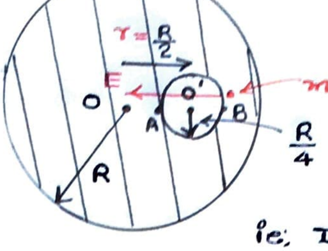
Eg: A solid sphere of Radius 'R' & density 'ρ' have a spherical cavity of radius '$\frac{R}{4}$' as shown in the figure. A particle of mass 'm' is released from rest from point B (inside the cavity) Find:

i) The position where this particle strikes the cavity.

ii) velocity of the particle at this instant.

Solⁿ:→



∵ Gravitational field inside the cavity

$$\vec{E} = -\frac{4}{3}\pi G\rho.\vec{r}$$

(along $\overrightarrow{OO'}$)

$$\Rightarrow E = \frac{4}{3}\pi G\rho.\frac{R}{2} = \frac{2}{3}\pi G\rho.R \quad ms^{-2}$$

ie; the acceleration of the particle inside the cavity —①

∴ The particle will accelerate along the straight line BO'A

from; $\vec{v}^2 = \vec{u}^2 + 2\vec{a}.\vec{s}$

$$v^2 = 0 + \cancel{2}.\frac{2}{3}\pi G\rho R.\frac{R}{\cancel{2}}$$

∴ speed when it strikes A;

$$v = \sqrt{\frac{2}{3}\pi G\rho R} \quad m/s.$$

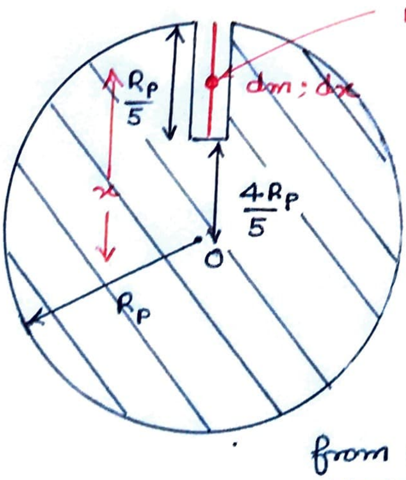
Eg: A planet of radius $R_p = \frac{R_e}{10}$ has the same mass density as Earth.
(ADV 2014) Scientists dig a well of depth $\frac{R_p}{5}$ on it and lower a wire of same length & linear mass density $10^{-3}\ kg.m^{-1}$ into it. If the wire is not touching anywhere, Find the force required by a person holding it from the Top. ($R_e = 6\times10^6 m$; $g = 10 ms^{-2}$)

Soln: field at a distance x from the center

$$E = \frac{GM_p.x}{R_p^3} \quad ms^{-2}$$

point sized element

$dm = \lambda.dx$ —①



so gravitational force on the element

$$dF = dm.E$$

$$dF = \frac{GM_p.x.\lambda.dx}{R_p^3} \quad —②$$

∵ $\rho_e = \rho_p$

$$\frac{M_e}{\frac{4}{3}\pi R_e^3} = \frac{M_p}{\frac{4}{3}\pi R_p^3}$$

$$\therefore \frac{M_p}{R_p^3} = \frac{M_e}{R_e^3} \quad —③$$

from ② & ③

$$dF = \frac{GM_e}{R_e^3}.\lambda.x.dx$$

$$\because \frac{GM_e}{R_e^2} = g$$

so: $dF = \frac{g \cdot \lambda}{R_e} \cdot x \cdot dx$

$$\int_0^F dF = \frac{g\lambda}{R_e} \cdot \int_{\frac{4R_p}{5}}^{R_p} x \cdot dx$$

$$\Rightarrow \int_0^F dF = \frac{g\lambda}{R_e} \cdot \left(\frac{x^2}{2}\right)_{\frac{4R_p}{5}}^{R_p}$$

$$\Rightarrow F = \frac{g\lambda}{2R_e} \cdot R_p^2 - \frac{16 R_p^2}{25}$$

$$\Rightarrow F = \frac{g\lambda}{2R_e} \times \frac{9R_p^2}{25}$$

$$\because R_p = \frac{R_e}{10}$$

$$\Rightarrow F = \frac{9g\lambda}{50 R_e} \cdot \frac{R_e^2}{100}$$

$$\Rightarrow F = \frac{9 \times 10 \times \cancel{10^{-3}} \times 6 \times \cancel{10^6}}{5 \times \cancel{10^3}}$$

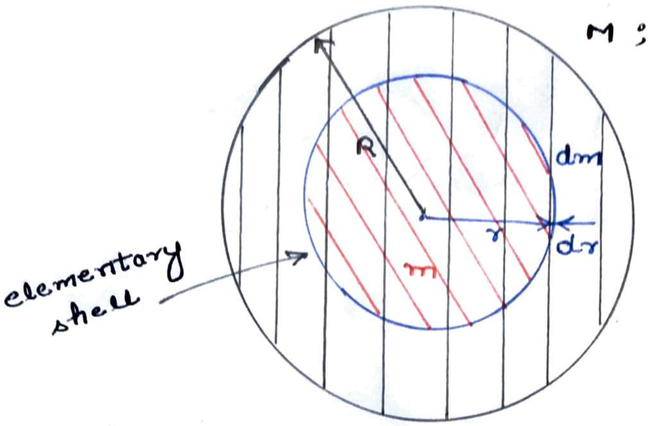
$$= 18 \times 6$$

gravitational pull on the wire $\Rightarrow F = 108\,N$

So force required to keep it hanging is 108 N.

Eg: A uniform sphere has mass M and radius R. Find the pressure P inside the sphere, caused by the gravitational compression, as a function of distance r from the center.

Solⁿ:→



M ; $\rho = \dfrac{M}{\frac{4}{3}\pi R^3}$

mass of the sphere of radius r (m) $= \rho \cdot \frac{4}{3}\pi r^3$

$= \dfrac{M}{\frac{4}{3}\pi R^3} \cdot \frac{4}{3}\pi r^3$

$m = \dfrac{M \cdot r^3}{R^3}$ —①

mass of the considered shell

$dm = \rho \cdot dV$

$= \dfrac{M}{\frac{4}{3}\pi R^3} \cdot 4\pi r^2 dr$

$dm = \dfrac{3 \cdot M \cdot r^2 \cdot dr}{R^3}$ —②

force on the element due to the sphere

$dF = dm \times E$

$= dm \cdot \dfrac{Gm}{r^2}$

$= \dfrac{3G \cdot M^2 \cdot r^5 \cdot dr}{r^2 \cdot R^6}$

$dF = \dfrac{3GM^2 \cdot r^3}{R^6} \cdot dr$

so pressure on the element.

$dP = \dfrac{dF}{A} = \dfrac{3GM^2}{R^6} \cdot \dfrac{r^3}{4\pi r^2} \cdot dr$

$\Rightarrow \displaystyle\int_0^P dP = \frac{3}{4\pi} \cdot \frac{GM^2}{R^6} \int_r^R r \cdot dr$

$\Rightarrow (P)_0^P = \dfrac{3}{4\pi} \cdot \dfrac{GM^2}{R^6} \left(\dfrac{r^2}{2}\right)_r^R$

$\Rightarrow P = \dfrac{3}{8\pi} \cdot \dfrac{GM^2}{R^6} (R^2 - r^2)$

at the center;

$r = 0$

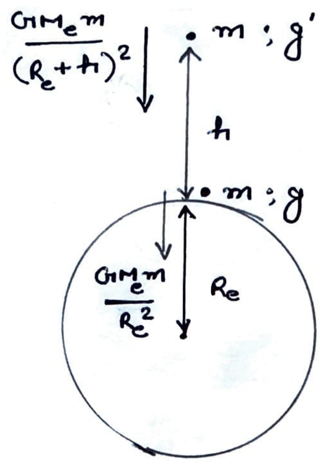
$P = \dfrac{3GM^2}{8\pi R^4}$

Variation of gravitational acceleration 'g':

Gravitational acceleration on the earth's surface

$$g = \frac{GM_e}{R_e^2} = 9.8\ m.s^{-2}$$

i) variation due to altitude (height):



weight of the object on the earth's surface $(W) = mg = \frac{GM_e m}{R_e^2}$ — ①

weight of the object at a height h above the earth's surface

$$(W') = mg' = \frac{GM_e m}{(R_e+h)^2} \quad — ②$$

eqn ② / ①

$$\frac{W'}{W} = \frac{g'}{g} = \frac{R_e^2}{(R_e+h)^2}$$

$$\Rightarrow g' = \frac{g \cdot R_e^2}{(R_e+h)^2} = g\left[\frac{R_e}{R_e+h}\right]^2 \quad — ③$$

$$= g\left[\frac{1}{1+\frac{h}{R_e}}\right]^2$$

gravitational acceleration at a ht. 'h' above the earth's surface

$$\Rightarrow \boxed{g' = \frac{g}{\left(1+\frac{h}{R_e}\right)^2}} \quad — ✪ \quad m.s^{-2}$$

if $h \ll R_e$

$$g' = g\left[1+\frac{h}{R_e}\right]^{-2}$$

expanding binomially and neglecting higher orders.

$$g' = g\left[1-\frac{2h}{R_e}\right] \quad — ✪ \quad ms^{-2}$$

extra point: as at $h \ll R_e$

$$g' = g\left[1-\frac{2h}{R_e}\right]$$

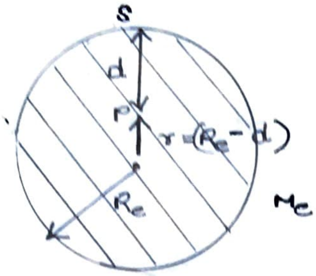
$$\Rightarrow g - g' = \frac{2gh}{R_e}$$

Relative error $\Rightarrow \frac{\Delta g}{g} = \frac{2h}{R_e}$

$$\therefore \%\ \text{error}\ \left(\frac{\Delta g}{g} \times 100\%\right) = \frac{2h}{R_e} \times 100\%$$

ii) <u>variation with depth</u>:

$\because$ intensity of gravitational field is the gravitational acceleration.



so gravitational acceleration on the earth's surface

$$g = E_S = \frac{GM_e}{R_e^2} \quad \text{—①}$$

and gravitational acceleration at depth 'd'

$$g'' = E_{in} = \frac{GM_e \cdot r}{R_e^3}$$

$$\Rightarrow g'' = \frac{GM_e}{R_e^3}(R_e - d) \quad \text{—②}$$

<u>eqn. ②/①</u>

$$\frac{g''}{g} = \frac{(R_e - d)}{R_e^3} \times R_e^2$$

$$\Rightarrow \frac{g''}{g} = \left(\frac{R_e - d}{R_e}\right)$$

gravitational acceler. at a depth 'd' below the earth's surface. $\Rightarrow \boxed{g'' = g\left(1 - \frac{d}{R_e}\right)}$ ms^{-2} —④

<u>Extra point</u>:→ $\because g'' = g\cdot\left(1 - \frac{d}{R_e}\right)$

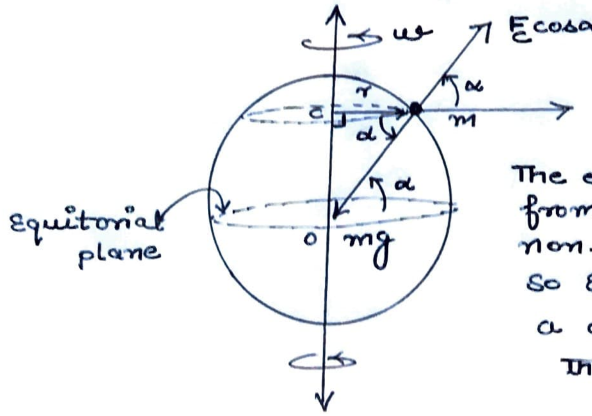
$$\Rightarrow g - g'' = g\frac{d}{R_e}$$

Relative Error $\Rightarrow \frac{\Delta g}{g} = \frac{d}{R_e}$

$\therefore$ % error $\left(\frac{\Delta g}{g} \times 100\%\right) = \frac{d}{R_e} \times 100\%$.

So the value of gravitational acceleration g decreases if we move above or below the earth's surface.

iii) <u>variation due to rotation of earth</u>:



$F_c = mr\omega^2 = mR\omega^2 \cos\alpha$

The earth is rotating about its axis from west to east. So the earth is a non-inertial frame of Reference, so every body on its surface feels a centrifugal force, here α is the latitude.

$\therefore$ effective wt. of the body $(mg') = mg - F_c \cos\alpha$

$$mg' = mg - mR\omega^2 \cos^2\alpha$$

gravitational acc. at latitude α $\Rightarrow$ $\boxed{g' = g - R\cdot\omega^2 \cos^2\alpha}$ $m\bar{s}^2$ —— (*)

note: i) at the equator ; $\alpha = 0°$

$$\text{so } g_{eq} = g - R\cdot\omega^2 \cos^2 0°$$

$$\Rightarrow \boxed{g_{eq} = g - R\omega^2 = g_{min}}$$

ii) at the poles ; $\alpha = 90°$

$$\text{so } g_p = g - R\omega^2 \cdot \cos^2 90°$$

$$\Rightarrow \boxed{g_P = g = g_{max}}$$

Eg: At what height above the earth's surface, the gravitational acc. becomes one-fourth of that on the surface.

Soln: $\because$ acceleration due to gravity at a ht. h above the earth's surface

$$g' = \frac{g}{\left(1+\frac{h}{R_e}\right)^2}$$

$$\Rightarrow \frac{g}{4} = \frac{g}{\left(1+\frac{h}{R_e}\right)^2}$$

$$\Rightarrow \left(1+\frac{h}{R_e}\right)^2 = 2^2$$

$$\Rightarrow 1+\frac{h}{R_e} = 2$$

$$\text{so } h = R_e.$$

Eg: if the acceleration due to gravity inside the earth to be kept constant, find the relation between the density 'ρ' & radius 'r' from the center of earth.

Soln: acceleration due to gravity at a distance r from the center = intensity of gravitational field.

$$\therefore g' = E_{in} = \frac{GM_e \cdot r}{R_e^3} = \text{const}$$

$$\text{as } M_e = \rho \cdot \frac{4}{3}\pi R_e^3$$

$$\text{so; } \frac{G \cdot \rho \cdot 4 \cdot \pi R^3 \cdot r}{3R_e^3} = \text{const}$$

$$\Rightarrow \rho \cdot r = \text{const}$$

$$\Rightarrow \rho \propto \frac{1}{r}$$