ALGEBRA OF MONOTONOUS FUNCTIONS:

Let increasing = I and decreasing = D (where both the functions take +ve values) and can not say anything = Ω

(i) **Addition**

(a) $I + I = I$ (b) $D + D = D$

(c) $I + D = \Omega$ (d) $D + I = \Omega$

(ii) **Negativity**

(a) $-I = D$ (b) $-D = I$

(iii) **Difference**

(a) $I - I = \Omega$ (b) $D - D = \Omega$

(c) $I - D = I$ (d) $D - I = D$.

(iv) **Product**

(a) $I \times I = I$ (b) $I \times D = \Omega$

(c) $D \times D = D$ (d) $D \times I = \Omega$.

(v) **Reciprocality**

(a) $1/I = D$ (b) $1/D = I$

(vii) **Composition**

(a) $I(I) = I$ (b) $I(D) = D$

(c) $D(I) = D$ (d) $D(D) = I$.

Decreasing

(a) e.g. $f(x)$ increasing, $g(x)$ decreasing

$f(f(x)) = h(x)$

$x_2 > x_1$

$\boxed{h(x_2) > h(x_1)}$

$f(f(x_2))$ $f(f(x_1))$

$\boxed{f(x_2) > f(x_1)}$

$\boxed{t_2 > t_1} \rightarrow f(t_2) > f(t_1)$

(b) $f(g(x)) = h(x)$

$x_2 > x_1$ $\boxed{h(x_2) < h(x_1)}$

$\boxed{f(g(x_2)) < f(g(x_1))}$

$\boxed{f(t_2) < f(t_1)}$

$g(x_2) < g(x_1)$

t_2 t_1 $t_2 < t_1$

$\Rightarrow f(f(f(f(f(x)))))$

$\Rightarrow g(g(x)) = I$

$f(f(g(x))) \rightarrow \downarrow$

Q) If $f(x)$ is strictly increasing function & $f(\alpha^2 - 3\alpha) > f(-2)$ then solve for α.

$f(x_2) > f(x_1)$ — $x_2 > x_1$

$\alpha^2 - 3\alpha > -2$

$\alpha^2 - 3\alpha + 2 > 0$

$(\alpha - 1)(\alpha - 2) > 0$

$+$ 1 $-$ 2 $+$

$(-\infty, 1) \cup (2, \infty)$

Q) Let $f(x)$ and $g(x)$ be two cont. functions defined from $R \to R$ such that $f(x_1) > f(x_2)$ and $g(x_1) < g(x_2)$ for all $x_1 > x_2$. Find the solution of $f(g(\alpha^2 - 2\alpha)) > f(g(3\alpha - 4))$

f is increasing

g is decreasing

$\alpha^2 - 2\alpha < 3\alpha - 4$

A) $(0, 4)$

B) $(-\infty, 1) \cup (4, \infty)$

C) $(1, 4)$

D) none of above

$x_1 > x_2$

$g(\alpha^2 - 2\alpha) > g(3\alpha - 4)$

$\alpha^2 - 2\alpha < 3\alpha - 4$

$\alpha^2 - 5\alpha + 4 < 0$

$(\alpha - 1)(\alpha - 4) < 0$

$+$ 1 $-$ 4 $+$

$(1, 4)$

$f(f(g(f(g(f(g(g(x))))))))$

Q) If $\phi(x) = f(x) + f(1 - x)$ and $f''(x) < 0$ for $0 \le x \le 1$, then $\phi(x)$ is decreasing in the interval

$f'(x)$ is decreasing

① $\phi'(x) = f'(x) - f'(1 - x) < 0$

$\Rightarrow f'(x) < f'(1 - x)$

② $x > 1 - x$

$2x > 1$

$\boxed{x > \frac{1}{2}}$

$\left[\frac{1}{2}, 1\right]$ $\frac{1}{2}, \infty$

A) $\left(0, \frac{1}{2}\right)$

B) $(0, 1)$

C) $\left(\frac{1}{2}, 1\right)$

D) ϕ

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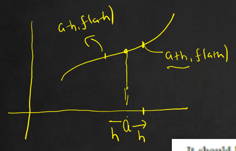
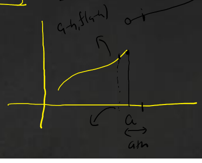
Monotonicity at a point:

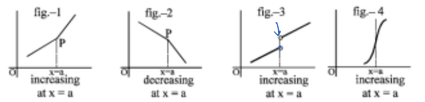
A function $f(x)$ is called strictly increasing function at a point $x=a$, if in a sufficiently small neighbourhood around $x=a$ it satisfies $\boxed{f(a-h) < f(a) < f(a+h)}$

Same analogy for decreasing. {strictly decreasing}

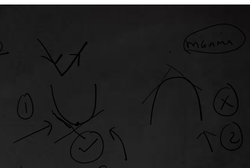




It should be noted that we can talk of monotonocity of $f(x)$ at $x = a$ only if $x = a$ lies in the domain of f, without any consideration of continuity or differentiability of $f(x)$ at $x = a$.



Q) Check monotonicity of $f(x) = x^3 - 3x + 2$



at $f'(x) = 3x^2 - 3$

① $x = 0$ $f'(0) = -3 < 0$ ↓

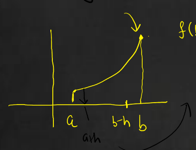
② $x = 1$ $f'(1) = 3(1^2 - 1) = 0$

$f'(1^+) = 3((1^+)^2 - 1) > 0$

$f'(1^-) = 3((1^-)^2 - 1) < 0$

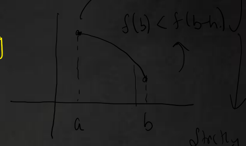
③ $x = 2$ $f'(2) = 3\times 2^2 - 3 = 9 > 0$ ↑

Note: ① If $x=a$ is a boundary point then use appropriate one sided inequality for test of monotonicity.



$f(b-h) < f(b)$ → strictly increasing

$f(a) < f(a+h)$



$f(a) > f(a+h)$

$f(b) < f(b-h)$ } strictly decreasing

② Test for increasing / decreasing at a point of differentiable function.

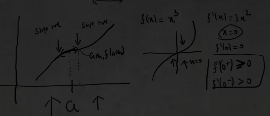
i d n.

① $f'(a) > 0$ strictly increasing at $x=a$

② $f'(a) < 0$ strictly decreasing at $x=a$

③ $f'(a) = 0$

a) if $f'(a^+) > 0$ and $f'(a^-) > 0$



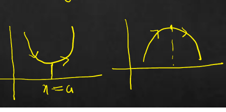
Self practice b) if $f'(a^+) < 0$ and $f'(a^-) < 0$ strictly decreasing

e.g $f(x) = -x^3$, $x = 0$.

$f(a+h) < f(a) < f(a-h)$

c) if $f'(a^+) \cdot f'(a^-) < 0$

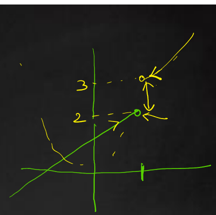
neither increasing nor decreasing



Q) find interval for λ for which function

$$f(x) = \begin{cases} x+1 & x<1 \\ \lambda & x=1 \\ x^2 - x + 3 & x>1 \end{cases}$$

is strictly increasing at $x=1$



A) $2 < \lambda < 3$

B) $2 \le \lambda \le 3$ ✓

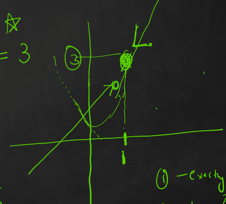
C) $2 < \lambda \le 3$

D) $2 \le \lambda < 3$

L.H.L $= 2$ R.H.L $= 3$

$f(1-h) < f(1) < f(1+h)$

$\boxed{2 < f(1) < 3}$



① – exactly ③

② – bigger ③

$2 \le \lambda \le 3$

Use of monotonicity to prove inequalities:

\# Comparing of two functions $f(x)$ and $g(x)$ can be done by analysing the monotonic behaviour of another function $h(x) = f(x) - g(x)$.

e.g $\sin x < x$ in $(0, \frac{\pi}{2})$

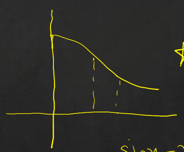
$f(x) = \sin x, \; g(x) = x.$

$h(x) = \sin x - x.$

$\boxed{h'(x) = \cos x - 1 < 0}$ $\rightarrow (0, \frac{\pi}{2})$

$h(0) = 0$

☆ $h(x)$ is strictly decreasing in $(0, \frac{\pi}{2})$



☆ $h(x) < h(0)$

$\sin x - x < 0$

$\boxed{\sin x < x}$

e.g $x < \tan x$ $(0, \frac{\pi}{2})$

$h(x) = x - \tan x$

$h'(x) = 1 - \sec^2 x < 0$

$h(x) \downarrow$

$h(x) < h(0)$

$x - \tan x < 0$

$x < \tan x$

Q) Find the monotonic behaviour of $f(x) = x^{1/x}$ and explain which is greater $\rightarrow$ π^e or e^π!

100^{101} or 101^{100}