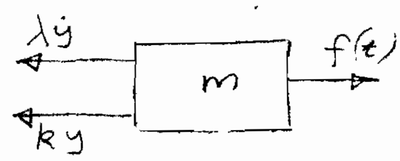
Mechanical Vibrations

Examples Paper 2 Solutions



sum of forces on the moving mass = mass × acceleration

$\therefore\ f(t) - \lambda\dot{y} - ky = m\ddot{y} \qquad \therefore\ \underline{m\ddot{y} + \lambda\dot{y} + ky = f(t)}$

Dividing by k to give $\quad \dfrac{\ddot{y}}{\omega_n^2} + \dfrac{2\zeta}{\omega_n}\dot{y} + y = x$

where $\underline{\omega_n^2 = \dfrac{k}{m}}$, $\underline{\zeta = \dfrac{\lambda}{2\sqrt{km}}}$ and $\underline{x = \dfrac{f(t)}{k}}$ (metres)

For $\omega = 0$, $Y = X = \underline{\dfrac{F}{k}}$ "static" stiffness

$\underline{\phi = 0}$ low frequency response is in phase

for $\omega \to \infty$, $\underline{Y \to 0}$

$\underline{\phi \to 180^\circ}$

These answers hold for any amount of damping

Q 2 Substitute $k = 400$ N/m, $m = 100$ kg and $\lambda = 80$ Ns/m in the above to give $\underline{\omega_n = 2 \text{ rad/s}}$
and $\underline{\zeta = 0.2}$

For $\omega = 1.8$ rad/s, $\dfrac{\omega}{\omega_n} = 0.9 \quad \therefore \dfrac{Y}{X} \approx 2.45$ (from the graph in Data Book, Case (a))

Input $X = \dfrac{F}{k} = \dfrac{10}{400} = 0.025$ m $\quad \therefore\ \underline{Y \simeq 0.061 \text{ m}}$

Phase lag $\underline{\phi \approx 62^\circ}$ (read from graph)

Increase λ to 400 Ns/m $\quad \therefore\ \underline{\zeta = 1.0} \quad \therefore \dfrac{Y}{X} = 0.55$

$\therefore\ \underline{Y =}\ 0.55 \times 0.025 = \underline{0.014 \text{ m}}$

Q 3 Compare given equation with Case (a) equation in Data Book with $i_r = y$ and $i_i = x$

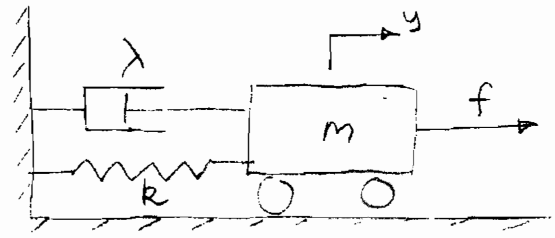
$I_r = Y$ and $I_i = X$

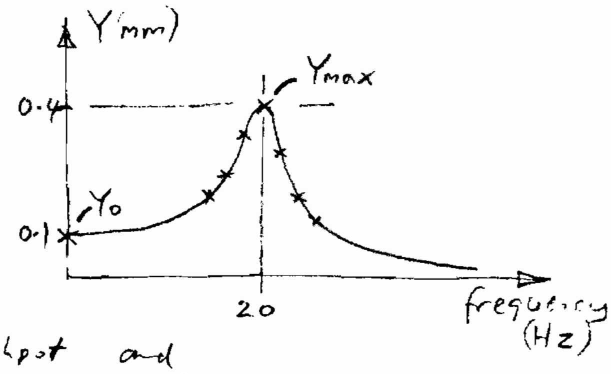
and with $\underline{\omega_n = 4 \text{ rad/s}}$ and $\underline{\zeta = 0.5}$

Case (a) graph of $\dfrac{Y}{X}$ for $\zeta = 0.5$ has a peak value of 1.16 at the damped resonance $\dfrac{\omega}{\omega_n} = 0.71$

The maximum error in the range $0 < \omega < \omega_n$ is therefore $\left|\dfrac{Y}{X} - 1\right| = \underline{16\ \%}$

Q4





Low frequency response : the dashpot and inertia forces are negligible $\therefore$ $F = kY_0$

$\therefore$ $\underline{k} = \frac{F}{Y_0} = \frac{5.8}{0.1 \times 10^{-3}} = \underline{\underline{58\ kN/m}}$

$\frac{Y_{max}}{X} \approx \frac{1}{2\zeta}$ (Q factor) for $c \ll 1$

$\frac{Y_0}{X} = 1$ $\quad\therefore$ $Q = \frac{1}{2\zeta} = \frac{Y_{max}}{Y_0} = 4$ $\quad\therefore$ $\underline{\underline{\zeta = 0.125}}$

$\zeta \ll 1$ $\therefore$ ok

Maximum amplitude occurs at the damped resonance

$\underline{\omega_{max}} = 20 \times 2\pi = \underline{126\ rad/s}$

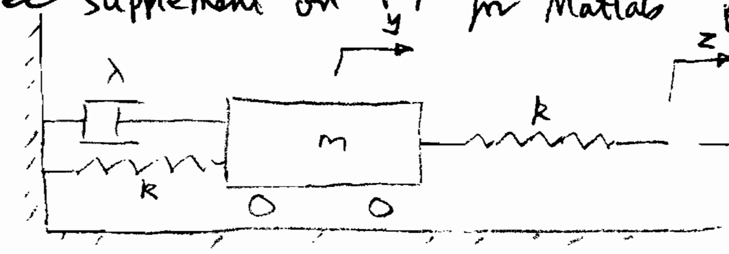
Undamped resonance $\underline{\omega_n} = \omega_{max} / \sqrt{1 - 2c^2} = \underline{\underline{128\ rad/s}}$

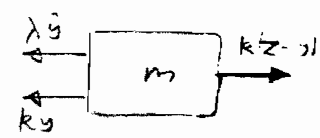
Then $\omega_n = \sqrt{\frac{k}{m}}$ $\quad\therefore$ $\underline{m} = k/\omega_n^2 = \frac{58000}{128^2} = \underline{\underline{3.5\ kg}}$

and $\zeta = \frac{\lambda}{2\sqrt{km}}$ $\quad\therefore$ $\underline{\lambda} = 2\zeta\sqrt{km} = \underline{\underline{113\ Ns/m}}$

See Supplement on P7 for Matlab plot

Q5





Consider the sum of forces acting on the mass

$$m\ddot{y} = k(z - y) - ky - \lambda\dot{y}$$

$$\therefore \quad m\ddot{y} + \lambda\dot{y} + 2ky = kz$$

$$\therefore \quad \underline{\underline{\frac{\ddot{y}}{\omega_n^2} + \frac{2\zeta}{\omega_n}\dot{y} + y = x}}$$

with $\underline{\underline{\omega_n = \sqrt{\frac{2k}{m}}}}$

and $\underline{\underline{\zeta = \frac{\lambda}{2\sqrt{2km}}}}$

and $\underline{\underline{x = \frac{z}{2}}}$

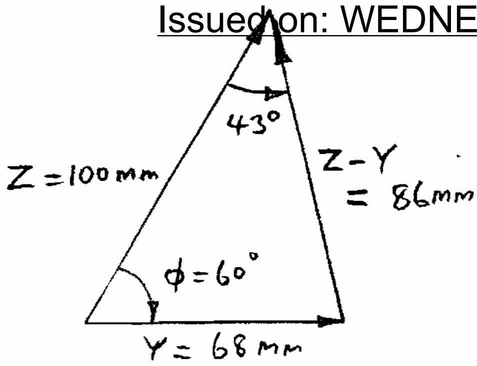
With $k = 0.5\ N/m$, $m = 1\ kg$, $\lambda = 0.8\ Ns/m$ $\Big\}$ $\omega_n = 1\ rad/s$, $\zeta = 0.4$

$\omega = 0.8\ rad/s$ $\quad\therefore$ $\frac{\omega}{\omega_n} = 0.8$ and using Case (a) curves from the Data Book $\frac{Y}{X} = 1.36$ and $\underline{\underline{\phi = 60^\circ}}$

$X = \frac{Z}{2} = 50\ mm$ $\quad\therefore$ $\underline{\underline{Y = 68\ mm}}$

The exciting force $f = k(z - y)$ ie the spring extension

$\therefore$ $F = k(Z - Y)$ (using complex algebra)



but $Z-Y$ can be measured directly from a phasor diagram

$$\therefore\ |F| = k\,|Z-Y| = 0.5 \times 0.086 = \underline{\underline{0.043\ N}}$$

and $\underline{\text{the phase lead is } 43°}$ measured

<u>Alternative method using Complex Arithmetic</u>

put $y = Y e^{i\omega t}$ & $z = Z e^{i\omega t}$

$$\therefore\ \left(-\left(\frac{\omega}{\omega_n}\right)^2 + i\,2\zeta\left(\frac{\omega}{\omega_n}\right) + 1\right) Y = \frac{Z}{2}$$

$$\therefore\ Z - Y = Z\left(1 - \frac{1}{2\left(1-\left(\frac{\omega}{\omega_n}\right)^2 + i\,2\zeta\left(\frac{\omega}{\omega_n}\right)\right)}\right)$$

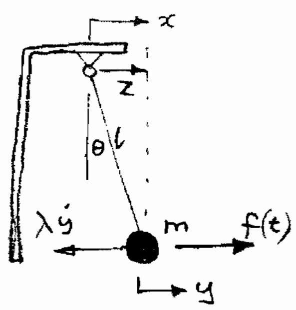
$$\therefore\ F = k(Z-Y) = \frac{kZ}{2}\left(\frac{1-2\left(\frac{\omega}{\omega_n}\right)^2 + i\,4\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2 + i\,2\zeta\frac{\omega}{\omega_n}}\right)$$

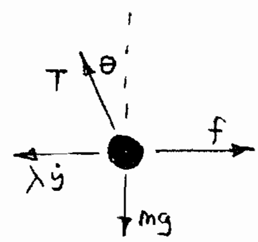
$$\therefore\ |F| = \frac{0.5 \times 0.1}{2}\left|\frac{-0.28 + 1.28i}{0.36 + 0.64i}\right| = \underline{\underline{0.0446}}\ N$$

and $\angle F = \angle\,(\uparrow\text{this thing}\uparrow) = \underline{\underline{41.7°}}$ (lead)

These answers are "exact" but more tiresome to obtain

Q6





The bob moves in circular motion and the centripetal acceleration is $l\dot{\theta}^2$. For small amplitude oscillations, θ and $\dot{\theta}$ are small $\therefore\ \dot{\theta}^2$ is second order so ignore it.

Resolve vertically : $T\cos\theta = mg$

but $\cos\theta \approx 1$ for small θ

$$\therefore\ T \approx mg$$

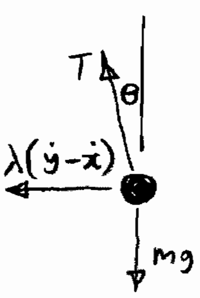
Now resolve horizontally : $f - \lambda\dot{y} - T\sin\theta = m\ddot{y}$

and $\sin\theta = \frac{y}{l}$

$$\therefore\ \underline{\underline{m\ddot{y} + \lambda\dot{y} + \frac{mg}{l}y = f(t)}} \quad \text{for small } \theta$$

Divide by $\frac{mg}{l}$ $\therefore\ \frac{\ddot{y}}{\omega_n^2} + \frac{2\zeta\dot{y}}{\omega_n}$ — Case (a)

with $\omega_n = \sqrt{\frac{g}{l}}$, $\zeta = \frac{\lambda}{2m}\sqrt{\frac{l}{g}}$ and $x = \frac{fl}{mg}$



For motion of the frame, redefine $\sin\theta = \frac{y-x}{l}$

$$m\ddot{y} + \lambda(\dot{y}-\dot{x}) + \frac{mg}{l}(y-x) = 0 \qquad (\text{for } f = 0)$$

$$\therefore\ \underline{\underline{\ddot{y} + \frac{\lambda}{m}\dot{y} + \frac{g}{l}y = \frac{g}{l}x}}$$

Q6 cont'd Divide by $\frac{g}{l}$ $\therefore$ $\underline{\underline{\frac{\ddot{y}}{\omega_n^2} + \frac{2\zeta\dot{y}}{\omega_n} + y = \frac{2\zeta\dot{x}}{\omega_n} + x}}$ case(c)

with ω_n & c as before and x is the anchor displacement

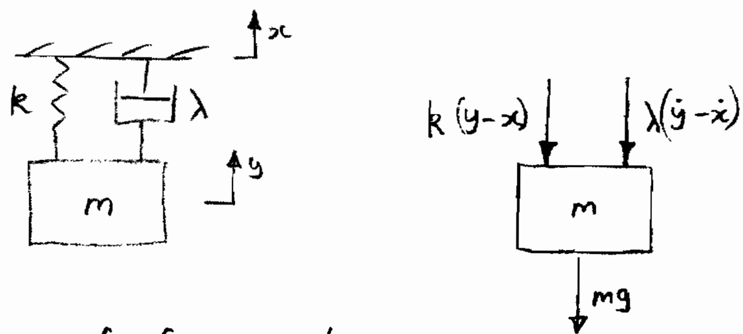
Now substitute $z = y - x$ $\therefore$ $\ddot{z} = \ddot{y} - \ddot{x}$ & $\dot{z} = \dot{y} - \dot{x}$

$$\therefore \frac{\ddot{z} + \ddot{x}}{\omega_n^2} + \frac{2\zeta}{\omega_n}(\dot{z} + \dot{x}) + z + x = \frac{2\zeta\dot{x}}{\omega_n} + x$$

$$\therefore \underline{\underline{\frac{\ddot{z}}{\omega_n^2} + \frac{2\zeta\dot{z}}{\omega_n} + z = -\frac{\ddot{x}}{\omega_n^2}}} \quad \text{— Case (b)}$$

and multiply by $\omega_n^2 = \frac{g}{l}$ to give $\underline{\underline{\ddot{z} + \frac{\lambda}{m}\dot{z} + \frac{g}{l}z = -\ddot{x}}}$

Q7



Sum of forces acting on mass:

$$-k(y-x) - \lambda(\dot{y} - \dot{x}) - mg = m\ddot{y} \quad \text{(upwards)}$$

$$\therefore m\ddot{y} + \lambda\dot{y} + k\left(y + \frac{mg}{k}\right) = \lambda\dot{x} + kx$$

Static deflection under gravity $k y_0 = -mg$ $\therefore$ $y_0 = \frac{-mg}{k}$

Replacing y by $y - y_0$ then gives

$$\underline{m\ddot{y} + \lambda\dot{y} + ky = \lambda\dot{x} + kx}$$

Substitute $z = y - x$, $\dot{z} = \dot{y} - \dot{x}$, $\ddot{z} = \ddot{y} - \ddot{x}$

$$\therefore m(\ddot{z} + \ddot{x}) + \lambda(\dot{z} + \dot{x}) + k(z + x) = \lambda\dot{x} + kx$$

$$\therefore \underline{m\ddot{z} + \lambda\dot{z} + kz = -m\ddot{x}}$$

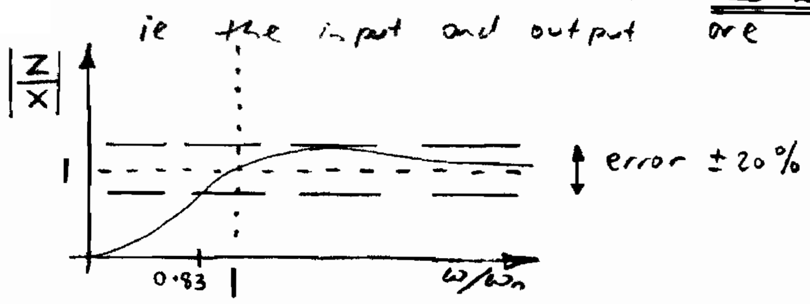
Divide by k $\therefore$ $\frac{\ddot{z}}{\omega_n^2} + \frac{2\zeta}{\omega_n}\dot{z} + z = -\frac{\ddot{x}}{\omega_n^2}$ (Case(b))

with $\omega_n = \sqrt{\frac{k}{m}}$ and $\zeta = \frac{\lambda}{2\sqrt{km}}$

At high frequencies, the mass acceleration is large ($\ddot{x} = -\omega^2 X \cos\omega t$ and ω is large) hence inertia forces dominate and the mass stays still ie $y \approx 0$

Then $z = y - x$ $\therefore$ $\underline{\underline{z \approx -x}}$

ie the input and output are equal and opposite.



From the graph, $\zeta = 0.47$ gives a value of $\left|\frac{Z}{X}\right|_{max} = 1.2$

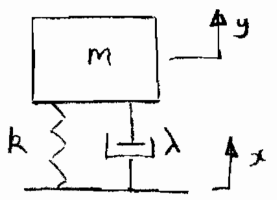
Q7 cont'd $\quad \left|\frac{Z}{X}\right| = 0.8$ when $\frac{\omega}{\omega_n} = 0.83$

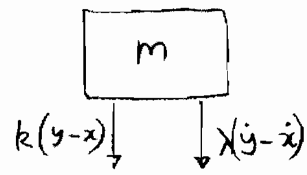
$\therefore$ the device is functional from $\underline{\underline{1.32\text{ Hz up to } \infty}}$ to within a 20% error

The damped resonance occurs at $\omega = \frac{\omega_n}{\sqrt{1-2c^2}} = 13.4$ rad/s $\equiv \underline{\underline{2.13\text{ Hz}}}$

Note that the device operates accurately well below resonance because there is sufficient damping present

Q8





Sum of forces

$m\ddot{y} = -k(y-x) - \lambda(\dot{y}-\dot{x})$

$\therefore \underline{m\ddot{y} + \lambda\dot{y} + ky = \lambda\dot{x} + kx}$

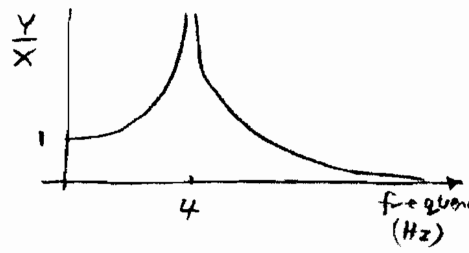
Divide by k to give $\frac{\ddot{y}}{\omega_n^2} + \frac{2\zeta}{\omega_n}\dot{y} + y = \frac{2\zeta}{\omega_n}\dot{x} + x$ (Case (c))

with $\omega_n = \sqrt{k/m}$

and $\zeta = \lambda / 2\sqrt{km}$

$$\left|\frac{Y}{X}\right| = \left|\frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right| \quad (\text{for } \zeta = 0)$$

$$= \frac{5\,\mu m}{40\,\mu m} \quad \text{for acceptable vibration}$$



$\therefore \frac{\omega}{\omega_n} = \sqrt{8+1} = 3$

Floor vibration at 12 Hz $\therefore$ $\underline{\underline{\text{natural frequency}}} = \frac{12}{3} = \underline{\underline{4\text{ Hz}}}$

$\omega_n = \sqrt{\frac{k}{m}} \quad \therefore \quad \underline{\underline{k}} = m\omega_n^2 = 100 \times (2\pi \times 4)^2 = \underline{\underline{63\text{ KN/m}}}$

Acceptable amplification at resonance $\left|\frac{Y}{X}\right| = \frac{200\,\mu m}{40\,\mu m} = 5$

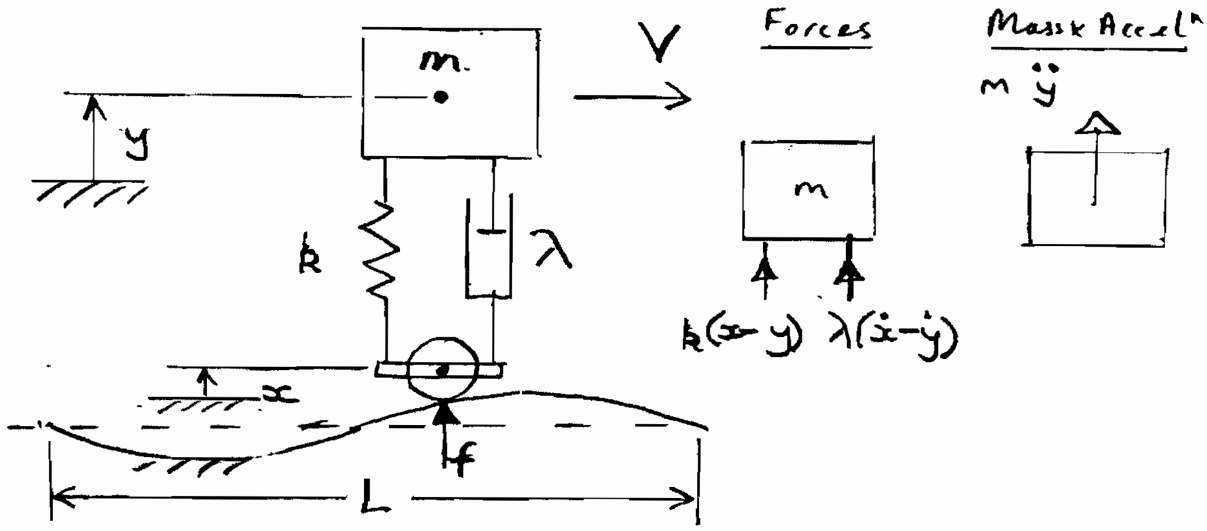
For light damping, $\left|\frac{Y}{X}\right| \approx \frac{1}{2\zeta} = 5 \quad \therefore \quad \underline{\underline{\zeta = 0.1}}$

$$\left|\frac{Y}{X}\right| = \frac{\sqrt{1 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}} \quad \text{from Data Book case (c)}$$

and for $\frac{\omega}{\omega_n} = 3$, $\zeta = 0.1$ $\quad \left|\frac{Y}{X}\right| = \frac{\sqrt{1+0.6^2}}{\sqrt{64+0.6^2}} = 0.145$

$\therefore \underline{\underline{Y}} = 0.145 \times 40\,\mu m = \underline{\underline{5.8\,\mu m}} > 5\,\mu m$ but not much

Q 9



$R(\uparrow)$ $k(x-y)+\lambda(\dot{x}-\dot{y}) = m\ddot{y}$ $\therefore$ $\underline{\underline{m\ddot{y}+\lambda\dot{y}+ky = \lambda\dot{x}+kx}}$

Write in form of case (c) in Mechanics Data Book

$$\underline{\frac{\ddot{y}}{\omega_n^2}+\frac{2\xi\dot{y}}{\omega_n}+y = \left(\frac{2\xi\dot{x}}{\omega_n}+x\right)}$$ where $\omega_n^2 = s/m$, and $c = \lambda/2\sqrt{sm}$

For $m = 500$ kg, $k = 20\,000$ N/m, $\lambda = 2000\sqrt{5}$ Ns/m;-

$\omega_n = 2\sqrt{10}$ rad/s, $\underline{\underline{\xi = 1/\sqrt{2} = 0{\cdot}71}}$

At $V = 50$ km/h $= 13.9$ m/s

$\omega = \dfrac{2\pi V}{L} = \dfrac{2\pi \times 13.9}{7{\cdot}5} = 11.6$ rad/s and $\left(\dfrac{\omega}{\omega_n}\right) = 1{\cdot}84$.

For $\xi = 0{\cdot}71$, Data Book curve, page 8, gives $\left(\dfrac{Y}{X}\right) = 0{\cdot}8$

$\therefore$ $Y = 0{\cdot}8 \times 0{\cdot}025 = \underline{\underline{0{\cdot}020\text{ m}}}$

$R(\uparrow)$ for entire system; $f = m\ddot{y}$

$\therefore$ $f = -m\omega^2 Y\cos(\omega t-\phi)$

$\therefore$ $\underline{\underline{f = -F\cos(\omega t-\phi)}}$ where $\underline{\underline{F = m\omega^2 Y}}$

For $V = 50$ km/h, $\omega = 11{\cdot}6$ rad/s and $Y = 0{\cdot}020$ m

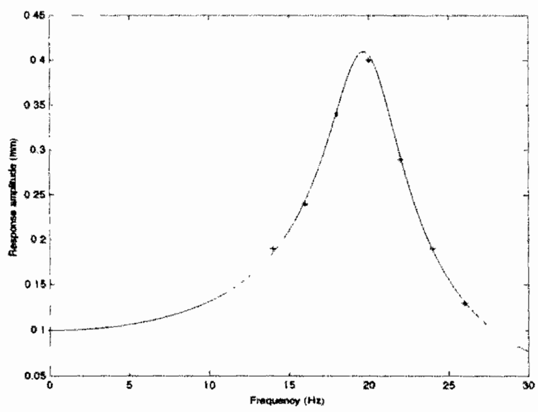
$\therefore$ $F = 500 \times (11{\cdot}6)^2 \times 0{\cdot}020$ $\therefore$ $\underline{\underline{F = 1350\text{ N}}}$

[Note that $F < mg$ $\therefore$ caravan remains in contact with road. Note also, curve on page 8 can be used directly to find F from X, by reading $\dfrac{F}{m\omega^2 X}$ for ordinate in place of $\left(\dfrac{Y}{X}\right)$].

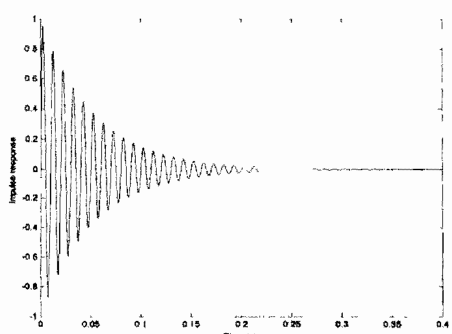
Q4 supplement: This Matlab program does the job:

```
%  Matlab example for IA vibration ex sheet 2 Q4
close all
clear all
fset=[0 14 16 18 20 22 24 26];
rset=[0.10 0.19 0.24 0.34 0.40 0.29 0.19 0.13];
figure(1)
plot(fset,rset,'*r')
% axis([0 30 0 0.5]);
faxis=0:0.01:30;
omega=2*pi*faxis;
k=5.8/0.1;
fres=20;
omegan=fres*2*pi;
mass=k/omegan^2;
zeta=0.123;
amp=5.8/k;
response=amp*abs(1./(1+2*i*zeta*omega/omegan-(omega/omegan).^2));
hold on
plot(faxis,response)
xlabel('Frequency (Hz)')
ylabel('Response amplitude (mm)')
print -dtiff Q4plot
```

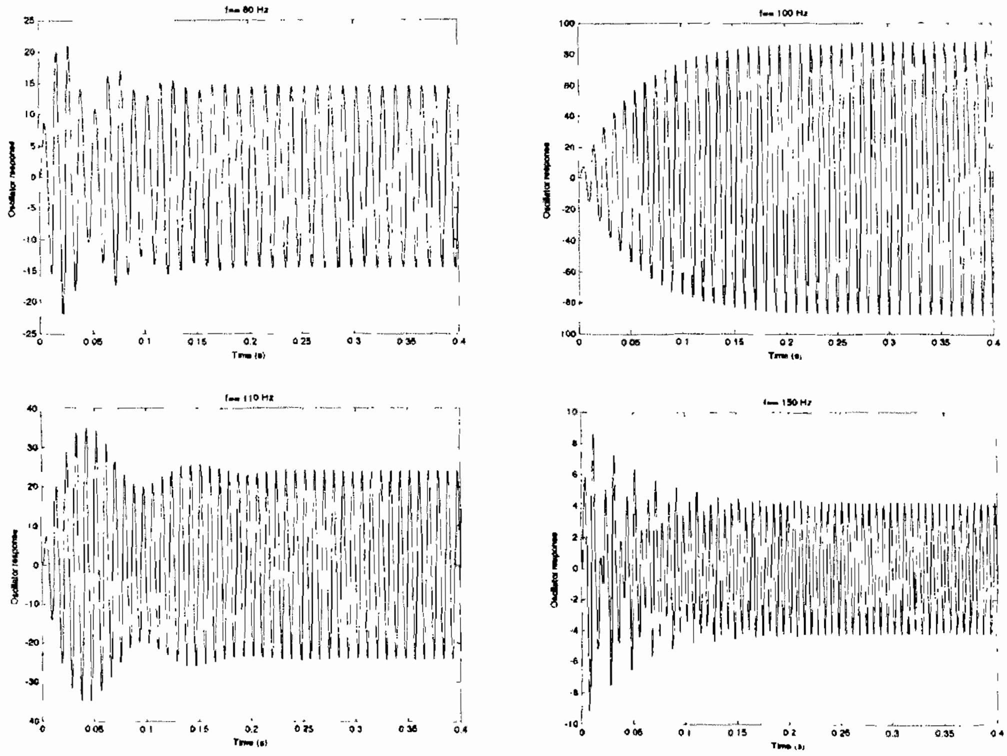
Here is the plot:



Q10.
Impulse response:



Here are responses at the frequencies asked for. The transient response and the steady harmonic response are clearly visible in all cases.



This Matlab program does the required job:

```
%  Matlab example for IA vibration ex sheet 2 Q10
close all
clear all
omegan=2*pi*100;
zeta=0.03;
dt=0.0003;
time=0:dt:0.4;
nt=length(time);
impulseresp=sin(omegan*time).*exp(-zeta*omegan*time);
plot(time,impulseresp)
xlabel('Time (s)')
ylabel('Impulse response')
f=110;
omega=2*pi*f;
force=sin(omega*time);
output=conv(force,impulseresp);
print -dtiff impresp
figure
plot(time,output(1:nt))
xlabel('Time (s)')
ylabel('Oscillator response')
title(sprintf('f== %g Hz',f))
print('-dtiff',['outputfile_' int2str(f)])
```