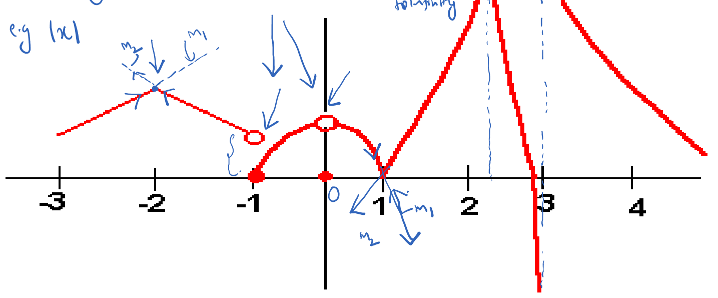
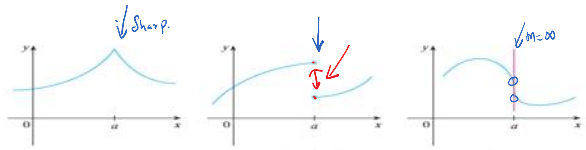
DIFFERENTIABILITY:

① Sharp point ② Discontinuity ③ Approaches to infinity

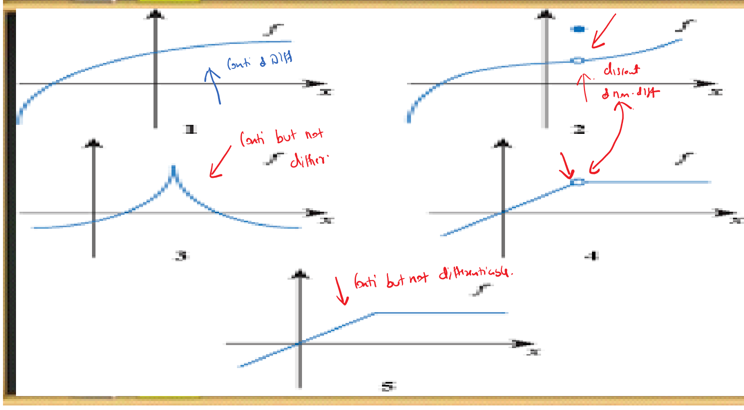
e.g $|x|$



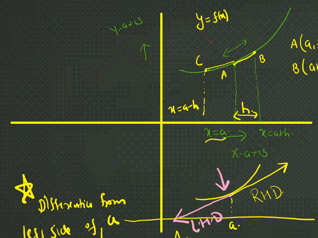
Non-differentiable function



(a) A corner (b) A discontinuity (c) A vertical tangent



FIRST PRINCIPLE AND GEOMETRICAL MEANING OF DERIVATIVE:



$A(a, f(a))$

$B(a+h, f(a+h))$

Rate of change

$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$

☆ $\frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{(a+h) - a}$

☆ $\frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{h}$

$h \to 0 \quad \Delta x \to dx, \ \Delta y \to dy$

☆ $\boxed{\frac{dy}{dx} = \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h}\right)} = R.H.D$

at $x=a$ {Diff from right side of a}

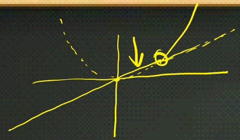
☆ Differentiation from left side of a

$C(a-h, f(a-h))$

$A(a, f(a))$

☆ $\boxed{\frac{dy}{dx} = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}} = L.H.D$

$a - h - a$



{If function is differentiable at $x=a$ then}

$\boxed{L.H.D = R.H.D = \text{finite \& exists}}$

Q) Check the differentiability of

$f(x) = \begin{cases} x & x<1 \\ x^2 & x \geq 1 \end{cases}$ at $x=1$

Conti:

$L.H.L = 1, \ R.H.L = 1 \ \ f(1) = 1$

Conti. at $x=1$

Diff: $a=1$

☆ $R.H.D = \lim_{h \to 0} \left(\frac{f(1+h) - f(1)}{h}\right) = \lim_{h \to 0} \frac{(1+h)^2 - 1}{h} = \lim_{h \to 0} \frac{\cancel{1} + h^2 + 2h - \cancel{1}}{h}$

$= 2$

$f(x) = x^2$

☆ $L.H.D = \lim_{h \to 0} \left(\frac{f(1-h) - f(1)}{-h}\right) = \lim_{h \to 0} \frac{\cancel{1} - h - \cancel{1}}{-h}$

$= 1$

$f(x) = x$

$\boxed{L.H.D \neq R.H.D}$

Conti at $x=1$ but not diff at $x=1$

DIFFERENTIATION USING FIRST PRINCIPLE:

e.g. $y = x^2 = f(x)$

Differ. at general point $x = x_0 = a_0$

$$f'(x) = \frac{dy}{dx} = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$$= \lim_{h\to 0} \frac{(x_0+h)^2 - x_0^2}{h} = \lim_{h\to 0} \frac{\cancel{x_0^2} + h^2 + 2x_0h - \cancel{x_0^2}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(h+2x_0)}{\cancel{h}} = 2x_0$$

☆ not a first principle:

$\boxed{\frac{d}{dx}(x^2) = 2x}$

$\left\{ \frac{d}{dx}(x^n) = nx^{n-1} \right\}$

e.g. $f(x) = \cos^2 x$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$\lim_{h\to 0} \left(\frac{\cos^2(x+h) - \cos^2 x}{h} \right)$$

$$= \lim_{h\to 0} \frac{\left(\frac{\cancel{1} + \cos(2x+2h)}{2} - \left(\frac{\cancel{1} + \cos 2x}{2} \right) \right)}{h}$$

$$\Rightarrow \lim_{h\to 0} \frac{(\cos(2x+2h) - \cos 2x)}{2h} = \lim_{h\to 0} \frac{-\cancel{2}\sin(h)\sin(2x+h)}{\cancel{2}h}$$

$$= \boxed{-\sin 2x}$$

e.g. $f(x) = \sin x$ $\quad f(x+h) = \sin(x+h)$, $f(x) = \sin x$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$\sin C - \sin D$

$$= 2\sin\left(\frac{C-D}{2}\right)\cos\left(\frac{C+D}{2}\right)$$

$$= \lim_{h\to 0} \left(\frac{\overset{C}{\sin(x+h)} - \overset{D}{\sin x}}{h} \right)$$

$$= \lim_{h\to 0} \left(\frac{\cancel{2}\sin\left(\frac{h}{2}\right)\cos\left(\frac{2x+h}{2}\right)}{\cancel{2}\left(\frac{h}{2}\right)} \right) = \cos x$$

Q) $f(x) = \sin\sqrt{x}$ $\quad f'(x) = \frac{1 \cos\sqrt{x}}{2\sqrt{x}}$

$f(x+h) = \sin(\sqrt{x+h})$

$$f'(x) = \lim_{h\to 0} \left(\frac{\sin(\overset{C}{\sqrt{x+h}}) - \sin\overset{D}{\sqrt{x}}}{h} \right)$$

$$= \lim_{h\to 0} \frac{\cancel{2}\sin\left(\frac{\sqrt{x+h}-\sqrt{x}}{2}\right)\cos\left(\frac{\sqrt{x+h}+\sqrt{x}}{2}\right)}{h} \left(\frac{\sqrt{x+h}-\sqrt{x}}{\cancel{2}} \right)$$

1

$$\frac{\sqrt{x+h}-\sqrt{x}}{2}$$

$$= \cos\sqrt{x} \cdot \frac{1}{2\sqrt{x}}$$

$$= \lim_{h\to 0} \cos\sqrt{x} \left(\frac{\sqrt{x+h}-\sqrt{x}}{h} \right) \times \frac{\sqrt{x+h}+\sqrt{x}}{(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{\cos\sqrt{x}\,(\cancel{x}+\cancel{h}-\cancel{x})}{\cancel{h}\,2\sqrt{x}}$$

$$= \frac{\cos\sqrt{x}}{2\sqrt{x}}$$

Q) $f(x) = \tan x$. $\quad f(x+h) = \tan(x+h)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$\boxed{\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}}$$

$$= \lim_{h\to 0} \frac{(\tan(x+h) - \tan x)}{h}$$

$$= \lim_{h\to 0} \frac{\left(\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x} \right)}{h}$$

$$\Rightarrow \lim_{h\to 0} \frac{(\overset{A}{\sin(x+h)}\overset{B}{\cos x} - \overset{A}{\cos(x+h)}\overset{B}{\sin x})}{h(\cos x \cdot \cos(x+h))}$$

$$\Rightarrow \lim_{h\to 0} \left(\frac{\sin(h)}{h} \right) \frac{1}{\cos x \cdot \cos x}$$

$$\Rightarrow \boxed{1 \cdot \sec^2 x}$$

H.W

$$f(x) = \begin{cases} A + Bx^2 & x < 1 \quad ① \\ 3Ax - B + 2 & x \geq 1 \quad ② \end{cases}$$

is differentiable at $x = 1$. then find A & B

☆ $f'(x) = \begin{cases} 2Bx & x < 1 \\ 3A & x \geq 1 \end{cases}$ ② L.H.D

Alt R.H.D

$\boxed{\begin{array}{l} L.H.D = 2B \\ R.H.D = 3A \end{array}}$ ②

Conti: at $x = 1$
L.H.L = R.H.L = $f(1)$

$A + B = 3A - B + 2$.

$0 = 2A - 2B + 2$

$A - B + 1 = 0$ — ①

$\times ②$ $\quad -2A + 2B - 2 = 0$

Differentiability: L.H.D = R.H.D

①

$a = 1$

$$L.H.D = \lim_{h\to 0} \frac{f(a-h) - f(a)}{-h}$$

$$= \lim_{h\to 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h\to 0} \frac{A + B(1-h)^2 - (3A - B + 2)}{-h}$$

$$= \lim_{h\to 0} \frac{A + B + Bh^2 - 2Bh - 3A + B - 2}{-h}$$

$$= \frac{(-2A + 2B - 2) + Bh^2 - 2Bh}{-h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(Bh - 2B)}{-\cancel{h}} = 2B$$

R.H.D = 3A

L.H.D = R.H.D

$3A = 2B$ — ②

0

from ① & ② $\boxed{A = 2, B = 3}$

Q) If $f(x) = \begin{cases} 3x-2 & 0 < x \le 1 \\ 2x^2 - x & 1 < x \le 2 \\ 5x-4 & x > 2 \end{cases}$, $f'(x) = \begin{cases} 3 & 0 < x \le 1 \\ 4x-1 & 1 < x \le 2 \\ 5 & x > 2 \end{cases}$

Cont | Diffe

$x = 1$: L.H.L = R.H.L = f(1); $\boxed{1 = 1 = 1}$ ✓ | $x=1$, L.H.D = 3, R.H.D = 3 = 4−1 ✓

$x = 2$: ✓ L.H.L = R.H.L = f(2); $\boxed{6 = 6 = 6}$ | $x=2$: L.H.D = 7, R.H.D = 5 ✗

$x=2$ Sharp point.

Q) find the L.H.D of $f(x) = [x] \sin \pi x$ at $x = 7$

A) 6π
B) -6π
C) 7π
D) -7π

$a = 7$

$$L.H.D = \lim_{h \to 0} \left(\frac{f(7-h) - f(7)}{-h} \right)$$

$f(7-h) = [7-h] \sin \pi (7-h)$

$f(7) = [7] \sin 7\pi = 0$

$$= \lim_{h \to 0} \frac{[7-h] \sin (7\pi - \pi h) - 0}{-h}$$

$\sin(7\pi - \theta) = (+) \sin\theta$

$$\lim_{h \to 0} \frac{[7-h] \sin \pi h - 0}{-h} = \lim_{h \to 0} \frac{6 \sin \pi h}{-h} = \boxed{-6\pi}$$

Q) Discuss Diffrn. of $f(x) = \begin{cases} \dfrac{x}{2x^2 + |x|} & x \ne 0 \\ 1 & x = 0 \end{cases}$ at $\boxed{x=0}$

$x \ne 0$ → $\begin{cases} \dfrac{x}{2x^2+x} & x > 0 \\ \dfrac{x}{2x^2 - x} & x < 0 \end{cases}$

Conti

$$L.H.L = \lim_{x \to 0^-} \left(\frac{x}{2x^2 - x} \right) = \lim_{x \to 0^-} \left(\frac{1}{2x-1} \right) = -1 = \frac{1}{0-1}$$

$f(0) = 1$

$\boxed{L.H.L \ne f(0)}$

discontinuous → non-differentia

Q) P.T L.H.D $f(x) = [x] \sin \pi x$ at $x = k$ is $\boxed{(-1)^k (k-1)\pi}$ {where k is an integer}

Q) find a+b if $f(x) = \begin{cases} e^{x^2} + x & x > 0 \\ ax + b & x \le 0 \end{cases}$ is differentiable at $x = 0$

Conti: $\boxed{\begin{matrix} L.H.L = e^0 + 0 = 1 \\ R.H.L = b \end{matrix}}$ $\boxed{b = 1}$

$f'(x) = \begin{cases} e^{x^2} \cdot 2x + 1 & x > 0 \\ a & x < 0 \end{cases}$

$\boxed{a + b = 2}$

differ: $\boxed{\begin{matrix} L.H.D = R.H.D \\ a = 0 + 1 \end{matrix}}$ $\boxed{a = 1}$

$x = 0$

Q) Discuss Differentia. at $x = 1$ for $f(x) = \begin{cases} [\cos \pi x] & x \le 1 \\ 2\{x\} - 1 & x > 1 \end{cases}$

$\boxed{L.H.D \ne R.H.D}$ not differentiable

Conti:

$x = 1$: $f(1) = -1$

$$L.H.L = \lim_{x \to 1^-} [\cos \pi x] = -1$$

$$R.H.L = \lim_{x \to 1^+} (2\{x\} - 1) = 0 - 1 = -1$$

$\boxed{L.H.L = R.H.L = f(1)}$

$\boxed{\text{Conti at } x=1}$

$f(1-h) = [\cos(\pi(1-h))]$

$\cos(\pi - \theta) = -\cos\theta$

$\boxed{f(1+h) = 2\{1+h\} - 1}$

Diffi:

$$L.H.D = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h \to 0} \left(\frac{\cos(\pi - \pi h) + 1}{-h} \right)$$

$$= \lim_{h \to 0} \left(\frac{1 - \cos \pi h}{-h} \right)$$

$$= \lim_{h \to 0} \frac{2 \sin^2 \frac{\pi h}{2}}{-h \times h} \times h = -2\left(\frac{\pi}{2}\right)^2 \times 0 = 0$$

$$R.H.D = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \quad (1)$$

$$= \lim_{h \to 0} \left(\frac{2\{1+h\} - 1 + 1}{h} \right) \quad (2)$$

$$= \lim_{h \to 0} \left(\frac{2\{h\}}{h} \right) = \lim_{h \to 0} \frac{2h}{h} = 2 \quad (3)$$

$0 < x < 1$, $\{x\} = x$

$[x] + \{x\} = x$

$\{x + I\} = \{x\}$

$y = x$

Q) If the function $g(x) = \begin{cases} k\sqrt{x+1} & 0 \le x \le 3 \\ mx + 2 & 3 < x \le 5 \end{cases}$ is differentiable at $x = 3$

find $(k+m)$

A) 2
B) $\frac{16}{5}$
C) $\frac{10}{3}$
D) 4

$\boxed{\frac{8}{5} + \frac{2}{5} = 2}$

Conti: $x = 3$

$L.H.L = k\sqrt{3+1} = 2k$

$R.H.L = 3m + 2$

$2k = 3m + 2$ — (1)

from (1) & (2)

$\boxed{k = \frac{8}{5}, m = \frac{2}{5}}$

$g'(x) = \begin{cases} k \cdot \dfrac{1}{2\sqrt{x+1}} & 0 \le x \le 3 \\ m & 3 < x \le 5 \end{cases}$

$L.H.D = \dfrac{k}{2 \times \sqrt{3+1}} = \dfrac{k}{4}$

$R.H.D = m$

$\boxed{k = 4m}$ — (2)

H.W Check conti & differentiability

of $f(x) = \begin{cases} x^2 \operatorname{sgn}[x] + \{x\} & 0 \le x < 2 \\ \sin x + |x-3| & 2 \le x \le 4 \end{cases}$

at $x=1$ & $x=2$

	Conti.	Differ.
$x=1$		
$x=2$		

Draw the graph of following function:

1. $y = \sin|x|$
2. $y = \left|\log|x|\right|$
3. $y = -\sin^{-1}(\sin x)$
4. $y = -\cos^{-1}(\sin x)$
5. $\min(\sin x, \cos x)$
6. $\max(\sin x, \cos x)$
7. $y = \max(\tan x, \cot x)$ in $(0, 2\pi)$
8. $y = \left||x-1| - 5\right|$
9. $\min\{|x|, |x-2|, 2-|x-1|\}$
10. $\max\{2\sin x, 1-\cos x\}$ $x \in (0, \pi)$