TRIGONOMETRIC LIMITS:

① $\lim_{x\to0}\left(\frac{\sin x}{x}\right)=1$

② $\lim_{x\to0}\left(\frac{\tan x}{x}\right)=1$

③ $\lim_{x\to0}\left(\frac{\sin mx}{x}\right)=m$

④ $\lim_{x\to0}\left(\frac{\tan mx}{x}\right)=m$

⑤ $\lim_{x\to0}\left(\frac{\sin mx}{\sin nx}\right)=\frac{m}{n}$

⑥ $\lim_{x\to0}\left(\frac{\sin mx}{\tan nx}\right)=\frac{m}{n}$

Proof: $\lim_{x\to0}\left(\frac{\sin mx}{x}\right)\left(\frac{x}{\sin nx}\right)=\frac{m}{n}$

⑦ $\lim_{x\to0}\frac{1-\cos x}{x^2}=\frac{1}{2}$

$\lim_{x\to0}\frac{2\sin^2\frac{x}{2}}{x^2}$

$\lim_{x\to0}2\left(\frac{\sin\frac{x}{2}}{x}\right)^2$

$=2\times\left(\frac{1}{2}\right)^2=2\times\frac{1}{4}=\frac{1}{2}$

③ Proof $\lim_{x\to0}\left(\frac{\sin mx}{x}\right)=m$

$\lim_{x\to0}m\frac{\sin(mx)}{(mx)}=m\lim_{x\to0}\frac{\sin(mx)}{mx}=m\lim_{t\to0}\left(\frac{\sin t}{t}\right)$

$t=mx$ $\quad=m$

$x\to0$

$t\to0$

Proof:

$\lim_{x\to0}\left(\frac{\sin x}{x}\right)=\lim_{x\to0}\frac{\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}\cdots\cdots\right)}{x}$

$=\lim_{x\to0}\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}\cdots\right)=1$

☆ $\boxed{\lim_{x\to0}\left(\frac{\tan x}{x}\right)=\lim_{x\to0}\frac{\left(x+\frac{x^3}{3}+\frac{2}{15}x^5+\frac{17}{315}x^7\cdots\cdots\right)}{x}}=1$

Alt)

$\lim_{x\to0}\left(\frac{\tan x}{x}\right)=\lim_{x\to0}\frac{\sin x}{\cos x\,x}=\lim_{x\to0}\frac{1}{\cos x}\cdot\lim_{x\to0}\left(\frac{\sin x}{x}\right)$

$=1$

① $\lim_{x\to0}\left(\frac{1-\cos x}{x}\right)=\lim_{x\to0}\frac{2\sin^2\frac{x}{2}}{x}=\lim_{x\to0}2\left(\frac{\sin\frac{x}{2}}{x}\right)\sin\frac{x}{2}=0$

$=\lim_{x\to0}2\cdot\lim_{x\to0}\left(\frac{\sin\frac{x}{2}}{x}\right)\lim_{x\to0}\sin\frac{x}{2}$

$=2\cdot\frac{1}{2}\cdot0=0$

② $\lim_{x\to0}\left(\frac{1-\cos x}{x^2}\right)=\frac{1}{2}$

③ $\lim_{x\to0}\left(\frac{1-\cos x}{x^3}\right)=\lim_{x\to0}\left(\frac{2\sin^2\frac{x}{2}}{x^3}\right)=\lim_{x\to0}2\left(\frac{\sin\frac{x}{2}}{x}\right)^2\left(\frac{1}{x}\right)$

$=2\times\frac{1}{4}\times\infty=\infty$

Q)

$\lim_{x\to0}\left(\frac{\tan x-\sin x}{x^3}\right)=\frac{1}{2}$

$\Rightarrow\lim_{x\to0}\frac{\sin x\,(1-\cos x)}{x^3\cos x}$

$\Rightarrow\lim_{x\to0}\left(\frac{\sin x}{x}\right)\left(\frac{1-\cos x}{x^2}\right)\frac{1}{\cos x}$

$\Rightarrow 1\cdot\frac{1}{2}\cdot\frac{1}{1}=\frac{1}{2}$

Problems on Trigonometric Limits:

① $\lim_{x\to 0}\frac{\sin(x^\circ)}{x} = \lim_{x\to 0}\frac{\sin\left(\overset{mx}{\frac{\pi}{180}x}\right)}{x} = \left(\frac{\pi}{180}\right)$

π radian $= 180^\circ$

$1^\circ = \frac{\pi}{180}$ radian

$x^\circ = \frac{\pi}{180}x$ rad

② $\lim_{x\to 0}\left(\frac{\sin(\overset{\text{radian}}{x})}{x}\right) = 1$

③ $\lim_{x\to 0}\frac{(1-\cos 2x)(\sin 5x)}{x^2(\sin 3x)}$ A) $\frac{5}{3}$ B) $\frac{5}{6}$ C) $\frac{10}{3}$ D) $\frac{5}{4}$

$= \left(\frac{5}{3}\right)\left(\lim_{x\to 0}\frac{1-\cos 2x}{x^2}\right) = \frac{5}{3}\lim_{x\to 0}\frac{2\sin^2 x}{x^2} = \frac{10}{3}\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2$

$= \frac{10}{3}\times 1^2 = \frac{10}{3}$

Q) $\lim_{x\to \frac{\pi}{4}}\left(\frac{\cos x - \sin x}{x-\frac{\pi}{4}}\right) = \lim_{x\to \frac{\pi}{4}}\frac{\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x\right)}{x-\frac{\pi}{4}}$

A) $\sqrt{2}$

B) $-\sqrt{2}$

C) 1

D) -1

$= \lim_{x\to \frac{\pi}{4}}\frac{\sqrt{2}\left(\sin\frac{\pi}{4}\cos x - \cos\frac{\pi}{4}\sin x\right)}{x-\frac{\pi}{4}}$

$\Rightarrow \lim_{x\to \frac{\pi}{4}}\sqrt{2}\,\frac{\sin\left(\frac{\pi}{4}-x\right)}{x-\frac{\pi}{4}}$ $\quad t = \frac{\pi}{4}-x$

$\Rightarrow \lim_{t\to 0}\sqrt{2}\left(\frac{\sin t}{-t}\right) = -\sqrt{2}$

Q) $\lim_{x\to 0}\frac{\sin(\pi\cos^2 x)}{x^2} \rightarrow \frac{\sin(\pi\cos^2 x)}{\pi\cos^2 x}\cdot\frac{\pi\cos^2 x}{x^2}$

$\sin(\pi-\theta) = \sin\theta$

A) 1 $= \lim_{x\to 0}\frac{\sin(\pi - \pi\cos^2 x)}{x^2}$

B) π $= \lim_{x\to 0}\frac{\sin(\pi\sin^2 x)}{x^2\,(\pi\sin^2 x)}\,\pi\sin^2 x$ $\quad t = \sin^2 x\cdot\pi$

C) $-\pi$

D) -1

$= \lim_{x\to 0}\pi\,\frac{\sin^2 x}{x^2} = \boxed{\pi}$

Q) Evaluate $\lim_{x\to 0} f(x)$ if $f(x) = \begin{cases}\frac{1-\cos 4x}{x^2} & x<0\\ 0 & x=0\\ [x] & x>0\end{cases}$

A) 2

B) 1

C) 8

D) does not exist

L.H.L $= \lim_{x\to 0^-}\left(\frac{1-\cos 4x}{x^2}\right) = \lim_{x\to 0^-}\frac{2\sin^2 2x}{x^2} = \lim_{x\to 0^-} 2\cdot\left(\frac{\sin 2x}{x}\right)^2 = 2\times 2^2 = 8$

R.H.L $= \lim_{x\to 0^+}[x] = 0$

L.H.L $\neq$ R.H.L

Standard Algebraic Limits;

$$\underset{x\to a}{Lt}\left(\frac{x^n-a^n}{x-a}\right) = na^{n-1} \qquad n\in R$$

$$f(x) = \frac{x^n-a^n}{x-a} \qquad f(a+h) = \frac{(a+h)^n-a^n}{\cancel{a}+h-\cancel{a}}$$

R.H.L: $\underset{x\to a^+}{Lt} f(x) = \underset{h\to 0}{Lt} f(a+h) = \underset{h\to 0}{Lt} \frac{(a+h)^n-a^n}{h}$

$$= \underset{h\to 0}{Lt} \frac{\cancel{a^n} + {}^nC_1 a^{n-1}h + {}^nC_2 a^{n-2}h^2 + \dots + {}^nC_{n-1} a^1 h^{n-1} + {}^nC_n a^0 h^n - \cancel{a^n}}{h}$$

$$= \underset{h\to 0}{Lt} \frac{\cancel{h}\left({}^nC_1 a^{n-1} + {}^nC_2 a^{n-2}h + \dots + {}^nC_{n-1} a^1 h^{n-2} + {}^nC_n h^{n-1}\right)}{\cancel{h}}$$

$$= {}^nC_1 a^{n-1} + 0 + 0 + 0 \dots + 0 = \boxed{na^{n-1}}$$

H.W $\boxed{\text{L.H.L;}}$

Q) $\underset{x\to 0}{Lt} \frac{(8+x)^{1/3}-2}{x} = \underset{x\to 0}{Lt} \frac{(x+8)^{1/3}-8^{1/3}}{(x+8)-8}$

A) $\frac{1}{4}$ B) $\frac{1}{3}$

C) $\frac{1}{6}$ ~~D)~~ $\frac{1}{12}$

$t = x+8$, $x\to 0$, $t\to 8$

$$= \boxed{\underset{t\to 8}{Lt} \frac{t^{1/3}-8^{1/3}}{t-8}} = na^{n-1}$$

$a=8$, $n=\frac{1}{3}$

$$= \frac{1}{3}\times 8^{1/3-1} = \frac{1}{3}\times 8^{-2/3} = \frac{1}{3}\times\frac{1}{4} = \frac{1}{12}$$

e.g

$\Rightarrow \underset{x\to 1}{Lt} \frac{x^{10}-1}{x-1} = 10\cdot 1^{10-1} = 10.$

$a=1$, $n=10$

$\Rightarrow \underset{x\to 8}{Lt} \frac{x^{1/3}-8^{1/3}}{x-8} = na^{n-1} = \frac{1}{3}\times 8^{1/3-1}$

$n=\frac{1}{3}$, $a=8$

$$= \frac{1}{3}\times 8^{-2/3} = \frac{1}{3}\times\frac{1}{2^2} = \frac{1}{12}.$$

Q) $\underset{x\to 2}{Lt}\left(\frac{x^n-2^n}{x-2}\right) = 80$ find n.

A) 4

B) 5

C) 6

D) 7

$\Downarrow$

$n(2^{n-1}) = 80$

$n\, 2^{n-1} = 16\times 5$

$\boxed{n=5}$

Q) $\underset{x\to 0}{Lt}\left(\frac{(1+x)^6-1}{(1+x)^2-1}\right)$

A) 1 C) 3

B) 2 D) 6

$t = 1+x$

$$\underset{t\to 1}{Lt} \frac{t^6-1}{t^2-1} = \frac{\underset{t\to 1}{Lt} \frac{t^6-1}{t-1}}{\underset{t\to 1}{Lt} \frac{t^2-1}{t-1}} = \frac{6\cdot 1^{6-1}}{2\cdot 1^{2-1}} = 3$$

Q) $\underset{x\to 2}{Lt} \frac{x^{10}-2^{10}}{x^5-2^5} = \frac{\underset{x\to 2}{Lt}\left(\frac{x^{10}-2^{10}}{x-2}\right)}{\underset{x\to 2}{Lt}\left(\frac{x^5-2^5}{x-2}\right)} = \frac{10\times 2^{10-1}}{5\times 2^{5-1}} = \frac{2\times 2^9}{2^4} = 2^6 = 64$

$n=10$ $a=2$

$n=5$ $a=2$

A) 1024 C) 32

B) 64 D) 16

$$(x^5)^2-(2^5)^2 = \frac{(\cancel{x^5-2^5})(x^5+2^5)}{\cancel{x^5-2^5}} = 64$$

EXPONENTIAL AND LOGARITHM LIMITS:

$\log_e 3 = \dfrac{\log_{10} 3}{\log_{10} e}$ $\quad \log_e 9 = \dfrac{\log_{10} 9}{\log_{10} e}$

① $\displaystyle \lim_{x\to 0} \frac{e^x - 1}{x} = 1$

☆ ② $\displaystyle \lim_{x\to 0} \frac{a^x - 1}{x} = \ln a$

e.g. $\displaystyle \lim_{x\to 0} \frac{2^x - 1}{x} = \ln 2$

☆ ③ $\displaystyle \lim_{x\to 0} \frac{e^{mx} - 1}{x} = m$

☆ ④ $\displaystyle \lim_{x\to 0} \frac{a^{mx} - 1}{x} = m \ln a$

$\displaystyle \lim_{x\to 0} \frac{e^x - 1}{x} = \ln e = 1$

Proof: ② $\displaystyle \lim_{x\to 0} \frac{\left(\cancel{1} + \frac{x \ln a}{1!} + \frac{x^2 (\ln a)^2}{2!} + \dots\right) - \cancel{1}}{\cancel{x}} = \lim_{x\to 0} \ln a + x(\ln a)^2 + x^2(\ln a)^3 \dots$

$= \ln a$

Proof: ④ $\displaystyle m \lim \left(\frac{a^{mx} - 1}{mx}\right) = m \lim_{t\to 0} \frac{a^t - 1}{t} = m \ln a$ $\quad t = mx$

Q) $\displaystyle \lim_{x\to 0} \frac{e^{2x} - 1}{x} = 2$

Q) $\displaystyle \lim_{x\to 0} \frac{5^x - 1}{x} = \ln 5$

Logarithms

☆ ① $\displaystyle \lim_{x\to 0} \frac{\log(1+x)}{x} = 1$

$t = \log(1+x) \Rightarrow e^t = 1 + x$

$x = e^t - 1$

$\displaystyle \lim_{t\to 0} \frac{t}{e^t - 1} = \frac{1}{\lim_{t\to 0} \left(\frac{e^t - 1}{t}\right)} = \frac{1}{1}$

☆ ② $\displaystyle \lim_{x\to 0} \frac{\log(1+mx)}{x} = m$

Q) $\displaystyle \lim_{x\to 0} \frac{e^{2x} - 1}{\sin 8x} = \frac{2}{8} = \frac{1}{4}$

Q) $\displaystyle \lim_{x\to 0} \frac{\ln(1+5x)}{\ln(1-7x)} = \frac{-5}{7}$

$t = x - 3$ $\quad x \to 3, t \to 0$

☆ $\displaystyle \lim_{x\to 3} \left(\frac{e^x - e^3}{x - 3}\right) = \lim_{x\to 3} e^3 \left(\frac{e^{x-3} - 1}{x - 3}\right) = \lim_{t\to} e^3 \left(\frac{e^t - 1}{t}\right)$

A) 1 B) e C) e^3 D) 0 $\quad = e^3 \cdot 1$

Q) $\displaystyle \lim_{x\to 0} \left(\frac{e^{2x} - 1}{e^{5x} - 1}\right) = \frac{2}{5}$

Q) $\displaystyle \lim_{x\to 0} \left(\frac{e^{2x} - 1}{\ln(1+7x)}\right) = \frac{2}{7}$

$\displaystyle \frac{\lim_{x\to 0} \left(\frac{e^{2x} - 1}{x}\right)}{\lim_{x\to 0} \frac{\ln(1+7x)}{x}} = \frac{2}{7}$