Engineering **FIRST YEAR**

Part 1A Paper 4: Mathematics

Examples paper 1

(Elementary exercises are marked †, problems of Tripos standard *)

Students are encouraged to use mathematical grammar correctly.

Revision question

Simplify the following expressions:

(a) $\dfrac{1}{1+\left\{\dfrac{\sin 2x}{1+\cos 2x}\right\}^2}$ (b) $\dfrac{\cos 3x-\cos 5x}{\sin 3x+\sin 5x}$

(c) $\cos 2x+1+\dfrac{8\sin^2(x/2)}{1+\tan^2(x/2)}$ (d) $\dfrac{\sin 4x+(\cos x+\sin x)^2-1}{\sin 3x}$

(Note that the Mathematics Data Book contains some trigonometric identities.)

Vectors

1† (i) If $\mathbf{a} = (1,4,6)$ and $\mathbf{b} = (2,-1,3)$, find $\mathbf{a} \,.\, \mathbf{b}$ and $\mathbf{a}\times\mathbf{b}$.

(ii) Find the value of s that makes $(1,1,3) \times (-1,-1, s)$ equal to zero.

2† (i) Find an equation for the line through the points (1,–5,2) and (6,3,–1).

(ii) Find an equation for the plane through the point (1,6,2) whose normal is parallel to the vector (1,–2,–3).

(iii) Find an equation for the plane through (4,0,2), (1,3,2) and (3,1,0) .

(iv) Find the relationship between α, β, and γ which makes

$$\mathbf{r} = \alpha\,\mathbf{a} + \beta\,\mathbf{b} + \gamma\,\mathbf{c}$$

represent a plane through $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

3 Find the directions of the normals to the planes

$$x - 3y + 5z = 3, \qquad 2x + y + z = 2$$

and hence find the direction of the line of intersection of the planes.

4 If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are the position vectors of points A, B, C respectively, show that the vector

$$\mathbf{a}\times\mathbf{b} + \mathbf{b}\times\mathbf{c} + \mathbf{c}\times\mathbf{a}$$

is perpendicular to the plane ABC.

5 Find a vector normal to both the lines

$$\mathbf{r}_1 = (1,2,3) + \lambda(4,5,6)$$
$$\mathbf{r}_2 = (2,3,2) + \mu(5,6,7)$$

(i) Hence find the shortest distance between the two lines. Check that your answer seems reasonable by plotting the lines with Matplotlib.

Hint: You will need to import and use a module from `mpl_toolkits`, e.g.

```
from mpl_toolkits.mplot3d import Axes3D
```

(ii) Calculate the shortest distance between the two lines with Python and find the points where the two lines are closest together.

See the URL

http://nbviewer.jupyter.org/github/CambridgeEngineering/Part-IA-ExamplesPapers-Python/blob/master/paper4/IA%20Paper%204%20Mathematics%2001.ipynb

6 (i) Simplify $(\mathbf{a}+\mathbf{b}) \times (\mathbf{a}-\mathbf{b})$.

(ii) For given non-zero vectors $\mathbf{t}$ and $\mathbf{a}$, find all position vectors $\mathbf{r}$ which satisfy the equation

$$\mathbf{t} \times \mathbf{r} = \mathbf{t} \times \mathbf{a}.$$

7† If $\mathbf{a} = (1,2,3)$, $\mathbf{b} = (-1,0,-1)$ and $\mathbf{c} = (0,1,2)$, evaluate $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ by (a) direct multiplication and (b) use of the expansion formula for a vector triple product.

8† Prove that $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} + (\mathbf{b} \times \mathbf{c}) \times \mathbf{a} + (\mathbf{c} \times \mathbf{a}) \times \mathbf{b} = 0.$

9* A vector $\mathbf{x}$ satisfies the equation

$$\mathbf{x} + (\mathbf{x}.\mathbf{a})\,\mathbf{b} = \mathbf{c} \qquad (1)$$

where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are constant vectors such that

$$\mathbf{a}.\mathbf{b} \neq -1.$$

(a) Assuming that $\mathbf{b}$ and $\mathbf{c}$ are not parallel, explain why any vector can be expressed in the form

$$\mathbf{x} = \alpha\mathbf{b} + \beta\mathbf{c} + \gamma\mathbf{b} \times \mathbf{c}.$$

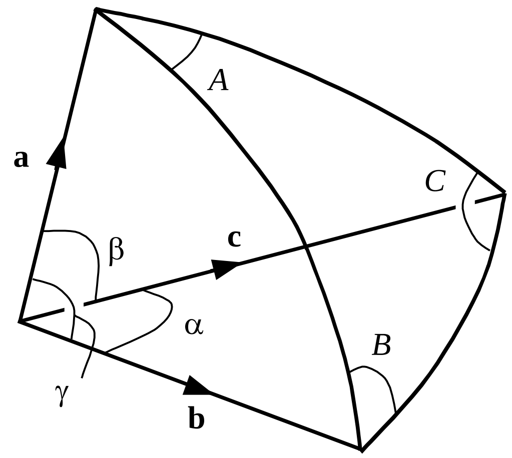
(b) Use this form for $\mathbf{x}$ in the equation (1) to determine possible values of α, β and γ, and hence find $\mathbf{x}$.

(c) Find $\mathbf{x}$, satisfying (1), when $\mathbf{b}$ and $\mathbf{c}$ are parallel, with $\mathbf{b} = \lambda\mathbf{c}$.

10* Show that

$$(\mathbf{a} \times \mathbf{b}).(\mathbf{a} \times \mathbf{c}) = (\mathbf{a}.\mathbf{a})\,(\mathbf{b}.\mathbf{c}) - (\mathbf{a}.\mathbf{b})\,(\mathbf{a}.\mathbf{c})$$

The *unit* vectors **a**, **b**, **c** define three points on the surface of a sphere of unit radius. These points are joined by arcs on the surface which lie in planes passing through the centre of the sphere, as shown. These three great circle arcs define a *spherical triangle*. A, B, C are the vertex angles on the surface of the sphere of this triangle and α, β, γ are the angles subtended by the arcs at the centre of the sphere, with the arc subtending α opposite to the vector **a** and the angle A, etc. Use the result obtained in the first part of this question to deduce the cosine formula for spherical triangles,



$$\cos\alpha = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos A.$$

Suitable past Tripos questions:

08 Q1 (short), 09 Q5 (not (c)) (long), 11 Q4 (long), 12 Q3 (short), 13 Q4 (long), 14 Q4a (long), 16 Q3 (short), 18 Q4 (long).

Answers

(Answers are not given for revision questions: your supervisor will check your answers.)

1 (i) 16 , (18,9,–9)

(ii) $s = -3$

2 (i) $\mathbf{r} = (1,-5,2) + \lambda\,(5,8,-3)$.

(ii) $x - 2y - 3z = -17$

(iii) $\mathbf{r} = (4,0,2) + \lambda\,(-3,3,0) + \mu\,(-1,1,-2)$ or $x + y = 4$

(iv) $\alpha + \beta + \gamma = 1$

(N.B. Parametric equations for lines and planes are not unique, so there are many alternative answers for parts (i) and (iii))

3 (1,–3,5), (2,1,1); (–8,9,7)

5 (–1,2,–1); $\dfrac{2}{\sqrt{6}}$

6 (i) $2\,\mathbf{b} \times \mathbf{a}$ (ii) Line $\mathbf{r} = \mathbf{a} + \lambda\,\mathbf{t}$

7 $(-6,4,-2)$

9 (b) $\mathbf{x} = -\dfrac{\mathbf{a.c}}{1+\mathbf{a.b}}\,\mathbf{b} + \mathbf{c}$

(c) $\mathbf{x} = \dfrac{\mathbf{c}}{1+\lambda\mathbf{a.c}}$

SJS/JPL September 2018