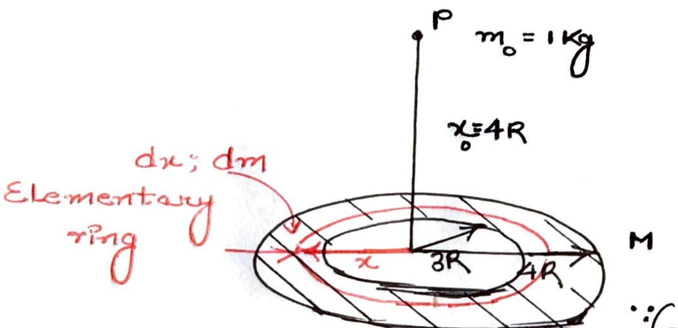
Eg: A Thin uniform disc of inner & outer radii 3R & 4R respectively. The work required to take a unit mass from P to ∞ will be:

i) $\frac{2GM}{7R}(4\sqrt{2}-5)$ ii) $-\frac{2GM}{7R}(4\sqrt{2}-5)$ iii) $\frac{GM}{4R}$ iv) $\frac{2GM}{5R}(\sqrt{2}-1)$

Soln:



mass of the elementary Ring

$$dm = \frac{M}{\pi(4R)^2 - \pi(3R)^2} \times dA$$

$$= \frac{M}{7\pi R^2} \cdot 2\pi x \cdot dx$$

$$\Rightarrow dm = \frac{2M x \cdot dx}{7R^2} \quad — ①$$

$$\because \left(W_{P\to\infty}\right)_{ext} = m_o \cdot \Delta V_{\infty P} = m_o(V_\infty - V_P)$$

as $V_\infty = 0$

$$\text{so } \left(W_{P\to\infty}\right)_{ext} = -m_o \cdot V_P \quad — ②$$

so potential at point P due to the elementary Ring

$$dV_P = -\frac{G\,dm}{\sqrt{x^2+x_o^2}} = \frac{-G \cdot 2M x \cdot dx}{7R^2\sqrt{16R^2+x^2}}$$

$$\int_0^{V_P} dV_P = -\frac{GM}{7R^2} \cdot \int_{3R}^{4R} \frac{2x\,dx}{\sqrt{16R^2+x^2}}$$

$$= -\frac{GM}{7R^2} \cdot \left[2 \cdot \sqrt{16R^2+x^2}\right]_{3R}^{4R}$$

$$= -\frac{2GM}{7R^2} \cdot \left[\sqrt{32R^2} - \sqrt{25R^2}\right]$$

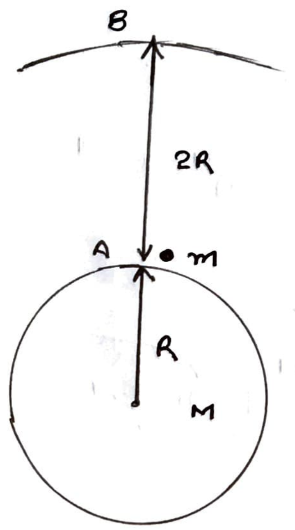
Gravitational potential at point P $\Rightarrow V_P = -\frac{2GMR}{7R^2} \cdot \left[4\sqrt{2} - 5\right]$ J/kg — ③

from ② & ③

$$\left(W_{P\to\infty}\right)_{ext} = \frac{2GMR(4\sqrt{2}-5)}{7R^2} \text{ J} = \frac{2GM(4\sqrt{2}-5)}{7R} \text{ J}$$

Eg: what is the minimum energy required to Launch a satellite of mass m from the surface of a planet of mass M & radius R in a circular orbit at an altitude of $2R$?

Eg:→



as the gravitational field is conservative, so from conservation of energy.

$$K_A + U_A = K_B + U_B$$

$$\Rightarrow K_A + m\cdot V_A = 0 + m\cdot V_B$$

$$\Rightarrow K_A = m\cdot\left[V_B - V_A\right] = \left(W_{ext}\right)_{A\to B}$$

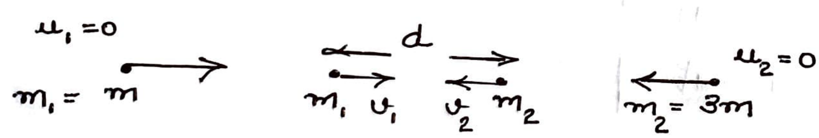
$$= m\cdot\left[\left(-\frac{GM}{3R}\right) - \left(-\frac{GM}{R}\right)\right]$$

$$= m\cdot\left[\frac{GM}{R} - \frac{GM}{3R}\right]$$

$$\Rightarrow K_A = \frac{2GMm}{3R} \text{ J}$$

Eg: Two particles of mass m & $3m$ are initially at rest and infinite distance apart. Both particles start moving due to mutual gravitation. At any instant their relative velocity of approach is $\sqrt{\frac{\eta Gm}{d}}$, where d is instantaneous separation, Find η.

Soln:→



as there is no external force on the system

so P_{sys} = constant

$$\Rightarrow P_i = P_f$$

$$\Rightarrow 0 = m_1 v_1 - m_2 v_2$$

$$\Rightarrow v_1 = 3v_2 \quad \text{—①}$$

from C.O.M.E.

$$K_i + U_i = K_f + U_f$$

$$0 + 0 = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 - \frac{Gm_1 m_2}{d}$$

$$\Rightarrow \frac{3Gm^2}{d} = \frac{1}{2}m \times 9v_2^2 + \frac{1}{2}\cdot 3m v_2^2$$

$$\Rightarrow \frac{3Gm}{d} = 6v_2^2$$

so $v_2 = \sqrt{\frac{Gm}{2d}}$ and $v_1 = 3\sqrt{\frac{Gm}{2d}}$

so $v_{rel} = v_1 + v_2$

$$= 3\sqrt{\frac{GM}{2d}} + \sqrt{\frac{GM}{2d}}$$

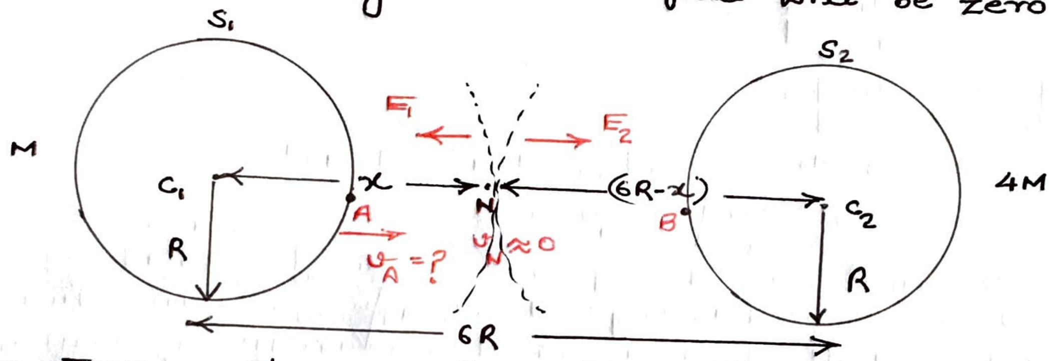
$$= 4\sqrt{\frac{GM}{2d}}$$

$$\therefore v_{rel} = \sqrt{\frac{16GM}{2d}} = \sqrt{\frac{8GM}{d}}$$

so $\eta = 8$.

eg: Two uniform solid spheres of mass M & 4M are of same radius R. Separation b/w their centers is 6R. Both the spheres are held fixed. A particle of mass m is projected from the surface of M towards 4M, find the min speed of projection so that it may reach on the second sphere's surface.

Solⁿ: There will be a point somewhere b/w the spheres where the gravitational field will be zero (Nutral point)



In other words a point where G. field of S_1 ends & that of S_2 starts. if the projected particle somehow crosses this point, the second sphere will attract it itself only. Therefore the speed of projection is atleast equal to the speed require to reach the null point.

Let the null point is at a dist. x from C_1;

so $\vec{E_N} = \vec{E_1} + \vec{E_2}$

$$\Rightarrow 0 = \frac{GM}{x^2}\cdot(-\hat{i}) + \frac{4GM}{(6R-x)^2}\cdot(\hat{i})$$

$$\Rightarrow \frac{4GM}{(6R-x)^2} = \frac{GM}{x^2}$$

$$\Rightarrow \frac{2}{6R-x} = \frac{1}{x}$$

$$\Rightarrow 2x = 6R - x$$

so $x = 2R$ —①

So, from C.O.M.E. b/w A & N:

$$K_A + U_A = K_N + U_N$$

$$\Rightarrow \frac{1}{2}mv_A^2 + \left[m\cdot V_{AS_1} + m\cdot V_{AS_2}\right] = \frac{1}{2}mv_N^2 + \left[m\cdot V_{NS_1} + m\cdot V_{NS_2}\right]$$

$$\Rightarrow \frac{1}{2}\cdot v_A^2 + \left[-\frac{GM}{R} - \frac{4GM}{5R}\right] = 0 + \left[-\frac{GM}{2R} - \frac{4GM}{4R}\right]$$

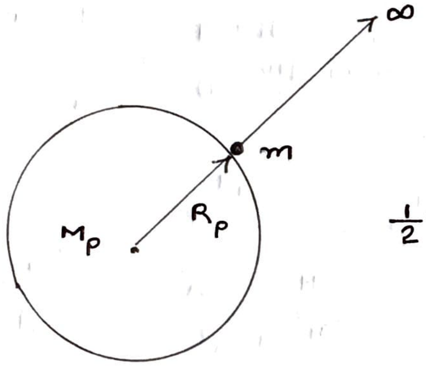
$$\Rightarrow \frac{v_A^2}{2} = \frac{GM}{R}\cdot\left[\cancel{1} + \frac{4}{5} - \frac{1}{2} - \cancel{1}\right]$$

$$\Rightarrow \frac{v_A^2}{2} = \frac{GM}{R}\cdot\left(\frac{8-5}{10}\right)$$

$$\Rightarrow v_A^2 = \frac{3GM}{5R}$$

$$\text{so } (v_A)_{min} = \sqrt{\frac{3GM}{5R}} \text{ m/s.}$$

Escape speed (v_e): The min. speed required to sent a body out of the gravity of a planet (say at ∞).



from conservation of Energy:

$$K_i + U_i = K_f + U_f$$

$$\frac{1}{2}mv^2 + \left(\frac{-GM_p\cdot m}{R_p}\right) = 0 + 0$$

$$\text{so } (v_p)_{escape} = \sqrt{\frac{2GM_p}{R_p}} \text{ m/s.}$$

Escape speed on the Earth's surface; $v_e = \sqrt{\frac{2GM_e}{R_e}}$

$$v_e = \sqrt{\frac{2GM_e\cdot R_e}{R_e^2}}$$

$$= \sqrt{2gR_e}$$

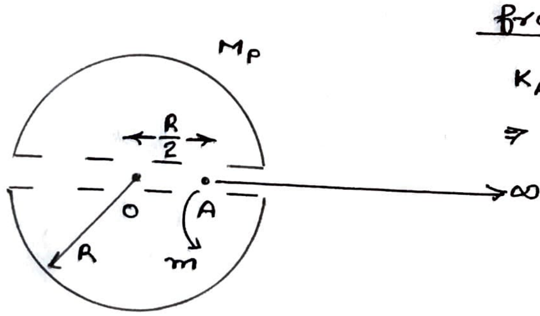
$$\Rightarrow v_e = 11.2 \text{ Km/sec}$$

note: i) it does not depends upon mass of the projected body.

ii) does not depends upon the angle of projection.

Eg: find the escape speed of particle of mass m_0 projected from point A inside the groove.

Soln:



from C.O.M.E.

$$K_A + U_A = K_\infty + U_\infty$$

$$\Rightarrow \frac{1}{2}mv_A^2 + m\cdot\left[-\frac{GM_p}{2R^3}\cdot\left(3R^2 - \frac{R^2}{4}\right)\right] = 0$$

$$\frac{v_A^2}{2} = \frac{11GM_p\cdot R^2}{8R^3}$$

$$\text{so } v_A = \sqrt{\frac{11GM}{4R}} \text{ m/s.}$$