LMVT(Lagrange's Mean Value Theorem)

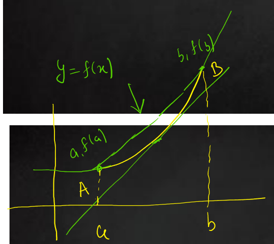
Let $f(x)$ be a function of x subject to following conditions;

(i) $f(x)$ is a continuous function of x in the interval $[a,b]$.

(ii) $f(x)$ is a differentiable function of x in (a,b)

Then their exist atleast one point $x = c \in (a,b)$

Such that $\boxed{f'(c) = \dfrac{f(b)-f(a)}{b-a}}$ $\quad m_{AB} = \dfrac{y_2-y_1}{x_2-x_1}$



① conti
② diff

Q) Verify LMVT of

$f(x) = x^3 - 5x^2 - 3x$ in the interval $[1,3]$ (a = 1, b = 3)

① conti ✓
② diff ✓

$f'(c) = \dfrac{f(b)-f(a)}{b-a} = \dfrac{-27+7}{3-1} = \dfrac{-20}{2} = -10$

$f(3) = 3^3 - 5\times3^2 - 3\times3 = 27 - 45 - 9$
$= -27$

$f(1) = 1 - 5 - 3 = -7$

$f'(x) = 3x^2 - 10x - 3$

$f'(c) = -10$

$3c^2 - 10c - 3 = -10$

$3c^2 - 10c + 7 = 0$

$3c^2 - 7c - 3c + 7 = 0$

$c(3c-7) - 1(3c-7) = 0$

$c = 1, \quad c = \frac{7}{3}$

$c = \frac{7}{3} \in (1,3)$

verified

Q) A value of c for which the conclusion of LMVT holds for $f(x) = \ln x$ in the interval $[1,3]$

A) $2\log_3 e$
B) $\frac{1}{2}\log_e 3$
C) $\log_3 e$
D) $\log_e 3$

$f'(c) = \dfrac{1}{c} = \dfrac{f(3)-f(1)}{3-1}$

$\dfrac{1}{c} = \dfrac{\ln 3 - \ln 1}{2}$

$c = \dfrac{2}{\log_e 3} = 2\log_3 e$

(H.W.) find value of c in LMVT for $f(x) = \ln \sin x$ in $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$ (a, b)

① ✓
② ✓

$f'(c) = \dfrac{f(b)-f(a)}{b-a}$

$= \dfrac{\ln\sin\frac{5\pi}{6} - \ln\sin\frac{\pi}{6}}{\frac{5\pi}{6} - \frac{\pi}{6}}$

$f'(x) = \dfrac{1}{\sin x}\cos x = \cot x$

$f'(c) = 0 = \cot c$

$\boxed{c = \frac{\pi}{2}}$

$f\left(\frac{5\pi}{6}\right) = \ln\sin\frac{5\pi}{6}$
$= \ln\sin\frac{\pi}{6}$

$f\left(\frac{\pi}{6}\right) = \ln\sin\frac{\pi}{6}$

$\sin(\pi - \theta) = \sin\theta$

Q) find 'c' of LMVT for which $f(x) = \sqrt{25 - x^2}$ in $[1,5]$

A) $\sqrt{13}$ B) $\cancel{\sqrt{15}}$ C) $\sqrt{17}$ D) $\sqrt{11}$

$f'(c) = \dfrac{(-2x)}{2\sqrt{25-x^2}}$

$= \dfrac{-c}{\sqrt{25-c^2}} = \dfrac{0 - \sqrt{24}}{5-1} = \dfrac{-2\sqrt{6}}{4\,2}$

$f(5) = \sqrt{25-25} = 0$

$f(1) = \sqrt{24}$

$\Rightarrow 2c = \sqrt{6}\sqrt{25-c^2}$

$4c^2 = 150 - 6c^2$

$10c^2 = 150$

$c^2 = 15$

$c = \sqrt{15}$

$-\sqrt{15}$ ✗

If $f(x) = 2x^2 + 3x + 5$ satisfies LMVT at $x = 2$ on close interval $[1, a]$ then find a.

$[1,a)$

$[1,1]$ ✗

$f'(2) = \dfrac{f(a)-f(1)}{a-1}$

$f'(x) = 4x + 3$

$f'(2) = 11$

$f(1) = 2 + 3 + 5 = 10$

$f(a) = 2a^2 + 3a + 5$

$11 = \dfrac{2a^2 + 3a + 5 - 10}{a-1}$

$11a - 11 = 2a^2 + 3a - 5$

$2a^2 - 8a + 6 = 0$

$a^2 - 4a + 3 = 0$

$a = 1$ ✗, $a = 3$

Q) If $2a+3b+6c=0$ then Prove that the equation $ax^2+bx+c=0$ has atleast one root in $(0,1)$.

Divide by 6: $\frac{a}{3}+\frac{b}{2}+c=0$

$\Rightarrow$ $f'(c)=0$ $\quad c \Rightarrow$ root of $f'(x)$

$f(x) = \frac{ax^3}{3}+\frac{bx^2}{2}+cx$ $\quad f'(c)=0$ in $(0,1)$

(1) Conti.
(2) diffn
(3) $f(a)=f(b)$

$f(0)=0$, $f(1)=\frac{a}{3}+\frac{b}{2}+c=0$

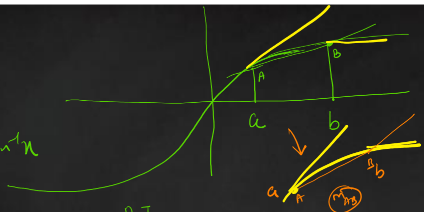
$f(0)=f(1)$

$f'(x)=ax^2+bx+c$

$(0,1)$

R.T

$4a^2+9b^2-c^2+12ab$



$f(x)=\tan^{-1}x$

P.T.

$$\frac{b-a}{1+b^2} < \tan^{-1}b-\tan^{-1}a < \frac{b-a}{1+a^2}$$

$\boxed{b>a>0}$

$$\frac{1}{1+b^2} < \left(\frac{\tan^{-1}b-\tan^{-1}a}{b-a}\right) < \frac{1}{1+a^2}$$

$f(x)=\tan^{-1}x$

$f'(b) < m_{ab} < f'(a)$

Cauchy's theorem:

If $f(x)$ and $g(x)$ are
(1) Continuous in $[a,b]$
(2) differentiable in (a,b)

then there exist $c \in (a,b)$ such that

$$\boxed{\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}}$$

$g'(c)\neq 0$.

LMVT:

(1) $f'(c) = \frac{f(b)-f(a)}{b-a}$

(2) $g'(c) = \frac{g(b)-g(a)}{b-a}$

Divide:

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Q) Let $f(x)$ and $g(x)$ be differentiable in $[0,2]$ such that $f(0)=2$, $g(0)=1$ & $f(2)=8$. Let there exist a real number 'c' in $(0,2)$ such that $f'(c)=3g'(c)$ then value of $g(2)$ must be

A) 2 C) 4

B) 3 D) 5

$$\frac{f'(c)}{g'(c)} = \frac{f(2)-f(0)}{g(2)-g(0)}$$

$$3 = \frac{8-2}{g(2)-1}$$

$g(2)-1=2$

$\boxed{g(2)=3}$

⇒ Interpretation of $\frac{dy}{dx}$ as rate measurer;

If a variable quantity y is some function of time 't' i.e $y = f(t)$ then small change in time Δt have a corresponding change Δy in y.

☆ average rate of change: $\boxed{\frac{\Delta y}{\Delta t} = \frac{y_2 - y_1}{t_2 - t_1}}$

When limit $\Delta t \to 0$ is applied then the rate of change becomes instantaneous & we get the rate of change w.r.t instant t.

$$\boxed{\lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \frac{dy}{dt}}$$

Q) If area of circles increases at the rate of $2\ cm^2/sec$ then find the rate at which area of inscribed square increases ☆

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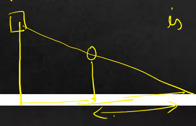
Q) ☆ A man 1.6m high walks at the rate of 30 m/minute from a lamp which is 4 meter above ground. How fast is the man's shadow is lenghting/increasing/decreasing

☆



Q) If the radius of circle be increasing at the uniform rate of 2 cm/sec. find rate of increasing area of circle at the instant when $r = 20cm$

A) $80\pi\ cm^2/sec$

B) $40\pi\ cm^2/sec$

C) $20\pi\ cm^2/sec$

D) $10\pi\ cm^2/sec$.

$$\frac{dr}{dt} = 2\ cm/sec$$

$$A = \pi r^2$$

$$\frac{dA}{dt} = \pi 2r \frac{dr}{dt}$$

$$= \pi \times 2 \times 20 \times 2$$

$$= 80\pi\ cm.$$

Q) on the curve $x^3 = 12y$. find the interval at which the abscissa changes at a faster rate than ordinate.

A) $(-2, 2)$

B) $(-2, 2) - \{0\}$

C) $(-1, 1)$

D) $(-1, 1) - \{0\}$

$$3x^2 = 12\frac{dy}{dx}$$

$$\frac{dx}{dt} > \frac{dy}{dt}$$

$$\frac{dx}{dy} = \frac{4}{x^2}$$

$x \neq 0$

$$\frac{dx}{dy} > 1$$

$$\frac{4}{x^2} > 1$$

$$x^2 < 4$$

$$(x-2)(x+2) < 0$$

$\oplus \quad -2 \quad \ominus \quad 2 \quad \oplus$

$(-2, 2) - \{0\}$

Q) A ballon which always remain spherical has a variable radius. find rate at which the volume is increasing with radius when later is $\boxed{r = 10\ cm}$ 10cm

A) 100π

B) 200π

C) 300π

D) 400π

$$\frac{dV}{dr}$$

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dv}{dr} = 4\pi r^2$$

$$= 4\pi \times 100$$

$$= \boxed{400\pi}$$