Engineering Tripos Part 1A

Solutions for Mathematics Paper 10

1. (i) $f(x,y) = x^2y^5$

$\partial f/\partial x = 2xy^5$; $\partial^2 f/\partial x^2 = \partial/\partial x\,[\partial f/\partial x] = 2y^3$

$\partial f/\partial y = 5x^2y^4$; $\partial^2 f/\partial y^2 = \partial/\partial y\,[\partial f/\partial y] = 20x^2y^4$

$\partial^2 f/\partial y\partial x = \partial/\partial y\,[\partial f/\partial x] = 10xy^4$

$\partial^2 f/\partial x\partial y = \partial/\partial x\,[\partial f/\partial y] = 10xy^4$

(ii) $f(x,y) = x \sin y$

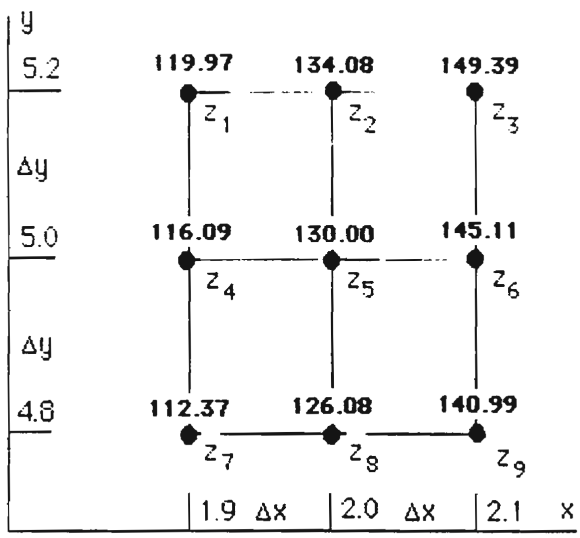
$\partial f/\partial x = \sin y$; $\partial^2 f/\partial x^2 = \partial/\partial x\,[\partial f/\partial x] = 0$

$\partial f/\partial y = x \cos y$; $\partial^2 f/\partial y^2 = \partial/\partial y\,[\partial f/\partial y] = -x\sin y$

$\partial^2 f/\partial y\partial x = \partial/\partial y\,[\partial f/\partial x] = \cos y$

$\partial^2 f/\partial x\partial y = \partial/\partial x\,[\partial f/\partial y] = \cos y$

2.



Approximate values are as follows:

$\partial f/\partial x$ at the centre point $= \{ [(z_6 - z_5)/\Delta x] + [(z_5 - z_4)/\Delta x] \}/2$

$$= [z_6 - z_4]/2\Delta x \qquad (1)$$

$\partial f/\partial y$ at the centre point $= \{ [(z_5 - z_8)/\Delta y] + [(z_2 - z_5)/\Delta y] \}/2$

$$= [z_2 - z_8]/2\Delta y \qquad (2)$$

$\partial^2 f/\partial x^2$ at the centre point $= \{ [(z_6 - z_5)/\Delta x] - [(z_5 - z_4)/\Delta x] \}/\Delta x$

$$= [z_4 + z_6 - 2z_5]/(\Delta x)^2 \qquad (3)$$

$\partial^2 f/\partial y^2$ at the centre point $= \{ [(z_2 - z_5)/\Delta y] - [(z_5 - z_8)/\Delta y] \}/\Delta y$

$$= [z_8 + z_2 - 2z_5]/(\Delta y)^2 \qquad (4)$$

$\partial^2 f/\partial y\partial x$ at the centre point $= \{ [(z_3 - z_1)/2\Delta x] - [(z_9 - z_7)/2\Delta x] \}/2\Delta y$

$$= [z_3 + z_7 - z_1 - z_9]/4\Delta x\Delta y \qquad (5)$$

$\partial^2 f/\partial x\partial y$ at the centre point $= \{ [(z_3 - z_9)/2\Delta y] - [(z_1 - z_7)/2\Delta y] \}/2\Delta x$

$$= [z_3 + z_7 - z_1 - z_9]/4\Delta x\Delta y \qquad (6)$$

Substituting values for z:

$\partial^2 f/\partial x^2$ at the centre point $= (116.09 + 145.11 - 260)/0.1^2 = 120$

$\partial^2 f/\partial y^2$ at the centre point $= (126.08 + 134.08 - 260)/0.2^2 = 4$

$\partial^2 f/\partial y\partial x$ at the centre point $= (149.39 + 112.37 - 119.97 - 140.99)/4(0.1)(0.2) = 10$

We see from (5) and (6) that these approximations for $\frac{\partial^2 f}{\partial x\,\partial y}$ and $\frac{\partial^2 f}{\partial y\,\partial x}$ are always equal. (The mixed 2^{nd} partials themselves are equal providing the limits as $\Delta x \to 0$ and $\Delta y \to 0$ are the same, no matter in which order they are taken. This can be shown to hold if the 2^{nd} mixed partials are continuous.)

[The data was constructed from $z = 10x^3 + xy^2$].

3. For what value of n is $\theta = t^n \exp\{-r^2/4t\}$ a solution of the equation

$$\partial/\partial r\,[r^2\partial\theta/\partial r] = r^2\,\partial\theta/\partial t?$$

LHS

$\partial\theta/\partial r = t^n \exp\{-r^2/4t\}\,[-2r/4t]$

$r^2\partial\theta/\partial r = -0.5t^{n-1}\,r^3\exp\{-r^2/4t\}$

$\partial/\partial r\,[r^2\partial\theta/\partial r] = -0.5t^{n-1}\,r^3\exp\{-r^2/4t\}\,[-2r/4t] - 1.5t^{n-1}r^2\exp\{-r^2/4t\}$

$= 0.25t^{n-2}\,r^4\exp\{-r^2/4t\} - 1.5t^{n-1}r^2\exp\{-r^2/4t\} \qquad (1)$

RHS

$\partial\theta/\partial t = t^n\exp\{-r^2/4t\}[r^2/4t^2] + nt^{n-1}\exp\{-r^2/4t\}$

$r^2\,\partial\theta/\partial t = 0.25t^{n-2}\,r^4\exp\{-r^2/4t\} + nt^{n-1}\,r^2\exp\{-r^2/4t\} \qquad (2)$

Comparing (1) and (2), $n = -1.5$

4.

(a) $w = xyz = (\cos\theta\sin\phi)(\sin\theta\sin\phi)(\cos\phi) = 0.5\sin 2\theta\sin^2\phi\cos\phi$

so $\partial w/\partial\theta|_\phi = \cos 2\theta\sin^2\phi\cos\phi$

(b) $\partial w/\partial\theta|_\phi = yz(-\sin\theta\sin\phi) + xz(\cos\theta\sin\phi) + xy(0)$

$= -\sin^2\theta\sin^2\phi\cos\phi + \cos^2\theta\sin^2\phi\cos\phi = \cos 2\theta\sin^2\phi\cos\phi$

5. An expression of the form

$$P(x,y)\,dx + Q(x,y)\,dy \qquad (1)$$

is a perfect differential if $P = \frac{\partial f}{\partial x}$ and $Q = \frac{\partial f}{\partial y}$ for some function $f(x,y)$ so that

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = P\,dx + Q\,dy,$$

which means that the integral along a path in the x-y plane depends only on the end points. A necessary and sufficient condition for (1) to be a perfect differential is that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

Given $dh = T\,ds + v\,dp$, note first that ds and dp are the two small quantities on the right hand side so T and v are implied functions of s and p. Let us write

$$dh = T(s,p)\,ds + v(s,p)\,dp.$$

Then we have $\left(\frac{\partial h}{\partial s}\right)_p = T$ and $\left(\frac{\partial h}{\partial p}\right)_s = v$. Equating the 2nd mixed partials gives

$$\left(\frac{\partial T}{\partial p}\right)_s = \frac{\partial^2 h}{\partial p\,\partial s} = \frac{\partial^2 h}{\partial s\,\partial p} = \left(\frac{\partial v}{\partial s}\right)_p .$$

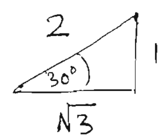
From $g = h - Ts$ we have $dg = dh - T\,ds - s\,dT = v\,dp - s\,dT$. As above we obtain:

$$\left(\frac{\partial v}{\partial T}\right)_p = -\left(\frac{\partial s}{\partial p}\right)_T .$$

6. $\nabla T = (4x - 3y)\underline{i} - 3x\underline{j} \Rightarrow \nabla T(1,1) = \underline{i} - 3\underline{j}$.

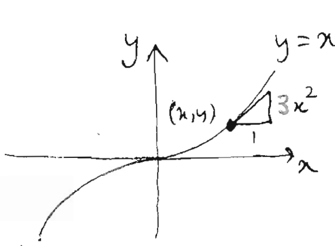
For a unit vector $\underline{b}$ the directional derivative $D_{\underline{b}}T = \underline{b}.\nabla T$

(a) $\underline{b} = (\sqrt{3}\,\underline{i} + \underline{j})/2$



$\Rightarrow$ temp. grad. $= D_{\underline{b}}T = (\sqrt{3} - 3)/2$

(b) $\underline{b} = (\underline{i} + 3\underline{j})/\sqrt{10}$



$\Rightarrow$ temp. grad. $= D_{\underline{b}} T = (1-9)/\sqrt{10} = -\dfrac{8}{\sqrt{10}}$

(c) largest gradient is when $\underline{b} = \nabla T / |\nabla T| = (\underline{i} - 3\underline{j})/\sqrt{10}$ and $D_{\underline{b}} T = |\nabla T| = \sqrt{10}$.

7. $\nabla w = \dfrac{\partial w}{\partial x}\underline{i} + \dfrac{\partial w}{\partial y}\underline{j} + \dfrac{\partial w}{\partial z}\underline{k} = (2z^2 - 3y - 4)\underline{i} - 3x\underline{j} + 4xz\underline{k}$

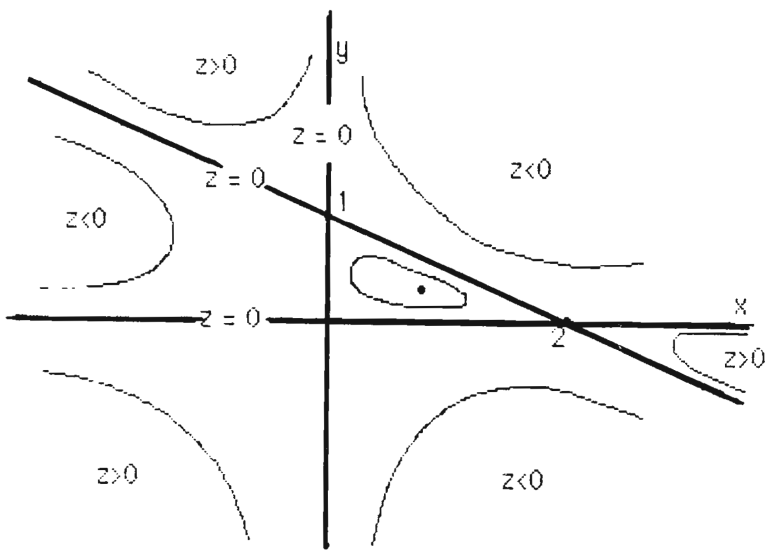
∇w is normal to a level surface $w(x,y,z) = $ const. at (x,y,z). Thus, the tangent plane is given by:

$$\nabla w \cdot (\underline{r} - \underline{r}_0) = 0$$

where $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$ and $\underline{r}_0$ is any vector in the tangent plane. $\nabla w(1,-1,2) = 7\underline{i} - 3\underline{j} + 8\underline{k}$ and $\underline{r}_0 = (1,-1,2)$ gives

$$7x - 3y + 8z = 26.$$

8.



The thick black lines are the $z=0$ contours corresponding to $x=0$, $y=0$ and $y = 1 - x/2$. By considering the sign of each factor in $z = xy(2-x-2y)$ in each region gives approximate shape of contours.

$z = xy(2 - x - 2y) = 2xy - x^2y - 2xy^2$ (1)

Differentiating partially:

$\partial z/\partial x = 2y(1 - x - y)$ (2)

$\partial z/\partial y = x(2 - x - 4y)$ (3)

Stationary points occur when:

$y(1 - x - y) = 0$ (4)

$x(2 - x - 4y) = 0$ (5)

Solving these simultaneous equations

From (4), $y = 0$ or $x + y = 1$

From (5), $x = 0$ or $x + 4y = 2$

There are four stationary points: (0,0); (2,0); (0,1); (2/3,1/3)

From contours we see that 1st three are saddles and the 4th is a maximum. Alternatively, proceeding analytically:

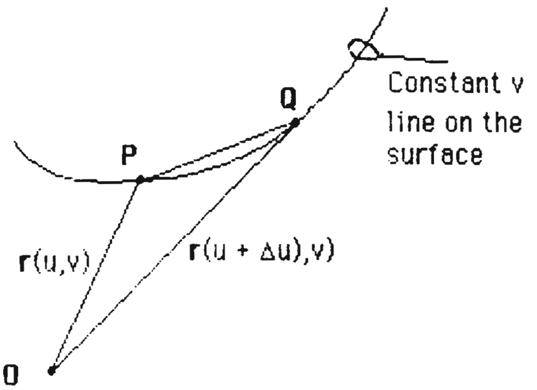
$$\frac{\partial^2 z}{\partial x^2} = -2y, \quad \frac{\partial^2 z}{\partial y^2} = -4x, \quad \frac{\partial^2 z}{\partial x \partial y} = 2 - 2x - 4y.$$

Define $\Delta = \frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$.

From p.5 of maths data book we deduce:

Point	$\partial^2 z/\partial x^2$	$\partial^2 z/\partial y^2$	$\partial^2 z/\partial x\partial y$	Δ	Deduction
(0,0)	0	0	2	-4	Saddle Point
(2,0)	0	-8	-2	-4	Saddle Point
(0,1)	-2	0	-2	-4	Saddle Point
(2/3,1/3)	-2/3	-8/3	-2/3	+4/3	Maximum

9. The figure shows a constant v line on the surface:



Then the vector **PQ** = **r**(u + Δu,v) - **r**(u,v)
The vector **PQ**/Δu = [**r**(u + Δu,v) - **r**(u,v)]/Δu is in the same direction as PQ. In the limit as $\Delta u \to 0$, the vector **PQ**/Δu becomes a (non unit length) tangent to the curve at point P, whereas the RHS becomes ∂**r**/∂u.

A similar argument for a constant u line shows that ∂**r**/∂v is a tangent to the constant u line on the surface. Thus ∂**r**/∂u and ∂**r**/∂v are tangents to the surface, but note carefully that they are **not unit** tangents.

For $\mathbf{r} = (u^2 + v)\mathbf{i} + 2uv\mathbf{j} + (u + v^2)\mathbf{k}$

$$\partial\mathbf{r}/\partial u = 2u\,\mathbf{i} + 2v\mathbf{j} + 1\,\mathbf{k} \qquad (1)$$

$$\partial\mathbf{r}/\partial v = 1\,\mathbf{i} + 2u\mathbf{j} + 2v\mathbf{k} \qquad (2)$$

Since these vectors are tangents to the surface at the point **r**(u,v), their cross product is in the direction of the normal to the surface at this point.
Then $\partial\mathbf{r}/\partial u \times \partial\mathbf{r}/\partial v = (4v^2 - 2u)\mathbf{i} + (1 - 4uv\mathbf{j} + (4u^2 - 2v)\mathbf{k}$ (3)
For u = -1 and v = -1, $\partial\mathbf{r}/\partial u \times \partial\mathbf{r}/\partial v = 6\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$. The length of this vector is 9, so the unit normal is $\mathbf{N} = (2/3)\mathbf{i} - (1/3)\mathbf{j} + (2/3)\mathbf{k}$.

The constant u and constant v lines on the surface intersect orthogonally when the tangent vectors to these curves are orthogonal. For this to happen the dot product of the tangents to these two curves must be zero. From (1) and (2):

$$\partial\mathbf{r}/\partial u \,.\, \partial\mathbf{r}/\partial v = 2u + 4uv + 2v \qquad (4)$$

For u = v = -1, $\partial\mathbf{r}/\partial u \,.\, \partial\mathbf{r}/\partial v = 0$, so the curves intersect orthogonally at this particular point. But not at every intersection!

MCS / PCW
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