Logarithmic Differentiation:

The Process of taking logarithm of function first and then differentiating is called the Logarithmic differentiation. This is used mainly for two situations mentioned below:

1) A function is a product or quotient of number of functions.

$y = (x-1)(x-2)(x-3)$ — Product — (1)

$\ln y = \ln (x-1)(x-2)(x-3)$

$\ln y = \ln(x-1) + \ln(x-2) + \ln(x-3)$

Diff. wrt x: $\frac{1}{y}\frac{dy}{dx} = \frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-3}$

$\frac{dy}{dx} = y\left(\frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-3}\right)$

(2)

$y = \left(\frac{(x-1)(x-2)}{(x-3)(x-4)}\right)^{1/2}$

$\ln y = \ln(x-1) + \ln(x-2) - \ln(x-3) - \ln(x-4)$

Diff. wrt x

$$\boxed{\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left(\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4}\right)}$$

Q) $y = \cos(x^x)$, $\frac{dy}{dx} = ?$ ($x^x \to t$)

$\frac{dy}{dx} = -\sin(t)\frac{dt}{dx}$

$\frac{dy}{dx} = -\sin(x^x)\left(x^x(1+\ln x)\right)$

$t = x^x$

$\ln t = x \ln x$

Diff wrt x

$\frac{1}{t}\frac{dt}{dx} = x \cdot \frac{1}{x} + 1 \cdot \ln x$

$\frac{dt}{dx} = x^x(1+\ln x)$.

2. A function is of the form $f(x)^{g(x)}$ where f(x) and g(x) are both differentiable functions.

e.g $y = x^x$

$\ln y = \ln x^x$

L.h.s → R.h.s Diff wrt x: $\frac{d}{dx}(\ln y) = \frac{d}{dx}(x \ln x)$

$\frac{1}{y}\frac{dy}{dx} = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1$

$\frac{dy}{dx} = y(\ln x + 1)$

$\frac{dy}{dx} = x^x(\ln x + 1)$

e.g $y = (\sin x)^{\cos x}$

$\ln y = \ln(\sin x)^{\cos x}$

$\ln y = \cos x \ln(\sin x)$ ($\sin x \to t$)

Diff. w.r.t x

$\frac{1}{y}\frac{dy}{dx} = -\sin x \ln(\sin x) + \cos x \frac{1}{t}\left(\frac{d}{dx}(\sin x)\right)$

$\frac{1}{y}\frac{dy}{dx} = -\sin x \ln(\sin x) + \frac{\cos x \cos x}{\sin x}$

$$\boxed{\frac{dy}{dx} = (\sin x)^{\cos x}\left(-\sin x \ln(\sin x) + \frac{\cos^2 x}{\sin x}\right)}$$

Q) $y = 10^{(10^x)} + x^{(10^x)} + 10^{(x^{10})}$
(u, v, w)

$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx}$

$\frac{d}{dx}(a^x) = a^x \ln a$

$\frac{d}{dx}a^{f(x)} = a^{f(x)} \ln a \frac{d(f(x))}{dx}$ ($f(x) \to t$)

$\frac{d}{dx}(a^{\sin x}) = a^{\sin x} \ln a \cos x$

$\frac{d}{dx}(3^{\tan x}) = 3^t \ln 3 \frac{dt}{dx}$

$= 3^{\tan x} \ln 3 \sec^2 x$

$V = x^{10^x}$

Q) $x^y + y^x = 4$.

find $\frac{dy}{dx} = ?$

$\Rightarrow \frac{d}{dx}(x^y + y^x) = \frac{d}{dx}(4)$ ($x^y \to u$, $y^x \to v$)

$\Rightarrow \frac{d}{dx}(u) + \frac{d}{dx}(v) = 0$ — (1)

$\Rightarrow x^y \ln x \frac{dy}{dx} + x^y \frac{y}{x} + y^x \ln y + y^x \frac{x}{y}\frac{dy}{dx} = 0$

☆ $\Rightarrow$ $$\boxed{\frac{dy}{dx} = -\frac{\left(x^{y-1}y + y^x \ln y\right)}{x^y \ln x + y^x x/y}}$$

$u = x^y$

Take log

$\ln u = \ln x^y = y \ln x$

Diff. w.r.t x

$\frac{1}{u}\frac{du}{dx} = (\ln x)\frac{dy}{dx} + y \cdot \frac{1}{x}\frac{dx}{dx}$

$\frac{du}{dx} = x^y\left(\ln x \frac{dy}{dx} + \frac{y}{x}\right)$ — (2)

$V = y^x$

Take log

$\ln v = x \ln y$

Diff w.r.t x

$\frac{1}{v}\frac{dv}{dx} = 1 \cdot \ln y + x \cdot \frac{1}{y}\frac{dy}{dx}$

$\frac{dv}{dx} = y^x\left(\ln y + \frac{x}{y}\frac{dy}{dx}\right)$. — (3)

$u = 10^{(10^x)}$ ($10^x \to t$)

$\frac{du}{dx} = 10^t \ln 10 \frac{d(t)}{dx}$

$= 10^{10^x} \ln 10 \frac{d(10^x)}{dx}$

$= 10^{10^x} \ln 10 \cdot 10^x \ln 10$

$= 10^{10^x} \cdot 10^x \cdot (\ln 10)^2$

$v = x^{(10^x)}$

$\ln v = 10^x \ln x$

Diff w.r.t x

$\frac{1}{v}\frac{dv}{dx} = 10^x \cdot \frac{1}{x} + 10^x \ln 10 \cdot \ln x$

$\frac{dv}{dx} = x^{(10^x)}\left(\frac{10^x}{x} + 10^x \ln 10 \ln x\right)$

$w = 10^{(x^{10})}$ ($x^{10} \to t$)

$\frac{dw}{dx} = 10^t \ln 10 \frac{dt}{dx}$

$\frac{dw}{dx} = 10^{x^{10}} \ln 10 \, d(x^{10})$

$= 10^{x^{10}} \ln 10 \, 10 \cdot x^9$

$= x^9 \cdot 10^{x^{10}} \cdot \ln 10$

DIFFERENTIATION OF PARAMETRIC FUNCTIONS:

$y = f(t)$, $x = g(t)$ $\quad \frac{dy}{dx} = ?$

$\frac{dy}{dt} = f'(t)$, $\frac{dx}{dt} = g'(t)$ $\quad$ Divide : $\boxed{\frac{dy}{dx} = \frac{f'(t)}{g'(t)}}$

$t \in$ parameter

Q) $x = a(\theta - \sin\theta)$ $\quad \theta \in$ parameter $\quad$ A) $\tan\theta/2$ B) $-\cot\theta/2$ ✓ C) $-\tan\theta/2$. D) $\cot\theta/2$

$y = a(1+\cos\theta)$

$\frac{dy}{d\theta} = a(-\sin\theta)$

$\frac{dx}{d\theta} = a(1-\cos\theta)$

Divide : $\frac{dy}{dx} = \frac{-\sin\theta}{1-\cos\theta} = -\frac{\cancel{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cancel{2}\sin^2\theta/2} = \boxed{-\cot\frac{\theta}{2}}$

e.g $\quad y = 2at$, $x = at^2$ $\longleftrightarrow$ $y^2 = 4ax$

$\frac{dy}{dt} = 2a$, $\frac{dx}{dt} = 2at$

$\boxed{\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}} = \frac{2a}{y}$

$\frac{d}{dx}(y^2) = \frac{d}{dx}(4ax)$

$2y\frac{dy}{dx} = 4a$

$\boxed{\frac{dy}{dx} = \frac{2a}{y}}$

Q) $\Rightarrow x = a\left(\cos t + \ln\left(\tan\frac{t}{2}\right)\right)$ $\quad$ A) $\tan t$ ✓ B) $\cot t$ C) $-\tan t$ D) $-\cot t$

$\Rightarrow y = a\sin t$ $\quad t \in$ parameter

$\frac{dy}{dx} = ?$ $\quad \frac{dy}{dt} = a\cos t$ — (1)

$\frac{dx}{dt} = a\left(-\sin t + \frac{\sec^2 t/2 \cdot 1/2}{\tan\frac{t}{2}}\right)$

$= a\left(-\sin t + \frac{\cancel{\cos t/2}}{2\sin t/2 \cos^{\cancel{2}} t/2}\right) = a\left(-\sin t + \frac{1}{\sin t}\right) = \frac{a(\cos^2 t)}{\sin t}$ — (2)

Divide (1) & (2)

$\frac{dy}{dx} = \frac{\cancel{a}\,\cancel{\cos t}\,\sin t}{\cancel{a}\cos^{\cancel{2}} t} = \tan t$

Q) $y = a\cos^3 t$ $\quad \frac{dy}{dx} = ?$ $\quad a \in$ parameter

$x = a\sin^3 t$ $\quad$ A) $-\cot t$ B) $-\tan t$ C) $\cot^3 t$ D) $\tan^3 t$

$t \in$ parameter

A) $-\cot t$ ✓

B) $-\tan t$

C) $\cot t$

D) $\tan t$

$\frac{dy}{dt} = a\,3\cos^2 t(-\sin t)$

$\frac{dx}{dt} = a\,3\sin^2 t\cos t$

$\frac{dy}{dx} = \frac{-3\cancel{a}\cos^{\cancel{2}} t\,\sin t}{3\cancel{a}\sin^{\cancel{2}} t\,\cancel{\cos t}} = -\cot t$

$\frac{dy}{da} = 1.\cos^3 t$, $\frac{dx}{da} = 1.\sin^3 t$

$\frac{dy}{dx} = \frac{\cos^3 t}{\sin^3 t} = \cot^3 t$.

Q) $x = \sqrt{a^{\sin^{-1} t}}$, $y = \sqrt{a^{\cos^{-1} t}}$ $\quad$ then select correct stat. for $\frac{dy}{dx}$

A) xy $\quad$ C) $-y/x$ ✓

B) $\frac{y}{x}$ $\quad$ D) $-x/y$

$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$. $\quad \frac{dy}{dt} = ?$

$\frac{dx}{dt} = \frac{1}{2\sqrt{a^{\sin^{-1}t}}}\frac{d}{dx}\left(a^{\sin^{-1}t}\right) = \frac{1}{2\sqrt{a^{\sin^{-1}t}}} \cdot a^{\sin^{-1}t}\ln a \cdot \frac{d}{dx}(\sin^{-1}t)$

$= \frac{1}{2\sqrt{a^{\sin^{-1}t}}} \cdot a^{\sin^{-1}t}\frac{\ln a}{\sqrt{1-x^2}}$

Multiply $\quad xy = \sqrt{a^{\pi/2}}$

Diff wrt x $\quad x.\frac{dy}{dx} + y = 0$ $\quad \boxed{\frac{dy}{dx} = \frac{-y}{x}}$