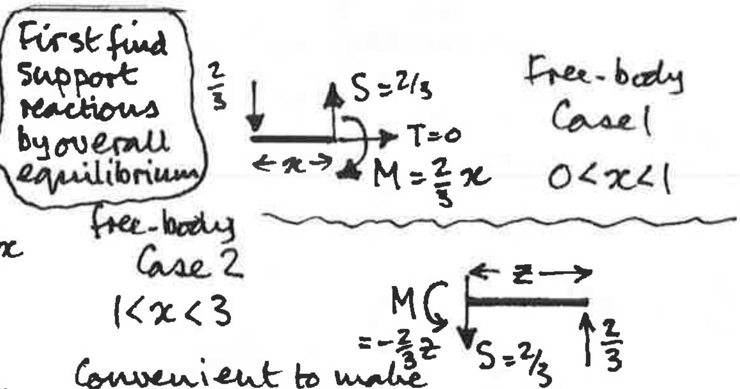
Note:
This plot is not a "bending-moment diagram" as such. It is the trace of the peak of the BM diagram as the load moves along the beam.

3



4 (a)





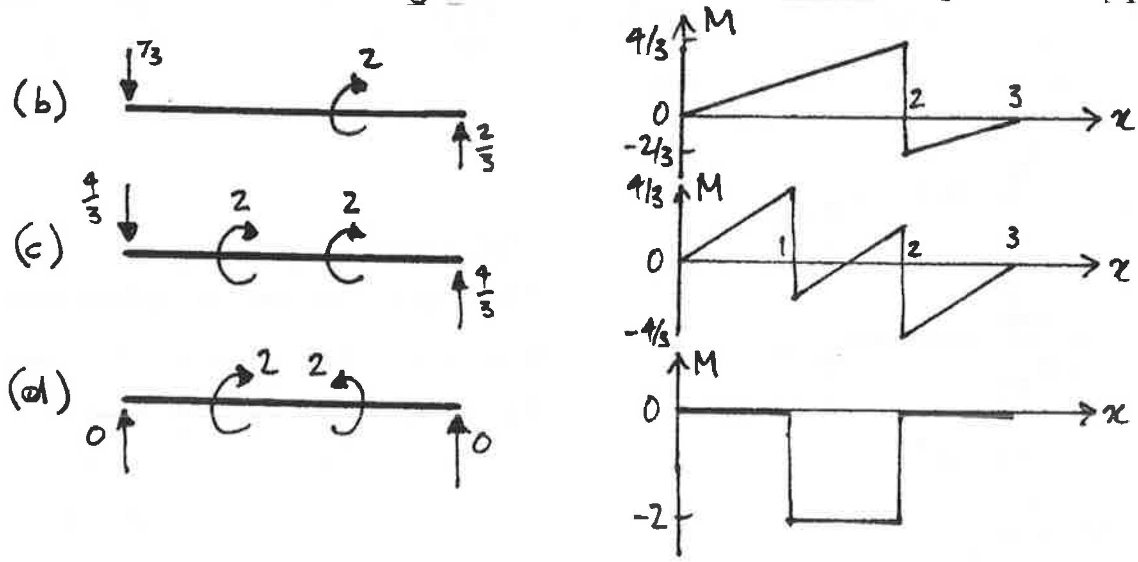
Convenient to make cut distance z (say) from R.H. end and to consider R.H. piece as free body.

Note; S is continuous, but M has discontinuity (jump) at place where couple is applied.

(b)

(c)

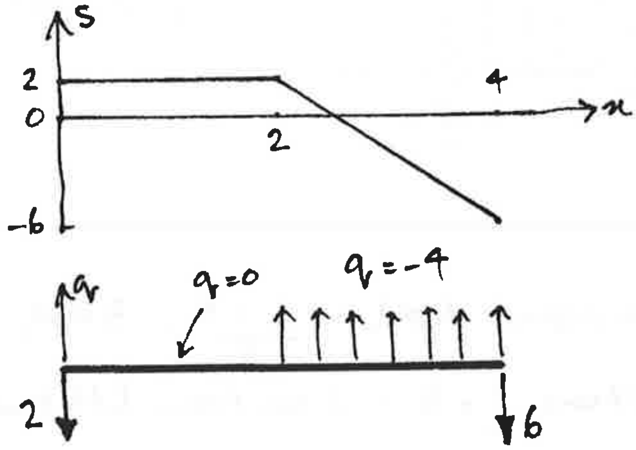
(d)



Note. Part (b): result is related to that of (a).
(c): use superposition of (b) and (a). Be careful to add separately in the three regions!
(d): this time (a) − (b). Observe that since end reactions are both zero, M = 0 in the two end portions.

5. For $0 < x < 2$, $M = 2x \therefore S = \frac{dM}{dx} = 2$; $q = \frac{dS}{dx} = 0$

For $2 < x < 4$, $M = 10x - 2x^2 - 8 \therefore S = \frac{dM}{dx} = 10 - 4x$; $q = \frac{dS}{dx} = -4$.



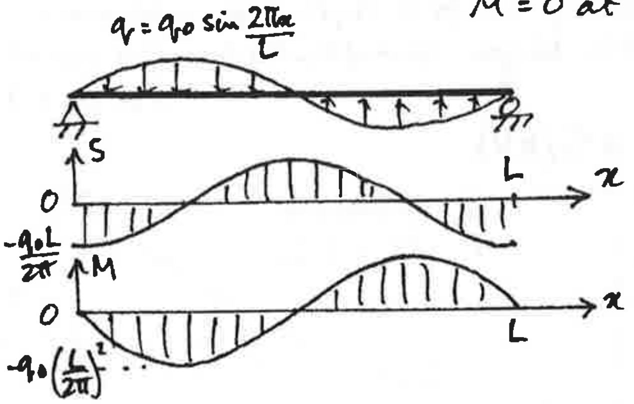
Note: easy enough to get expressions for q. We need to think carefully about point forces, which occur where there is a jump in S. Here we get the end forces from S. (Good idea to check equilibrium overall).

6 $\frac{dS}{dx} = q_0 \sin \frac{2\pi x}{L}$

Int: $S = -q_0 \frac{L}{2\pi} \cos \frac{2\pi x}{L} + C_1 = \frac{dM}{dx}$

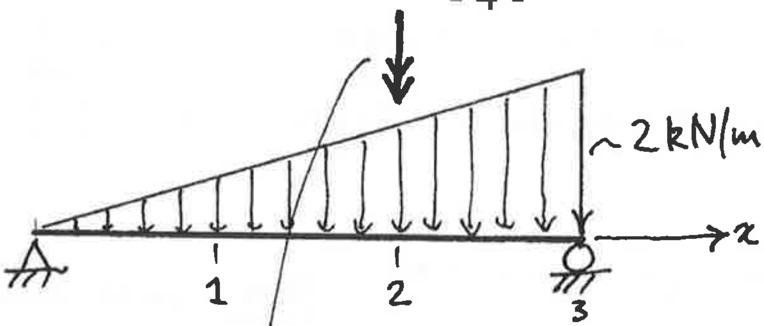
Int: $M = -q_0 \left(\frac{L}{2\pi}\right)^2 \sin \frac{2\pi x}{L} + C_1 x + C_2$.

Use BC's to find C_1, C_2: $M = 0$ at $x = 0 \Rightarrow 0 = 0 + C_1 . 0 + C_2 \therefore C_2 = 0$
$M = 0$ at $x = L \Rightarrow 0 = 0 + C_1 . L \therefore C_1 = 0$



Note. It would be straightforward to generate $S(x)$, $M(x)$ for a loading $q_0 \sin \frac{n\pi x}{L}$, n an integer.

7.

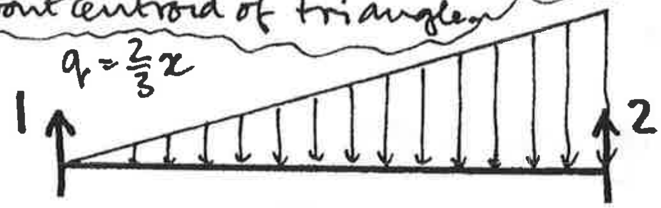


Beam and supports

Can find resultant by integration; but simpler to say Resultant = mean load × length — or, equivalently use area of triangle formula. Similarly, location of resultant can be found by taking moments of forces about (say) LH end ... but again, students should know standard result about centroid of triangle.

Resultant load $= \frac{2 \times 3}{2} = 3\,\text{kN}$,

acting $\frac{2}{3} \times 3 = 2\,\text{m}$ from LH end



End reactions by moments.

forces acting on beam.

Equil. eq^n: $\frac{dS}{dx} = q = \frac{2}{3}x \quad \therefore S = \frac{x^2}{3} + C$

At LH end, $S = -1$ (upward force at LH end of beam; so $S < 0$, by comparison with sign-convention picture on p.1)

$\therefore C = -1$ and

$S = \frac{x^2}{3} - 1$: see plot at right.

Now use 2nd Equil. eq^n:

$\frac{dM}{dx} = S = \frac{x^2}{3} - 1$

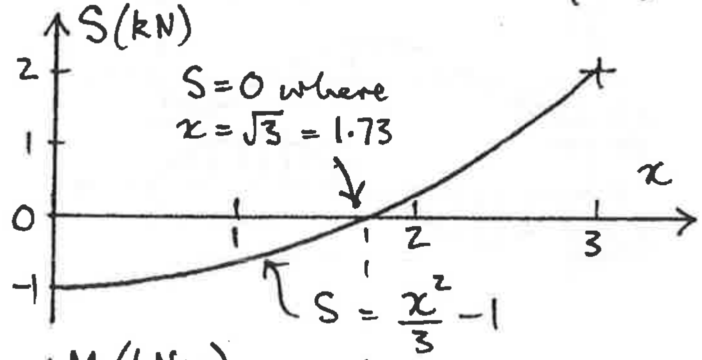
$\therefore M = \frac{x^3}{9} - x + \cancel{C}$

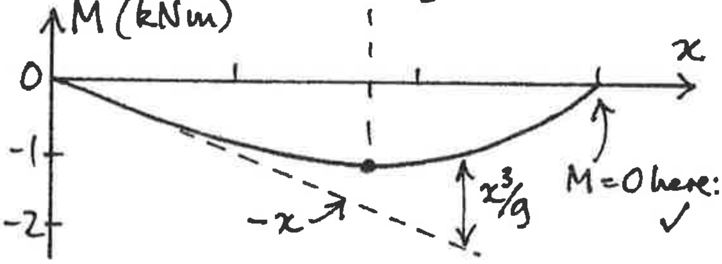
But $M(0) = 0$ (S.S. end)

$\therefore C = 0$

M is max/min where $S = 0$, ie at $x = \sqrt{3}$:

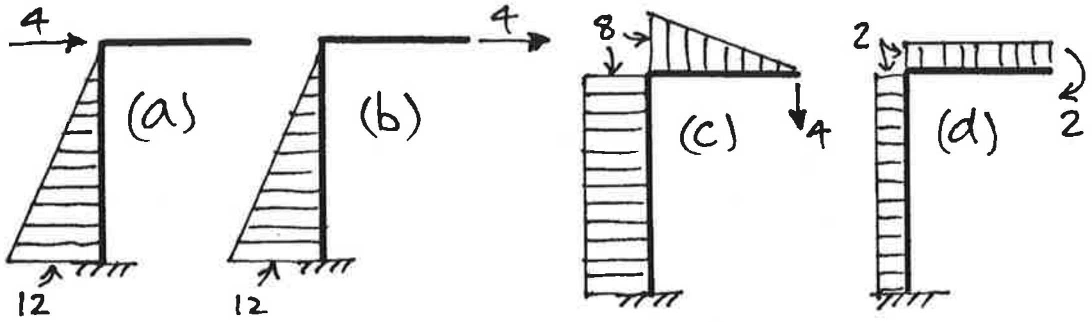
$M_{min} = \frac{1}{\sqrt{3}} - \sqrt{3} = -\frac{2}{\sqrt{3}} = \underline{\underline{-1.15}}\ \text{kNm}$





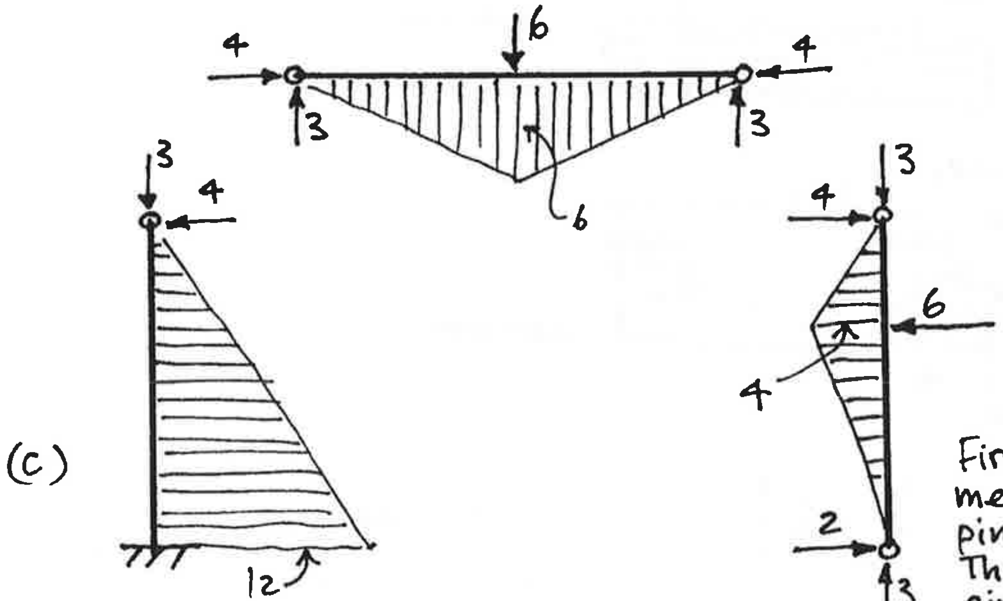
Note: slope of M = value of S ...
slope of S = " " q

8.

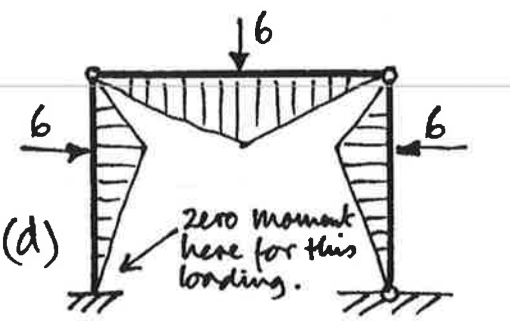


Notes. (1) Make sure to distinguish between the frame itself (bold here) and the BM plot which is offset from it. The "shading" lines are drawn ⊥ member.
(a) Vertical is a cantilever; no moment in horizontal.
(b) Same as (a) as far as BM is concerned — though not for Tension (which is not asked for here).
(c) Now horiz. is a cantilever under tip load, while vertical has uniform BM.
(d) Entire frame under uniform BM.
In every case, of course, the answer is found by making a cut; but by now it should not be necessary to draw free-body diagrams for such simple cases. Also, it should be clear which side of the member is in tension on account of BM.

9.

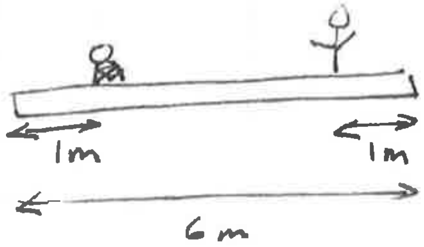


First separate the members at the pins.
Then apply the given external forces and find the end reactions ⊥ to the top and RH beams, by overall equilibrium of each. The horiz. force 4→ at top of RH member requires 4← at RH end of top beam; and horiz. equil. of top beam requires 4→ at other end; and so 4← on LH cantilever. Similarly for vertical forces 3↑,↓. Then when pieces are fully in equilibrium, work out bending moments



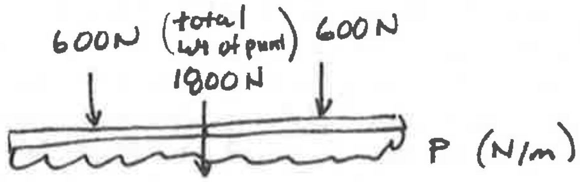
For (d) the only changes on account of the new load are in the lower part of the L.H. column.

10.



symmetric loading
→ uniform p

a)



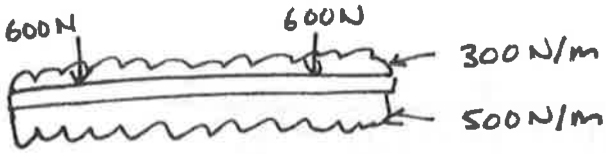
uniform punt wt

$$\frac{1800\,N}{6\,m} = 300\,N/m$$

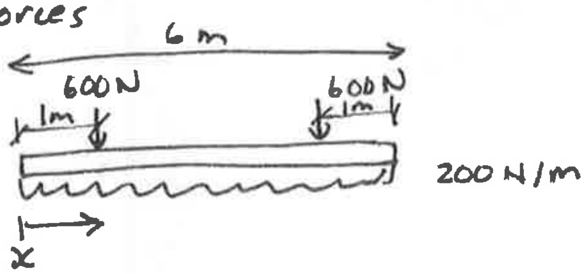
$\Sigma V = 0$ $\quad 600 \times 2 + 1800 = p \times 6$

$$p = 500\,N/m$$

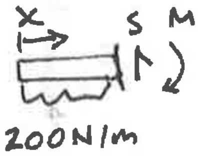
b) forces acting on beam (punt)



Net forces



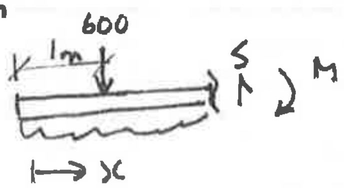
FBD
$x < 1m$



$S = -200x$

$M = -\frac{200x^2}{2} = -100x^2$

$1m < x < 5m$

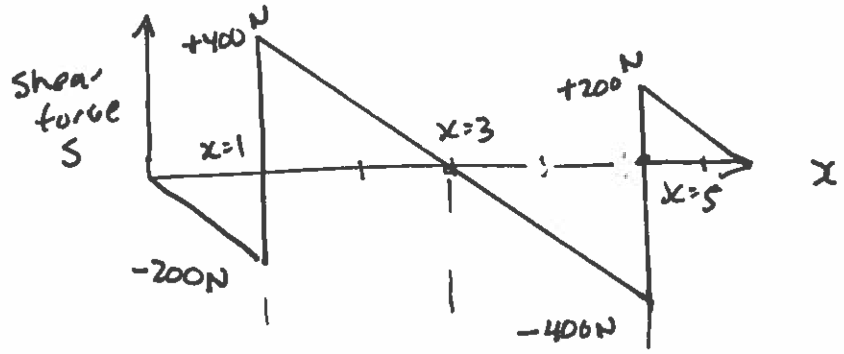


$S = -200x + 600$

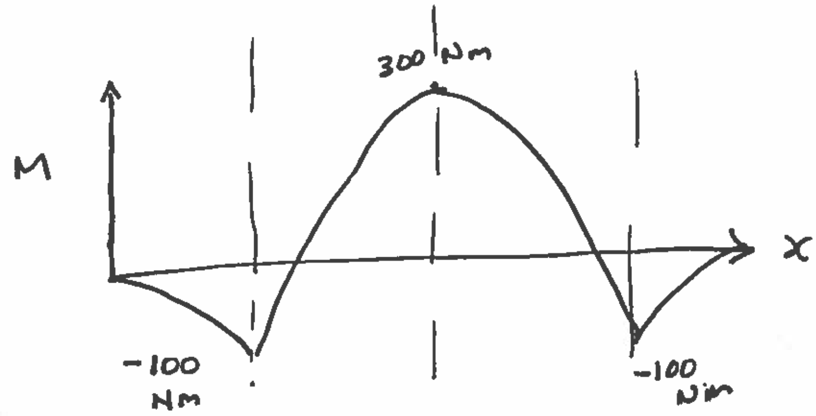
$M = -100x^2 + 600(x-1)$

10. b) cont'd

SFD



BMD



M_{max} when $\frac{dM}{dx} = 0 \rightarrow S = 0$

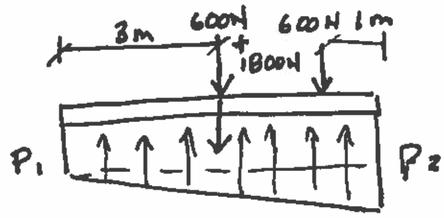
$\therefore \quad -200x + 600 = 0$

$x = 3\text{ m}$

$M = -100(3)^2 + 600(3-1)$

$= 300 \text{ N.m}$

c) loading is no longer symmetric



$\Sigma V = 0 \quad \left(\frac{P_1 + P_2}{2}\right) \times 6 = 3000\text{ N}$ (N/m, m)

$\therefore \quad P_1 + P_2 = 1000 \quad (1)$

$\Sigma M_{\text{C}} = 0 \quad 600 \times 2 = \underbrace{\left(\frac{P_2 - P_1}{2}\right) \times 6}_{\text{resultant force}} \times 1$

$\therefore \quad P_2 - P_1 = 400 \quad (2)$

(1) + (2) $\Rightarrow P_2 = 700\text{ N/m}$

$P_1 = 300\text{ N/m}$

$p = 300 + 200x/3 \text{ N/m}$

11.

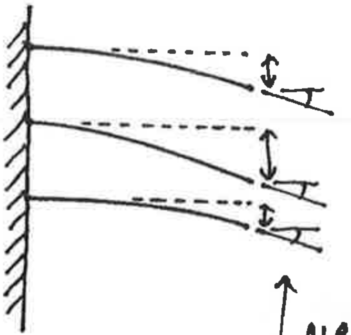
$$\kappa = \frac{d\psi}{ds} \text{ (defn) } \simeq \frac{d\psi}{dx} \quad \text{when } |\psi| \ll 1$$

$$\therefore d\psi = \kappa\, dx \;;\; \int d\psi = \int \kappa\, dx \;;\; \psi_{x=0.5} - \underset{=0}{\psi_{x=0}} = \int_0^{0.5} \kappa\, dx.$$

$$\therefore \psi_{x=0.5} = \int_0^{0.5} \kappa\, dx$$

Case (b): $= \underbrace{0.005 \times 20}_{\kappa} \times 0.5 = 0.05 \text{ rad} = 2.87^\circ$

Cases (c), (d): Same answer because area under κ curve is the same as in (b).



(b) uniform curvature

(c) curvature concentrated near root – more deflection

(d) curvature concentrated near tip – less deflection

NB end rotation same for all 3 cases

Note. 1st part very simple – here written out at length for the benefit of those who don't see how to do it immediately. In last part it should be obvious that curvature concentrated at root tends to increase the deflection, other things being equal.

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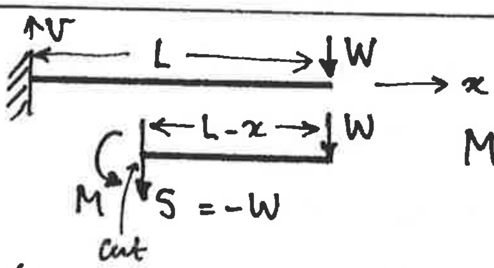
12. (a)



$M = W(L-x)$

$M = W(L-x)$ (Statics)

$\kappa = \frac{W}{B}(L-x)$ (use of elastic law $M = B\kappa$ for initially straight beam)

$\therefore \frac{d^2v}{dx^2} = \frac{W}{B}(x-L)$ (Since $\kappa = -\frac{d^2v}{dx^2}$)

Int: $\frac{dv}{dx} = \frac{W}{B}\left(\frac{x^2}{2} - Lx + \cancel{C_1}\right)$ BC: $\frac{dv}{dx} = 0$ at $x=0$ $\therefore C_1 = 0$

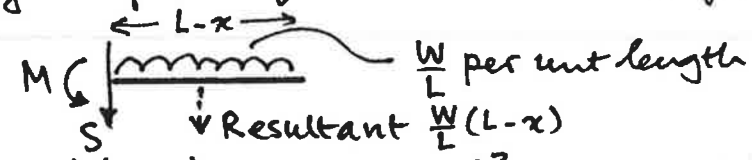
Int: $v = \frac{W}{B}\left(\frac{x^3}{6} - \frac{Lx^2}{2} + \cancel{C_2}\right)$ BC: $v = 0$ at $x=0$ $\therefore C_2 = 0$

At $x = L$, $\quad v = \dfrac{W}{B}\left(\dfrac{L^3}{6} - \dfrac{L^3}{2}\right) = -\dfrac{WL^3}{3B}$ $\quad$ (− because ↓)

rotation, $\dfrac{dv}{dx} = \dfrac{W}{B}\left(\dfrac{L^2}{2} - L^2\right) = -\dfrac{WL^2}{2B}$ $\quad$ (− because ↻; +ve sense of $\dfrac{dv}{dx}$ is ↺)

Note. These agree with values given in Data Book, p.6; but there $B = EI$ and the signs of the quantities are not defined.

(b)



$$M = \underbrace{\frac{W}{L}(L-x)}_{\text{resultant force}}\underbrace{\left(\frac{L-x}{2}\right)}_{\text{"lever arm"}} = \frac{W}{2L}(L-x)^2$$

Using elastic law and putting $K = -\dfrac{d^2v}{dx^2}$:

$$-\frac{d^2v}{dx^2} = \frac{W}{2LB}\left(L^2 - 2Lx + x^2\right)$$

Int: $\quad -\dfrac{dv}{dx} = \dfrac{W}{2LB}\left(L^2x - Lx^2 + \dfrac{x^3}{3} + \cancel{C_1}\right)$ $\qquad$ BC: $\dfrac{dv}{dx} = 0$ at $x = 0$ $\Rightarrow C_1 = 0$

Int: $\quad -v = \dfrac{W}{2LB}\left(L^2\dfrac{x^2}{2} - \dfrac{Lx^3}{3} + \dfrac{x^4}{12} + \cancel{C_2}\right)$ $\qquad$ BC: $v = 0$ at $x = 0$ $\Rightarrow C_2 = 0$

At $x = L$, $v_{tip} = -\dfrac{WL^3}{8B}$ $\quad$ (ie ↓)

Also $\quad -\left.\dfrac{dv}{dx}\right)_{tip} = \dfrac{W}{2LB}\left(L^3 - L^3 + \dfrac{L^3}{3}\right) = \dfrac{WL^2}{6B}$ $\quad \therefore \left.\dfrac{dv}{dx}\right)_{tip} = -\dfrac{WL^2}{6B}$ (as in data book).

Note: Exactly the same method in each case. Here all of the integration constants come out at zero, but this will not always be the case. In (a) we used $\dfrac{d^2v}{dx^2} = -K = -\dfrac{M}{B}$, while in (b) we used $-\dfrac{d^2v}{dx^2} = K = \dfrac{M}{B}$

Probably safest to use one of these, consistently.

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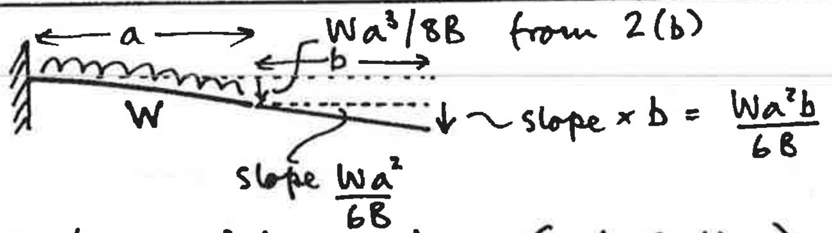
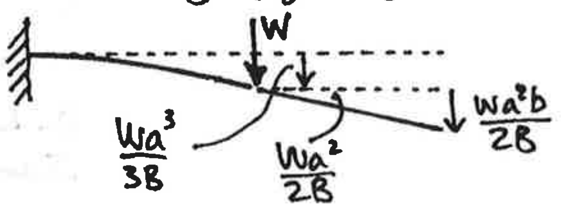
13. (a)



End defl ↓ $= \dfrac{Wa^2}{B}\left(\dfrac{a}{8} + \dfrac{b}{6}\right)$ $\qquad$ End rotation ↻ $= \dfrac{Wa^2}{6B}$

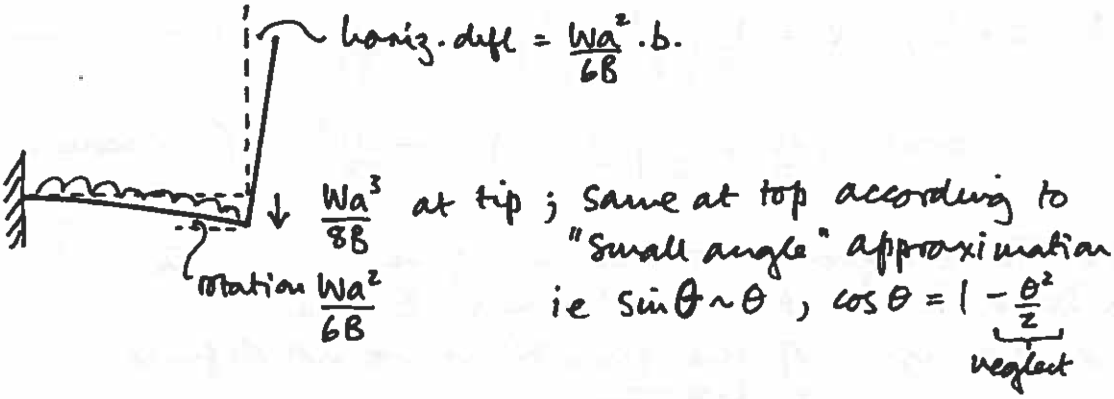
(b)



$\Big\}\downarrow \dfrac{Wa^2}{B}\left(\dfrac{a}{3} + \dfrac{b}{2}\right)$

End rotation ↻ $= \dfrac{Wa^2}{2B}$

(c)



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14. (a)

$$M = -q_0 \frac{L^2}{\pi^2} \sin\frac{\pi x}{L}$$

$$\therefore \quad \kappa = -\frac{q_0 L^2}{B\pi^2} \sin\frac{\pi x}{L} \qquad \therefore \frac{d^2 v}{dx^2} = \frac{q_0 L^2}{B\pi^2} \sin\frac{\pi x}{L}$$

$$\text{Int:} \quad \frac{dv}{dx} = -\frac{q_0 L^3}{B\pi^3} \cos\frac{\pi x}{L} + C_1$$

$$\text{Int:} \quad v = -\frac{q_0 L^4}{B\pi^4} \sin\frac{\pi x}{L} + C_1 x + C_2$$

(N.B. here we have to get both C_1 and C_2 from boundary conditions on v; unlike Q12, where we had a slope condition)

BC's $v = 0$ at $x = 0 \Rightarrow 0 = 0 + 0 + C_2 \quad \therefore C_2 = 0$
$v = 0$ at $x = L \Rightarrow 0 = 0 + C_1 L \quad \therefore C_1 = 0$

So $$v = -\frac{q_0 L^4}{B\pi^4} \sin\frac{\pi x}{L}$$

(d) $v_{centre} = -\frac{q_0 L^4}{B\pi^4}$ (ie ↓, as expected).

(e) For $q = q_0 \sin\frac{n\pi x}{L}$, we simply need to replace π in calculations above by $(n\pi)$: hence

$$v = \frac{-q_0 L^4 \sin\left(\frac{n\pi x}{L}\right)}{n^4 B\pi^4}$$; i.e. amplitude $\frac{1}{n^4}$ × previous one.

Notes (1) Check dimensions of formulas.
(2) for L^4 in expression for v comes from the fact that we integrate $\sin\frac{\pi x}{L}$ four times : twice to get M, then from κ twice to get v.

Simon Guest
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