CHAIN RULE:

It is used for differentiation of Composite functions.

$y = \sin(x^2) \longleftarrow y = f(g(x))$

$f(x) = \sin x, \quad g(x) = x^2$

$y = (\sin x)^2 \longleftarrow y = f(g(x))$

$f(x) = x^2, \quad g(x) = \sin x$

DIFFERENTIATION OF COMPOSITE FUNCTION:

e.g ① $\Rightarrow \quad y = \sin(x^2)$, $x^2 = t$

$$\frac{dy}{dx} = \cos t \frac{d}{dx}(x^2) = (\cos x^2)(2x).$$

e.g ② $\Rightarrow \quad y = \sin(\cos(\ln x))$, $\cos(\ln x) = t$

$$\frac{dy}{dx} = \cos t \frac{d}{dx}(\cos(\ln x)), \quad \ln x = u$$

$$\frac{dy}{dx} = \cos t\,(-\sin u)\frac{d}{dx}(\ln x)$$

$$= -\cos(\cos(\ln x))\sin(\ln x)\cdot\frac{1}{x}.$$

$\Rightarrow$ e.g ③ $y = \sin^{-1}(2x+1) = f(g(x))$, $2x+1 = t$

$f(x) = \sin^{-1}(x)$

$g(x) = 2x+1.$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-t^2}}\frac{d}{dx}(2x+1)$$

$$\boxed{\frac{dy}{dx} = \frac{2}{\sqrt{1-(2x+1)^2}}}$$

$y = \sin(\cos(x^2))$ $\qquad f(g(h(x))) =$

$f(x) = \sin x, \quad g(x) = \cos x, \quad h(x) = x^2$

$y = ((\cos(\sin x))^2$ $\qquad f(g(h(x)))$

$f(x) = x^2 \quad g(x) = \cos x \quad h(x) = \sin x$

Q) $y = \ln(\sin^{-1}(x^2))$. $\quad \sin^{-1}(x^2) = t$

$\frac{dy}{dx} = ?$

$$\frac{dy}{dx} = \frac{1}{t}\frac{d}{dx}(\sin^{-1}(x^2)), \quad x^2 = u$$

$$= \frac{1}{t}\frac{1}{\sqrt{1-u^2}}\frac{d}{dx}(x^2)$$

$$= \frac{1}{\sin^{-1}(x^2)}\cdot\frac{1}{\sqrt{1-x^4}}\cdot 2x$$

Q) $y = \sin^{-1}((x-1)^2)$ $\quad dy/dx = ?$, $(x-1)^2 = t$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-t^2}}\frac{d}{dx}(x-1)^2, \quad x-1 = u$$

$$= \frac{2u}{\sqrt{1-t^2}}\frac{d}{dx}(x-1)$$

$$= \frac{2(x-1)\cdot 1}{\sqrt{1-(x-1)^4}} =$$

Q) $y = (2x^2+3)^{5/3}(x+5)^{-1/3}$ $\quad \frac{dy}{dx}$

$$\frac{dy}{dx} = (x+5)^{-1/3}\frac{d}{dx}((2x^2+3))^{5/3} + (2x^2+3)^{5/3}\frac{d}{dx}(x+5)^{-1/3}, \quad 2x^2+3 = t, \quad x+5 = u$$

$$= (x+5)^{-1/3}\frac{5}{3}t^{2/3}\frac{d}{dx}(2x^2+3) + (2x^2+3)^{5/3}\frac{-1}{3}u^{-4/3}\frac{d}{dx}(x+5)$$

$$= (x+5)^{-1/3}\frac{5}{3}(2x^2+3)^{2/3}(4x+0) - (2x^2+3)^{5/3}(x+5)^{-4/3}\cdot 1.$$

Q) $y = \sqrt{a+\sqrt{a+x}}$ $\quad \frac{dy}{dx} = ?$

$y = (a+\sqrt{a+x})^{1/2}$, $a+\sqrt{a+x} = t$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{t}}\frac{d}{dx}(a+\sqrt{a+x}), \quad a+x = u$$

$$= \frac{1}{2\sqrt{t}}\frac{1}{2\sqrt{u}}\frac{d}{dx}(a+x) = \frac{1}{4}\frac{1}{\sqrt{a+\sqrt{a+x}}\sqrt{a+x}}.$$

$y = \sqrt{x} \quad \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

$y = \sqrt{\sin x} \quad \frac{dy}{dx} = \frac{\cos x}{2\sqrt{\sin x}}$

$y = \sqrt{\ln x} \quad \frac{dy}{dx} = \frac{1}{2\sqrt{\ln x}}\frac{1}{x}$

Differentiation of Implicit Functions:

Implicit Functions:
The functions in which y(dependent variable) cannot be represented explicitly in terms of x(independent variable) is called an
Implicit Functions.

$x+y=1 \Rightarrow \boxed{y=1-x} \rightarrow$ explicit fn.

$xy^2+\sin y=\ln y \longrightarrow$ Implicit fn.

$\Rightarrow x^2=4ay \longrightarrow \boxed{y=\frac{x^2}{4a}} \Rightarrow$ explicit

$\Rightarrow \sin y = x^2+4 \Rightarrow \boxed{y=\sin^{-1}(x^2+4) \Rightarrow \text{explicit}}$

e.g

$\sin \overset{t}{y} + \ln \overset{t}{y} + x = e^x + \ln x$

$\sin(\overset{t}{x^2+1})$

$\cos t \frac{d}{dx}(x^2+1)$

① Diff. w.r.t 'x' (independent variable)

$\cos t \frac{dy}{dx} + \frac{1}{t}\frac{dy}{dx} + \frac{dx}{dx} = \frac{d}{dx}(e^x) + \frac{d}{dx}(\ln x)$

$\Rightarrow \cos y \frac{dy}{dx} + \frac{1}{y}\frac{dy}{dx} + 1 = e^x + \frac{1}{x}$

Collect all the terms containing $\frac{dy}{dx}$:

$$\boxed{\frac{dy}{dx} = \frac{e^x+\frac{1}{x}-1}{\cos y+\frac{1}{y}}}$$

Note:

$\frac{d}{dx}\cos \overset{t}{u} = -\sin t \frac{dt}{dx}$
$= -\sin u \frac{du}{dx}$

$\frac{d}{dx}\sin(\overset{t}{y}) = \cos t \frac{dy}{dx}$
$= \cos y \frac{dy}{dx}$

$\frac{d(\sin(\overset{t}{y^2}))}{dx} = \cos t \frac{d(y^2)}{dx}$
$= \cos y^2 \, 2y\left(\frac{dy}{dx}\right)$

$\frac{d(\sin(\overset{t}{y^2}))}{dy} = \cos t \, 2y\left(\frac{dy}{dy}\right)$
$= \cos y^2 \, 2y$

Q) If $xy+y^2=\sin(\overset{t}{\ln y})+\sin^{-1}x$

$\frac{dy}{dx}=?$

Diff. w.r.t x

$x\cdot\frac{dy}{dx}+y\cdot 1+2y\frac{dy}{dx} = \cos t \frac{d}{dx}(\ln y) + \frac{1}{\sqrt{1-x^2}}$

$x\frac{dy}{dx}+y+2y\frac{dy}{dx} = \cos(\ln y)\frac{1}{y}\frac{dy}{dx} + \frac{1}{\sqrt{1-x^2}}$

Collect terms $\frac{dy}{dx}$

$$\boxed{\frac{dy}{dx} = \frac{y-\frac{1}{\sqrt{1-x^2}}}{\frac{\cos(\ln y)}{y}-2y-x}}$$

Q) Find $\frac{dy}{dx}$ if $xy=\sin x+y^2$

Diff w.r.t x

$t=y$

$\frac{d}{dx}(xy) = \frac{d}{dx}(\sin x) + \frac{d}{dx}(\overset{t}{y^2})$

$y\cdot\frac{dx}{dx}+x\cdot\frac{dy}{dx} = \cos x + 2y\frac{dy}{dx}$

Collect the term containing $\frac{dy}{dx}$

$\frac{y-\cos x}{2y-x} = \frac{dy}{dx}$

Logarithmic Differentiation:

The Process of taking logarithm of function first and then differentiating is called the Logarithmic differentiation. This is used mainly for two situations mentioned below:

1) A function is a product or quotient of number of functions.

$y = (x-1)(x-2)(x-3) \xrightarrow{\text{Product}} (1)$

$\ln y = \ln(x-1)(x-2)(x-3)$

$\ln y = \ln(x-1) + \ln(x-2) + \ln(x-3)$

Diff. wrt x: $\frac{1}{y}\frac{dy}{dx} = \frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-3}$

(2) $\frac{dy}{dx} = y\left(\frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-3}\right)$

$y = \left(\frac{(x-1)(x-2)}{(x-3)(x-4)}\right)^{1/2}$

$\ln y = \ln(x-1) + \ln(x-2) - \ln(x-3) - \ln(x-4)$

Diff. wrt x

$\boxed{\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left(\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4}\right)}$

2. A function is of the form $f(x)^{g(x)}$ where f(x) and g(x) are both differentiable functions.

e.g $y = x^x$

$\ln y = \ln x^x$

L.h.s → R.h.s

Diff wrt x: $\frac{d}{dx}(\ln y) = \frac{d}{dx}(x \ln x)$

$\frac{1}{y}\frac{dy}{dx} = 1 \cdot \ln x + x \cdot \frac{1}{x}$

$= \ln x + 1$

$\frac{dy}{dx} = y(\ln x + 1)$

$\frac{dy}{dx} = x^x(\ln x + 1)$

e.g $y = (\sin x)^{\cos x}$

$\ln y = \ln(\sin x)^{\cos x}$

$\ln y = \cos x \ln(\underbrace{\sin x}_{t})$

Diff. w.r.t x

$\frac{1}{y}\frac{dy}{dx} = -\sin x \ln(\sin x) + \cos x \frac{1}{t}\frac{d}{dx}(\sin x)$

$\frac{1}{y}\frac{dy}{dx} = -\sin x \ln(\sin x) + \frac{\cos x \cos x}{\sin x}$

$\boxed{\frac{dy}{dx} = (\sin x)^{\cos x}\left(-\sin x \ln(\sin x) + \frac{\cos^2 x}{\sin x}\right)}$

Q) $x^y + y^x = 4$

find $\frac{dy}{dx} = ?$