Engineering **FIRST YEAR**

Part IA Paper 4: Mathematics

Examples paper 7

(Elementary exercises are marked †, problems of Tripos standard *)

Steps and impulses

1† What familiar quantities do the following functions represent?

(i) A time-varying force $f(t) = F\delta(t)$.

(ii) A line distribution of loading $w(x) = P\delta(x-a)$ per unit length.

2† Explain briefly the meaning of $\delta(t)$ and $\delta(t-a)$. If $f(t)$ is a continuous function, explain why $\int \delta(t-a) f(t) dt = f(a)$, provided $t = a$ lies in the range of integration.

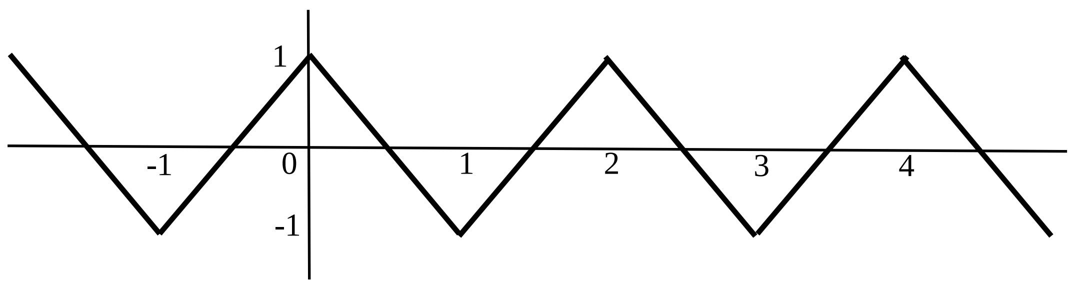
Evaluate

(i) $\int_0^5 \delta(t-3) dt$; (ii) $\int_{-1}^1 \delta(x) \sin x \, dx$;

(iii) $\int_0^1 \delta(x - \pi/4) \exp[\cos(1/x)] \, dx$; (iv) $\int_0^1 \delta(x + \pi/4) \cos x \, dx$.

3† Calculate and sketch the first and second derivatives of the following functions, paying particular attention to points of discontinuity.

(i) the periodic triangular wave, a portion of which is plotted below:



(ii) $f(t) = \begin{cases} 0 & t < 0 \\ \sin \Omega t & t \geq 0 \end{cases}$

4 Consider the following differential equations:

(i) $\dfrac{dy}{dt}+3y=f(t)$; (ii) $\dfrac{d^2y}{dt^2}+4y=f(t)$

when the input is a unit step function, i.e. $f(t)=\begin{cases}0 & t<0\\ 1 & t\geq 0\end{cases}$, and the output $y(t)$ is zero for $t<0$. Explain why $y(t)$ is continuous at $t=0$ in (i), and both $y(t)$ and $\dot{y}(t)$ are continuous at $t=0$ in (ii). Calculate the solution $y(t)$ in each case. Hence calculate the solution of each equation with input $f(t)=\delta(t)$ when the output $y(t)$ is zero for $t<0$.

5* For a certain system the output $y(t)$ to an input $f(t)$ satisfies the differential equation

$$\frac{dy}{dt}+\alpha y=f(t).$$

(i) Find the step response of the system, by solving the equation with $f(t)=\begin{cases}0 & t<0\\ 1 & t\geq 0\end{cases}$ and $y=0$ for $t\leq 0$.

(ii) Hence write down the output of the system to input $f(t)=\begin{cases}0 & t<0\\ 1/T & 0\leq t\leq T\\ 0 & t>T\end{cases}$ with $y=0$ for $t\leq 0$.

Verify that this becomes the impulse response in the limit $T\to 0$.

Convolution

6 For the system whose input-output characteristics are described by the equation studied in question 4(i), find by convolution the output $y(t)$ in response to input

(i) $f(t)=\begin{cases}0 & t<0\\ e^{-2t} & t\geq 0\end{cases}$;

(ii) $f(t)=\begin{cases}0 & t<0\\ \sin t & t\geq 0\end{cases}$.

(Assume that the output of the system is zero for $t<0$.)

In both cases, verify by substitution that the result you obtain really does satisfy the differential equation of question 4(i).

7* For a certain linear system, the output $y(t)$ for an input $f(t)$ satisfies the differential equation

$$\frac{1}{12}\left(2\frac{d^2y}{dt^2}+11\frac{dy}{dt}+12y\right)=f(t).$$

(i) Find the step response of the system, by solving the equation with $f(t)=\begin{cases}0 & t<0\\ 1 & t\geq 0\end{cases}$

and $y=\dfrac{dy}{dt}=0$ for $t=0$.

(ii) Find the impulse response of the system.

(iii) Hence find by convolution the system response to input $f(t)=\begin{cases}0 & t<0\\ e^{-t} & t\geq 0\end{cases}$

assuming $y=\dfrac{dy}{dt}=0$ for $t=0$.

(iv) Suppose the differential equation describes a car's suspension, with the input $f(t)$ representing the height of the road and the output $y(t)$ representing the height of the chassis. The car is to be driven over a rough track with height profile given in the `f.dat`. The file `f.dat` contains 10000 numbers, which give the height profile $f(t)$. Each number is the height of the track at 50 ms intervals, as the car drives over it at constant speed. Use Python to plot (a) the road height as a function of time, (b) the suspension's step response, (c) the suspension's impulse response and (d) the height of the chassis as the car drives over the track. You might find it illuminating to plot $f(t)$ and $y(t)$ on the same axes, zooming in on, say, the first 25 s.

Access the following URL through the online version of this paper in the *IA Examples Paper repository.*

http://nbviewer.jupyter.org/github/CambridgeEngineering/Part-IA-ExamplesPapers-Python/blob/master/paper4/IA%20Paper%204%20Mathematics%2007.ipynb

8* An elastic band with tension T is fixed at points $x=0, L$. A transverse force $f(x)$ per unit length is applied, and small transverse displacements of the band $y(x)$ are produced.

(i) A point force $f(x)=\delta(x-a)$ is applied at $x=a$ on the band. Sketch the form of the displacement of the band. Using a force-balance argument (and assuming that the displacement is small), show that the displacement is approximately

$$g(x,a)=\begin{cases}\dfrac{(L-a)x}{LT} & x\leq a\\[2ex] \dfrac{a(L-x)}{LT} & x\geq a\,.\end{cases}$$

(ii) Explain briefly why the displacement in response to a general applied force is

$$y(x)=\int_0^L g(x,a)\,f(a)\,da\,.$$

(iii) Hence find the displacement of the band when a uniform force is applied over half of the length so that $f(x)=\begin{cases}F & 0\leq x\leq L/2\\ 0 & L/2<x\leq L\,;\end{cases}$

Past Questions

2018Q7, 2017 Q8, 2016 Q7, 2015 Q7, 2014 Q6, 2013 Q7, 2012 Q6, 2011 Q6

Answers

1 (i) An impulse of magnitude F at $t = 0$.

(ii) A point load of magnitude P at $x = a$.

2 (i) 1; (ii) 0; (iii) $\exp[\cos(4/\pi)] = 1.3407$; (iv) 0.

3 (i) $\frac{df}{dt}$ is a square wave, with values $-2,\quad 0 \le t < 1$ and $2,\quad 1 \le t < 2$ then periodic.

$$\frac{d^2 f}{dt^2} = \cdots + 4\delta(t+1) - 4\delta(t) + 4\delta(t-1) - 4\delta(t-2)\cdots$$

(ii) $\frac{df}{dt} = \begin{cases} 0 & t < 0 \\ \Omega\cos\Omega t & t > 0 \end{cases}$ $\qquad \frac{d^2 f}{dt^2} = \Omega\delta(t) + \begin{cases} 0 & t < 0 \\ -\Omega^2 \sin\Omega t & t \ge 0 \end{cases}$

4 (i) Step response $y = \begin{cases} 0 & t \le 0 \\ \dfrac{1 - e^{-3t}}{3} & t \ge 0 \end{cases}$; impulse response $y = \begin{cases} 0 & t < 0 \\ e^{-3t} & t \ge 0 \end{cases}$

(ii) Step response $y = \begin{cases} 0 & t \le 0 \\ \dfrac{1 - \cos 2t}{4} & t \ge 0 \end{cases}$; impulse response $y = \begin{cases} 0 & t < 0 \\ \dfrac{\sin 2t}{2} & t \ge 0 \end{cases}$

5 (i) $y = \begin{cases} 0 & t \le 0 \\ \dfrac{1 - e^{-\alpha t}}{\alpha} & t \ge 0 \end{cases}$; (ii) $y = \begin{cases} 0 & t < 0 \\ \dfrac{1 - e^{-\alpha t}}{\alpha T} & 0 \le t \le T \\ \dfrac{\left(e^{\alpha T} - 1\right) e^{-\alpha t}}{\alpha T} & t > T \end{cases}$

6 (i) $e^{-2t} - e^{-3t}$; (ii) $\left(3\sin t - \cos t + e^{-3t}\right)/10$

7 (i) $y = 3e^{-4t}/5 - 8e^{-3t/2}/3 + 1$

(ii) $y = 12\left(e^{-3t/2} - e^{-4t}\right)/5$

(iii) $4\left(e^{-4t} - 6e^{-3t/2} + 5e^{-t}\right)/5$

8 (iii) $y = \begin{cases} \dfrac{Fx(3L - 4x)}{8T} & 0 \le x \le L/2 \\ \dfrac{FL(L - x)}{8T} & L/2 \le x \le L \end{cases}$

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