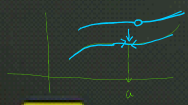
CONTINUITY :

Definition:

A function $f(x)$ is Said to be continuous at $x = a$

if $\boxed{\lim_{x\to a^-}(f(x)) = \lim_{x\to a^+}(f(x)) = f(a) \qquad L.H.L = R.H.L = \text{function at } x=a \text{ Value.}}$



e.g $f(x) = \frac{\sin x}{x}$ at $x=0$ is it continuous? not continuous

$\Rightarrow L.H.L = R.H.L = \lim_{x\to 0}\left(\frac{\sin x}{x}\right) = 1$

$f(0) = \frac{\sin 0}{0} =$

e.g $f(x) = \frac{\sin x}{x}$, $x = \frac{\pi}{2}$

$x = 2$, continuous?

$L.H.L = \lim_{x\to 2^-} \frac{\sin 2}{2} = R.H.L$

$f(2) = \frac{\sin 2}{2}$

$\frac{\sin 2}{2} \to 1$

$1 = 57^\circ$

$\frac{\sin 114^\circ}{2}$

Q) $\Rightarrow$ Discuss the continuity at $\boxed{x=0}$

$$\begin{cases} \frac{1-\cos x}{x^2} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$

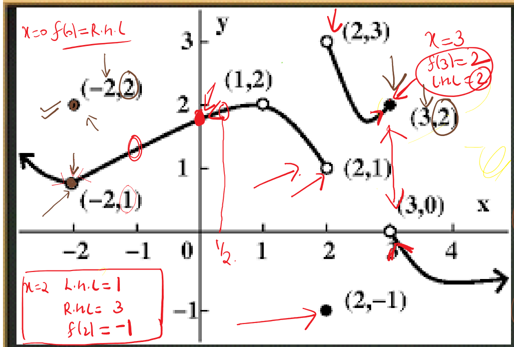
$f(0) = 1$

$L.H.L = R.H.L$

$\lim_{x\to a} f(x) = f(a)$

☆ $\lim_{x\to 0} f(x) \neq f(0)$.

$\boxed{\text{Discontinuous}}$



① $\lim_{x\to -2} f(x) = 1$

Is function conti. at $x = -2$

$f(-2) = 2$

② Is function conti at $x = 0$? Yes

③ $\lim_{x\to 3^-} f(x) = 2 = f(3)$

④ If f(x) is conti at $x = 3$?

$R.H.L = 0$ at $x = 3$

$x = 2$, $x = 1$

$x = 1/2$

Q) If the function is $\boxed{\text{Continuous}}$ at $x = 1$ find a & b.

$$f(x) = \begin{cases} ① \; 3ax+b & x>1 \\ ② \; 11 & x=1 \\ ③ \; 5ax-2b & x<1 \end{cases}$$

$L.H.L = \lim_{x\to 1^{(-)}}(5ax-2b) = 5a-2b$

$R.H.L = \lim_{x\to 1^{(+)}}(3ax+b) = 3a+b$

$f(1) = 11$

$\boxed{5a-2b=11 \quad 3a+b = 11}$ $\boxed{a=3 \quad b=2}$ ☆

Q) Let $f(x)$ be the function

$f(x) = \left(\frac{\sqrt{4+x}-2}{x}\right) (x \neq 0)$

what choice of $f(0)$ will make it continuous at $x=0$

$x = 0$

$\lim_{x\to 0} f(x) = f(0) = \frac{1}{4}$

$\lim_{x\to 0} f(x) = f(0) = \frac{1}{4}$

$\lim_{x\to 0}\left(\frac{\sqrt{4+x}-2}{x}\right) = \frac{\frac{1}{2\sqrt{4+x}} - 0}{1}$

$= \frac{1}{2\times 2} = \frac{1}{4}$

PROBLEMS

Q) Let $f(x) = \begin{cases} \dfrac{1-\cos 4x}{2x^2} & x<0 \\ a & x=0 \\ \dfrac{\sqrt{x}}{\sqrt{16+x}-4} & x>0 \end{cases}$

find value of a such that $f(x)$ is conti. at $x=0$

A) 4

B) 8

C) 12

D) No value of a ✓

$$\text{L.H.L} = \text{R.H.L} = f(0)$$

$$\boxed{4 \neq \infty \neq a}$$

$$\text{L.H.L} = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1-\cos 4x}{2x^2}$$

$$= \lim_{x\to 0^-} \frac{\cancel{2}\sin^2 2x}{\cancel{2}x^2} = \lim_{x\to 0^-}\left(\frac{\sin 2x}{x}\right)^2$$

$$= 2^2 = 4$$

$$\text{R.H.L} = \lim_{x\to 0^+} \frac{\sqrt{x}}{\sqrt{16+x}-4} \times \frac{(\sqrt{16+x}+4)}{\sqrt{16+x}+4}$$

$$= \lim_{x\to 0^+} \frac{\sqrt{x}\,(\sqrt{16}+4)}{\cancel{16}+x-\cancel{16}} = 8 \lim_{x\to 0^+}\left(\frac{\sqrt{x}}{x}\right) \to \infty$$

NON-REMOVABLE TYPE

Q) The function $f(x) = \dfrac{\log(1+ax) - \log(1-bx)}{\cancel{\sin x}\ x}$ $(x \neq 0)$

is not defined at $x=0$. find $f(0)$ so that f is continuous at $x=0$

A) $a-b$

B) a

C) b

D) $a+b$ ✓

$$\lim_{x\to 0} f(x) = f(0)$$

$$\lim_{x\to 0}\left(\frac{\log(1+ax)}{x} - \frac{\log(1-bx)}{x}\right)$$

$$= a-(-b) = a+b$$

Q) if $f(x) = \begin{cases} |x-a| \sin\left(\dfrac{1}{x-a}\right) & x \neq a \\ 0 & x=a \end{cases}$

$0 \times [-1,1] = 0$

$$|x-a| = \begin{cases} x-a & x>0 \\ -(x-a) & x<0 \end{cases}$$

Prove that $f(x)$ is conti. at $x=a$

$$\boxed{x \sin \frac{1}{x}}$$

$$f(x) = \begin{cases} (x-a)\sin\left(\dfrac{1}{x-a}\right) & x>a \\ 0 & x=a \\ -(x-a)\sin\left(\dfrac{1}{x-a}\right) & x<a \end{cases}$$

$$\text{R.H.L} = \lim_{x\to a^+} (x-a)\sin\left(\frac{1}{x-a}\right) \quad [-1,1]$$

$$= 0 \times \_\_ = 0$$

$$f(a) = 0$$

$$\text{L.H.L} = 0 \times \{\_\_\} = 0$$

Types of Continuity;

Continuous at $x=a$,

① if $f(x)$ is continuous at $x=a$

$$\text{L.H.L} = \text{R.H.L} = f(a)$$

{ Right as well as left continuous }

② Left Continuous; $\text{L.H.L} = f(a)$

③ Right Continuous: $\text{R.H.L} = f(a)$.

Q) Examine the Continuity of function

$$f(x) = \begin{cases} \dfrac{e^{1/x}-1}{e^{1/x}+1} & x \neq 0 \\ 0 & x=0 \end{cases}$$

at $x=0$.

$$\boxed{\begin{aligned} \lim_{x\to 0^-}\left(\frac{1}{x}\right) &= -\infty \\ \lim_{x\to 0^+}\left(\frac{1}{x}\right) &= \infty \end{aligned}}$$

$$\lim_{x\to 0^-}\left(\frac{1-e^{-1/x}}{1+e^{-1/x}}\right) \quad \infty$$

$$\text{L.H.L} = \lim_{x\to 0^-}\left(\frac{e^{1/x}-1}{e^{1/x}+1}\right) = \left(\frac{e^{-\infty}-1}{e^{-\infty}+1}\right) = \frac{0-1}{0+1} = -1$$

$$\text{R.H.L} = \lim_{x\to 0^+}\left(\frac{e^{1/x}-1}{e^{1/x}+1}\right) = 1$$

$$\boxed{f(0) = 0}$$

$$\boxed{\text{R.H.L} \neq \text{L.H.L} \neq f(0)}$$

e.g $\lim_{x\to 1} [x] =$ $\text{L.H.L} = \lim_{x\to 1^-} [x] = 0$

$$f(1) = 1$$

$$\text{R.H.L} = [1^+] = 1$$

Right Continuous

e.g $\lim_{x\to 1} \{x\} =$

Right Conti at $x=1$.

TYPES OF DISCONTINUITY:

① Disconti. of I kind : (Removable type)

If $\lim_{x \to a} f(x)$ exist L.H.L = R.H.L $\neq f(a)$

or $f(a)$ is not defined

then we will choose/change such that

$\boxed{L.H.L = R.H.L = f(a)}$

☆ e.g $f(x) = \frac{\sin x}{x}$ at $\boxed{x=0}$

redefine $\boxed{f(0)=1}$

② Disconti. of II kind : (non-removable type)

If L.H.L $\neq$ R.H.L or L.H.L, R.H.L $\rightarrow$ not defined

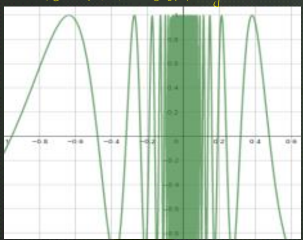
Limit does not exist

③ Oscillation discontinuity :

If $f(x)$ oscillates between two finite values at $x=a$ then $f(x)$ has oscillation discontinuity at $x=a$

e.g $f(x) = \sin \frac{\pi}{x}$

$x \to 0$



④ Jump of discontinuity :

If L.H.L $\rightarrow$ exists R.H.L $\rightarrow$ exists L.H.L $\neq$ R.H.L

e.g at $x=1$

$f(x) = [x]$ L.H.L = 0 , R.H.L = 1

$|L.H.L - R.H.L| = |0-1| = 1$

$f(x) = \{x\}$

then Jump of disconti. :

☆ $= |L.H.L - R.H.L|$

☆ $\lim_{x \to 0} \frac{|\sin x|}{x}$

L.H.L $= \lim_{x \to 0^-} \frac{-\sin x}{x} = -1$

R.H.L $= \lim_{x \to 0^+} \left(\frac{\sin x}{x}\right) = 1$

②

THEOREM ON CONTINUITY OF FUNCTION AT X=a:

① If f & g are two functions which are continuous at $x=a$ then functions defined as

✓ $F_1(x) = f_1(x) \pm g_1(x)$

✓ $F_2(x) = f_1(x) \cdot g_1(x)$

✓ $F_3(x) = \frac{f_1(x)}{g_1(x)}$ $\{g_1(a) \neq 0\}$

are all continuous at $x=a$.

e.g $f(x) = \frac{x + \sin x}{\cos x}$ — conti. at $x=0$

at $\boxed{x=0}$

e.g $f(x) = \frac{\sin x}{x}$ $\boxed{x=0}$ disconti. at $x=0$

$\boxed{x=1}$ continuous at $x=1$.

② $f(x)$ is conti. & $g(x)$ is disconti. at $x=a$.

☆ then $F_1(x) = f(x) \pm g(x) \longrightarrow$ discontinuous.

e.g $f(x) = x$, $g(x) = [x]$ $f(x) \pm g(x) \rightarrow$ discontinuous

$\boxed{x=1}$

C
d
m

$F_2(x) = f(x) \cdot g(x)$ may or may not be

e.g $x[x] = 0$

$\boxed{x \to 0}$

L.H.L $= (0^-)[0^-]$
$= 0 \times (-1)$
$= 0$

R.H.L $= (0^+)([0^+]) = 0$

☆ $F_3(x) = \frac{f(x)}{g(x)} \rightarrow$ may not may be.

③ $f(x)$ is disconti. at $x=a$ then
$g(x)$ is disconti.

$\boxed{L.H.L, R.H.L, f(a)}$

① $f(x) \pm g(x)$
② $f(x) \cdot g(x)$
③ $\frac{f(x)}{g(x)}$

may or may not be

e.g $f(x) = [x]$ $g(x) = \{x\}$ $f(x) + g(x) = [x] + \{x\}$
$= x$

$\boxed{x=1}$ $f(x)$ is disconti. $x=1$
$g(x)$ is disconti. at $x=1$

at $x=1$ conti. at $\boxed{x=1}$

THEOREMS CONT...

4. Every Polynomial
$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x^1 + a_0 x^0$$ is

Continuous at every point on real number line.

5. Every rational polynomial is continuous at every point where its denominator is not zero.

e.g. $f(x) = \dfrac{x^2+x+1}{(x-1)(x-2)}$ Points where f(x) is discontinuous: $x=1, x=2$

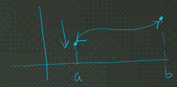
6. Log, Exponential, Trigonometric, $\boxed{\text{Inverse trigonometric}}$, modulus function are all continuous in their $\boxed{\text{domain}}$.

☆ $\tan x$ $\boxed{x = \frac{\pi}{2}}$ discontinuous $x = \frac{\pi}{3}$ continuous

CONTINUITY OF A FUNCTION IN AN INTERVAL:

1. Continuity of a function in open interval (a,b);

A function is said to be continuous in (a,b) if f is continuous at each and every point in (a,b)



2. Continuity of a function in closed interval $[a,b]$;

A function is said to be continuous in close interval $[a,b]$ if it satisfies following condi-

(i) f is continuous in (a,b)

(ii) f is right continuous at $x=a$
$$\lim_{x\to a^+} f(x) = f(a)$$

(iii) f is left conti. at $x=b$
$$\lim_{x\to b^-} f(x) = f(b)$$

Note: Generally continuity should be checked at following points.

1. Continuity of a function should be checked at the points where definition of function changes.

$$f(x) = \begin{cases} x+2 & x \geq 0 \\ 1-x & x < 0 \end{cases}$$

☆ $\boxed{x=0}$
L.H.L $= 1$
R.H.L $= 2$

$[x]$, $\{x\}$, $\{2x\}$

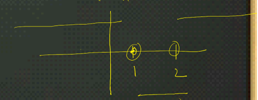
2. Continuity of $\{f(x)\}$ & $[f(x)]$ should be checked at all the points where f(x) is an integer.

3. Continuity of sgn(f(x)) should be checked at all the points where $f(x) = 0$.

$$\text{Sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

4. In case of composite functions $f(g(x))$ continuity should be checked at points of possible discontinuity of g(x) and at the points where $g(x) = c$ & $x = c$ is the discontinuity point of f(x).

☆ Sgn(f(x))
☆ Sgn((x-2)(x-1))
$x = 2, x = 1$



Sgn((x-1)(x-2)(x-3))

PROBLEMS ON CONTINUITY IN AN INTERVAL:

Q) Discuss the continuity of following functions:

① $f(x) = [\log_e x]$ $\quad 0 < x < 10$

$x = 0$

$\log_e x = 0$ $\quad x = e^0$ $\quad x = 1.$

$\log_e x = 1$ $\quad x = e$

$\log_e x = -1$ $\quad x = e^{-1}$

$e^{-2}, e^{-1}, e^0, e^1, e^2, \ldots$

$\frac{1}{e^4}, \frac{1}{e^3}, \frac{1}{e^2}, \frac{1}{e}, 1, e, e^2, e^3$

$2.7 \quad (2.7)^2 \quad (2.7)^3$

② $f(x) = [\sin^{-1} x]$

$\boxed{-\sin 1, 0, \sin 1}$

$-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$

$-1.57 \qquad 1.57$

$x = -\sin 1 \leftarrow \sin^{-1} x = -1$ $\qquad \sin^{-1} x = 1$ $\quad x = \sin 1$

$x = 0 \leftarrow \sin^{-1} x = 0$

③ $f(x) = [\cot^{-1} x]$ $\qquad 0 < \cot^{-1} x < \pi$

$\cot^{-1} x = 1, \quad \cot^{-1} x = 2, \quad \cot^{-1} x = 3$

$x = \cot 1, \quad x = \cot 2, \quad x = \cot 3$

Q) find number of points of discon

for $f(x) = \operatorname{sgn}(x^2 - 3x + 2) + [x - 3]$ in $[0, 4]$

$= \operatorname{sgn}\underbrace{((x-1)(x-2))}_{f_1(x)} + \underbrace{[x]}_{f_2(x)} - 3$

Right limit

	$x=0$	$x=1$	$x=2$	$x=3$	$x=4$
$f_1(x)$	C	d	d	C	C
$f_2(x)$	C	d	d	d	d

$x = 0$ continuous

discontinuous $x = 3$ $\qquad$ disconti $x = 4$

4

Q) find the number of points of discontinuity of $f(x) = \left[\frac{2}{1+x^2}\right]$ $\quad x \ge 0$

for range:

$y = \frac{2}{1+x^2}$ $\qquad (0, 2]$

$y + yx^2 = 2$

$x^2 = \left(\frac{2-y}{y}\right) \ge 0$

(-1) $\quad = \frac{y-2}{y} \le 0$

$+ \quad - \quad +$ $\quad 0 \quad 2$

$\frac{2}{1+x^2} = 1$, $\quad x^2 + 1 = 2$, $\quad x^2 = 1$, $\quad \boxed{x = \pm 1}$

$\frac{2}{1+x^2} = 2$, $\quad x^2 + 1 = 1$, $\quad \boxed{x = 0}$

$\{0, 1\}$

---

at $x = 1$ $\quad f(1) = 0 + [1] - 3 = -2$

LHL $= \operatorname{sgn}((1^- - 2)(1^- - 1)) + [1^-] - 3 = -2$ $\qquad$ LHL $\ne$ RHL

-ve -ve

$1 + 0 - 3$

discontinuous

RHL $= \operatorname{sgn}((1^+ - 2)(1^+ - 1)) + [1^+] - 3 = -1 + 1 - 3 = -3$

-ve +ve

$x = 2$ $\quad f(2) =$

RHL $= 0$

LHL $= -3$

disconti at $x = 2$.

Q) ① find the sum of points of discontinuity of $\operatorname{sgn}(x(x^2 + x + 1)) = f(x)$

$x = 0$

number of points of disconti

② $f(x) = \operatorname{sgn}(\sin x)$ in $0 \le x \le 4\pi$

$0, \pi, 2\pi, 3\pi, 4\pi$

Q) If $f(x) = \operatorname{sgn}(\sin x + a)$ is continuous for all values of x. then find the the interval of a

$-1 \le \sin x \le 1$

$\sin x = -a$

$-1 \le -a \le 1$

$\boxed{-1 \le a \le 1}$

$\boxed{a > 1 \text{ or } a < -1}$

A) $(-\infty, -1] \cup [1, \infty)$

B) $(-\infty, -1) \cup (1, \infty)$

C) $[-1, 1]$

D) $(-1, 1)$

$a = 2$

$\operatorname{sgn}(\sin x + 2)$

$\operatorname{sgn}(\sin x - 5)$

$\operatorname{sgn}(\sin x + 1)$

Q) If $f(x) = \begin{cases} [\sin \pi x] & 0 \le x < 1 \\ \operatorname{sgn}\left(x - \frac{5}{4}\right)\left\{x - \frac{2}{3}\right\} & 1 \le x \le 2 \end{cases}$

H.W

$x = 0, \quad x = \frac{1}{2}, \quad x = 1,$

$x = \frac{5}{4}, \quad \cancel{x = \frac{2}{3}}, \quad x = \frac{5}{3}$

$\left[\frac{2}{1+x^2}\right]$ $\quad$ $Sgn\,((x-1)(x-2))$ $\quad$ $[\sin x]$

Continuity of Composite function:

$\boxed{x=-2}$ $\quad$ $1-e^{-\frac{(x-1)}{x+2}}=0$ $\quad$ $\boxed{x=1}$

e.g $f(x)=\dfrac{1}{1-e^{-\left(\frac{x-1}{x+2}\right)}}$ $\quad$ $f(c)$ $\quad$ $\boxed{x=1}$

Points of disconti.

$\dfrac{x-1}{x+2}=0$

$g(x)=\dfrac{1}{1-e^{-x}}$ $\quad$ $h(x)=\dfrac{x-1}{x+2}$

$\boxed{x=-2}$

$f(x)=g(h(x))$

$X=c$

$1-e^{-c}=0$ $\quad$ $1=e^{-c}\Rightarrow \boxed{c=0}$

Q) If $f(x)=\dfrac{1}{1-x}$ $\longrightarrow$ $x=1$ discontinuous

then find no. of points of disconti

of $f(f(f(x)))=$

$f(f(x))=\dfrac{1}{1-f(x)}=\dfrac{1}{1-\frac{1}{1-x}}=\dfrac{(1-x)}{1-x-1}$ $\quad$ $x=0$ — disconti.

$=\dfrac{x-1}{x}$

$\boxed{x=1,\ x=0}$

Points of disconti.

$f(f(f(x)))=\dfrac{1}{1-f(f(x))}=\dfrac{1}{1-\frac{x-1}{x}}=\dfrac{x}{\cancel{x}-\cancel{x}+1}$

$\boxed{f(f(f(x)))=x}$