Use of monotonicity to prove inequalities:

\# Comparing of two functions $f(x)$ and $g(x)$ can be done by analysing the monotonic behaviour of another function $h(x) = f(x) - g(x)$.

e.g. $\sin x < x$ in $(0, \frac{\pi}{2})$

$f(x) = \sin x$, $g(x) = x$.

$h(0) = 0$

$h(x) = \sin x - x$.

$\boxed{h'(x) = \cos x - 1 < 0}$ $\rightarrow (0, \frac{\pi}{2})$

☆ $h(x)$ is strictly decreasing in $(0, \frac{\pi}{2})$

☆ $h(x) < h(0)$

$\sin x - x < 0$

☆ $\boxed{\sin x < x}$

e.g. $x < \tan x$ $(0, \frac{\pi}{2})$

$h(x) = x - \tan x$

$h'(x) = 1 - \sec^2 x < 0$

$h(x) \downarrow$

$h(x) < h(0)$

$x - \tan x < 0$ — $x < \tan x$

Q) for $x \in (0, \frac{\pi}{2})$ Prove that

$\sin x > x - \frac{x^3}{6}$

$f(x) = \sin x - x + \frac{x^3}{6}$

$(0, \frac{\pi}{2})$

$f'(x) = \boxed{\cos x - 1 + \frac{x^2}{2}}$

$f''(x) = -\sin x + x$

$(0, \frac{\pi}{2})$ $= x - \sin x > 0$

$f''(x) > 0 \rightarrow f'(x)$ increasing

$f'(x) > f'(0)$

$\cos x - 1 + \frac{x^2}{2} > 0$

$\boxed{f(x) \text{ is increasing}}$ $\leftarrow$ $\boxed{f'(x) > 0}$ $(0, \frac{\pi}{2})$

$f(x) > f(0)$

$\sin x - x + \frac{x^3}{6} > 0$

$\boxed{\sin x > x - \frac{x^3}{6}}$

Q) Find the monotonic behaviour of $f(x) = x^{1/x}$ and explain which is greater

① π^e or e^π.

② 100^{101} or 101^{100}

$\pi^e < e^\pi$

$(\pi^e)^{1/\pi e} < (e^\pi)^{1/\pi e}$

$\pi^{1/\pi} < e^{1/e}$

$f(x) = x^{1/x}$ $f(e) = e^{1/e}$ $f(\pi) = \pi^{1/\pi}$

$f(e) > f(\pi)$

Diff wrt x:

$y = x^{1/x}$

$\ln y = \frac{\ln x}{x}$

$\frac{1}{y}\frac{dy}{dx} = \frac{x \cdot \frac{1}{x} - \ln x}{x^2}$

$f'(x) = \boxed{\frac{dy}{dx} = \frac{y(1 - \ln x)}{x^2}}$

$f'(x) = \frac{x^{1/x}(1 - \ln x)}{x^2}$

$x = -2$, $(-2)^{-1/2} = \text{Img}$

$x = -3$, $(-3)^{-1/3} \rightarrow \text{Real}$

$x > 0$

Increasing: $1 - \ln x > 0$

$\ln x < 1 \Rightarrow x < e$

Decreasing: $1 - \ln x < 0$

$\ln x > 1 \Rightarrow \boxed{x > e}$

$x = e$

e π 100 101

$101^{100} > 100^{101}$

$100^{\frac{1}{100}} > 101^{\frac{1}{101}}$

$f(100) > f(101)$

① ②

$\frac{1}{100 \times 101}$

True or false.

① for all $x \in (0, 1)$ $\boxed{e^x < 1 + x}$ — $e^x - 1 - x < 0$

$f(x) = e^x - (1 + x)$

$f'(x) = e^x - 1 > 0$

$f(x)$ is increasing in $(0, 1)$

$\boxed{e^x > 1, x \in (0,1)}$

$f(0) = 1 - 1 = 0$

$f(x) > f(0)$

$e^x - (1 + x) > 0$

$\boxed{e^x > 1 + x}$ $\boxed{\text{false}}$

② $\ln(1 + x) > x$ for all $x \in (0, 1)$

$f(x) = \ln(1 + x) - x$

$f'(x) = \frac{1}{1 + x} - 1$

$= \frac{1 - 1 - x}{1 + x} = \frac{-x}{1 + x} < 0$

$f(0) = 0$

$f'(x) < 0$

$f(x)$ decreasing

$\boxed{f(x) < f(0)}$

$\ln(1 + x) - x < 0$

$\boxed{\ln(1 + x) < x}$

$\boxed{\text{false}}$

③ $\ln x > x$ for $x \in (0, 1)$

$f(x) = \ln x - x$

$f'(x) = \frac{1}{x} - 1 = \frac{1 - x}{x} > 0$

$f(x)$ increasing

$f(1) = \ln 1 - 1$

$f(1) = -1$

$f(x) > f(0)$ ⊗ $f(x) < f(1)$

$\ln x - x < -1$

$\ln x < x - 1$

$\boxed{\text{false}}$ $\boxed{x > \ln x}$ ☆ $\boxed{x > \ln x + 1}$

$x > 2 + 1$ $x > 2$

Roll's theorem:

Let $f(x)$ be a function of x subject to the following condition:

1. $f(x)$ is continuous function of x in $[a,b]$
2. $f'(x)$ exists for every point in (a,b)
3. $f(a)=f(b)$

then there exists atleast one point $\boxed{c\in(a,b)}$ such that $\boxed{f'(c)=0}$

Q) Consider the function $f(x)=e^{-2x}\sin 2x$ over the interval $[0,\frac{\pi}{2}]$. A real no. $c\in(0,\frac{\pi}{2})$ as guaranteed by Roll's theorem such that $f'(c)=0$ is

A) $\pi/8$ B) $\pi/6$. $\longrightarrow f'(c)=0$

C) $\pi/4$ D) $\pi/3$.

$f'(x) = 2\cos 2x\, e^{-2x} + e^{-2x}(-2)\sin 2x$

$= 2e^{-2x}(\cos 2x - \sin 2x) = 0$

$\Rightarrow \cos 2x = \sin 2x$

$\tan 2x = 1$

$2x = \frac{\pi}{4}$

$c = ?$

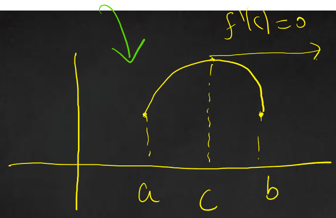
$\boxed{x=\pi/8}$

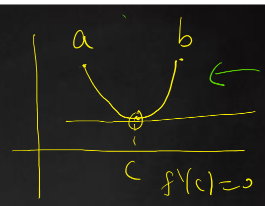
Q) verify Roll's theorem:

1.2 $f(x)=\cos\left(\frac{1}{x}\right)$ in $[-1,1]$ — No

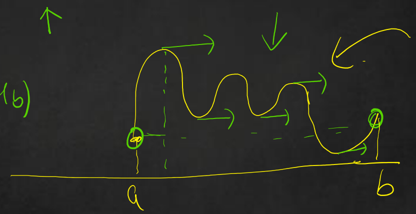
1.3 $f(x)=|x+1|$ in $[1,4]$ — No

1.4 $f(x)=(x-1)(x-2)$ in $[1,2]$





$f(a)=f(b)$



Q) verify Roll's theorem for

$f(x) = x^2+2x-8$

in $[-4,2]$ ($a=-4$, $b=2$)

1. ✓ S-1 ; Conti in $[-4,2]$
✓ S-2 ; diff. in $(-4,2)$ $f(-4)=f(2)$ ✓
✓ S-3 ; $f(-4) = (-4)^2+2(-4)-8 = 0$, $f(2) = 4+4-8 = 0$

$f'(c)=0$

$c\in[-4,2]$

2. $f'(x) = 2x+2 = 0$

$x=c$

$f'(c) = 2c+2 = 0$ $\boxed{c=-1}$

$c\in(-4,2)$

$\boxed{\text{verified}}$