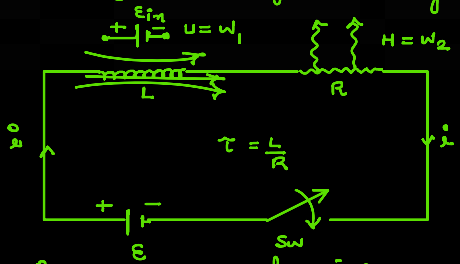
Self & Mutual Induction

22 September 2020 17:00

Q:→ In the following circuit find the work done by the battery in one time constant after closing the switch.



Sol^n:→ instantaneous charging current in the circuit:

$$i = \frac{\varepsilon}{R}\cdot\left\{1 - e^{-\frac{Rt}{L}}\right\} \quad \text{— (1)}$$

∵ conservative part of work done by the battery get stored in form of P.E. on the axis of the inductor coil.

$$W_1 = U = \frac{1}{2}Li^2$$

$$= \frac{1}{2}\cdot L\cdot\frac{\varepsilon^2}{R^2}\cdot\left(1 - e^{-\frac{Rt}{L}}\right)^2$$

at $t = \frac{L}{R}$ or τ

$$= \frac{1}{2}\cdot L\cdot\frac{\varepsilon^2}{R^2}\cdot\left\{1 - e^{-\frac{R}{L}\cdot\frac{L}{R}}\right\}^2$$

$$= \frac{\varepsilon^2 L}{2R^2}\cdot\left\{1 - e^{-1}\right\}^2$$

$$\therefore W_1 = \frac{\varepsilon^2 L}{2R^2}\cdot\left\{1 + e^{-2} - 2e^{-1}\right\} \quad \text{— (1)}$$

non-conservative part of work get lost in form of heat across the resistor

$$\therefore W_2 = H = \int_0^{\tau} i^2 R\cdot dt$$

$$= \int_0^{\tau}\frac{\varepsilon^2}{R^2}\cdot\left(1 - e^{-\frac{Rt}{L}}\right)^2\cdot R\cdot dt$$

$$= \frac{\varepsilon^2}{R}\cdot\int_0^{\tau}\left\{1 + e^{-\frac{2t}{\tau}} - 2\cdot e^{-\frac{t}{\tau}}\right\}\cdot dt$$

$$= \frac{\varepsilon^2}{R}\cdot\left[t + \frac{e^{-\frac{2t}{\tau}}}{\left(\frac{-2}{\tau}\right)} - \frac{2\cdot e^{-\frac{t}{\tau}}}{\left(\frac{-1}{\tau}\right)}\right]_0^{\tau}$$

$$= \frac{\varepsilon^2}{R}\cdot\left[\left\{\tau - \frac{\tau}{2}\cdot e^{-2} + 2\tau\cdot e^{-1}\right\} - \left\{0 - \frac{\tau}{2}\cdot e^{0} + 2\tau\cdot e^{0}\right\}\right]$$

$$= \frac{\varepsilon^2}{R}\cdot\left[\tau - \frac{\tau}{2}\cdot e^{-2} + 2\tau\cdot e^{-1} + \frac{\tau}{2} - 2\tau\right]$$

$$= \frac{\varepsilon^2}{R}\cdot\left[-\frac{\tau}{2} - \frac{\tau}{2}\cdot e^{-2} + 2\tau\cdot e^{-1}\right]$$

$$= \frac{\varepsilon^2}{R}\cdot\frac{L}{2R}\cdot\left\{-1 - e^{-2} + 4e^{-1}\right\}$$

$$\Rightarrow W_2 = \frac{\varepsilon^2 L}{2R^2}\left\{-1 - e^{-2} + 4e^{-1}\right\} \quad \text{— (2)}$$

eq^n (1) + (2)

$$\therefore W = W_1 + W_2$$

$$= \frac{\varepsilon^2 L}{2R^2}\cdot\left\{1 + e^{-2} - 2e^{-1} - 1 - e^{-2} + 4\cdot e^{-1}\right\}$$

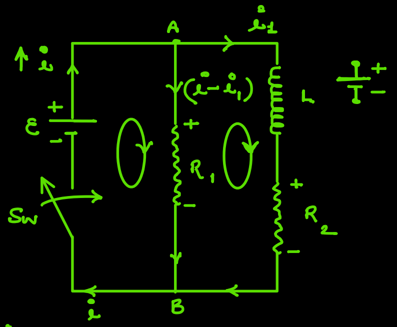
$$= \frac{\varepsilon^2 L}{2R^2}\cdot 2\cdot e^{-1}$$

$$\therefore W = \frac{\varepsilon^2 L}{eR^2} \text{ Joules}$$

Q:→ find the time constant & steady-state current in the following circuit

$\therefore \quad W = \frac{\varepsilon \cdot L}{eR^2}$ Joules

Q:→ find the time constant & steady-state current in the following circuit.



$\varepsilon in = \left| -L \cdot \frac{di_1}{dt} \right|$

Soln:→ KVL in loop 1 :→

$$-(i-i_1)\cdot R_1 + \varepsilon = 0$$

$$\Rightarrow (i-i_1)\cdot R_1 = \varepsilon \quad \text{—①}$$

KVL in loop 2 :→

$$\Rightarrow -L\cdot\frac{di_1}{dt} - i_1\cdot R_2 + (i-i_1)\cdot R_1 = 0$$

from eqn ①;

$$-L\cdot\frac{di_1}{dt} - i_1\cdot R_2 + \varepsilon = 0$$

$$\Rightarrow L\cdot\frac{di_1}{dt} = \varepsilon - i_1\cdot R_2$$

$$\Rightarrow \int_0^{i_1} \frac{di_1}{(\varepsilon - i_1 R_2)} = \int_0^t \frac{dt}{L}$$

$$\Rightarrow \left\{ -\frac{1}{R_2}\cdot \log_e(\varepsilon - i_1 R_2) \right\}_0^{i_1} = \left(\frac{t}{L}\right)_0^t$$

$$\Rightarrow \log_e(\varepsilon - i_1\cdot R_2) - \log_e \varepsilon = -\frac{R_2}{L}\cdot(t-0)$$

$$\Rightarrow \log_e\left\{\frac{\varepsilon - i_1 R_2}{\varepsilon}\right\} = -\frac{R_2\cdot t}{L}$$

$$\Rightarrow \left(\frac{\varepsilon - i_1 R_2}{\varepsilon}\right) = e^{-\frac{R_2\cdot t}{L}}$$

$$\Rightarrow \varepsilon - i_1\cdot R_2 = \varepsilon\cdot e^{-\frac{R_2\cdot t}{L}}$$

$$\Rightarrow i_1 = \frac{\varepsilon}{R_2}\cdot\left\{1 - e^{-\frac{R_2\cdot t}{L}}\right\}$$

here $\boxed{\tau = \frac{L}{R_2}}$

from eqn ①

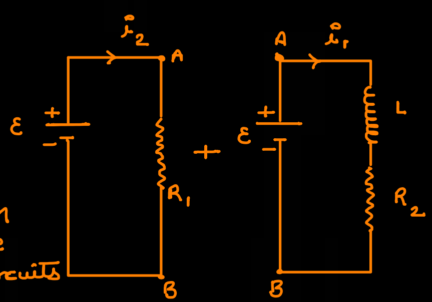
$$i\cdot R_1 = \varepsilon + i_1\cdot R_1$$

$$\therefore \quad i = \frac{\varepsilon}{R_1} + \frac{\varepsilon}{R_2}\cdot\left(1 - e^{-\frac{R_2}{L}\cdot t}\right)$$

inst. current from the battery.

Method 2:→

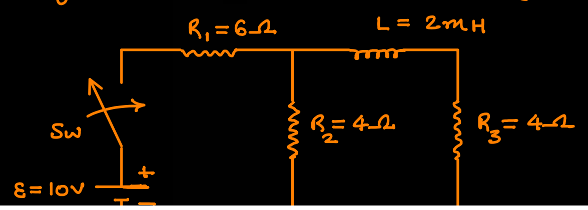
Given circuit can be written as superposition of these 2 circuits

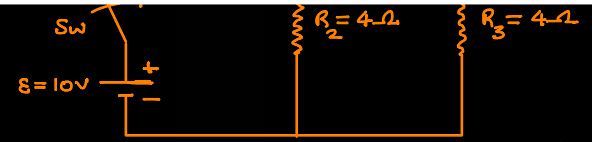


$i_2 = \frac{\varepsilon}{R_1}$; $i_1 = \frac{\varepsilon}{R_2}\cdot\left(1 - e^{-\frac{R_2\cdot t}{L}}\right)$

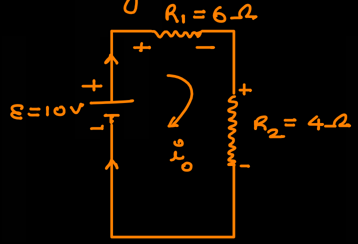
$$\therefore \quad i = i_1 + i_2 = \frac{\varepsilon}{R_1} + \frac{\varepsilon}{R_2}\cdot\left(1 - e^{-\frac{R_2 \cdot t}{L}}\right)$$

Q: find the ratio of the initial to the steady state currents





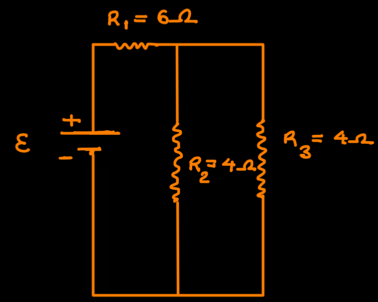
Sol^n: at $t = 0s$; initially inductor is open circuit



$$i_0 = \frac{\mathcal{E}}{R_{eq}} = \frac{\mathcal{E}}{R_1 + R_2} = \frac{10}{10}$$

$$i_0 = 1\ Amp \text{ — (1)}$$

at $t \to \infty$; inductor coil becomes short circuit

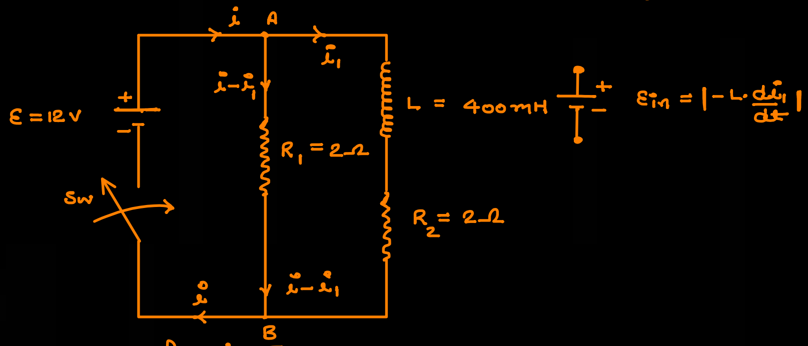


$$i_\infty = \frac{\mathcal{E}}{R_{eq}} = \frac{\mathcal{E}}{R_1 + R_{23}} = \frac{10}{6+2}$$

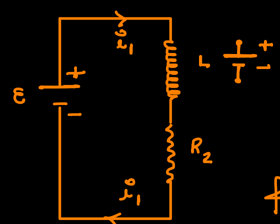
$$i_\infty = \frac{5}{4}\ A \text{ — (2)}$$

$$\therefore \frac{i_0}{i_\infty} = \frac{4}{5}$$

Q:→ find the P.D. across the inductor coil as a fn. of time t.



Sol^n: as P.D across A & B is E ;
we can consider the circuit as :



here: $i_1 = \frac{\mathcal{E}}{R_2} \cdot \left(1 - e^{-R_2 \cdot \frac{t}{L}}\right)$

& $\mathcal{E}_{in} = \left| -L \cdot \frac{di_1}{dt} \right|$

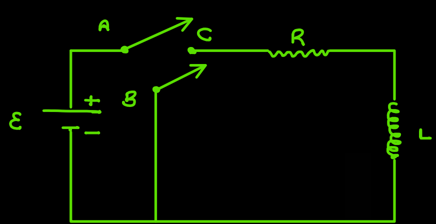
$$= \left| -L \cdot \frac{\mathcal{E}}{R_2} \cdot \left(0 - e^{-R_2 \frac{t}{L}} \times -\frac{R_2}{L}\right) \right|$$

$$\Rightarrow \mathcal{E}_{in} = \mathcal{E} \cdot e^{-R_2 t / L}$$

$$= 12 \cdot e^{-\frac{2 \times t}{0.4}}$$

$$\mathcal{E}_{in} = 12 \cdot e^{-5t}$$

Q:→ initially point A & C were connected for a long time, then it is disconnected & B & C are connected at $t = 0$. Find the ratio of the voltages about the resistor & the inductor at $t = \frac{L}{R}$.



Sol^n:→ when A & C were connected for a long time (charging

Sol^n:→ when A & C were connected for a long time (charging state)

$$i = \frac{\varepsilon}{R}\cdot\left(1 - e^{-\frac{R\cdot t}{L}}\right)$$

at $t \to \infty$ (steady state)

$$i = \frac{\varepsilon}{R} = i_0 \quad \text{— (1)}$$

when B & C are connected (Discharging state)

Discharging current $(i) = i_0 \cdot e^{-\frac{Rt}{L}}$

∴ P.D. across the resistor $(\Delta V_R) = i \cdot R$

$$= i_0 \cdot R \cdot e^{-\frac{Rt}{L}}$$

$$\Delta V_R = \varepsilon \cdot e^{-Rt/L} \quad \text{— (2)}$$

inst. P.D. across the inductor

$$\Delta V_L = \left| -L\cdot\frac{di}{dt} \right|$$

$$= \left| -L\cdot\frac{\varepsilon}{R}\cdot e^{-\frac{R\cdot t}{L}} \times -\frac{R}{L} \right|$$

$$\Delta V_L = \varepsilon \cdot e^{-Rt/L} \quad \text{— (3)}$$

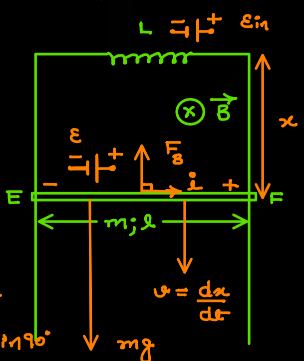
$$\therefore \frac{\Delta V_R}{\Delta V_L} = 1$$

Q:→ A smooth conducting frame ABCD is kept in the verticle plane. A conducting rod of mass m can slide over it remaining always horizontal. There is no resistance in the loop & its inductance is L. Initially no current was there in the circuit. The rod is allowed to fall under gravity & a uniform magnetic field exists perpendicular to the plane.

find: i) maximum velocity of the rod.

ii) Maximum current in the circuit

Sol^n :→



as the rod & inductor are in parallel

$$\therefore (\varepsilon_{in})_{S.I.} = (\varepsilon_{in})_{Mot.}$$

$$\Rightarrow L\cdot\frac{di}{dt} = B\cdot v\cdot l\sin 90^\circ$$

$$\Rightarrow L\cdot\frac{di}{dt} = B\cdot v\cdot l$$

$$\Rightarrow L\cdot\frac{di}{dt} = B\cdot\frac{dx}{dt}\cdot l$$

$$\Rightarrow L\cdot\int_0^i di = B\cdot l\cdot\int_0^x dx$$

$$\Rightarrow L\cdot i = B\cdot l\cdot x \quad \text{— (1)}$$

∵ $mg - F_B = m\cdot a$

$$mg - i\cdot B\cdot l\sin 90^\circ = m\cdot a \quad \text{— (2)}$$

as v becomes maximum

$a = 0$

$$mg - iBl = 0$$

current when velocity becomes max. $\Rightarrow i = \frac{mg}{B\cdot l}$ — (3)

from (1)

$$x_M = \frac{L}{B\cdot l}\cdot\frac{mg}{B\cdot l}$$

position where velocity becomes max. $x_M = \frac{mgL}{B^2\cdot l^2}$ — (4)

from (1) & (2)

$$mg - \frac{B^2\cdot l^2}{L}\cdot x = m\cdot v\cdot\frac{dv}{dx}$$

$$mg\int_0^{x_M} dx - \frac{B^2\cdot l^2}{L}\int_0^{x_M} x\cdot dx = m\cdot\int_0^{v_M} v\cdot dv$$

$$\Rightarrow mg\cdot x_M - \frac{B^2\cdot l^2}{L}\cdot\frac{x_M^2}{2} = m\cdot\frac{v_M^2}{2}$$

$$\Rightarrow mg\cdot x_M - \frac{B^2 l^2}{L}\cdot\frac{x_M^2}{2} = m\cdot\frac{v_M^2}{2}$$

from ④

$$mg\cdot\frac{mg\cdot L}{B^2 l^2} - \frac{B^2 l^2}{L}\times\frac{m^2 g^2 L^2}{2\cdot B^4 l^4} = \frac{m v_M^2}{2}$$

$$\Rightarrow \frac{m^2 g^2 L}{B^2 l^2} - \frac{m^2 g^2 L}{2\cdot B^2 l^2} = \frac{m v_M^2}{2}$$

$$\Rightarrow \frac{m^2 g^2 L}{2\cdot B^2 l^2} = \frac{m v_M^2}{2}$$

$$\therefore v_M^2 = \frac{m\cdot g^2\cdot L}{B^2 l^2}$$

$$\therefore \boxed{v_M = \frac{g\cdot\sqrt{mL}}{Bl}} \text{ m/s}$$

Max. velocity

from ① & ④

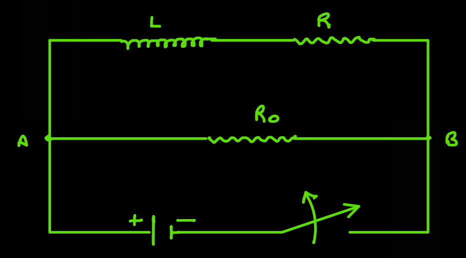
$$i_M = \frac{B\cdot l\cdot x_M}{L}$$

$$= \frac{Bl}{L}\cdot\frac{mg\cdot L}{B^2 l^2}$$

Max. current $\boxed{i_M = \frac{mg}{Bl}}$

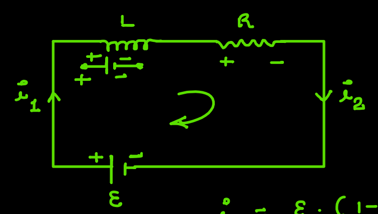
Q:→ A coil of inductance L = 2μH & resistance R = 1Ω is connected as shown in the fig. EMF of the battery is ε = 3V & the resistance R_0 is of 2Ω. Find the heat generated in resistor R_0 after the switch is disconnected.

Soln:→



During the charging state:

as the L-R & ε are directly in parallel



$$i_1 = \frac{\varepsilon}{R}\cdot\left(1 - e^{-\frac{R\cdot t}{L}}\right)$$ instantaneous charging current

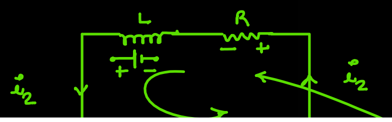
at steady state;

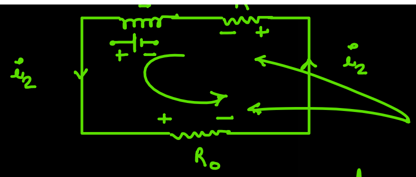
$t \to \infty$

$$i_1 = \frac{\varepsilon}{R}\cdot\left(1 - e^{-\infty}\right)$$

$$(i_1)_{Max} = \frac{\varepsilon}{R} \quad \text{—①}$$; Max. current through the inductor

During the discharging state;





Series : $R_{eq} = R + R_0$

instantaneous discharging current ;

$$i_2 = (i_1)_{Max} \cdot e^{-R_{eq}\cdot\frac{t}{L}}$$

$$i_2 = \frac{\varepsilon}{R} \cdot e^{-(R_0+R)\cdot\frac{t}{L}} \quad — ②$$

$\therefore$ heat generated across R_0 during the discharge

$$H = \int_0^\infty i_2^2 \cdot R_0 \cdot dt$$

$$= \frac{\varepsilon^2}{R^2} \cdot R_0 \cdot \int_0^\infty e^{-2(R_0+R)\cdot\frac{t}{L}} \cdot dt$$

$$= \frac{\varepsilon^2 \cdot R_0}{R^2} \cdot \left\{ \frac{e^{-2(R_0+R)\cdot\frac{t}{L}}}{\frac{-L}{2(R_0+R)}} \right\}_0^\infty$$

$$= \frac{\varepsilon^2 \cdot R_0}{2R^2} \cdot \frac{L}{(R_0+R)} \cdot \left\{ e^0 - e^{-\infty} \right\}$$

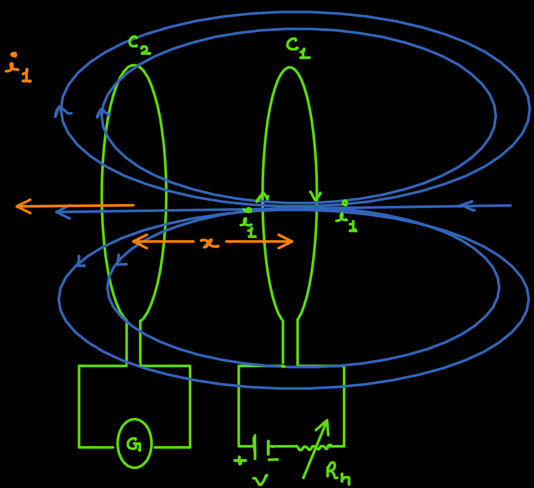
$$= \frac{\varepsilon^2 \cdot R_0 \cdot L}{2R^2 \cdot (R_0+R)}$$

$$= \frac{9 \times 2 \times 2 \times 10^{-6}}{2 \times 1^2 \times 3}$$

$$\therefore H = 6\,\mu J$$

<u>Mutual Inductance</u> :→ In this type of EMI, EMF & current are induced in any coil by changing the current in any nearby coil.

$N_2\phi_2 \propto i_1$



Magnetic Induction at the center of C_2

$B_2 \propto i_1$

$$B_2 = \frac{\mu_0}{2} \cdot \frac{N_1 \cdot i_1 \cdot R_1^2}{(R_1^2 + x^2)^{3/2}}$$

$\Rightarrow B_2 \propto i_1$

flux linked to the 2^{nd} coil $\propto$ current in the 1^{st} coil

$$N_2\phi_2 \propto i_1$$

$$N_2 \cdot \phi_2 = M_{21} \cdot i_1 \quad — ①$$

coefficient of mutual induction of C_2 w.r.t C_1

$$\boxed{M_{21} = \frac{N_2 \cdot \phi_2}{i_1}}$$

Henry ; $[ML^2T^{-2}A^{-2}]$

it depends upon the shape, size, material & the surrounding medium.

<u>from faraday's Law</u> ;

$$\varepsilon_2 = -N_2 \cdot \frac{d\phi_2}{dt} \quad — ②$$

Differentiating eqn ① ;

Differentiating eqn ①;

$$N_2 \cdot \frac{d\phi_2}{dt} = M_{21} \cdot \frac{di_1}{dt} \quad \text{—③}$$

from ② & ③

Mutually induced EMF in C_2 $\boxed{\varepsilon_2 = -M_{21} \cdot \frac{di_1}{dt}}$ volt —④

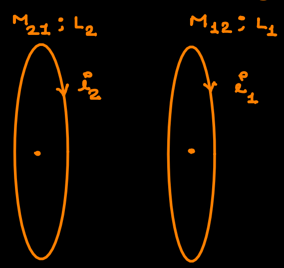
current induced in C_2 due to mutual induction

$$\boxed{i_2 = \frac{\varepsilon_2}{R_2} = -\frac{M_{21}}{R_2} \cdot \frac{di_1}{dt}} \quad \text{—⑤}$$

note: 1) due to change in current in C_1, self induction takes place in C_1 itself.

self induced EMF in C_2 ; $\varepsilon_1 = -L_1 \cdot \frac{di_1}{dt}$

2) if currents in both the nearby coils are changing.



total induced EMFs in the coils

$$\varepsilon_1 = (\varepsilon_1)_{self} + (\varepsilon_1)_{mutual}$$

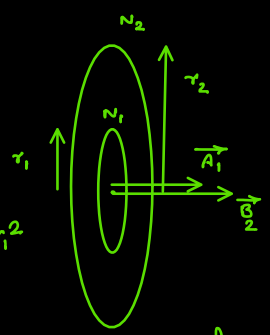
$$\therefore \varepsilon_1 = \left(-L_1 \cdot \frac{di_1}{dt}\right) + \left(-M_{12} \cdot \frac{di_2}{dt}\right)$$

& $\varepsilon_2 = (\varepsilon_2)_{self} + (\varepsilon_2)_{mutual}$

$$= \left(-L_2 \cdot \frac{di_2}{dt}\right) + \left(-M_{21} \cdot \frac{di_1}{dt}\right)$$

Q: find the coefficient of mutual induction of the following two concentric & coplanar coils. Here $r_1 << r_2$

Soln :→



flux linked to each turn of C_1

$$\phi_1 = \vec{B_2} \cdot \vec{A_1}$$

$$= B_2 \cdot A_1 \cdot \cos 0^\circ$$

$$= \frac{\mu_0}{2} \cdot \frac{N_2 \cdot i_2}{r_2} \cdot \pi r_1^2$$

so coeff. of mutual induction of C_1 w.r.t. C_2 $(M_{12}) = \frac{N_1 \cdot \phi_1}{i_2}$

$$M_{12} = \frac{\mu_0 \pi \cdot N_1 \cdot N_2 \cdot r_1^2}{r_2} \text{ H} \quad \text{—①}$$

flux linked to each turn of C_2 $(\phi_2) = \vec{B_1} \cdot \vec{A_1}$

{area of the smaller coil will be again taken into account}

$$\phi_2 = \frac{\mu_0}{2} \cdot \frac{N_1 \cdot i_1}{r_1} \times \pi r_1^2$$

∴ coeff. of mutual induction of C_2 w.r.t. C_1

$$M_{21} = \frac{N_2 \cdot \phi_2}{i_1} = \frac{\mu_0 \cdot N_1 N_2 \cdot \pi r_1^2}{2 r_1} \text{ H} \quad \text{—②}$$

eqn ① / ②

$M_{..}$ $r_{..}$

eqn 1 & 2

$$\frac{M_{12}}{M_{21}} = \frac{r_1}{r_2}$$