Part IA Paper 4: Mathematical Methods

Examples Paper 6

Elementary exercises are marked †, problems of Tripos standard ∗.
Answers can be found at the back of the paper.

Revision Questions

Evaluate the following integrals:

(a) $\displaystyle\int x^{-1/2}\,dx$ (b) $\displaystyle\int x^{-1}\,dx$ (c) $\displaystyle\int \frac{x^5}{1+x^6}\,dx$

(a) $\displaystyle\int x^2 \ln x\,dx$ (b) $\displaystyle\int \sin^4 x\,\cos x\,dx$ (c) $\displaystyle\int_{-\pi}^{\pi} \sin x\left(x^2\cos x + x^6\right)\,dx$

Eigenvalues and eigenvectors

1. Find the eigenvalues and eigenvector(s) of the following matrices:

(i) $\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

(iv) $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ (v) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (vi) $\begin{bmatrix} 3 & -4 & 1 \\ -4 & 8 & -4 \\ 1 & -4 & 3 \end{bmatrix}$

2. If $\underline{y}$ is the reflection of the vector $\underline{x}$ in the plane through the origin with unit normal $\underline{n}$, show that

$$\underline{y} = \underline{x} - (2\underline{x}\cdot\underline{n})\underline{n},$$

and hence find the 3×3 matrix R describing the transformation, i.e. find the matrix R such that $\underline{y} = R\underline{x}$. By a geometric argument, what are the eigenvalues and corresponding eigenvectors of R?

3. Construct the symmetric matrix A which has the eigenvalues $\lambda_i = 3, 1, 1/2$ and corresponding eigenvectors

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}.$$

Find also the symmetric matrix B with the *same* eigenvectors, but with the corresponding eigenvalues equal to $1/\lambda_i$. Evaluate the matrix AB and comment on your result.

4. A is a real symmetric 2×2 matrix with eigenvalues λ_1 and λ_2, and $f(\lambda) = |A - \lambda I|$ is the polynomial whose roots are the eigenvalues. By considering relevant coefficients in the polynomial $f(\lambda)$, prove that

(a) $A_{11} + A_{22} = \lambda_1 + \lambda_2$ (b) $|A| = \lambda_1 \lambda_2$

5. † Find the eigenvalues and eigenvectors of the 2×2 real, symmetric matrix A where

$$A = \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}.$$

Hence, find a matrix R which changes the coordinate system so that the new axes are aligned with the eigenvectors of A. Calculate the version of the matrix A in these new coordinates, and verify that it agrees with the form derived in the lectures.

6. A 3×3 real, symmetric matrix A has eigenvalues λ_i, $i = 1, 2, 3$, and corresponding normalised eigenvectors $\underline{u}_i$. The eigenvalues are real, distinct, and arranged in the order

$$\lambda_1 < \lambda_2 < \lambda_3.$$

Explain why any vector $\underline{x}$ can be expressed in the form $\underline{x} = \alpha \underline{u}_1 + \beta \underline{u}_2 + \gamma \underline{u}_3$, and hence show that

(a) $\underline{x}^t \underline{x} = \alpha^2 + \beta^2 + \gamma^2$, (b) $\underline{x}^t A \underline{x} = \lambda_1 \alpha^2 + \lambda_2 \beta^2 + \lambda_3 \gamma^2$.

Use these expressions to show that

$$\lambda_1 \leq \frac{\underline{x}^t A \underline{x}}{\underline{x}^t \underline{x}} \leq \lambda_3$$

for all vectors $\underline{x}$. When is equality achieved?

7. $*$ Using the matrix A of Question 6, and again expressing a general vector $\underline{x}$ in terms of the eigenvectors of A, what is $A^n \underline{x}$? What happens to $A^n \underline{x}$ as n gets large? Using the result of Question 3, derive a similar result for $(A^{-1})^n \underline{x}$.

With

$$A = \begin{bmatrix} 3 & -4 & 1 \\ -4 & 8 & -4 \\ 1 & -4 & 3 \end{bmatrix} \quad \text{and} \quad \underline{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

use Python/Numpy to calculate $A\underline{x}$, $A^2\underline{x}$, $A^3\underline{x}$ and $A^4\underline{x}$, and hence obtain an approximation for the eigenvalue of A with the largest absolute value, and the corresponding eigenvector. Experiment with higher powers of A, and compare your result with the exact answer, which was calculated in Question 1(vi).

Hints

After importing NumPy (as np), enter `A = np.array([[3 -4 1],[-4 8 -4],[1 -4 3]]` to create the matrix A. `np.linalg.matrix_power(A,3)` can be used to raise a matrix to a power (power of three in this case). A Jupyter notebook which implements this can be found at

`https://github.com/CambridgeEngineering/PartIA-Paper4-Mathematics`

Answers

1. (i) eigenvalues 2 and -3, eigenvectors $[2/\sqrt{5}, 1/\sqrt{5}]^t$ and $[1/\sqrt{5}, -2/\sqrt{5}]^t$

   (ii) eigenvalues 4.618 and 2.382, eigenvectors $[0.526, 0.851]^t$ and $[0.851, -0.526]^t$

   (iii) eigenvalues 2 and 3, eigenvectors $[1, 0]^t$ and $[0, 1]^t$

   (iv) eigenvalues 2 and 3, eigenvectors $[1, 0]^t$ and $[1/\sqrt{2}, 1/\sqrt{2}]^t$

   (v) eigenvalues 1 and 1 (repeated), eigenvector $[1, 0]^t$

   (vi) eigenvalues 0,2 and 12, eigenvectors $[1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}]^t$, $[1/\sqrt{2}, 0, -1/\sqrt{2}]^t$ and $[-1/\sqrt{6}, 2/\sqrt{6}, -1/\sqrt{6}]^t$

2. $I - 2\underline{n}\underline{n}^t$

3. $A = \dfrac{1}{12}\begin{bmatrix} 23 & 13 & 2 \\ 13 & 23 & -2 \\ 2 & -2 & 8 \end{bmatrix} \quad B = \dfrac{1}{6}\begin{bmatrix} 5 & -3 & -2 \\ -3 & 5 & 2 \\ -2 & 2 & 10 \end{bmatrix} \quad AB = I$

5. Eigenvalues 2 and 6, eigenvectors $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, $\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, $R = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$.

6. Equality when $\underline{x}$ is a multiple of $\underline{u}_1$ or $\underline{u}_3$

7. $A\underline{x} = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}$, $A^2\underline{x} = \begin{bmatrix} 26 \\ -48 \\ 22 \end{bmatrix}$, $A^3\underline{x} = \begin{bmatrix} 292 \\ -576 \\ 284 \end{bmatrix}$, $A^4\underline{x} = \begin{bmatrix} 3464 \\ -6912 \\ 3448 \end{bmatrix}$.

   Approximations for eigenvalue 12.00, eigenvector $[0.4092, -0.8165, 0.4073]^t$.

MCS & GNW
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