Part IA Paper 4: Mathematical Methods

EXAMPLES PAPER 11

Laplace Transforms

Elementary exercises are marked †, problems of Tripos standard ∗

A table of Laplace Transforms is given on p. 24 of the Mathematics Data Book.
Note that the Mathematics Data Book denotes the Laplace transform of $y(t)$ by $\bar{y}(s)$.

†1. Resolve the following expressions into partial fractions.

(a) $\dfrac{1}{(x-3)(x-7)}$, (b) $\dfrac{(x+7)}{(x+3)(x-7)^2}$, (c) $\dfrac{(x^2+3x+4)}{x\,(x^2+4)}$.

†2. From first principles, calculate the Laplace transforms of the following functions given that these functions are zero for $t < 0$.

(a) $y(t) = t$.

(b) $y(t) = e^{at}\sin\omega t$.

3. Show that if $Y_n(s)$ is the Laplace transform of t^n, then $Y_n(s) = \dfrac{n}{s}Y_{n-1}(s)$. Hence show that $Y_n(s) = \dfrac{n!}{s^{n+1}}$.

4. For each part, convert the Laplace transform, $Y(s)$, to partial fraction form and find the corresponding inverse transform, $y(t)$.

(a) $Y(s) = \dfrac{1}{(s+1)(s+2)(s+3)}$,

(b) $Y(s) = \dfrac{1}{(s+1)(s+2)^2}$,

(c) $Y(s) = \dfrac{1}{(s+1)(s^2+4)}$.

∗5. Solve the following differential equations using Laplace transforms.

(a) $\ddot{y} + 4\dot{y} + 3y = e^{-t}$, $\quad y(0) = \dot{y}(0) = 1$,

(b) $\ddot{y} - y = \sin t$, $\quad y(0) = 1,\ \dot{y}(0) = 0$,

(c) $\ddot{y} + y = t^3$, $\quad y(0) = \dot{y}(0) = 1$.

{Each part of this question is of Tripos standard}.

†6. Solve the following pair of simultaneous differential equations using Laplace transforms for $u(t)$ and $v(t)$ given $u = v = 0$ at $t = 0$:

$$\dot{u} + av = b,$$
$$\dot{v} - au = 0,$$

where a and b are constants.

$*$7. A charged particle of mass m with electrical charge e is released at time $t = 0$ from rest at the origin of a Cartesian coordinate system. The particle moves under the influence of a uniform electric field E and a uniform magnetic flux of density B acting in such directions that the particle moves in the plane $z = 0$ and the equations of motion of the particle in this plane are:

$$m\ddot{x} + eB\dot{y} = eE,$$
$$m\ddot{y} - eB\dot{x} = 0.$$

Find x and y as functions of t

(a) by integration of the functions $u(t)$ and $v(t)$ of Question 6 and noting that $u = \dot{x}$ and $v = \dot{y}$,

(b) by direct application of Laplace transforms to the equations of motion.

$*$8. (a) What are the Laplace transforms of $y_1(t) = 1$ and $y_2(t) = \delta(t)$ if the lower limit in the Laplace transform integral is (i) 0^- (usual case), (ii) 0^+ (unusual)?

(b) Using Laplace transforms, find $y(t)$ for $t \geq 0$ if

$$\dot{y} + y = \dot{u} + 2u,$$

where $y(0^-) = 0$ and u is the unit step function. What is $y(t)$ if $y(0^+) = 0$ instead?

9. Let $h(t) = \int_0^t f(\tau)g(t-\tau)d\tau$, where $f(t) = t$ and $g(t) = e^t$. Find $H(s)$ by: (i) first evaluating the convolution integral, (ii) first taking Laplace transforms.

Answers.

1. (a) $\frac{-1}{4(x-3)} + \frac{1}{4(x-7)}$ (b) $\frac{1}{25(x+3)} - \frac{1}{25(x-7)} + \frac{35}{25(x-7)^2}$, (c) $\frac{1}{x} + \frac{3}{(x^2+4)}$.

2. (a) $1/s^2$, (b) $\omega/\left((s-a)^2 + \omega^2\right)$.

4. (a) $\frac{1}{2}\frac{1}{s+1} - \frac{1}{s+2} + \frac{1}{2}\frac{1}{s+3}$, $y(t) = \frac{1}{2}e^{-t} - e^{-2t} + \frac{1}{2}e^{-3t}$,
(b) $\frac{1}{s+1} - \frac{1}{s+2} - \frac{1}{(s+2)^2}$, $y(t) = e^{-t} - e^{-2t} - te^{-2t}$,
(c) $\frac{1}{5}\left\{\frac{1}{s+1} + \frac{1}{2}\frac{2}{s^2+4} - \frac{s}{s^2+4}\right\}$, $y(t) = \frac{1}{5}\left\{e^{-t} + \frac{1}{2}\sin 2t - \cos 2t\right\}$.

5. (a) $y(t) = \frac{7}{4}e^{-t} + \frac{1}{2}te^{-t} - \frac{3}{4}e^{-3t}$.
(b) $y(t) = -\frac{1}{2}\sin t + \frac{1}{4}e^{-t} + \frac{3}{4}e^{t}$.
(c) $y(t) = t^3 - 6t + 7\sin t + \cos t$.

6. $u(t) = \frac{b}{a}\sin at$, $v(t) = \frac{b}{a}\{1 - \cos at\}$.

7. $x(t) = \frac{E}{B}\left\{\frac{1-\cos at}{a}\right\}$, $y(t) = \frac{E}{B}\left\{t - \frac{\sin at}{a}\right\}$, where $a = \frac{eB}{m}$.

8. (a) (i) $\frac{1}{s}$, 1, (ii) $\frac{1}{s}$, 0. (b) $2 - e^{-t}$, $2 - 2e^{-t}$.

9. $\frac{1}{s^2(s-1)}$.

Suitable past Tripos questions: Maths IA Q8 2001-4, Q6 2005, Q9 2006, Q7 2007, Q6 2008, Q7 2009, Q6 2010, Q9 2011-12, Q6 2013, Q10 2014, Q6 2015-16, Q7 2017, Q6 2018.

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