ENGINEERING **FIRST YEAR**

Part IA Paper 1: Mechanical Engineering

MECHANICAL VIBRATIONS

Examples paper 3

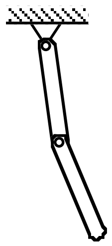
Straightforward questions are marked with a †
Tripos standard questions are marked * .

Systems with two or more degrees of freedom

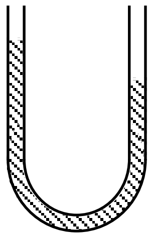
Throughout this examples paper, assume that displacements are small and neglect the effects of damping.

Free vibration

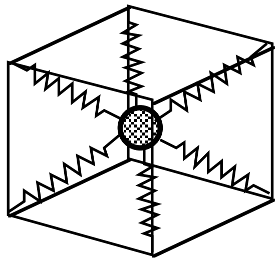
†1. How many degrees of freedom have each of the dynamic systems shown in Fig. 1?



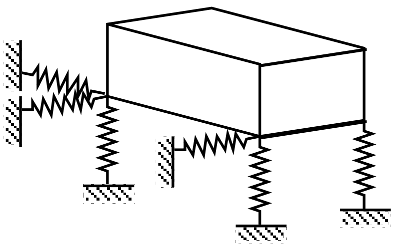
(a) Planar double pendulum



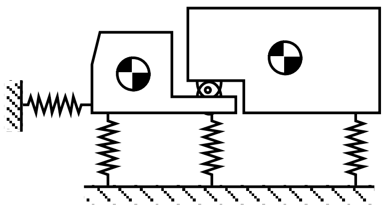
(b) U-tube manometer



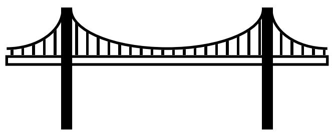
(c) Point mass in rigid frame



(d) Spring-mounted engine block



(e) Planar articulated-vehicle model with pin joint



(f) Tacoma Narrows bridge

Fig. 1

†2. For the system shown in Fig. 2, show that the equation of free motion can be written in the form

$$M \ddot{\underline{y}} + K \underline{y} = 0$$

where $$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad K = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \quad \text{and} \quad \underline{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

If the displacements in the j^{th} normal mode of free vibration are given by

$$\underline{y}_j = \underline{Y}_j \cos\omega_j t ,$$

deduce that each natural frequency ω_j and its corresponding normal mode $\underline{Y}_j$ must satisfy

$$[K - \omega_j^2 M] \underline{Y}_j = 0 .$$

Hence find the natural frequencies and normal modes for the case $k_1 = k_2 = k$ and $m_1 = m_2 = m$.
Sketch the two mode shapes.

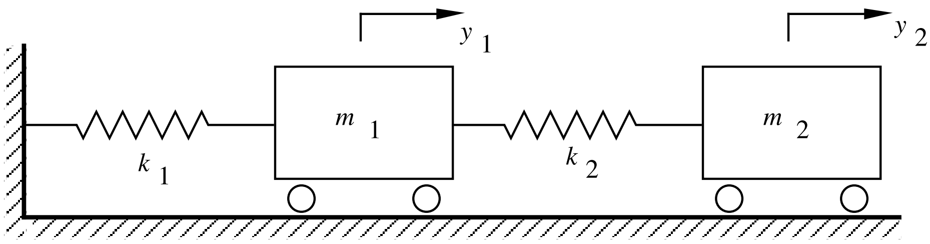


Fig. 2

3. Two rigid discs with moments of inertia J and $2J$ are mounted on three light elastic shafts of torsional stiffness k, k, and $2k$ as shown in Fig. 3. The angles of rotations of the discs from their equilibrium positions are θ_1 and θ_2 respectively. Show that the equation of free torsional vibration of the system can be written in the form

$$M\ddot{\underline{\theta}} + K\underline{\theta} = 0$$

where $M = \begin{bmatrix} J & 0 \\ 0 & 2J \end{bmatrix}$, $K = \begin{bmatrix} 2k & -k \\ -k & 3k \end{bmatrix}$ and $\underline{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$.

Using the method outlined in Q2 above, find the natural frequencies and normal modes. Sketch the mode shapes.

Type in the following Matlab program to check your answers:

```
K=[2 -1;-1 3];
M=[1 0;0 2];
[V,D]=eig(K,M);
for n=1:2
  mode=V(:,n);
  mode=mode/mode(1);
  freq2=D(n,n);
  disp(sprintf('Mode %i has squared frequency %g and mode [%g, %g]',n,freq2,mode))
end
```

Do you understand the program? "eig" calculates eigenvalues and vectors: type "help eig" in the Matlab command window to see details.

Save your program, to use again in question 5.

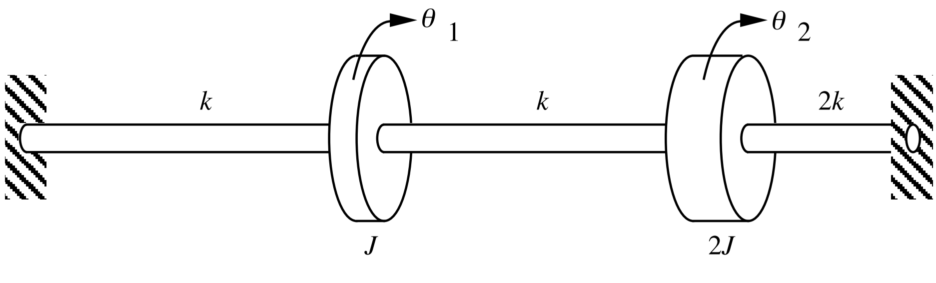


Fig. 3

†4. Determine by inspection the normal modes of vibration of the systems shown in Fig. 4. Hence calculate the natural frequencies of each system.

Describe the motion of the two masses in Fig. 4(b) for the following sets of initial conditions:

(i) $y_1 = y_2 = 1$ and $\dot{y}_1 = \dot{y}_2 = 0$;

(ii) $y_1 = 1$, $y_2 = 0$ and $\dot{y}_1 = \dot{y}_2 = 0$;

(iii)* $y_1 = y_2 = 0$ and $\dot{y}_1 = 1$, $\dot{y}_2 = 0$;

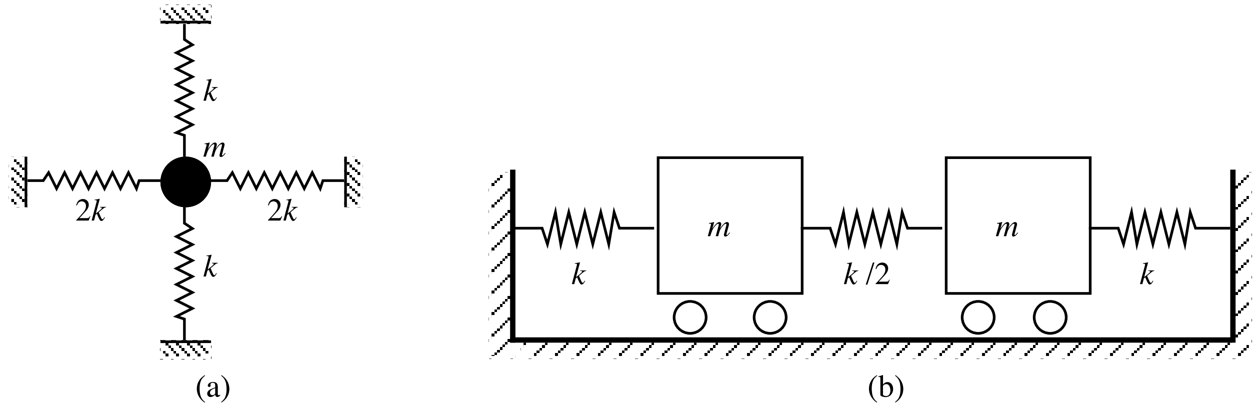


Fig. 4

*5. Two equal masses m are attached to a third mass M by two equal springs each of stiffness k as shown in Fig. 5. Derive the equations of motion for the system in terms of the three coordinates y_1, y_2 and y_3 as shown.

Calculate the three natural frequencies and corresponding normal modes, and sketch the mode shapes. (Hint: two of the natural frequencies and mode shapes can be determined easily by inspection.)

What is the significance of a zero natural frequency?

Modify the program from question 3 to check your answers. Try different ratios M/m.

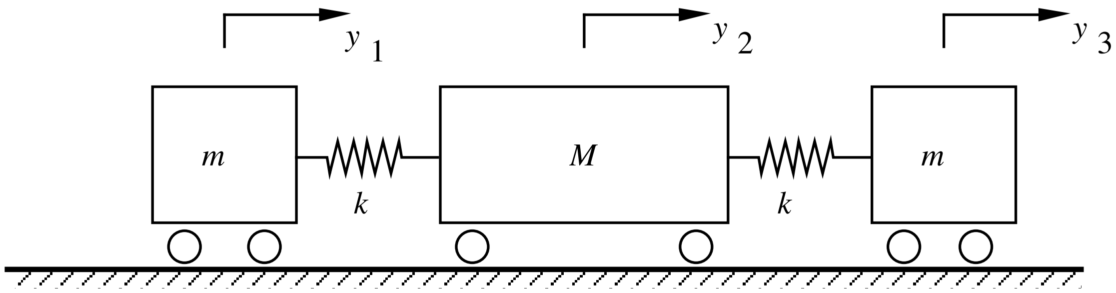


Fig. 5

<u>Forced harmonic vibration</u>

(Ignore any transient response throughout)

6. Show that the equation of motion for the system shown in Fig. 6 may be written in the form

$$M\,\underline{\ddot{y}} + K\,\underline{y} = \underline{f}$$

where $M = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$, $K = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$, $\underline{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ and $\underline{f} = \begin{bmatrix} f \\ 0 \end{bmatrix}$.

For free motion ($f = 0$) find the natural frequencies and normal modes (by inspection).

When $f = F\cos\omega t$, the forced harmonic response is given by $\underline{y} = \underline{Y}\cos\omega t$. Deduce that

$$\underline{Y} = \left[K - \omega^2 M\right]^{-1} \begin{bmatrix} F \\ 0 \end{bmatrix}.$$

Hence derive expressions for the response amplitudes Y_1 and Y_2 and sketch their variation with frequency ω . (It is easiest to plot the non-dimensional quantities kY_1/F and kY_2/F against non-dimensional-frequency squared $\omega^2 m/k$. Give due regard to signs.)

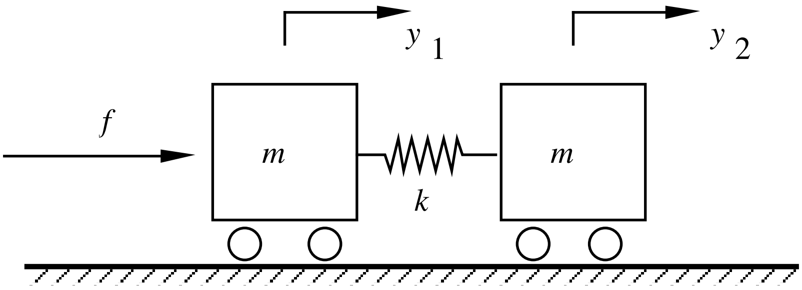


Fig. 6

*7. Figure 7 shows a model of a caravan suspension. The mass of the caravan body is m_1 and that of the axle assembly is m_2 . The corresponding vertical displacements are y_1 and y_2 . The stiffness of the suspension springs and of the tyres are k_1 and k_2 respectively. Road roughness is represented by the displacement x of the bottom of the tyre spring as shown.

Show that the equation of motion may be written in the form

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} + \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1+k_2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ k_2 x \end{bmatrix} .$$

A particular caravan has $m_1 = 500$ kg , $k_1 = 20$ kN/m and $m_2 = 40$ kg , $k_2 = 160$ kN/m . It is towed at a constant velocity V along a bumpy road whose surface profile varies sinusoidally with an amplitude X of 25 mm and a wavelength L of 1.25 m so that the tyre displacement input is

$$x = X\cos\omega t \text{ , where } \omega = \frac{2\pi V}{L} .$$

The caravan displacement and that of its axle are

$$y_1 = Y_1 \cos\omega t \qquad \text{and} \qquad y_2 = Y_2 \cos\omega t .$$

Find the amplitude of caravan vibration Y_1 when the velocity $V = 50$ km/h . Find also the amplitude of the axle vibration Y_2 . Why is the axle motion so large? [Hint: calculate the two natural frequencies.] Comment on the use of shock absorbers.

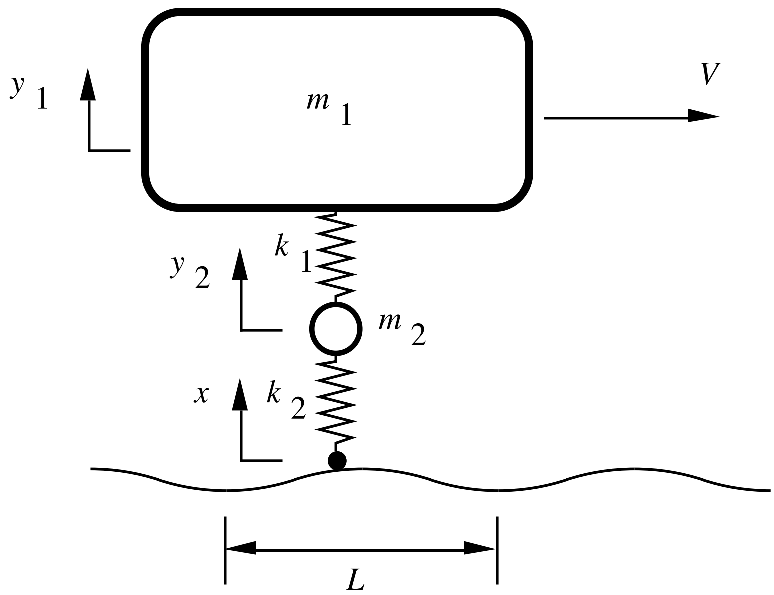


Fig. 7

*8. The vibration of a pedestrian footbridge in its fundamental bending mode may be modelled as a mass $m_1 = 10{,}000$ kg supported on a spring of stiffness $k_1 = 2.7$ MN/m so that the natural frequency of the bridge in this mode is 2.6 Hz.

A vibration absorber is attached at the centre of the bridge to prevent excessive amplitudes of vibration when hooligans jump up and down on the bridge. The absorber comprises a mass m_2 supported on a spring of stiffness k_2 as shown in Fig. 8.

Derive the matrix equation for vertical motion when the hooligan exerts a force f on the bridge.

If $f = F \cos\omega t$, derive an expression for the amplitude of vibration of the bridge Y_1 as a function of ω.

It is desired that there is no bridge motion when it is excited at $\omega = \Omega$. Determine the required absorber stiffness in terms of the absorber mass and Ω.

For the case $m_2 = 1000$ kg and $k_2 = 270$ kN/m, calculate the two natural frequencies of the system. Sketch the variation of $|k_1 Y_1 / F|$ with ω, both with and without the absorber.

What is the effect of adding damping to the absorber?

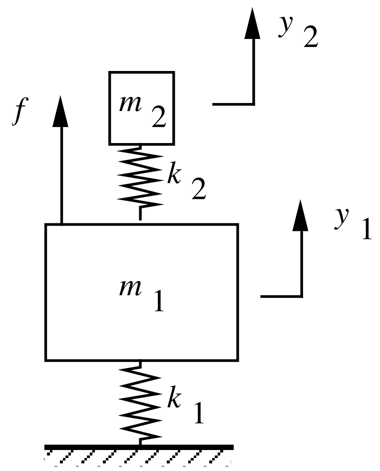


Fig. 8

Eight such absorbers are fitted to the cycle bridge over the Cambridge railway station. Each absorber comprises a 100 kg sprung mass in an oil-filled enclosure – the oil provides damping.

Answers

1. $2,\ 1,\ 3,\ 6,\ 4,\ \infty$

2. $\omega_1{}^2 = 0.382\,\dfrac{k}{m}$, $\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1.618 \end{bmatrix}$ $\qquad$ $\omega_2{}^2 = 2.618\,\dfrac{k}{m}$, $\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.618 \end{bmatrix}$

3. Use the Matlab program!

4. (a) $\omega_1{}^2 = \dfrac{4k}{m}$, $\omega_2{}^2 = \dfrac{2k}{m}$ $\qquad$ (b) $\omega_1{}^2 = \dfrac{k}{m}$, $\omega_2{}^2 = \dfrac{2k}{m}$

   (i) $y_1 = y_2 = \cos \omega_1 t$; $\qquad$ (ii) $y_1 = 0.5\ (\cos \omega_1 t + \cos \omega_2 t)$;

   (iii) $y_1 = 0.5\ (\dfrac{1}{\omega_1} \sin \omega_1 t + \dfrac{1}{\omega_2} \sin \omega_2 t)$; $y_2 = 0.5\ (\dfrac{1}{\omega_1} \sin \omega_1 t - \dfrac{1}{\omega_2} \sin \omega_2 t)$

5. $$\begin{bmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_3 \end{bmatrix} + \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

   Use the Matlab program to check answers.

   Zero natural frequency corresponds to rigid-body motion. Mode shape is always [1, 1, 1, ...]

6. $\omega_1{}^2 = 0$, $\underline{Y}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (rigid body motion) $\qquad$ $\omega_2{}^2 = \dfrac{2k}{m}$, $\underline{Y}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

   $Y_1 = -\dfrac{1 - \omega^2 m/k}{\omega^2 m/k\,[2 - \omega^2 m/k]}\,\dfrac{F}{k}$ $\qquad$ $Y_2 = \dfrac{-1}{\omega^2 m/k\,[2 - \omega^2 m/k]}\,\dfrac{F}{k}$

7. 2.24 mm $\qquad$ –270 mm $\qquad$ input at 11.1 Hz is just above the 'wheel-hop' frequency (10.7 Hz)

8. $$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} + \begin{bmatrix} k_1{+}k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}$$

   $Y_1 = \dfrac{(k_2 - \omega^2 m_2)F}{\Delta}$ $\qquad$ where $\Delta = m_1 m_2 \omega^4 - [m_2(k_1 + k_2) + m_1 k_2]\omega^2 + k_1 k_2$

   $k_2 = m_2 \Omega^2$ $\qquad$ $\omega_1 = 14.0$ rad/s $\qquad$ $\omega_2 = 19.2$ rad/s

   $$\frac{k_1 Y_1}{F} = \frac{(\Omega^2 - \omega^2)\Omega^2}{(\omega_1{}^2 - \omega^2)(\omega_2{}^2 - \omega^2)}$$

   Damping reduces peakiness of the two resonances. Instead of large bridge motion, we have large motion of the absorber mass. It is often easier to add damping to an absorber than to a bridge.

For further practice, the following Tripos questions from Paper 1 are suitable:
2013 Q12; 2012 Q11; 2011 Q12; 2010 Q10; 2009 Q12; 2008 Q9; 2007 Q12