ALGEBRA OF LIMITS:

Let f & g be two functions such that

$\boxed{\lim_{x\to a} f(x) = l}$ & $\boxed{\lim_{x\to a} g(x) = m}$ then

$\lim_{x\to \frac{\pi}{2}} (\sin x + x^2) = \left(\lim_{x\to \frac{\pi}{2}} \sin x\right) + \left(\lim_{x\to \frac{\pi}{2}} x^2\right)$

(1) $\lim_{x\to a} k f(x) = k \lim_{x\to a} f(x) = kl.$

(2) $\lim_{x\to a} \left(f(x) \pm g(x)\right) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = l \pm m.$

(3) ☆ $\lim_{x\to a} \left(f(x)\cdot g(x)\right) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right) = l\cdot m$

(4) ☆ $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{l}{m} \quad (m \neq 0)$

(5) $\lim_{x\to a} \left(\frac{1}{f(x)}\right) = \frac{1}{\lim_{x\to a} (f(x))} = \frac{1}{l} \quad (l \neq 0)$

eg $\lim_{x\to 2} \frac{1}{x^2-1} = \frac{1}{3}$

$\frac{1}{\lim_{x\to 2} (x^2-1)} = \frac{1}{3}$

(6) $\lim_{x\to a} (f(x))^n = \left(\lim_{x\to a} f(x)\right)^n = l^n \quad \boxed{n \in N}$

eg $\lim_{x\to 2} (x^2+1)^2 = 25 = \left(\lim_{x\to 2} (x^2+1)\right)^2 = \underbrace{5^2}_{=25}$

PROBLEMS CONT..

$\sqrt{16} = \pm 4$ ✗ $\sqrt{9^2} = \pm 3$ ✗ $+3$ $\sqrt{x^2} = |x|$

Q) $$\lim_{x \to -3} \left(\frac{x^3 + 27}{\sqrt{x^2+7} - 4}\right) \times \frac{\sqrt{x^2+7} + 4}{\sqrt{x^2+7} + 4}$$

$\sqrt{4^2} = 4$

$\sqrt{(-4)^2} = |-4| = 4.$

$\left(\lim_{x \to a} f(x) \cdot g(x)\right)$

$$\Rightarrow \lim_{x \to -3} \frac{(x+3)(x^2+9-3x)\left(\sqrt{x^2+7} + 4\right)}{x^2+7-16 - x^2-9}$$

$f(x)$, $g(x)$

$$\Rightarrow \lim_{x \to -3} \frac{\cancel{(x+3)}(x^2+9-3x)\left(\sqrt{x^2+7} + 4\right)}{\cancel{(x+3)}(x-3)} = \frac{(9+9+9)(4+4)}{-3-3}$$

$h(x)$

$$= \frac{\cancel{27}^{\,9} \times \cancel{8}^{\,4}}{-\cancel{6}} = \boxed{-36}$$

Q) $$\lim_{x \to a} \left(\frac{\sqrt{x-b} - \sqrt{a-b}}{x^2 - a^2}\right) \times \frac{\sqrt{x-b} + \sqrt{a-b}}{\sqrt{x-b} + \sqrt{a-b}}$$

$$= \lim_{x \to a} \frac{(x-\cancel{b}) - (a-\cancel{b})}{\cancel{(x-a)}(x+a)\left(\sqrt{x-b} + \sqrt{a-b}\right)}$$

a

$$= \frac{1}{2a \; 2\sqrt{a-b}} = \frac{1}{4a\sqrt{a-b}}$$

Q) $$\lim_{x \to 7} \left(\frac{4 - \sqrt{9+x}}{1 - \sqrt{8-x}}\right) \times \frac{\left(4 + \sqrt{9+x}\right)}{\left(1 + \sqrt{8-x}\right)} \frac{\left(1 + \sqrt{8-x}\right)}{\left(4 + \sqrt{9+x}\right)}$$

$\frac{0}{0}$

$$= \lim_{x \to 7} \left(\frac{16 - (9+x)}{1 - (8-x)}\right) \frac{\left(1 + \sqrt{8-x}\right)}{\left(4 + \sqrt{9+x}\right)}$$

$$= \lim_{x \to 7} \frac{(7-x)}{(x-7)} \frac{\left(1 + \sqrt{8-x}\right)}{\left(4 + \sqrt{9+x}\right)} = -\frac{\left(1 + \sqrt{8-7}\right)}{4 + \sqrt{9+7}} = -\frac{2}{8}$$

$$= \frac{-1}{4}.$$

Q) $$\lim_{x \to 2} \left(\frac{1}{(x-2)} - \left(\frac{4}{x^3 - 2x^2}\right)\right)$$

A) 0

B) 2

C) 1 ✓

D) does not exist

$(\infty - \infty)$

$$\lim_{x \to 2} \frac{x^3 - 2x^3 - 4(x-2)}{(x-2)(x^2)(x-2)}$$

$$= \lim_{x \to 2} \frac{(x^2-4)\cancel{(x-2)}}{\cancel{(x-2)}x^2(x-2)} = \lim_{x \to 2} \frac{\cancel{(x-2)}(x+2)}{x^2\cancel{(x-2)}}$$

$$= \frac{4}{4} = 1$$

Q) $$\lim_{x \to 1} \left(\frac{1}{x^2+x-2} - \frac{x}{x^3-1}\right)$$

$(\infty - \infty)$

A) $\frac{1}{3}$

B) $\frac{-1}{9}$ ✓

C) $\frac{1}{9}$

D) $\frac{-1}{3}$

$$\lim_{x \to 1} \left(\frac{1}{(x-1)(x+2)} - \frac{x}{(x-1)(x^2+x+1)}\right) = \lim_{x \to 1} \frac{1}{(x-1)} \left(\frac{1}{x+2} - \frac{1}{x^2+x+1}\right)$$

$$\lim_{x \to 1} \left(\frac{1}{x-1}\right)\left(\frac{\cancel{x^2}+x+1-\cancel{x^2}-2x}{(x+2)(x^2+x+1)}\right)$$

$$= \lim_{x \to 1} \frac{(-x+1)}{(x-1)(x+2)(x^2+x+1)} = \lim_{x \to 1} \frac{-\cancel{(x-1)}}{\cancel{(x-1)}(x+2)(x^2+x+1)} = \frac{-1}{3 \times 3} = \frac{-1}{9}$$

STANDARD SERIES EXPANSION:

SOME IMPORTANT EXPANSIONS

- In finding limits, use of expansions of following functions are useful:

(1) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + ...$

(2) $a^x = 1 + x\log a + \frac{(x\log a)^2}{2!} + ...$

(3) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$

(4) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ..., |x| < 1$

(5) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - ...$, where $|x| < 1$

(6) $(1+x)^{\frac{1}{x}} = e^{1 - \frac{x}{2} + \frac{x^2}{3}} ... = e\left(1 - \frac{x}{2} + \frac{11}{24}x^2 - ....\right)$

(7) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ....$

(8) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$

(9) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + ...$

(10 $\sin^{-1} x = x + 1^2 . \frac{x^3}{3!} + 3^2 . 1^2 . \frac{x^5}{5!} + ...$

(11) $\cos^{-1} x = \left(\frac{\pi}{2}\right) - \sin^{-1} x$

(12) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$

TRIGONOMETRIC LIMITS:

(1) $\displaystyle \underset{x\to 0}{\text{Lt}}\left(\frac{\sin x}{x}\right)=1$

(2) $\displaystyle \underset{x\to 0}{\text{Lt}}\left(\frac{\tan x}{x}\right)=1$

(3) $\displaystyle \underset{x\to 0}{\text{Lt}}\left(\frac{\sin mx}{x}\right)=m$

(4) $\displaystyle \underset{x\to 0}{\text{Lt}}\left(\frac{\tan mx}{x}\right)=m$

(5) $\displaystyle \underset{x\to 0}{\text{Lt}}\left(\frac{\sin mx}{\sin nx}\right)=\frac{m}{n}$

(6) $\displaystyle \underset{x\to 0}{\text{Lt}}\left(\frac{\sin mx}{\tan nx}\right)=\frac{m}{n}$

Proof: $\displaystyle \underset{x\to 0}{\text{Lt}}\left(\frac{\sin mx}{x}\cdot\frac{x}{\sin nx}\right)=\frac{m}{n}$

(7) $\displaystyle \underset{x\to 0}{\text{Lt}}\ \frac{1-\cos x}{x^2}=\frac{1}{2}$

$$\underset{x\to 0}{\text{Lt}}\ \frac{2\sin^2\frac{x}{2}}{x^2}$$

$$\underset{x\to 0}{\text{Lt}}\ 2\left(\frac{\sin\frac{x}{2}}{x}\right)^2$$

$$=2\times\left(\frac{1}{2}\right)^2=2\times\frac{1}{4}=\frac{1}{2}$$

(3) Proof $\displaystyle \underset{x\to 0}{\text{Lt}}\left(\frac{\sin mx}{x}\right)=m$

$$\underset{x\to 0}{\text{Lt}}\ m\frac{\sin(mx)}{(mx)}=m\ \underset{x\to 0}{\text{Lt}}\ \frac{\sin(mx)}{mx}=m\ \underset{t\to 0}{\text{Lt}}\left(\frac{\sin t}{t}\right)$$

$t=mx$ $\qquad = m$

$x\to 0$

$t\to 0$

Proof:

$$\underset{x\to 0}{\text{Lt}}\left(\frac{\sin x}{x}\right)=\underset{x\to 0}{\text{Lt}}\left(\frac{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}\cdots\cdots}{x}\right)$$

$$=\underset{x\to 0}{\text{Lt}}\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}\cdots\right)=1$$

☆ $\displaystyle \boxed{\underset{x\to 0}{\text{Lt}}\left(\frac{\tan x}{x}\right)=\underset{x\to 0}{\text{Lt}}\left(\frac{x+\frac{x^3}{3}+\frac{2}{15}x^5+\frac{17}{315}x^7\cdots\cdots}{x}\right)=1}$

Alt) $\displaystyle \underset{x\to 0}{\text{Lt}}\left(\frac{\tan x}{x}\right)=\underset{x\to 0}{\text{Lt}}\ \frac{\sin x}{\cos x\, x}=\underset{x\to 0}{\text{Lt}}\ \frac{1}{\cos x}\cdot\underset{x\to 0}{\text{Lt}}\left(\frac{\sin x}{x}\right)$

$$=1$$

(1) $\displaystyle \underset{x\to 0}{\text{Lt}}\left(\frac{1-\cos x}{x}\right)=\underset{x\to 0}{\text{Lt}}\ \frac{2\sin^2\frac{x}{2}}{x}=\underset{x\to 0}{\text{Lt}}\ 2\left(\frac{\sin\frac{x}{2}}{x}\right)\sin\frac{x}{2}=0$

$$=\underset{x\to 0}{\text{Lt}}\ 2\cdot\underset{x\to 0}{\text{Lt}}\left(\frac{\sin\frac{x}{2}}{x}\right)\underset{x\to 0}{\text{Lt}}\ \sin\frac{x}{2}$$

$$=2\cdot\frac{1}{2}\cdot 0=0$$

(2) $\displaystyle \underset{x\to 0}{\text{Lt}}\left(\frac{1-\cos x}{x^2}\right)=\frac{1}{2}$

(3) $\displaystyle \underset{x\to 0}{\text{Lt}}\left(\frac{1-\cos x}{x^3}\right)=\underset{x\to 0}{\text{Lt}}\left(\frac{2\sin^2\frac{x}{2}}{x^3}\right)=\underset{x\to 0}{\text{Lt}}\ 2\cdot\left(\frac{\sin\frac{x}{2}}{x}\right)^2\left(\frac{1}{x}\right)$

$$=2\times\frac{1}{4}\times\infty=\infty$$

Q) $\displaystyle \underset{x\to 0}{\text{Lt}}\left(\frac{\tan x-\sin x}{x^3}\right)=\frac{1}{2}$