ANSWERS

LEVEL II

1. (b)	2. (d)	3. (a)	4. (b)	5. (d)
6. (b)	7. (c)	8. (c)	9. (c)	10. (b)
11. (c)	12. (c)	13. (b)	14. (c)	15. (a)
16. (b)	17. (b)	18. (a)	19. (a, d)	20. (c)
21. (a, c)	22. (a)	23. (b, c)	24. (a)	25. (c)
26. (b)	27. (b)	28. (b)	29. (a)	30. (d)
31. (c)	32. (c)	33. (c)	34. (b)	35. (c)
36. (d)	37. (a)	38. (b)	39. (b)	40. (d)
41. (d)				

INTEGER TYPE QUESTIONS

1. 1	2. 3	3. 2	4. 3	5. 3
6. 4	7. 6	8. 4	9. 3	10. 5

COMPREHENSIVE LINK PASSAGES

Passage I :	1. (a)	2. (b)	3. (c)
Passage II:	1. (c)	2. (d)	3. (c)
Passage III:	1. (c, d)	2. (a, c, d)	3. (b, c, d)
Passage IV:	1. (b)	2. (a)	3. (c)
Passage V:	1. (c)	2. (a)	3. (b)
Passage VI:	1. (b)	2. (a)	3. (c)
Passage VII:	1. (c)	2. (b)	3. (c)

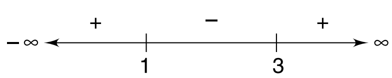
MATRIX MATCH

1. (A)→(P), (B)→(Q), (C)→(P), (D)→(Q)
2. (A)→(R), (B)→(Q), (C)→(Q), (D)→(P)
3. (A)→(R), (B)→(P), (C)→(Q), (D)→(S)
4. (A)→(R,T), (B)→(P), (C)→(Q), (D)→(S)
5. (A)→(Q), (B)→(P), (C)→(S), (D)→(R)
6. (A)→(Q), (B)→(P), (C)→(S), (D)→(T)
7. (A)→(R), (B)→(P), (C)→(S), (D)→(Q)
8. (A)→(R), (B)→(P), (C)→(S), (D)→(Q)
9. (A)→(R), (B)→(P), (C)→(Q), (D)→(S)
10. (A)→(Q), (B)→(T), (C)→(P, R), (D)→(P, Q, R, S)

HINTS AND SOLUTIONS

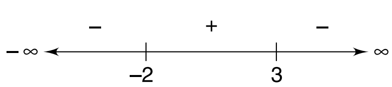
Level I

1. Given
$$f(x) = 2x^3 - 12x^2 + 18x + 5$$
$$= 6x^2 - 24x + 18$$
$$= 6(x^2 - 4x + 3)$$
$$= 6(x^2 - 4x + 3)$$
$$= 6(x - 1)(x - 3)$$



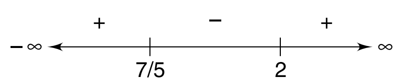
By the sign scheme, $f(x)$ is strictly increases in $(-\infty, 1) \cup (3, \infty)$ and strictly decreases in $(1, 3)$

2. Given
$$f(x) = 5 + 36x + 3x^2 - 2x^3$$
$$\Rightarrow \quad f'(x) = 36 + 6x - 6x^2$$
$$= -6(x^2 - x - 6)$$
$$= -6(x - 3)(x + 2)$$



By the sign scheme, we can say that, $f(x)$ is strictly increases in $(-2, 3)$ and strictly decreases in $(-\infty, -2) \cup (3, \infty)$.

3. Given $f(x) = (x - 1)^3 (x - 2)^2$
$$\Rightarrow \quad f'(x) = 3(x - 1)^2(x - 2)^2 + 2(x - 1)^3(x - 2)$$
$$= (x - 1)^2(x - 2)(3(x - 2) + 2(x - 1))$$
$$= (x - 1)^2(x - 2)(5x - 7)$$



By the sign scheme, we can say that, $f(x)$ is strictly increase in $\left(-\infty, \frac{7}{5}\right) \cup (2, \infty)$ and strictly decreases in $\left(\frac{7}{5}, 2\right)$

4. Given
$$f(x) = 2x^3 - 3x^2 + 6x + 10$$
$$\Rightarrow \quad f'(x) = 6x^2 - 6x + 6$$
$$= 6(x^2 - x + 2) > 0, \text{ for all } x \text{ in } R$$

Thus, the function $f(x)$ is strictly increases for all x in R.

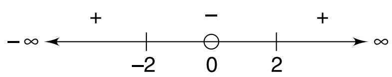
5. Given
$$f(x) = 2x^3 + 3x^2 + 12x + 20$$
$$\Rightarrow \quad f'(x) = 6x^2 + 6x + 12$$
$$= 6(x^2 + x + 2) > 0 \text{ for all } x \text{ in } R$$

Thus, $f(x)$ is strictly increases in $(-\infty, \infty)$

6. Given $f(x) = \frac{x}{2} + \frac{2}{x}$

$\Rightarrow$ $f'(x) = \frac{1}{2} - \frac{2}{x^2} = \frac{x^2 - 4}{2x^2}$

$= \frac{(x-2)(x+2)}{2x^2}$

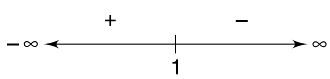


By the sign scheme, we can say that, $f(x)$ is strictly increases in $(-\infty, -2) \cup (2, \infty)$and strictly decreases in $(-2, 0) \cup (0, 2)$

7. Given $f(x) = 5x^{3/2} - 3x^{5/2}$, $x > 0$

$\Rightarrow$ $f'(x) = 15x^{1/2} - 15x^{3/2}$

$= 15\sqrt{x}(1 - x) - 15\sqrt{x}(x - 1)$



By the sign scheme, we can say that, $f(x)$ is strictly increases in $(0, 1)$ and strictly decreases in $(1, \infty)$

8. Given $f(x) = \log\left(x + \sqrt{1 + x^2}\right)$

$\Rightarrow$ $f'(x) = \frac{1}{\left(x + \sqrt{x^2+1}\right)}\left(1 + \frac{1 \times 2x}{2\sqrt{x^2+1}}\right)$

$\Rightarrow$ $f'(x) = \frac{1}{\left(x + \sqrt{x^2+1}\right)}\left(1 + \frac{x}{\sqrt{x^2+1}}\right)$

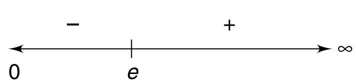
$= \frac{1}{\left(x + \sqrt{x^2+1}\right)}\left(\frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1}}\right)$

$= \frac{1}{\sqrt{x^2+1}} > 0 \ \forall \ x \in R$

Thus, $f(x)$ is strictly increases in $(-\infty, \infty)$

9. Given $f(x) = \frac{x}{\log x}$

$\Rightarrow$ $f'(x) = \frac{\log x - 1}{(\log x)^2}$



By the sign scheme, we can say that, $f(x)$ is strictly increases in (e, ∞) and strictly decreases in $(0, e)$.

10. Given $f(x) = \cot^{-1}x - \log\left(x + \sqrt{x^2+1}\right)$

$\Rightarrow$ $f'(x) = 1 + \frac{1}{1+x^2} - \frac{1}{\left(x + \sqrt{x^2+1}\right)}$

$\times \frac{\left(x + \sqrt{x^2+1}\right)}{\sqrt{x^2+1}}$

$\Rightarrow$ $f'(x) = 1 + \frac{1}{1+x^2} - \frac{1}{\sqrt{x^2+1}} > 0 \ \forall \ x \in R$

Thus, $f(x)$ is strictly increases in $(-\infty, \infty)$

11. Given $f(x) = -x^2 + mx + 1$

$\Rightarrow$ $f'(x) = -2x + m$

Since f is strictly increasing, so $f'(x) > 0$

$\Rightarrow$ $-2x + m > 0$

$\Rightarrow$ $m > 2x$

$\Rightarrow$ $m > 2, \ \forall \ x \in [1, 2]$

Hence, the least value of m is 2.

12. Given $f(x) = \sin x - bx + c$

$\Rightarrow$ $f'(x) = \cos x - b$

Since f is strictly decreasing, so $f < 0$

$\Rightarrow$ $\cos x - b < 0$

$\Rightarrow$ $b > \cos x$

$\Rightarrow$ $b > 1$

Hence, $b \in (1, \infty)$

13. Given $f(x) = e^{2x} - (a + 1)e^x + 2x$

$\Rightarrow$ $f'(x) = 2e^{2x} - (a + 1)e^x + 2$

$= 2(e^x)^2 - (a + 1)e^x + 2$

Since f is strictly increasing, so $f'(x) > 0$

$\Rightarrow$ $2(e^x)^2 - (a + 1)e^x + 2 > 0$

$\Rightarrow$ $2e^x - (a + 1)e^x + \frac{2}{e^x} > 0$

$\Rightarrow$ $2\left(e^x + \frac{1}{e^x}\right) - (a + 1) > 0$

$\Rightarrow$ $(a + 1) < 2\left(e^x + \frac{1}{e^x}\right)$

$\Rightarrow$ $(a + 1) < 2.2 = 4$

$\Rightarrow$ $a < 3$

Hence, the value of a is $\in (-\infty, 3)$

14. Given $f(x) = \left(\frac{a^2 - 1}{3}\right)x^3 + (a - 1)x^2 + 2x + 1$

$\Rightarrow$ $f'(x) = 3\left(\frac{a^2 - 1}{3}\right)x^2 + 2(a - 1)x + 2$

$\Rightarrow \quad f'(x) = (a^2 - 1)x^2 + 2(a - 1)x + 2$

Since f is strictly increasing, so $f'(x) > 0$

$\Rightarrow \quad (a^2 - 1)x^2 + 2(a - 1)x + 2 > 0$

$\Rightarrow \quad (a^2 - 1) > 0 \ \& \ 4(a - 1)^2 - 8(a^2 - 1) < 0$

$\Rightarrow \quad a^2 > 1 \ \& \ (a - 1)^2 - 2(a^2 - 1) < 0$

$\Rightarrow \quad (a + 1)(a - 1) > 0 \ \& \ (a + 3)(a - 1) > 0$

$\Rightarrow \quad a \in (-\infty, -1) \cup (1, \infty)$

and $\quad a \in (-\infty, -3) \cup (1, \infty)$

Hence, the values of a are $a \in (-\infty, -3) \cup (1, \infty)$

15. Given $\quad f(x) = (a + 2)x^3 - 3ax^2 + 9ax - 1$

$\Rightarrow \quad f'(x) = 3(a + 2)x^2 - 6ax + 9a$

Since f is strictly decreasing for all x in R, so

$f'(x) < 0$

$\Rightarrow \quad 3(a + 2)x^2 - 6ax + 9a < 0$

$\Rightarrow \quad (a + 2)x^2 - 2ax + 3a < 0$

Thus, $\quad (a + 2) < 0$ and $4a^2 - 12a(a + 2) < 0$

$\Rightarrow \quad a < -2$ and $a^2 - 3a(a + 2) < 0$

$\Rightarrow \quad a < -2 \ \& \ a(a + 3) > 0$

$\Rightarrow \quad a \in (-\infty, -2) \ \& \ a \in (-\infty, -3) \cup (0, \infty)$

Thus, $\quad a \in (-\infty, -3)$

16. Given $\quad f(x) = \dfrac{e^x}{x - 1}$

$$\Rightarrow \quad f'(x) = \frac{(x - 1)e^x - e^x \cdot 1}{(x - 1)^2} = \frac{e^x(x - 2)}{(x - 1)^2}$$

Also, $\quad D_f = R - \{1\}$

Since $x = 1$ is not an interior point in the domain of f, so, $x = 1$ is not a critical point of f.

Thus, the critical point of f is $x = 2$.

17. Given $f(x) = \dfrac{5x^2 - 18x + 45}{x^2 - 9}$

$$\Rightarrow \quad f'(x) = \frac{(x^2 - 9)(10x - 18) - (5x^2 - 18x + 45)\cdot 2x}{(x^2 - 9)^2}$$

$$\Rightarrow \quad f'(x) = \frac{18(x^2 - 10 + 9)}{(x^2 - 9)^2}$$

$$\Rightarrow \quad f'(x) = \frac{18(x - 1)(x - 9)}{(x^2 - 9)^2}$$

Also, $\quad D_f = R - \{-3, 3\}$

Since $x = -3, 3$ are not an interior point of the domain of f, so $x = -3, 3$ are not the critical point of f.

Thus, the critical points of f are $x = 1$ and 9.

18. Given $\quad f(x) = x^{4/5}(x - 4)^2$

$$\Rightarrow \quad f'(x) = 2x^{4/5}(x - 4) + \frac{4}{5x^{1/5}}(x - 4)^2$$

$$\Rightarrow \quad f'(x) = \frac{10x(x - 4) + 4(x - 4)^2}{5x^{1/5}}$$

$$\Rightarrow \quad f'(x) = \frac{(x - 4)(10x + 4x - 16)}{5x^{1/5}}$$

$$\Rightarrow \quad f'(x) = \frac{2(x - 4)(7x - 8)}{5x^{1/5}}$$

Also, $\quad D_f = R$

Thus, the critical points of f are $x = 0, \frac{8}{7}, 4$

19. Given $\quad f(x) = x + \cos^{-1}x + 1$

$$\Rightarrow \quad f'(x) = \frac{1 - 1}{\sqrt{1 - x^2}}$$

Now, $\quad f'(x) = 0 \Rightarrow 1 - \dfrac{1}{\sqrt{1 - x^2}} = 0$

$$\Rightarrow \quad \frac{1}{\sqrt{1 - x^2}} = 1$$

$\Rightarrow \quad x = 0.$

Also, $\quad D_f = [-1, 1]$

Thus, the critical points of 'f' is $x = 0$.

20. Given $\quad f(x) = \sqrt{x^2 - 6x + 15}$

$$\Rightarrow \quad f'(x) = \frac{1}{2\sqrt{x^2 - 6x + 15}} \times (2x - 6)$$

$$\Rightarrow \quad f'(x) = \frac{1}{2\sqrt{x^2 - 6x + 15}} \times 2(x - 3)$$

$$\Rightarrow \quad f'(x) = \frac{(x - 3)}{\sqrt{x^2 - 6x + 15}}$$

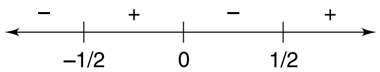
Also $\quad D_f = R$

Thus, the critical points of f is $x = 3$.

21. Given $\quad f(x) = 2x^2 - \ln|x|$

$$\Rightarrow \quad f'(x) = 4x - \frac{1}{x} = \frac{4x^2 - 1}{x}$$

$$= \frac{(2x - 1)(2x + 1)}{x}.$$

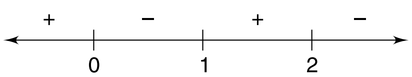


By the sign scheme for $f'(x)$, we have, $f(x)$ is increasing in $\left[-\frac{1}{2}, 0\right) \cup \left[\frac{1}{2}, \infty\right)$ and decreasing in $\left(-\infty, -\frac{1}{2}\right] \cup \left(0, \frac{1}{2}\right]$.

22. Given $\quad f(x) = |x - 1|/x^2$

$$\Rightarrow \quad f(x) = \begin{cases} \dfrac{x-1}{x^2} = \dfrac{1}{x} - \dfrac{1}{x^2} & : x \geq 1 \\ \dfrac{1-x}{x^2} = \dfrac{1}{x^2} - \dfrac{1}{x} & : x < 1 \end{cases}$$

$$\Rightarrow \quad f'(x) = \begin{cases} -\dfrac{1}{x^2} + \dfrac{2}{x^3} = \dfrac{2-x}{x^3} & : x \geq 1 \\ -\dfrac{2}{x^3} + \dfrac{1}{x^2} = \dfrac{x-2}{x^3} & : x < 1 \end{cases}$$



By the sign scheme for the function $f'(x)$, we have $f(x)$ is increasing in $(-\infty, 0) \cup [1, 2]$ and decreases in $(0, 1] \cup [2, \infty)$.

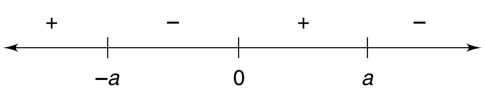
23. Given $\quad f(x) = x^2 e^{-x^2/a^2}, \ a > 0$

$$\Rightarrow \quad f'(x) = 2xe^{-x^2/a^2} + x^2 \cdot e^{-x^2/a^2} \times \left(-\frac{2x}{a^2}\right)$$

$$= 2xe^{-x^2/a^2}\left(1 - \frac{x^2}{a^2}\right)$$

$$= -2xe^{-x^2/a^2}\left(\frac{(x-a)(x+a)}{a^2}\right)$$

$$= \left(-\frac{2}{a^2}\right)e^{-x^2/a^2} \times x(x-a)(x+a)$$



Now, by the sign scheme for the function $f'(x)$, we have $f(x)$ is increases in $(-\infty, -a] \cup [0, a]$

24. Given $\quad x^3 = 3x + 1$

Let $\quad f(x) = x^3 - 3x - 1$

$$\Rightarrow \quad f'(x) = 3x^2 - 3 = 3(x^2 - 1)$$

$$\Rightarrow \quad f'(x) = 3(x^2 - 1) < 0, \ \forall \ x \in (-1, 1)$$

Thus $f(x)$ is strictly decreases in $(-1, 1)$

Now, $\quad f(-1) = -1 + 3 - 1 = 1 > 0$

and $\quad f(1) = 1 - 3 - 1 = -2 < 0$

Thus, the curve $y = f(x) = x^3 - 3x - 1$ will cut the x-axis exactly one point in $(-1, 1)$

25. Let $\quad f(x) = e^x - 1 - x - \dfrac{x^2}{2}$

$$\Rightarrow \quad f'(x) = e^x - 1 - x < 0, \ \forall \ x \in (-1, 1)$$

Thus, $f(x)$ is strictly decreases in $(-1, 1)$

Now, $\quad f(-1) = \dfrac{1}{e} - 1 + 1 - \dfrac{1}{2} = \dfrac{1}{e} - \dfrac{1}{2} < 0$

and $\quad f(1) = e - 1 - 1 - \dfrac{1}{2} = e - \dfrac{5}{2} > 0$

Thus, the equation $e^x = 1 + x + \dfrac{x^2}{2}$ has a real root in $(-1, 1)$

26. As we know that $\tan^{-1} x$ & e^x, both are strictly increasing for all x in R.
Therefore $f(x) = \tan^{-1}(e^x)$ is strictly increasing for all x in R.

27. As we know that $\tan^{-1} x$ is strictly increasing for all x in R and $(\log_{1/3} x)$ is strictly decreasing for all $x \in R^+$.
Therefore, $f(x) = \tan^{-1}(\log_{1/3} x)$ is strictly decreasing for all $x \in R$.

28. As we know that $\cot^{-1} x$ is strictly decreasing for all x in R and $(\log_4 x)$ is increasing for all $x \in R^+$.
Therefore, $f(x) = \cot^{-1}(\log_4 x)$ is strictly decreasing for all $x \in R^+$.

29. As we know that $(\log_{1/10} x)$ is strictly decreasing for all $x \in R^+$ and $(\cot^{-1} x)$ is strictly decreasing for all x.
Thus $f(x) = \cot^{-1}(\log_{1/10} x)$ is strictly increasing for all $x > 0$.

30. Let $\quad f(x) = 3x - x^2$ and $g(x) = \sqrt{x}$

Now, $\quad f'(x) = 3 - 2x$

By the sign scheme, f is strictly inc. in $\left(-\infty, \dfrac{3}{2}\right)$ and strictly decreasing in $\left(\dfrac{3}{2}, \infty\right)$.

Also, g is srictly increasing in $[0, \infty)$.

Now, $\quad D_f = [0, 3]$

Thus, the function $y = \sqrt{3x - x^2}$ is strictly increasing in $\left(0, \dfrac{3}{2}\right)$ and strictly decreasing in $\left(\dfrac{3}{2}, 2\right)$.

31. Let $g(x) = \tan^{-1}$ and $h(x) = (\sin x + \cos x)$

Now, $h'(x) = (\cos x - \sin x)$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x\right)$$

$$= \sqrt{2}\cos\left(x + \frac{\pi}{4}\right)$$

$h(x)$ is strictly increasing if $h'(x) > 0$

$\Rightarrow \quad \sqrt{2}\cos\left(x + \frac{\pi}{4}\right) > 0$

$\Rightarrow \quad \cos\left(x + \frac{\pi}{4}\right) > 0$

$\Rightarrow \quad 0 < \left(x + \frac{\pi}{4}\right) < \frac{\pi}{2} \;\&\; \frac{3\pi}{2} < \left(x + \frac{\pi}{4}\right) < 2\pi$

$\Rightarrow \quad -\frac{\pi}{4} < x < \frac{\pi}{4}$ and $\frac{5\pi}{4} < x < \frac{7\pi}{4}$

Thus, the given function $f(x)$ is strictly increases in $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right) \cup \left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$ and strictly decreasing in $\left(0, \frac{\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right) \cup \left(\frac{7\pi}{4}, 2\pi\right)$.

32. Let $g(x) = \frac{\log x}{x}$ and $h(x) = \log x$

Now, $g'(x) = \frac{1 - \log x}{x^2}$

By the sign scheme, $g(x)$ is strictly increasing in $(0, e)$ and strictly decreasing in (e, ∞)

Also, $h(x)$ is strictly increasing for all $x > 0$.

Also, $D_f = (1, \infty)$

Thus, $f(x)$ is strictly increasing in $(1, e)$ and strictly decreasing in (e, ∞)

33. Let $g(x) = \sin x + \cos x$ and $h(x) = \log x$

Since $h(x)$ is an increasing function, $f(x)$ will be decreases if $g(x)$ decreases.

Now, $g'(x) = \cos x - \sin x$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x\right)$$

$$= \sqrt{2}\left(\cos\left(x + \frac{\pi}{4}\right)\right)$$

Since $g(x)$ decreases, so $g'(x) < 0$

$\Rightarrow \quad \cos\left(x + \frac{\pi}{4}\right) < 0$

$\Rightarrow \quad 2n\pi + \frac{\pi}{2} < \left(x + \frac{\pi}{4}\right) < 2n\pi + \frac{3\pi}{2}$

$\Rightarrow \quad 2n\pi + \frac{\pi}{4} < x < 2n\pi + \frac{5\pi}{4}$

$\Rightarrow \quad 2n\pi + \frac{\pi}{4} < \log x < 2n\pi + \frac{5\pi}{4}$

$\Rightarrow \quad e^{\left(2n\pi + \frac{\pi}{4}\right)} < x < e^{\left(2n\pi + \frac{5\pi}{4}\right)}$

34. Let $g(x) = \log_e x$ and $h(x) = \cos x$

Here, $g(x)$ is strictly increases for all $x > 0$

Also, $h(x)$ is strictly decreases in $(0, \pi)$

Again, for the domain of the function, $\cos x > 0$

$\Rightarrow \quad x \in \left(0, \frac{\pi}{2}\right)$

Therefore, the function $f(x)$ is strictly decreases in $\left(0, \frac{\pi}{2}\right)$.

35. Let $g(x) = \sin x + \cos x$ and $h(x) = \sin x$

Now, $g'(x) = \cos x - \sin x$

$$= \sqrt{2}\cos\left(x + \frac{\pi}{4}\right)$$

When $g'(x) > 0$. $\sqrt{2}\cos\left(x + \frac{\pi}{4}\right) > 0$

$\Rightarrow \quad -\frac{\pi}{2} < \left(x + \frac{\pi}{4}\right) < \frac{\pi}{2}$

$\Rightarrow \quad -\frac{\pi}{2} - \frac{\pi}{4} < x < \frac{\pi}{2} - \frac{\pi}{4}$

$\Rightarrow \quad -\frac{3\pi}{2} < x < \frac{\pi}{4}$

When $g'(x) < 0$, $\frac{\pi}{2} < \left(x + \frac{\pi}{4}\right) < \frac{3\pi}{2}$

$\Rightarrow \quad \frac{\pi}{2} - \frac{\pi}{4} < x < \frac{3\pi}{2} - \frac{\pi}{4}$

$\Rightarrow \quad \frac{\pi}{4} < x < \frac{5\pi}{4}$

Thus, $f(x)$ is strictly increases in $\left(0, \frac{\pi}{2}\right)$ and strictly decreasing in $\left(\frac{\pi}{4}, \pi\right)$.

36. Let $f(x) = \log(1 + x) - x + \frac{x^2}{2}$

$\Rightarrow \quad f'(x) = \frac{1}{1 + x} - 1 + x$

$\Rightarrow \quad f'(x) = \frac{1}{1 + x} - (1 - x)$

$\Rightarrow \quad f'(x) = \frac{1 - 1 + x^2}{1 + x} = \frac{x^2}{x + 1} > 0 \;\forall\; x \in R^+$

$\Rightarrow \quad f(x)$ is strictly increasing in $(0, \infty)$

Thus $\quad f(x) > f(0)$

$\Rightarrow \quad \log(1 + x) - x + \dfrac{x^2}{2} > 0$

$\Rightarrow \quad \log(1 + x) > x - \dfrac{x^2}{2}$

Hence, the result.

37. Consider $f(x) = \log(1 + x) - \dfrac{x}{x + 1}$

$\Rightarrow \quad f'(x) = \dfrac{1}{1 + x} - \dfrac{1}{(x + 1)^2}$

$\Rightarrow \quad f'(x) = \dfrac{x + 1 - 1}{(x + 1)^2} = \dfrac{x}{(x + 1)^2}$

$\Rightarrow \quad f'(x) > 0$ for all $x > 0$

Thus, $f(x)$ is strictly increasing

$\Rightarrow \quad f(x) > f(0)$

$\Rightarrow \quad \log(1 + x) - \dfrac{x}{x + 1} > 0$

$\Rightarrow \quad \log(1 + x) > \dfrac{x}{x + 1}$ for all $x > 0$

Hence, the result.

38. Let $\quad f(x) = (1 + x)\log(1 + x) - e^x + 1$

$\Rightarrow \quad f'(x) = \dfrac{(1 + x)}{(1 + x)} + \log(1 + x) \cdot 1 - e^x$

$\Rightarrow \quad f'(x) = 1 + \log(1 + x) - e^x$

$\Rightarrow \quad f'(x) < 0$ for all $x < 0$

Thus, $f(x)$ is strictly decreasing function

$\Rightarrow \quad f'(x) < f(0)$

$\Rightarrow \quad (1 + x)\log(1 + x) - e^x + 1 < 0$

$\Rightarrow \quad (1 + x)\log(1 + x) < e^x - 1$

Hence, the result.

39. We have $f(x) = 2x\tan^{-1}x - \log(1 + x^2)$

$\Rightarrow \quad f'(x) = 2\tan^{-1}x + \dfrac{2x}{1 + x^2} - \dfrac{2x}{1 + x^2}$

$\Rightarrow \quad f'(x) = 2\tan^{-1}x > 0$ for all x in R^+

Thus, $f(x)$ is strictly increasing in R^+

$\Rightarrow \quad f(x) > f(0)$

$\Rightarrow \quad 2x\tan^{-1}x - \log(1 + x^2) > 0$

$\Rightarrow \quad 2x\tan^{-1}x > \log(1 + x^2)$

Hence, the result.

40. Let $\quad f(x) = 1 + x\log\left(x + \sqrt{x^2 + 1}\right) - \sqrt{1 + x^2}$

$$f'(x) = \log\left(x + \sqrt{x^2 + 1}\right) + \left(\frac{x}{x + \sqrt{x^2 + 1}}\right)$$
$$\left(1 + \frac{x}{\sqrt{x^2 + 1}}\right) - \frac{x}{\sqrt{1 + x^2}}$$

$$f'(x) = \log\left(x + \sqrt{x^2 + 1}\right) + \frac{x}{\sqrt{1 + x^2}} - \frac{x}{\sqrt{1 + x^2}}$$

$$f'(x) = \log\left(x + \sqrt{x^2 + 1}\right)$$

$$f'(x) = \log\left(x + \sqrt{x^2 + 1}\right) \geq 0, \ \forall x \geq 0$$

Thus, $f(x)$ is increasing in $[0, \infty)$

$f(x) \geq f(0)$

$1 + x\log\left(x + \sqrt{x^2 + 1}\right) - \sqrt{x^2 + 1} \geq 0$

$\Rightarrow \quad 1 + x\log\left(x + \sqrt{x^2 + 1}\right) \geq \sqrt{x^2 + 1}$

Hence, the result.

41. Let $\quad f(x) = x - \sin x$

$\Rightarrow \quad f'(x) = 1 - \cos x > 0, \ \forall x \in \left(0, \dfrac{\pi}{2}\right)$

Thus, $f(x)$ is strictly increasing in $\left(0, \dfrac{\pi}{2}\right)$

$\Rightarrow \quad f(x) > f(0)$

$\Rightarrow \quad x - \sin x > 0$

$\Rightarrow \quad x > \sin x$

$\Rightarrow \quad \cos x < \cos(\sin x)$ ...(i)

Also, for all x in $\left(0, \dfrac{\pi}{2}\right)$, $0 < \cos x < 1$

$\Rightarrow \quad \cos x < 1$

$\Rightarrow \quad \cos x > \sin(\cos x)$ ...(ii)

From (i) and (ii), we get,

$$\sin(\cos x) < \cos x < \cos(\sin x)$$

42. Given $\quad f(x) = \log x - Bx^2$

$\Rightarrow \quad f'(x) = \dfrac{1}{x} - 2Bx = \dfrac{1 - 2Bx^2}{x}$

The critical points of 'f' are

$$x = 0, \frac{1}{\sqrt{2B}}, -\frac{1}{\sqrt{2B}}$$

Now, $\quad f'(x) > 0, \ \forall x \in \left(0, \dfrac{1}{\sqrt{2B}}\right)$

and $\quad f'(x) < 0, \ \forall x \in \left(\frac{1}{\sqrt{2B}}, \infty\right)$

Now, $\quad \log x < Bx^2$ for $x > 0$

It holds good for $x = \frac{1}{\sqrt{2B}}$

Thus $\quad \log\left(\frac{1}{\sqrt{2B}}\right) < B, \frac{1}{2B} = \frac{1}{2}$

$\Rightarrow \quad -\log(\sqrt{2B}) < \frac{1}{2}$

$\Rightarrow \quad \log(\sqrt{2B}) < -\frac{1}{2}$

$\Rightarrow \quad \sqrt{2B} < e^{-\frac{1}{2}}$

$\Rightarrow \quad 2B < e^{-1}$

$\Rightarrow \quad B > \frac{1}{2e}$

Thus, the least value of B is $\frac{1}{2e}$

43. Let $\quad f(x) = ax^2 + \frac{b}{x} - c$

$\Rightarrow \quad f'(x) = 2ax - \frac{b}{x^2}$

Now, $\quad f'(x) = 0$ gives $2ax - \frac{b}{x^2} = 0$

$\Rightarrow \quad 2ax^3 = b$

$\Rightarrow \quad x = \left(\frac{b}{2a}\right)^{1/3}$

Thus, the least value of $f(x)$ occurs at $x = \left(\frac{b}{2a}\right)^{1/3}$

we have $a\left\{\frac{b}{2a}\right\}^{2/3} + \frac{b}{\left\{\frac{b}{2a}\right\}^{1/3}} \geq c$

$\Rightarrow \quad a\left(\frac{b}{2a}\right) + b \geq c \cdot \left(\frac{b}{2a}\right)^{1/3}$

$\Rightarrow \quad \left(\frac{3b}{2}\right)^3 \geq \frac{b}{2a} \cdot c^2$

$\Rightarrow \quad \frac{27b^3}{8} \geq \frac{b}{2a} \cdot c^2$

$\Rightarrow \quad 27b^2 a \geq 4c^3$

$\Rightarrow \quad 27ab^2 \geq 4c^3$

Hence, the result.

44. We have $f(x) = x^5 + 5x - 6$

$\Rightarrow \quad f'(x) = 5x^4 + 5$

$\Rightarrow \quad f''(x) = 20x^3$

$\Rightarrow \quad f'''(x) = 60x^2$ is exists for all x

Now, $\quad f''(x) = 0$ gives $x = 0$

By the sign scheme for $f''(x) = 0$, we have, $f(x)$ is concave down in $(-\infty, 0)$ and concave up in $(0, \infty)$.

45. We have $f(x) = x^4 - 5x^3 - 15x^2 + 30$

$\Rightarrow \quad f'(x) = 4x^3 - 15x^2 - 30x$

$\Rightarrow \quad f''(x) = 12x^2 - 30x - 30$

Now, $\quad f''(x) = 0$ gives $12x^2 - 30x - 30 = 0$

$\Rightarrow \quad 6x^2 - 15x - 15 = 0$

$\Rightarrow \quad 2x^2 - 5x - 5 = 0$

$\Rightarrow \quad x = \frac{5 \pm \sqrt{25 + 40}}{4}$

$\Rightarrow \quad x = \frac{5 \pm 8}{4} = \frac{13}{4}, -\frac{3}{4}$

By the sign scheme for the function $f''(x)$, the function $f(x)$ is concave down in $\left(-\frac{3}{4}, \frac{13}{4}\right)$ and concave up in $\left(-\infty, -\frac{3}{4}\right) \cup \left(\frac{13}{4}, \infty\right)$

46. We have $f(x) = (\sin x + \cos x)e^x$

$f'(x) = (\sin x + \cos x)e^x + e^x(\cos x - \sin x)$

$f'(x) = e^x(\sin x + \cos x + \cos x - \sin x)$

$f'(x) = 2e^x \cos x$

$f''(x) = 2(e^x \cos x - e^x \sin x)$

$f''(x) = 2e^x(\cos x - \sin x)$

Now, $\quad f''(x) = 0$ gives $2e^x(\cos x - \sin x) = 0$

$\Rightarrow \quad \tan x = 1$

$\Rightarrow \quad x = \frac{\pi}{4}, \frac{5\pi}{4}$

By the sign scheme for the function $f''(x) = 0$, we have $f(x)$ is concave down in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ and concave up in $\left(0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right)$

47. Given curve is $y = f(x) = Ax^2 + Bx + C$

$f'(x) = 2Ax + B$

$f''(x) = 2A$

Thus, the curve concave up if $f''(x) > 0$ and concave down if $f''(x) < 0$ i.e. concave up if $A > 0$ and concave down if $A < 0$.

48. We have $f(x) = x^4 - 4x^3 + x - 10$

$\Rightarrow \quad f'(x) = 4x^3 - 12x^2 + 1$

$\Rightarrow \quad f''(x) = 12x^2 - 24x = 12x(x - 2)$

Now, $\quad f''(x) = 0$ gives $x = 0$ and $x = 2$.

when $\quad x = 0, y = -10$

when $\quad x = 2$, y $= -24$

Thus, the point of inflection are (0, 10) and (2, −24)

49. We have $y = f(x) = (x - 2)^{2/3} + 10$

$\Rightarrow \quad f'(x) = \frac{2}{3}(x - 2)^{-1/3} = \frac{2}{3(x-2)^{1/3}}$

$\Rightarrow \quad f''(x) = \frac{2}{9(x-2)^{4/3}}$

Thus, $\quad f''(x)$ does not exist at $x = 2$.

when $\quad x = 2, y = 0 + 10 = 10$

Thus, the inflection point is (2, 10)

50. We have $f(x) = x^4 - 6x^3 + 12x^2 - 8x + 3$

$\Rightarrow \quad f'(x) = 4x^3 - 18x^2 + 24x - 8$

$\Rightarrow \quad f''(x) = 12x^2 - 36x + 24$

$= 12(x^2 - 3x + 2)$

$= 12(x - 1)(x - 2)$

Now, $\quad f''(x) = 0$ gives $x = 1, 2$

when $\quad x = 1, y = 2$

when $\quad x = 2, y = 3$

Thus, the point of inflection are (1, 2) and (2, 3).

51. Given $\quad y = f(x) = x^2 - \frac{1}{6x^3}$

$\Rightarrow \quad f'(x) = 2x + \frac{1}{2x^4}$

$\Rightarrow \quad f''(x) = 2 - \frac{2}{x^5}$

Now, $\quad f''(x) = 0$ gives $2 - \frac{2}{x^5} = 0$

Thus, $\quad x = 1$

when $\quad x = 1, y = \frac{5}{6}$

Thus, the point of inflection is $\left(1, \frac{5}{6}\right)$.

52. We have $y = f(x) = e^{-x^2}$

$\Rightarrow \quad f'(x) = e^{-x^2} \times -2x$

$\Rightarrow \quad f'(x) = -2e^{-x^2} \times x$

$\Rightarrow \quad f''(x) = -2\left(e^{-x^2} \cdot 1 + x \cdot e^{-x^2} \cdot (-2x)\right)$

$\Rightarrow \quad f''(x) = 2e^{-x^2}(2x^2 - 1)$

Now, $\quad f''(x) = 0$ gives $2e^{-x^2}(2x^2 - 1) = 0$

$\Rightarrow \quad x = \pm \frac{1}{\sqrt{2}}$

when $\quad x = \frac{1}{\sqrt{2}}, y = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$

when $\quad x = -\frac{1}{\sqrt{2}}, y = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$.

Thus, the point of inflection are

$$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{e}}\right) \& \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{e}}\right)$$

Level III

1. Given $\quad f(x) = 4x - \tan 2x$

$f'(x) = 4 - 2\sec^2 x$

Now, $\quad f'(x) > 0$ gives $4 - 2\sec^2 2x > 0$

$\Rightarrow \quad \sec^2 2x < 2$

$\Rightarrow \quad (\sec 2x + \sqrt{2})(\sec 2x - \sqrt{2}) < 0$

$\Rightarrow \quad -\sqrt{2} < \sec 2x < \sqrt{2}$

$\Rightarrow \quad -\frac{\pi}{4} < 2x < \frac{\pi}{4}$

$\Rightarrow \quad -\frac{\pi}{8} < x < \frac{\pi}{8}$

Thus, the length of the longest interval

$$= \left(\frac{\pi}{8} - \left(-\frac{\pi}{4}\right)\right) = \frac{\pi}{4}$$

2. We have $x^3 + 2x^2 + 5x + 2\cos x = 0$

Let $\quad f(x) = x^3 + 2x^2 + 5x + 2\cos x$

$\Rightarrow \quad f'(x) = 3x^2 + 4x + 5 - 2\sin x$

Let $\quad g(x) = 3x^2 + 4x + 5$ and $h(x) = 2\sin x$

Max value of $g(x)$ is $-\frac{16-60}{6} = \frac{44}{6} = \frac{22}{3}$ and max value of $h(x)$ is 2.

Thus, $\quad f'(x) > 0$

$\Rightarrow \quad f(x)$ is strictly increasing function

Also, $\quad f(0) = 2 > 0$ and $f(2\pi) > 0$

Therefore $f(x)$ has no real root.