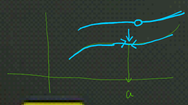
CONTINUITY :

Definition:

A function f(x) is Said to be continuous at $x = a$

if
$$\boxed{\lim_{x\to a^-}(f(x)) = \lim_{x\to a^+}(f(x)) = f(a)}$$
$$\text{L.H.L} = \text{R.H.L} = \text{function at } x = a \text{ value.}$$



e.g $f(x) = \frac{\sin x}{x}$ at $x=0$ is it continuous? → not continuous

$\Rightarrow$ L.H.L = R.H.L = $\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = 1$

$f(0) = \frac{\sin 0}{0} =$

e.g $f(x) = \frac{\sin x}{x}$, $x = \frac{\pi}{2}$

$x = 2$, continuous?

L.H.L = $\lim_{x\to 2^-}\frac{\sin 2}{2}$ = R.H.L

$f(2) = \frac{\sin 2}{2}$

$\frac{\sin 2}{2}$ → 1

$1 = 57^\circ$

$\frac{\sin 114^\circ}{2}$

Q) $\Rightarrow$ Discuss the continuity at $\boxed{x=0}$

$$\begin{cases} \frac{1-\cos x}{x^2} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

$$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \frac{1}{2}$$

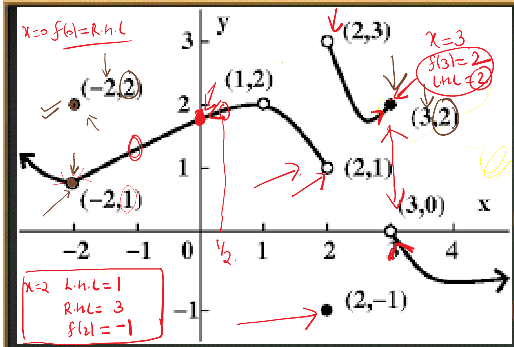
$f(0) = 1$

L.H.L = R.H.L

$\lim_{x\to a} f(x) = f(a)$

$\lim_{x\to 0} f(x) \neq f(0)$.

$\boxed{\text{Discontinuous}}$



① $\lim_{x\to -2} f(x) = 1$

Is function conti. at $x = -2$

$f(-2) = 2$

② Is function conti at $x = 0$? Yes

③ $\lim_{x\to 3^-} f(x) = 2 = f(3)$

④ If f(x) is conti at $x = 3$?

R.h.L = 0 at $x = 3$

$x = 2$, $x = 1$

$x = 1/2$

Q) If the function is $\boxed{\text{continous}}$ at $x = 1$ find a & b.

$$f(x) = \begin{cases} 3ax + b & x > 1 \\ 11 & x = 1 \\ 5ax - 2b & x < 1 \end{cases}$$

$$\text{L.H.L} = \lim_{x\to 1^{(-)}}(5ax - 2b) = 5a - 2b$$
$$\text{R.H.L} = \lim_{x\to 1^{(+)}}(3ax + b) = 3a + b$$
$$f(1) = 11$$

$\boxed{\begin{matrix} 5a - 2b = 11 \\ 3a + b = 11 \end{matrix}}$ $\boxed{\begin{matrix} a = 3 \\ b = 2 \end{matrix}}$

Q) Let f(x) be the function

$$f(x) = \left(\frac{\sqrt{4+x} - 2}{x}\right) (x \neq 0)$$

What choice of f(0) will make it continuous at $x = 0$

$$\lim_{x\to 0} f(x) = f(0) = \frac{1}{4}$$

$$\lim_{x\to 0} f(x) = f(0) = \frac{1}{4}$$

$$\lim_{x\to 0}\left(\frac{\sqrt{4+x} - 2}{x}\right) = \frac{\frac{1}{2\sqrt{4+x}} - 0}{1} = \frac{1}{2\times 2} = \frac{1}{4}$$

PROBLEMS

Q) Let $f(x) = \begin{cases} \dfrac{1-\cos 4x}{2x^2} & x<0 \\ a & x=0 \\ \dfrac{\sqrt{x}}{\sqrt{16+x}-4} & x>0 \end{cases}$

find value of a such that $f(x)$ is conti. at $x=0$

A) 4

B) 8

C) 12

D) No value of a

Non-Removable Type

$$\boxed{\begin{array}{cccc} L.H.L = & R.H.L & = f(0) \\ 4 \neq & \infty & \neq a \end{array}}$$

$$L.H.L = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1-\cos 4x}{2x^2}$$

$$= \lim_{x\to 0^-} \frac{2\sin^2 2x}{2x^2} = \lim_{x\to 0^-}\left(\frac{\sin 2x}{x}\right)^2$$

$$= 2^2 = 4$$

$$R.H.L = \lim_{x\to 0^+} \frac{\sqrt{x}}{\sqrt{16+x}-4} \times \frac{(\sqrt{16+x}+4)}{\sqrt{16+x}+4}$$

$$= \lim_{x\to 0^+} \frac{\sqrt{x}\,(\sqrt{16}+4)}{\cancel{16}+x-\cancel{16}} = 8 \lim_{x\to 0^+}\left(\frac{\sqrt{x}}{x}\right) \to \infty$$

Q) The function $f(x) = \dfrac{\log(1+ax)-\log(1-bx)}{\sin x}$ $(x \neq 0)$ is not defined at $x=0$. find $f(0)$ so that f is continuous at $x=0$

A) $a-b$

B) a

C) b

D) $a+b$

$$\lim_{x\to 0} f(x) = f(0)$$

$$\lim_{x\to 0}\left(\frac{\log(1+ax)}{x} - \frac{\log(1-bx)}{x}\right)$$

$$= a-(-b) = a+b$$

Q) if $f(x) = \begin{cases} |x-a|\sin\left(\dfrac{1}{x-a}\right) & x \neq a \\ 0 & x=a \end{cases}$

$$|x-a| \begin{cases} x-a & x>0 \\ -(x-a) & x<0 \end{cases}$$

Prove that $f(x)$ is conti. at $x=a$

$$0 \times [-1,1] = 0$$

$$f(x) = \begin{cases} (x-a)\sin\left(\dfrac{1}{x-a}\right) & x>a \\ 0 & x=a \\ -(x-a)\sin\left(\dfrac{1}{x-a}\right) & x<a \end{cases}$$

$$\boxed{x \sin \frac{1}{x}}$$

$$R.H.L = \lim_{x\to a^+} (x-a)\sin\left(\frac{1}{x-a}\right) \quad [-1,1]$$

$$= 0 \times \text{—} = 0$$

$$f(a) = 0$$

$$L.H.L = 0 \times \{\text{—}\} = 0$$

Types of Continuity;

Continuous at $x=a$

① if $f(x)$ is Continuous at $x=a$

$L.H.L = R.H.L = f(a)$

Right as well as left Continuous

② Left Continuous : $L.H.L = f(a)$

③ Right Continuous : $R.H.L = f(a)$.

Q) Examine the Continuity of function

$$f(x) = \begin{cases} \dfrac{e^{1/x}-1}{e^{1/x}+1} & x \neq 0 \\ 0 & x=0 \end{cases}$$ at $x=0$.

$$\boxed{\begin{array}{c} \lim_{x\to 0^-}\left(\frac{1}{x}\right) = -\infty \\ \lim_{x\to 0^+}\left(\frac{1}{x}\right) = \infty \end{array}}$$

$$\lim_{x\to 0^-}\left(\frac{1-e^{-1/x}}{1+e^{-1/x}}\right) \quad \infty$$

$$L.H.L = \lim_{x\to 0^-}\left(\frac{e^{1/x}-1}{e^{1/x}+1}\right) = \left(\frac{e^{-\infty}-1}{e^{-\infty}+1}\right) = \frac{0-1}{0+1} = -1$$

$$R.H.L = \lim_{x\to 0^+}\left(\frac{e^{1/x}-1}{e^{1/x}+1}\right) = 1$$

$$\boxed{f(0) = 0}$$

$$\boxed{R.H.L \neq L.H.L \neq f(0)}$$

e.g

$$\lim_{x\to 1}[x] = \quad L.H.L = \lim_{x\to 1^-}[x] = 0$$

$$f(1) = 1$$

$$R.H.L = [1^+] = 1$$

Right Continuous

e.g

$$\lim_{x\to 1}\{x\} =$$

Right Conti at $x=1$.

TYPES OF DISCONTINUITY:

① Disconti. of I kind : (Removable type)

If $\lim_{x\to a} f(x)$ exist $\underline{L.H.L = R.H.L} \neq f(a)$

or $f(a)$ is not defined

then we will choose/change such that

$\boxed{L.H.L = R.H.L = f(a)}$

☆ e.g $f(x) = \frac{\sin x}{x}$ at $\boxed{x=0}$

redefine $\boxed{f(0)=1}$

② Disconti. of II kind : (non-removable type)

If $L.H.L \neq R.H.L$ or $L.H.L, R.H.L \to$ not defined

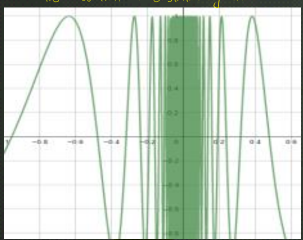
Limit does not exist

③ Oscillation discontinuity :

If $f(x)$ oscillates between two finite values at $x=a$ then $f(x)$ has oscillation discontinuity at $x=a$

e.g $f(x) = \sin\frac{\pi}{x}$

$x \to 0$



④ Jump of discontinuity :

$\boxed{\text{If } \begin{array}{l} L.H.L \to \text{exists} \\ R.H.L \to \text{exists} \\ L.H.L \neq R.H.L \end{array}}$

e.g at $x=1$

$L.H.L = 0$, $R.H.L = 1$

$f(x) = [x]$ $|L.H.L - R.H.L| = |0-1| = 1$

$f(x) = \{x\}$

then Jump of disconti. :

☆ $= |L.H.L - R.H.L|$

☆ $\lim_{x\to 0} \frac{|\sin x|}{x}$

$L.H.L = \lim_{x\to 0^-} \frac{-\sin x}{x} = -1$

$R.H.L = \lim_{x\to 0^+} \left(\frac{\sin x}{x}\right) = 1$

②

THEOREM ON CONTINUITY OF FUNCTION AT X=a:

① If f & g are two functions which are continuous at $x=a$ then functions defined as

✓ $F_1(x) = f_1(x) \pm g_1(x)$

✓ $F_2(x) = f_1(x) \cdot g_1(x)$

✓ $F_3(x) = \frac{f_1(x)}{g_1(x)}$ $\{g_1(a) \neq 0\}$

are all continuous at $x=a$.

e.g $f(x) = \frac{x + \sin x}{\cos x}$ — conti at $x=0$

at $\boxed{x=0}$

e.g $f(x) = \frac{\sin x}{x}$ $\boxed{x=0}$ disconti at $x=0$

$\boxed{x=1}$ continuous at $x=1$.

② $f(x)$ is conti. & $g(x)$ is disconti. at $x=a$.

☆ then $f_1(x) = f(x) \pm g(x) \longrightarrow$ discontinuous.

e.g $f(x) = x$, $g(x) = [x]$ $f(x) \pm g(x) \to$ discontinuous

$\boxed{x=1}$

c
d
m

$f_2(x) = f(x) \cdot g(x)$ may or may not be

e.g $x[x] = 0$ $\boxed{x \to 0}$

$L.H.L = (0^-)[0^-]$

$= 0 \times (-1)$

$= 0$

$R.H.L = (0^+)([0^+]) = 0$

☆ $F_3(x) = \frac{f(x)}{g(x)} \to$ may not may be.

③ $f(x)$ is disconti. at $x=a$ then

$g(x)$ is disconti

$\boxed{L.H.L, R.H.L, f(a)}$

① $f(x) \pm g(x)$

② $f(x) \cdot g(x)$

③ $\frac{f(x)}{g(x)}$

may or may not be

e.g $f(x) = [x]$ $g(x) = \{x\}$ $f(x) + g(x) = [x] + \{x\}$

$\boxed{x=1}$ $f(x)$ is disconti. $x=1$ $= x$

$g(x)$ is disconti at $x=1$

at $x=1$ conti at $\boxed{x=1}$

THEOREMS CONT...

(4) Every Polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0 x^0$$ is

Continuous at every point on real number line.

(5) Every rational polynomial is continuous at every point where its denomiator is not zero.

e.g

$$f(x) = \frac{x^2+x+1}{(x-1)(x-2)}$$

Points where $f(x)$ is discontinuous: $x=1, x=2$

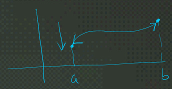
(6) Log, Exponential, Trigonometric, $\boxed{\text{Inverse trigonometric,}}$ modulus function are all continuous in their $\boxed{\text{domain}}$.

☆ $\tan x$ $\boxed{x = \frac{\pi}{2}}$ discontinuous $\qquad x = \frac{\pi}{3}$ continuous

CONTINUITY OF A FUNCTION IN AN INTERVAL:

(1) Continuity of a function in open interval (a,b);

A function is said to be continuous in (a,b) if f is continuous at each and every point in (a,b)



(2) Continuity of a function in closed interval $[a,b]$;

A function is said to be continous in close interval $[a,b]$ if it satisfies following condi-

(i) f is continuous in (a,b)

(ii) f is right continuous at $x=a$

$$\lim_{x \to a^+} f(x) = f(a)$$

(iii) f is lef conti. at $x=b$

$$\lim_{x \to b^-} f(x) = f(b)$$