DIFFERENTIATION OF PARAMETRIC FUNCTIONS:

$y = f(t)$, $x = g(t)$ $\qquad \frac{dy}{dx} = ?$

$\frac{dy}{dt} = f'(t)$, $\frac{dx}{dt} = g'(t)$ $\qquad$ Divide : $\boxed{\frac{dy}{dx} = \frac{f'(t)}{g'(t)}}$

$t \in$ parameter

Q) $x = a(\theta - \sin\theta)$ $\qquad \theta \in$ parameter $\qquad$ A) $\tan\theta/2$ B)✓ $-\cot\theta/2$ C) $-\tan\theta/2$. D) $\cot\theta/2$

$y = a(1 + \cos\theta)$

$\frac{dy}{d\theta} = a(-\sin\theta)$

$\frac{dx}{d\theta} = a(1 - \cos\theta)$

Divide : $\frac{dy}{dx} = \frac{-\sin\theta}{1-\cos\theta} = -\frac{\cancel{2}\cancel{\sin\frac{\theta}{2}}\cos\frac{\theta}{2}}{\cancel{2}\sin^2\theta/2}$

$= \boxed{-\cot\frac{\theta}{2}}$

e.g. $\quad y = 2at$ $\qquad \longleftrightarrow \qquad y^2 = 4ax$

$x = at^2$

$\frac{dy}{dt} = 2a$, $\frac{dx}{dt} = 2at$

$\boxed{\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}} = \frac{2a}{y}$.

$\frac{d}{dx}(y^2) = \frac{d}{dx}(4ax)$

$2y\frac{dy}{dx} = 4a$

$\boxed{\frac{dy}{dx} = \frac{2a}{y}}$

Q) $\Rightarrow x = a\left(\cos t + \ln\left(\tan\frac{t}{2}\right)\right)$ $\qquad$ A)✓ $\tan t$ B) $\cot t$ C) $-\tan t$ D) $-\cot t$

$\Rightarrow y = a\sin t$ $\qquad t \in$ parameter

$\frac{dy}{dx} = ?$ $\qquad \frac{dy}{dt} = a\cos t$ — (1)

$\frac{dx}{dt} = a\left(-\sin t + \frac{\sec^2 t/2 \cdot 1/2}{\tan\frac{t}{2}}\right)$

$= a\left(-\sin t + \frac{\cancel{\cos t/2}}{2\sin t/2\,\cos^{\cancel{2}} t/2}\right) = a\left(-\sin t + \frac{1}{\sin t}\right) = \frac{a(\cos^2 t)}{\sin t}$ — (2)

Divide (1) & (2)

$\frac{dy}{dx} = \frac{\cancel{a}\,\cancel{\cos t}\,\sin t}{\cancel{a}\cos^{\cancel{2}} t}$

$= \tan t$

Q) $y = a\cos^3 t$ $\qquad \frac{dy}{dx} = ?$ $\qquad a \in$ parameter

$x = a\sin^3 t$ $\qquad$ A) $-\cot t$ B) $-\tan t$ C) $\cot^3 t$ D) $\tan^3 t$

$t \in$ parameter

A)✓ $-\cot t$

B) $-\tan t$

C) $\cot t$

D) $\tan t$

$\frac{dy}{dt} = a\,3\cos^2 t(-\sin t)$

$\frac{dx}{dt} = a\,3\sin^2 t\cos t$

$\frac{dy}{dx} = \frac{-\cancel{3a}\cos^{\cancel{2}} t\,\cancel{\sin t}}{\cancel{3a}\sin^{\cancel{2}} t\,\cancel{\cos t}} = -\cot t$

$\frac{dy}{da} = 1.\cos^3 t$, $\frac{dx}{da} = 1.\sin^3 t$

$\frac{dy}{dx} = \frac{\cos^3 t}{\sin^3 t} = \cot^3 t$.

Q) $x = \sqrt{a^{\sin^{-1}t}}$, $y = \sqrt{a^{\cos^{-1}t}}$ $\qquad$ then select correct stat. for $\frac{dy}{dx}$

A) xy $\qquad$ C)✓ $-y/x$

B) $\frac{y}{x}$ $\qquad$ D) $-x/y$.

$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$. $\qquad \frac{dy}{dt} = ?$

$\frac{dx}{dt} = \frac{1}{2\sqrt{a^{\sin^{-1}t}}}\frac{d}{dx}\left(a^{\sin^{-1}t}\right) = \frac{1}{2\sqrt{a^{\sin^{-1}t}}} \cdot a^{\sin^{-1}t}\ln a \cdot \frac{d}{dx}(\sin^{-1}t)$

$= \frac{1}{2\sqrt{a^{\sin^{-1}t}}} \cdot a^{\sin^{-1}t}\frac{\ln a}{\sqrt{1-x^2}}$

Multiply. $xy = \sqrt{a^{\pi/2}}$

Diff wrt x $\quad x.\frac{dy}{dx} + y. = 0$ $\qquad \frac{dy}{dx} = \frac{-y}{x}$

Differentiation Using Substitution:

In certain Situations as mentioned below, Substitution simplifies differentiation for each of the following Expressions with appropriate substitution:

① # $\sqrt{x^2+a^2} \rightarrow x = a\tan\theta$ or $a\cot\theta$

$\sqrt{a^2\tan^2\theta + a^2} = a\sec\theta$

$\sqrt{a^2\cot^2\theta + a^2} = a\,\mathrm{cosec}\,\theta$

② # $\sqrt{a^2-x^2} \rightarrow x = a\sin\theta \rightarrow = a\cos\theta$

or $a\cos\theta \rightarrow = a\sin\theta$

③ $\sqrt{x^2-a^2} \rightarrow x = a\sec\theta \rightarrow = a\tan\theta$

$a\,\mathrm{cosec}\,\theta \rightarrow = a\cot\theta$

④ $\sqrt{\dfrac{a+x}{a-x}} \rightarrow x = a\cos 2\theta$

$\sqrt{\dfrac{a+a\cos 2\theta}{a-a\cos 2\theta}} = \dfrac{\sqrt{2a\cos^2\theta}}{\sqrt{2a\sin^2\theta}} = \cot\theta$

$\sqrt{\dfrac{a-x}{a+x}} \rightarrow \tan\theta$

Q) $y = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ find $\dfrac{dy}{dx} = ?$

$x = \tan\theta$

$y = \sin^{-1}\left(\dfrac{2\tan\theta}{1+\tan^2\theta}\right)$

$y = \sin^{-1}(\sin 2\theta)$ — Prop-3

$y = 2\theta = 2\tan^{-1}x$

$dy/dx = \dfrac{2}{1+x^2}$

Q) $y = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ $0<x<1$

$\boxed{x = \tan\theta}$ → $0 < \tan\theta < 1$ ($\tan\frac{\pi}{4}$)

$0 < \theta < \dfrac{\pi}{4}$

$\boxed{0 < 2\theta < \dfrac{\pi}{2}}$

$2\theta \in [0, \pi]$

$\cos^{-1}(\cos 2\theta) = 2\theta$.

$y = \cos^{-1}\left(\dfrac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$\boxed{y = \cos^{-1}(\cos 2\theta)}$

$y = 2\theta$

$y = 2\tan^{-1}x$

$\dfrac{dy}{dx} = \dfrac{2}{1+x^2}$

Q) $y = \sin^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ $0<x<1$

$\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}$.

$x = \tan\theta$

$y = \sin^{-1}(\cos 2\theta)$

$y = \dfrac{\pi}{2} - \cos^{-1}(\cos 2\theta)$

$y = \dfrac{\pi}{2} - 2\theta = \dfrac{\pi}{2} - 2\tan^{-1}x$

$\dfrac{dy}{dx} = \dfrac{-2}{1+x^2}$

Q) $y = \sec^{-1}\left(\dfrac{1}{2x^2-1}\right)$ $0 < x < \dfrac{1}{\sqrt{2}}$

$\cos 2\theta = 2\cos^2\theta - 1$

$x = \cos\theta$

$y = \sec^{-1}\left(\dfrac{1}{2\cos^2\theta - 1}\right)$

$0 < \cos\theta < \dfrac{1}{\sqrt{2}}$

$y = \sec^{-1}(\sec 2\theta)$

$\Rightarrow \cos\frac{\pi}{2} < \cos\theta < \cos\frac{\pi}{4}$.

$y = 2\theta = 2\cos^{-1}x$

$\boxed{\dfrac{\pi}{2} > \theta > \dfrac{\pi}{4}}$

$\boxed{\pi > 2\theta > \dfrac{\pi}{2}}$

$\boxed{\dfrac{dy}{dx} = \dfrac{-2}{\sqrt{1-x^2}}}$

Q) if $y = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$ find $\dfrac{dy}{dx} = ?$

A) $\dfrac{1}{1+x^2}$

B) $\dfrac{2}{1+x^2}$

C) $\dfrac{1}{2(1+x^2)}$

D) $\dfrac{1}{4(1+x^2)}$

$x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$y = \tan^{-1}\left(\dfrac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$

$y = \tan^{-1}\left(\dfrac{\sec\theta - 1}{\tan\theta}\right)$

$y = \tan^{-1}\left(\dfrac{1-\cos\theta}{\sin\theta}\right)$

$y = \tan^{-1}\left(\dfrac{\cancel{2}\sin^2\frac{\theta}{2}}{\cancel{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$

$\boxed{y = \tan^{-1}\left(\tan\frac{\theta}{2}\right) \rightarrow \text{Depends on interval of } x'}$

$y = \dfrac{\theta}{2} = \dfrac{1}{2}\tan^{-1}x$

$\boxed{\dfrac{dy}{dx} = \dfrac{1}{2}\,\dfrac{1}{1+x^2}}$

Q) $y = \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)$ $\dfrac{-1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}} \rightarrow \dfrac{-1}{\sqrt{3}} < \tan\theta < \dfrac{1}{\sqrt{3}}$

$-\dfrac{\pi}{6} < \theta < \dfrac{\pi}{6}$

$-\dfrac{\pi}{2} < 3\theta < \dfrac{\pi}{2}$

$x = \tan\theta$.

$y = \tan^{-1}\left(\dfrac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\right)$

$y = \tan^{-1}(\tan 3\theta)$

$y = 3\theta = y = 3\tan^{-1}x$

$\boxed{\dfrac{dy}{dx} = \dfrac{3}{1+x^2}}$

Differentiation of One function w.r.t other:

Let $y = f(x)$, $z = g(x)$ $\longrightarrow$ $\frac{dy}{dx} = f'(x)$, $\frac{dz}{dx} = g'(x)$

$$\boxed{\begin{gathered}\text{Differentiate } f(x) \text{ w.r.t } g(x)\\ \frac{d(f(x))}{d(g(x))} = \frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{f'(x)}{g'(x)}\end{gathered}}$$

e.g find Differentiation of $\sin x$ w.r.t $\cos x$.

$U = \sin x$, $V = \cos x$

$$\frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{\cos x}{-\sin x} = -\cot x$$

(2) Derivative of $\ln x$ w.r.t e^x $\longrightarrow$ $\frac{d(\ln x)}{d(e^x)} = \frac{\frac{1}{x}\cancel{dx}}{e^x \cdot \cancel{dx}} = \frac{1}{xe^x}$

$U = \ln x$, $V = e^x$

$$\frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{1/x}{e^x} = \frac{1}{xe^x}$$

(3) Derivative of $(\sin^3 x)$ w.r.t $(\sin^2 x)$ $\longrightarrow$ $\frac{d(\sin^3 x)}{d(\sin^2 x)} = \frac{3\sin^2 x \cos x\, dx}{2\sin x \cos x\, dx} = \frac{3}{2}\sin x$

$U = \sin^3 x$, $V = \sin^2 x$

$$\frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{3\sin^{\cancel{2}} x \cancel{\cos x}}{2\cancel{\sin x}\cancel{\cos x}} = \frac{3}{2}\sin x.$$

$$\boxed{d(f(x^2)) = f'(x^2)2x\,dx}$$

$d(\sin u) = \cos u\, du$

$\Rightarrow d(\tan x) = \sec^2 x(dx)$

$\frac{d}{dx}(\tan x) = \sec^2 x$

Q) find Diff. of $\sin(m\cos^{-1}x)$ w.r.t $\cos(m\sin^{-1}x)$

$U =$ $\qquad$ $V =$ $\qquad$ $m \in$ constant.

$$\frac{du}{dx} = \cos(m\cos^{-1}x)\frac{dt}{dx} = \frac{\cos(m\cos^{-1}x)(m)(-1)}{\sqrt{1-x^2}}$$

$$\frac{du}{dx} = \frac{-\cos(m\cos^{-1}x)(m)}{\sqrt{1-x^2}}$$

$$\frac{dv}{dx} = -\sin(m\sin^{-1}x)\frac{dt}{dx}$$

$$\frac{dv}{dx} = -\frac{\sin(m\sin^{-1}x)(m)1}{\sqrt{1-x^2}}$$

$$\boxed{\frac{du}{dv} = \frac{\cos(m\cos^{-1}x)}{\sin(m\sin^{-1}x)}}$$

$$= \frac{\sqrt{1-\sin^2(m\cos^{-1}x)}}{\sqrt{1-\cos^2(m\sin^{-1}x)}} = \boxed{\frac{\sqrt{1-u^2}}{\sqrt{1-v^2}}}$$

Q) if $f(x^3) = U$, $g(x^2) = V$

$f'(x) = \cos x \longrightarrow f'(x^3) = \cos x^3$

$g'(x) = \sin x \rightsquigarrow g'(x^2) = \sin(x^2)$

find $\frac{du}{dv} = ?$

A) $\frac{3}{2}x\frac{\cos x}{\sin x^2}$

B) $\frac{3}{2}x\cot x^3$

✓C) $\frac{3}{2}x\frac{\cos(x^3)}{\sin x^2}$

D) none of above

Divide

$$\frac{du}{dx} = \frac{d(f(x^3))}{dx} = \frac{f'(x^3)\,d(x^3)}{dx} = f'(x^3)3x^2$$

$$\frac{dv}{dx} = \frac{d(g(x^2))}{dx} = \frac{g'(x^2)d(x^2)}{dx} = g'(x^2)2x$$

$$\frac{du}{dv} = \frac{3}{2}x\frac{f'(x^3)}{g'(x^2)} = \frac{3}{2}x\frac{\cos(x^3)}{\sin(x^2)}$$

Q) Differentiate $\frac{2bt}{1+t^2}$ w.r.t $\frac{a(1-t^2)}{1+t^2}$

$t \in$ Parameter

$t = \tan\theta$

$U = b\left(\frac{2t}{1+t^2}\right) \Rightarrow U = b\sin 2\theta$, $\quad \frac{du}{d\theta} = 2b\cos 2\theta$

$V = a\left(\frac{1-t^2}{1+t^2}\right) \Rightarrow V = a\cos 2\theta$, $\quad \frac{dv}{d\theta} = -2a\sin 2\theta$

$$\frac{du}{dv} = \frac{du/d\theta}{dv/d\theta} = -\frac{b}{a}\cot 2\theta$$

$$= -\frac{b}{a}\frac{(1-\tan^2\theta)}{2\tan\theta}$$

$$\boxed{\frac{du}{dv} = \frac{b}{a}\frac{(t^2-1)}{2t}}$$