EXPONENTIAL AND LOGARITHM LIMITS:

$\log_e 3 = \frac{\log_{10} 3}{\log_{10} e}$ $\log_e 9 = \frac{\log_{10} 9}{\log_{10} e}$

① $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

☆ ② $\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a$

e.g. $\lim_{x\to 0} \frac{2^x - 1}{x} = \ln 2$

☆ ③ $\lim_{x\to 0} \frac{e^{mx} - 1}{x} = m$

☆ ④ $\lim_{x\to 0} \frac{a^{mx} - 1}{x} = m \ln a$

$\lim_{x\to 0} \frac{e^x - 1}{x} = \ln e = 1$

Proof: ②

$$\lim_{x\to 0} \frac{\left(\cancel{1} + \frac{x \ln a}{1!} + \frac{x^2 (\ln a)^2}{2!} + \dots\right) - \cancel{1}}{\cancel{x}} = \lim_{x\to 0} \ln a + x(\ln a)^2 + x^2(\ln a)^3 \dots$$

$= \ln a$

Proof: ④

$t = mx$

$$m \lim \left(\frac{a^{mx} - 1}{mx}\right) = m \lim_{t\to 0} \frac{a^t - 1}{t} = m \ln a$$

Q) $\lim_{x\to 0} \frac{e^{2x} - 1}{x} = 2$

Q) $\lim_{x\to 0} \frac{5^x - 1}{x} = \ln 5$

Logarithms

☆ ① $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$

$t = \log(1+x) \Rightarrow e^t = 1 + x$

$x = e^t - 1$

$$\lim_{t\to 0} \frac{t}{e^t - 1} = \frac{1}{\lim_{t\to 0}\left(\frac{e^t - 1}{t}\right)} = \frac{1}{1}$$

☆ ② $\lim_{x\to 0} \frac{\log(1+mx)}{x} = m$

Q) $\lim_{x\to 0} \frac{e^{2x} - 1}{\sin 8x} = \frac{2}{8} = \frac{1}{4}$

Q) $\lim_{x\to 0} \frac{\ln(1+5x)}{\ln(1-7x)} = \frac{-5}{7}$

$t = x - 3$ $x \to 3,\ t \to 0$

☆ $$\lim_{x\to 3} \left(\frac{e^x - e^3}{x - 3}\right) = \lim_{x\to 3} e^3 \left(\frac{e^{x-3} - 1}{x - 3}\right) = \lim_{t\to} e^3 \left(\frac{e^t - 1}{t}\right)$$

A) 1 B) e C) e^3 D) 0

$= e^3 \cdot 1$

Q) $\lim_{x\to 0} \left(\frac{e^{2x} - 1}{e^{5x} - 1}\right) = \frac{2}{5}$

Q) $\lim_{x\to 0} \left(\frac{e^{2x} - 1}{\ln(1+7x)}\right) = \frac{2}{7}$

$$\frac{\lim_{x\to 0} \left(\frac{e^{2x} - 1}{x}\right)}{\lim_{x\to 0} \frac{\ln(1+7x)}{x}} = \frac{2}{7}$$

PROBLEMS ON TRIGO, LOG AND EXPONENTIAL LIMITS;

$\text{Lt}_{x\to0}\frac{a^x-1}{x} = \ln a$

Q) $\lim_{x\to0}\frac{\sin x\,(e^{2x}-1)}{(1-\cos x)} =$

A) 1

B) 2

C) 4

D) 8

$\lim_{x\to0}\frac{\overbrace{\left(\frac{\sin x}{x}\right)}^{f(x)}\overbrace{\left(\frac{e^{2x}-1}{x}\right)}^{g(x)}}{\underbrace{\left(\frac{1-\cos x}{x^2}\right)}_{h(x)}}$

$= \frac{1\cdot 2}{1/2} = 4.$

$\text{Lt}_{x\to0}\left(\frac{e^x-e^{-x}}{x}\right) = \text{Lt}_{x\to0}\frac{e^x-e^{-x}}{\left(\frac{\sin x}{x}\right)x} \to \tan x$

$= \text{Lt}_{x\to0}\frac{e^x-e^{-x}}{\left(\frac{\tan x}{x}\right)x} = \text{Lt}_{x\to0}\frac{e^x-e^{-x}}{\left(\frac{1-e^x}{x}\right)x}$

$= \text{Lt}_{x\to0}\frac{e^x-e^{-x}}{\left(\frac{\ln(1+x)}{x}\right)x}$

$\text{Lt}_{x\to0}\frac{\sin x-\tan x}{\left(\frac{(1-e^x)^3}{x^3}\right)x^3} = \text{Lt}_{x\to0}\left(\frac{\sin x-\tan x}{-x^3}\right)$

A) 0

B) 1

C) 4

D) 2

$= \text{Lt}_{x\to0}\frac{\left(e^x-\frac{1}{e^x}\right)}{x}$

$= \text{Lt}_{x\to0}\frac{1}{e^x}\left(\frac{e^{2x}-1}{x}\right)$

$= \text{Lt}_{x\to0}\frac{1}{e^x}\cdot\text{Lt}_{x\to0}\frac{e^{2x}-1}{x}$

$= 1\cdot 2 = 2.$

1

2

Q) $\lim_{x\to0}\frac{3^x-7^x}{x} = \text{Lt}_{x\to0}\left(\frac{3^x-1}{x}\right)-\left(\frac{7^x-1}{x}\right)$

$= \ln 3-\ln 7$

$= \ln\frac{3}{7}$

A) $\frac{3}{7}$

B) $\ln\frac{7}{3}$

C) $\ln\frac{3}{7}$

D) $\frac{7}{3}$

Q) $\text{Lt}_{x\to0}\frac{(e^{3x}-1)}{\ln(1+4x)} = \text{Lt}_{x\to0}\frac{\left(\frac{e^{3x}-1}{x}\right)}{\frac{\ln(1+4x)}{x}} = 3/4.$

Q) $\text{Lt}_{x\to0}\frac{\ln(5+x)-\ln(5-x)}{x}$

$\boxed{\ln ab = \ln a+\ln b}$

$\text{Lt}_{x\to0}\frac{\ln(1+mx)}{x} = m$

A) 0

B) $\frac{2}{5}$

C) $\frac{4}{5}$

D) 10

$\Rightarrow \text{Lt}_{x\to0}\frac{\ln\left(5\left(1+\frac{x}{5}\right)\right)-\ln 5\left(1-\frac{x}{5}\right)}{x}$

$\Rightarrow \text{Lt}_{x\to0}\ \cancel{\ln 5}+\underbrace{\frac{\ln\left(1+\frac{x}{5}\right)}{x}}_{m=\frac{1}{5}}-\cancel{\ln 5}-\underbrace{\frac{\ln\left(1-\frac{x}{5}\right)}{x}}_{m=-\frac{1}{5}}$

$= \frac{1}{5}-\left(-\frac{1}{5}\right) = \boxed{\frac{2}{5}}$

Q) $\text{Lt}_{x\to0}\left(\frac{10^x-2^x-5^x+1}{x\tan x}\right) = \text{Lt}_{x\to0}\frac{2^x(5^x-1)-1(5^x-1)}{x\tan x}$

A) $\ln 10$

B) $\ln 2\cdot\ln 5$

C) $\frac{\ln 5}{\ln 2}$

D) $\ln 5/2$

$= \text{Lt}_{x\to0}\frac{\left(\frac{2^x-1}{x}\right)(5^x-1)}{\left(\frac{\tan x}{x}\right)x}$

$= \frac{\ln 2\cdot\ln 5}{1}$

$= \ln 2\cdot\ln 5.$

Q) $\text{Lt}_{x\to0}\left(\frac{27^x-9^x-3^x+1}{\sqrt{2}-\sqrt{1+\cos x}}\right)\times\frac{\sqrt{2}+\sqrt{1+\cos x}}{\sqrt{2}+\sqrt{1+\cos x}}$

A) $4\sqrt{2}(\ln 3)$

B) $4\sqrt{2}(\ln 3)^2$

C) $8\sqrt{2}(\ln 3)^2$

D) $8\sqrt{2}(\ln 3)$

$= \text{Lt}_{x\to0}\frac{\left(\frac{9^x-1}{x}\right)\left(\frac{3^x-1}{x}\right)}{\left(\frac{1-\cos x}{x^2}\right)}\left(\sqrt{2}+\sqrt{1+\cos x}\right)$ $\quad\sqrt{2}$

$= \left(\frac{(\ln 9)\ln 3}{1/2}\right)2\sqrt{2} = 4\sqrt{2}(\ln 9)(\ln 3)$

$= 4\sqrt{2}(\ln 3^2)(\ln 3)$

$= 8\sqrt{2}(\ln 3)^2.$

Q) $\text{Lt}_{x\to0}\frac{\left(e^x-e^{x\cos x}\right)}{x\left(\frac{x+\sin x}{x}\right)} = \text{Lt}_{x\to0}\frac{e^x-e^{x\cos x}}{x\left(1+\frac{\sin x}{x}\right)}$

3

A) 1

B) 2

C) 3

D) 0

$= \text{Lt}_{x\to0}\frac{e^x-e^{x\cos x}}{2x}$

$= \text{Lt}_{x\to0}\frac{\left(\frac{e^x}{e^{x\cos x}}-1\right)e^{x\cos x}}{2x}$

$\text{Lt}_{x\to0}\frac{e^{(x-x\cos x)}-1}{x(x-x\cos x)}\,x(1-\cos x)$ $\quad\text{Lt}_{x\to0}\frac{e^x-1}{x}$ $\quad\text{Lt}_{x\to0}\frac{e^{\sin x}-1}{\sin x}=1$

$= 1-\cos 0$

$= 1-1 = 0$

LIMITS TENDS TO INFINITY

e.g

$\Rightarrow \lim_{x\to\infty} (x) = n.d\,(\infty)$

$\Rightarrow \lim_{x\to\infty} (x^2+1) = n.d\,(\infty)$

$\Rightarrow \lim_{x\to\infty} \frac{1}{x} = 0$

$\Rightarrow \lim_{x\to\infty} \frac{1}{x^2+1} = 0$

$\Rightarrow \lim_{x\to\infty}\left(\frac{Ax+B}{Cx+D}\right) = \lim_{x\to\infty} \frac{\cancel{x}\left(A+\frac{B}{x}\right)}{\cancel{x}\left(C+\frac{D}{x}\right)} = \frac{A}{C}$

Linear

Linear

☆

$\Rightarrow \lim_{x\to\infty}\left(\frac{Ax+B}{px^2+qx+r}\right) = \lim_{x\to\infty} \frac{\cancel{x}\left(A+\frac{B}{x}\right)}{x^{\cancel{2}}\left(p+\frac{q}{x}+\frac{r}{x^2}\right)} = 0$ ✓

Linear

quadratic

$\Rightarrow \lim_{x\to\infty}\left(\frac{px^2+qx+r}{Ax+B}\right) = n.d\,(\infty)$

HW

Q)

① $\lim_{x\to\infty}\left(\frac{x^2+x-1}{3x^2+2x+4}\right)^{\left(\frac{3x^2+x}{x-2}\right)}$

② $\lim_{x\to\infty}\left(\sqrt{x^2+x+1} - \sqrt{x^2+1}\right) =$

$\infty - \infty$

③ $\lim_{n\to\infty}\left(\frac{1.n + 2.(n-1) + 3(n-2) + \cdots + n.1}{1^2+2^2+3^2+\cdots+n^2}\right)$

Q)

① $\lim_{x\to\infty} \frac{x^3+1}{x^2} = \infty$

② $\lim_{x\to\infty} \frac{x+1}{2x^2} = 0$

③ $\lim_{x\to\infty} \frac{x^2+x+1}{2x^2+x-1} = \lim_{x\to\infty} \frac{\cancel{x^2}\left(1+\frac{1}{x}+\frac{1}{x^2}\right)}{\cancel{x^2}\left(2+\frac{1}{x}-\frac{1}{x^2}\right)}$

$= \frac{1}{2}.$

④ $\lim_{n\to\infty} \frac{(1+2+3+\cdots+n)}{n^2} = \lim_{n\to\infty} \frac{n(n+1)}{2n^2} = \lim_{n\to\infty} \frac{\cancel{n}\left(1+\frac{1}{n}\right)}{2\cancel{n}}$

$= \frac{1}{2}.$

⑤ $\lim_{n\to\infty} \frac{(1^2+2^2+\cdots+n^2)}{n^3} = \lim_{n\to\infty} \boxed{\frac{n(n+1)(2n+1)}{6\,n^3} = \frac{1.1.2}{6\cdot 1}}$

$= \frac{1}{3}$

⑥ $\lim_{n\to\infty}\left(\frac{1^3+2^3+3^3+\cdots+n^3}{n^4}\right) =$

$= \lim_{n\to\infty} \frac{n^2(n+1)^2}{4\,n^4} = \frac{1}{4}$

$\lim_{x\to\infty} \frac{\sqrt{x^2+1} - (x^3+1)^{1/3}}{(x^4+1)^{1/4} - (x^4+1)^{1/5}}$

A) 0 ✓

B) 1

C) 2/3

D) 4/3

$\lim_{x\to\infty} \frac{\cancel{x}\left(\sqrt{1+\frac{1}{x^2}} - \left(1+\frac{1}{x^3}\right)^{1/3}\right)}{\cancel{x}\left(\left(1+\frac{1}{x^4}\right)^{1/4} - \left(\frac{x^4}{x^5}+\frac{1}{x^5}\right)^{1/5}\right)}$

x^5

$= \frac{\sqrt{1+0} - (1+0)^{1/3}}{(1+0)^{1/4} - (0+0)^{1/5}}$

$= \frac{1-1}{1-0} = \frac{0}{1}$

$= 0$