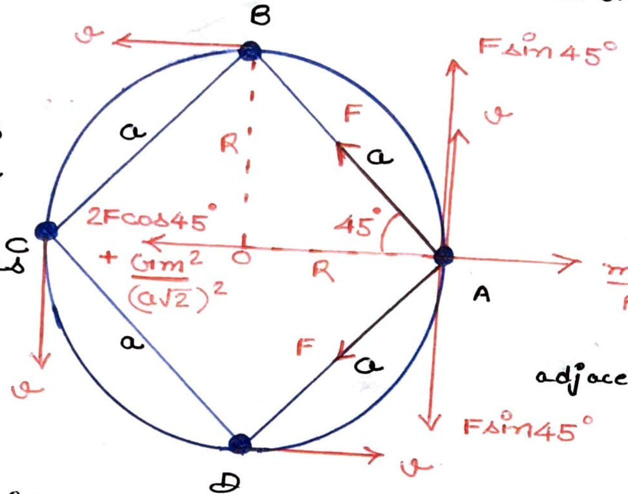
Eg (2019 April Mains): 4 identical particles each of mass M are located at the corners of a square of side length a. What should be their speed, if each of them revolves under the influence of other in a circular orbit circumscribing a square?

Soln:→

as all the 4 masses are symmetrically located under same circumstances so speed of every mass will be same.



here; $\sin 45° = \frac{R}{a}$

so Radius of the circular path

$R = \frac{a}{\sqrt{2}}$ —①

Gravit. force b/w any two particles kept adjacent; $F_B = F_D = \frac{GM^2}{a^2}$ —②

& $F_C = \frac{GM^2}{(a\sqrt{2})^2}$

considering mass M kept at A;

at equilibrium;

$$\frac{Gm^2}{2a^2} + 2F\cos 45° = \frac{mv^2}{R}$$

$$\Rightarrow \frac{\sqrt{2}}{\sqrt{2}} \times \frac{Gm^2}{2a^2} + \frac{2 \cdot Gm^2}{a^2} \times \frac{1}{\sqrt{2}} = \frac{m}{a} \times \sqrt{2} \times v^2 \Rightarrow v = \left\{ \frac{Gm}{2\sqrt{2}a} \cdot (2\sqrt{2}+1) \right\}^{\frac{1}{2}} \text{ m/s}$$

so $v = 1.16 \times \sqrt{\frac{GM}{a}}$ m/s (speed of each particle)

Eg (Adv 15) A Large spherical mass M is fixed at one position and 2 identical masses m are kept on a line passing through the center of M. The point masses are connected by a rigid light rod of length l and this assembly is free to move along the line connecting them. All 3 masses interact only due to their mutual gravitation. When the point mass m nearer to M is at a distance r = 3L, the tension in the rod is 0 for $m = K \cdot \frac{M}{288}$. Find K.

Method 1; for A: $F_{CA} - (F_{BA} + T) = ma$ —①

for B; $F_{CB} + F_{AB} + T = ma$ —②

as $T = 0$ —③

from ①, ② & ③

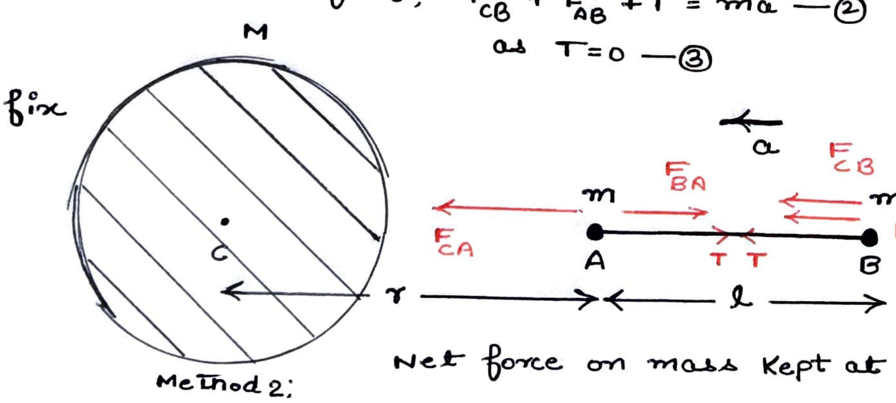
$\Rightarrow F_{CA} - F_{BA} = F_{CB} + F_{AB}$

$\Rightarrow \frac{GMm}{9L^2} - \frac{Gm^2}{L^2} = \frac{GMm}{16L^2} + \frac{Gm^2}{L^2}$

$\Rightarrow \frac{M \times 7}{144L^2} = \frac{2m}{L^2}$

$\Rightarrow \frac{7M}{144} = \frac{KM}{288} \times 2$

so K = 7



Method 2;

Net force on mass kept at A;

$$(F_A)_{net} = F_{CA} - F_{BA}$$

$$\Rightarrow (F_A)_{net} = \frac{GMm}{r^2} - \frac{Gm^2}{L^2} \text{ —①}$$

Net force on mass Kept at B $(F_B)_{net} = F_{CB} + F_{AB}$

$$(F_B)_{net} = \frac{GMm}{(r+L)^2} + \frac{Gm^2}{L^2} \quad \text{--- (2)}$$

$\because$ Tension in the rod $(T) = (F_A)_{net} - (F_B)_{net}$

$$\Rightarrow 0 = \left(\frac{GMm}{r^2} - \frac{Gm^2}{L^2}\right) - \left(\frac{GMm}{(r+L)^2} + \frac{Gm^2}{L^2}\right)$$

$$\Rightarrow \frac{2Gm^2}{L^2} = GMm\left[\frac{1}{9L^2} - \frac{1}{16L^2}\right]$$

$$\Rightarrow 2m = \frac{M \times 7}{144}$$

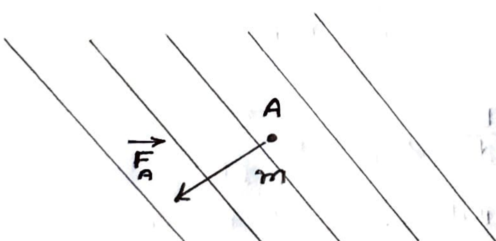
$$\Rightarrow m = \frac{7 \cdot M}{288}$$

as: $m = \dfrac{K \cdot M}{288}$

so $K = 7$

Gravitational Field :→ Every mass has its sphere of influence inside which any other mass experience force of Gravitation, this region is called gravitational field. It is conservative in nature.

Gravitational field Intensity $(\vec{E})$: Gravitational field intensity at any point inside the gravitational field is the gravitational force experienced by a unit mass kept at that point.

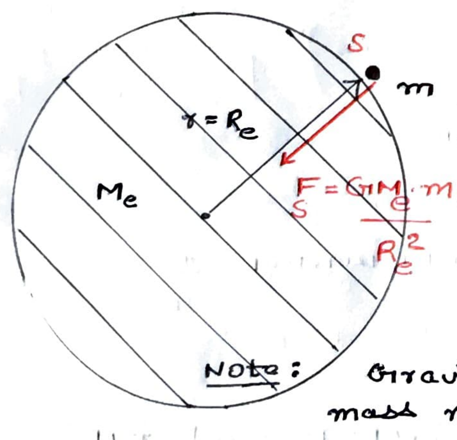


$$\boxed{\vec{E}_A = \frac{\vec{F}_A}{m}}$$ $N\text{-}Kg^{-1}$ or $m.s^{-2}$

D.F. $[LT^{-2}]$

note: gravitational acceleration at any point near the Earth's surface is nothing else but the intensity of the Earth's gravitational field.

field intensity at the Earth's surface:



$$E_s = \frac{F_s}{m} = \frac{GM_e \cdot m}{R_e^2 \cdot m} = \frac{GM_e}{R_e^2}$$

$$= \frac{6.67 \times 10^{-11} \times 6 \times 10^{24}}{(6.4 \times 10^6)^2}$$

$$= 9.8 \ m/s^2 \text{ (approx)}$$

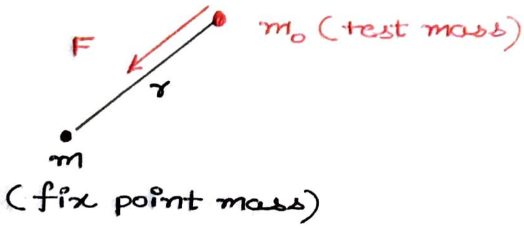
$$\therefore E_s = g = \frac{GM_e}{R_e^2} \quad \text{--- (★)}$$

Note: Gravitational force applied by Earth on a mass near its surface is called weight.

so $F = \dfrac{GM_e m}{r^2} = mg$ or W

Gravitational field intensity (E) due to different masses

i) point mass (m):



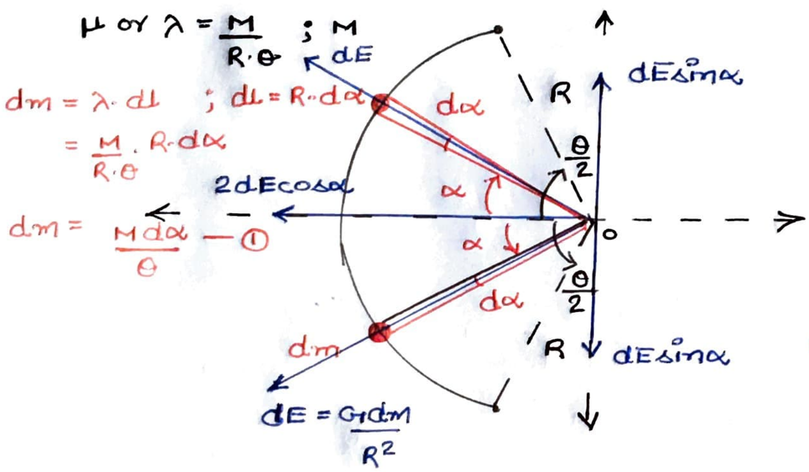
$\because \quad E = \dfrac{F}{m_o}$

$= \dfrac{Gmm_o}{r^2 \cdot m_o}$

$\Rightarrow \boxed{E = \dfrac{Gm}{r^2}} \quad m \cdot s^{-2}$

$\Rightarrow \vec{E} = -\dfrac{Gm}{r^2} \cdot \hat{r}$ or $-\dfrac{GM}{r^3} \cdot \vec{r}$

ii) Due to a circular arc:



field at the center o due to both the considered elements,

$dE_x = 2dE\cos\alpha$

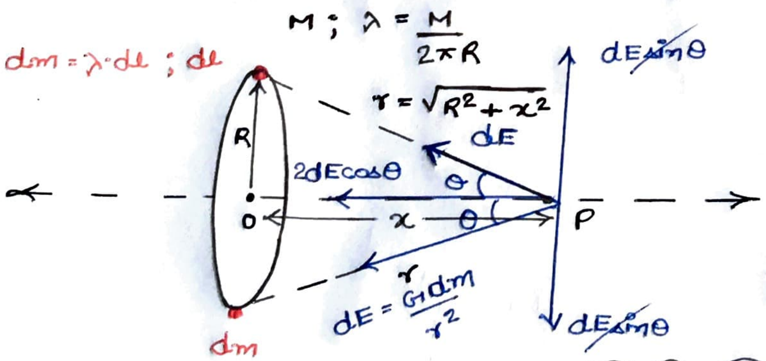
$= 2 \cdot \dfrac{G\,dm}{R^2} \cdot \cos\alpha$

$\Rightarrow \displaystyle\int_0^{E_\theta} dE_x = \dfrac{2 \cdot G}{R^2} \cdot \dfrac{M}{\theta} \cdot \int_0^{\theta/2} \cos\alpha \cdot d\alpha$

$= \dfrac{2GM}{R^2 \theta} \cdot \left[\sin\alpha\right]_0^{\theta/2}$

$$\boxed{E_\theta = \frac{2GM}{R^2} \cdot \frac{\sin(\theta/2)}{\theta} \quad \text{or} \quad E_\theta = \frac{2G\lambda}{R} \cdot \sin\left(\frac{\theta}{2}\right)} \quad m \cdot s^{-2}$$

iii) on the axis of a uniform circular Ring:



field at P due to both elements

$dE_x = 2dE\cos\theta$

$= 2 \cdot \dfrac{G\,dm}{(R^2+x^2)} \cdot \dfrac{x}{\sqrt{R^2+x^2}}$

$\displaystyle\int_0^{E_x} dE_x = \dfrac{2G\lambda \cdot x}{(R^2+x^2)^{3/2}} \cdot \int_0^{\pi R} dl$

$$\boxed{E_x = \frac{2G\lambda \cdot \pi R \cdot x}{(R^2+x^2)^{3/2}} = \frac{GMx}{(R^2+x^2)^{3/2}}} \quad m \cdot s^{-2}$$

i) at the center; $x = 0$

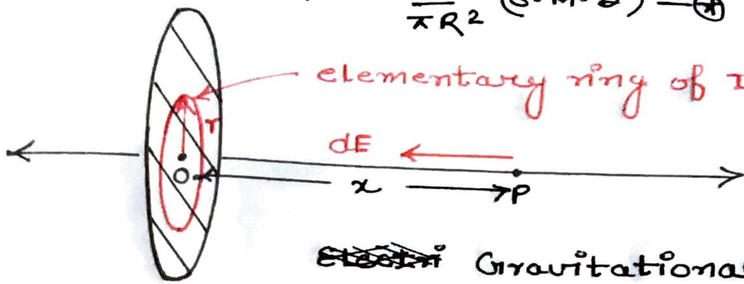
$E_{center} = 0$

ii) at $x = \pm R/\sqrt{2}$

$E_{max} = \dfrac{2}{3\sqrt{3}} \cdot \dfrac{GM}{R^2}$

iv) Gravitational field on the axis of a uniform disc

$M \; ; \; \sigma = \dfrac{M}{\pi R^2}$ (S·M·D) — ⊛



$\because \; dA = 2\pi r \cdot dr$

$\therefore \; dm = \sigma \cdot dA = \dfrac{2M \cdot r \cdot dr}{R^2}$ — ①

~~Electri~~ Gravitational field at point P, due to the considered ring.

$$dE = \frac{G \cdot dm \cdot x}{(r^2 + x^2)^{\frac{8}{2}}}$$

$$\int_0^{E_x} dE = \int_0^R \frac{2G \cdot M \cdot x \cdot r \cdot dr}{R^2 \cdot (r^2 + x^2)^{3/2}}$$

$$= \frac{GMx}{R^2} \cdot \left[\frac{-2}{\sqrt{r^2 + x^2}} \right]_0^R$$

$$= \frac{2GMx}{R^2} \cdot \left[\frac{1}{x} - \frac{1}{\sqrt{R^2 + x^2}} \right]$$

$$\Rightarrow E_x = \frac{2GM}{R^2} \cdot \left[1 - \frac{x}{\sqrt{R^2 + x^2}} \right] \text{ or } 2\pi G\sigma \left[1 - \frac{x}{\sqrt{R^2 + x^2}} \right] \; m s^{-2}$$

at the center of the disc

$x = 0$

$$\boxed{E_{max} = \frac{2GM}{R^2} \text{ or } 2\pi G\sigma} \; m s^{-2}$$