Electro-Static Potential Energy

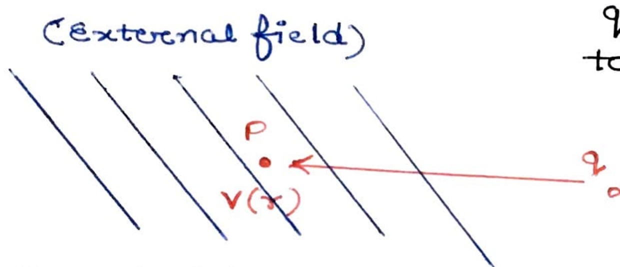
Potential Energy of a system of particles is defined only in conservative fields. As Electric field is conservative, we define potential energy in it.

" Potential energy in a conervative field is defined as the work done in assembling the system in a given configuration against the interactive forces b/w the particles."

Electro-static potential Energy is defined into two ways.

i) Electrostatic Interaction Energy :→ it is the external work required to assemble the charges from infinity (∞) to a given configuration.

a) E.P.E of an isolated point charge inside an external Electric field:



Let a positive point charge q is brought from ∞ to a point P, inside an external field where Electric potential is V(r).

∵ Electric potential at $P(V) = \frac{[W_{\infty \to P}]_{ext}}{q}$ — ①

∵ external force must be equal & opposite of the electric force $\Rightarrow (W_{\infty \to P})_{ext} = -(W_{\infty \to P})_{elect}$

also Electric force is conservative; $\Rightarrow (W_{\infty \to P})_{elect} = -\Delta U$

$$\Rightarrow (W_{\infty \to P})_{ext} = -(-\Delta U) = \Delta U \quad \text{— ②}$$

from ① & ②;

$$V = \frac{\Delta U}{q}$$

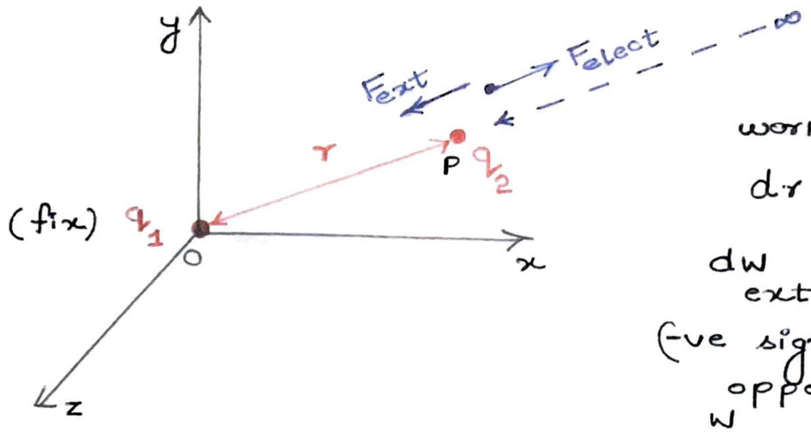
$\Rightarrow \Delta U$ or $U(r) - U(\infty) = q \cdot V(r)$

as $U(\infty) = 0$; ∞ is always taken at 0 P.E.

so $\boxed{U(r) = q \cdot V(r)}$ Joules

note; if the brought charge would be -ve, then;
$U(r) = -q \cdot V(r)$ J.

b) Electro-static P.E. of a pair of two point charges.



work done to bring q_2 dr closer to q_1 :

$$dW_{ext} = -\int \vec{F}_{ext} \cdot \vec{dr}$$

(-ve sign showing work is done opposite to the field)

$$\Rightarrow \int_0^W dw_{ext} = -\int_\infty^r \frac{K \cdot q_1 \cdot q_2}{r^2} \cdot dr \cdot \cos 0^\circ$$

$$\Rightarrow (W_{ext})_0^{W_{ext}} = -K \cdot q_1 \cdot q_2 \cdot \left[-\frac{1}{r}\right]_\infty^r$$

$$\Rightarrow W_{ext} - 0 = K \cdot q_1 \cdot q_2 \cdot \left[\frac{1}{r} - \frac{1}{\infty}\right]$$

$$\Rightarrow W_{ext} = \frac{K \cdot q_1 \cdot q_2}{r} \text{ Joule}$$

as $W_{ext} = \Delta U = U_r - U_\infty = U_{12} - 0$

so $$\boxed{U_{12} = \frac{K \cdot q_1 \cdot q_2}{r} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1 \cdot q_2}{r}} \text{ J}$$

Second Method :→ Let us consider q_2 enters in the field of q_1 so work done to bring q_2 from ∞ to the point P

$$W_{\infty \to P} = q_2 \times V_P$$

$$= q_2 \times \frac{K \cdot q_1}{r}$$

$$= \frac{K \cdot q_1 \cdot q_2}{r}$$

$$W_{\infty \to P} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1 \cdot q_2}{r}$$

so $$U_{12} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1 \cdot q_2}{r} \text{ J.}$$

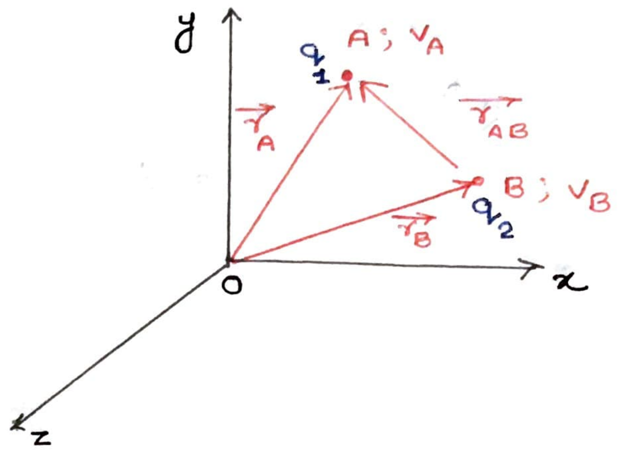
Note: Electric potential Energy of 2 like charges is +ve and 2 unlike charges is -ve.

ie: $+q_1 \xleftarrow{\;r\;} \rightarrow +q_2$

$-q_1 \leftarrow r \rightarrow -q_2$ $\qquad U = \frac{K q_1 \cdot q_2}{r}$

$-q_1 \leftarrow r \rightarrow +q_2$ $\qquad U = -\frac{K \cdot q_1 \cdot q_2}{r}$

c) Electric Potential Energy of a pair of 2 point charges Kept in an external field.

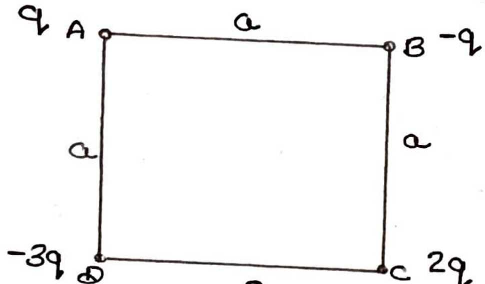


$$U_{total} = U_1 + U_2 + U_{12}$$

$$\boxed{U_{Total} = q_1 \cdot V_A + q_2 \cdot V_B + \frac{K \cdot q_1 \cdot q_2}{r_{AB}} \ J}$$

Eg: Four point charges $q, -q, 2q$ & $-3q$ are Kept at the 4 vertices of a square of side Length 'a'. Find the electric potential energy of a system. (or the work done to assemble the system by bringing charges from ∞)

Sol^n:



$$U_{total} = U_{AB} + U_{BC} + U_{CD} + U_{DA} + U_{AC} + U_{BD}$$

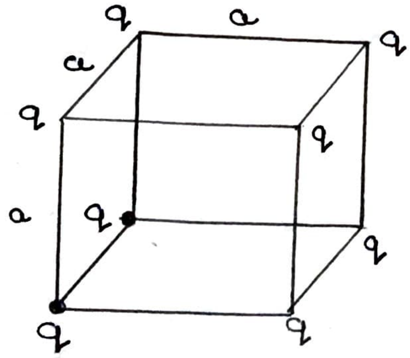
$$= \frac{K \cdot q \cdot -q}{a} + \frac{K \cdot -q \cdot 2q}{a} + \frac{K \cdot 2q \cdot -3q}{a} + \frac{K \cdot -3q \cdot q}{a} + \frac{K \cdot q \cdot 2q}{a\sqrt{2}} + \frac{K \cdot -q \cdot -3q}{a\sqrt{2}}$$

$$\Rightarrow U = \frac{K \cdot q^2}{a} \left[-1 - 2 - 6 - 3 + \frac{2}{\sqrt{2}} + \frac{3}{\sqrt{2}} \right]$$

$$\Rightarrow U = \left(W_{\infty \to sq.} \right)_{ext} = \frac{K \cdot q^2}{a} \left[-12 + \frac{5}{\sqrt{2}} \right] J$$

Eg: 8 identical point charges each 'q' are brought from ∞ to the vertices of a cube of side length 'a' calculate the E.P.E. of this system.

Sol^n:



There will be 3x4 pairs of charges at a separation 'a', 3x4 pairs at a separation '$a\sqrt{2}$' & 4 pairs at a separation '$a\sqrt{3}$'

$$\therefore U_{total} = 12 \cdot U_{r=a} + 12 \cdot U_{r=a\sqrt{2}} + 4 \cdot U_{r=a\sqrt{3}}$$

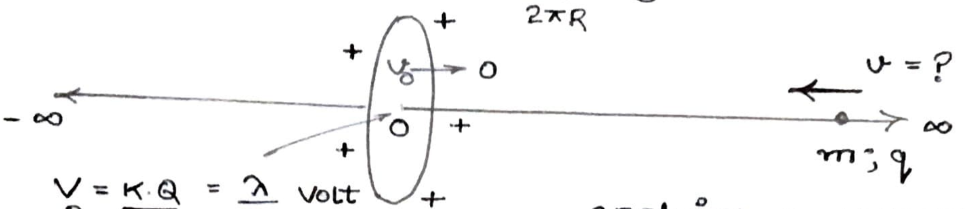
$$= 12 \cdot \frac{Kq^2}{a} + 12 \cdot \frac{Kq^2}{a\sqrt{2}} + 4 \cdot \frac{Kq^2}{a\sqrt{3}}$$

$$= \frac{Kq^2}{a} \cdot \left[12 + 6\sqrt{2} + \frac{4}{\sqrt{3}} \right] J$$

Eg: A charge particle of mass 'm' & charge 'q' is projected from ∞ to the center of a uniformly charged Ring of radius 'R' & linear charge density 'λ', along its axis. Find the min. speed of projection so that it may reach again ∞ on the other side of the ring.

Sol^n:

$$Q : \lambda = \frac{Q}{2\pi R} \quad \text{—①}$$



$V_\infty = 0$ volt

$$V_0 = \frac{K \cdot Q}{R} = \frac{\lambda}{2\pi} \text{ Volt}$$

applying conservation of Energy b/w ∞ & point O;

$$K_\infty + U_\infty = K_0 + U_0$$

$$\Rightarrow q \cdot V_\infty + \frac{1}{2} m v^2 = \frac{1}{2} m \cdot v_0^2 + q \cdot V_0$$

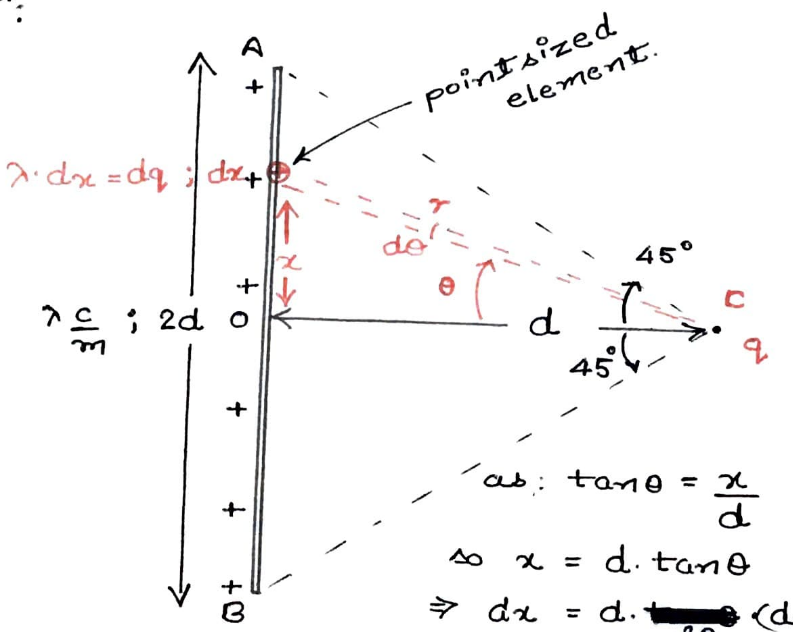
as; $V_\infty = 0$; $v_0 = 0$

$$\Rightarrow 0 + \frac{1}{2} m v^2 = 0 + \frac{q \cdot \lambda}{2\pi}$$

$$\text{so} \quad v = \sqrt{\frac{q \cdot \lambda}{\pi m}} \text{ m/s}$$

Eg: calculate the Electric potential energy of the charged rod & the point charge.

Sol^n:



Electric Potential at P due to the considered element

$$dV = \frac{K \cdot dq}{r}$$

$$dV = \frac{K \cdot \lambda \cdot dx}{r} \quad \text{—①}$$

as: $\cos\theta = \frac{d}{r}$

so $r = d \cdot \sec\theta$ —②

as: $\tan\theta = \frac{x}{d}$

so $x = d \cdot \tan\theta$

$\Rightarrow dx = d \cdot \sec^2\theta \, (d\theta)$ ③

from ①, ② & ③

$$dV = K\lambda \cdot \frac{\cancel{d} \cdot \cancel{\sec^2}\theta \cdot d\theta}{\cancel{d \sec\theta}}$$

$$\Rightarrow \int_0^V dV = K \cdot \lambda \cdot \int_{-\pi/4}^{\pi/4} \sec\theta \cdot d\theta$$

$$\Rightarrow (V)_0^V = K \cdot \lambda \cdot \left[\log_e (\sec\theta + \tan\theta) \right]_{-\frac{\pi}{4}}^{\pi/4}$$

$$= K \cdot \lambda \cdot \left[\log_e (\sqrt{2} + 1) - \log_e (\sqrt{2} - 1) \right]$$

$$= K \cdot \lambda \cdot \left[\log_e \left(\frac{\sqrt{2}+1}{\sqrt{2}-1} \right) \right] = K \cdot \lambda \cdot \log_e (\sqrt{2}+1)^2$$

$$\text{so} \quad V = 2K\lambda \log_e (\sqrt{2}+1) \text{ volt} \quad \text{— ④}$$

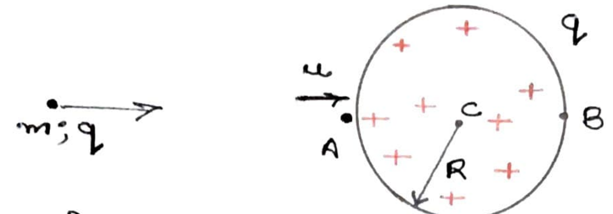
So P.E.

$$U = q \cdot V$$

$$= 2Kq\lambda \log_e (\sqrt{2}+1)]$$

$$U = \frac{q \cdot \lambda}{2\pi\varepsilon} \cdot \log_e (\sqrt{2}+1) \text{ Joules.}$$

eg: A bullet of mass 'm' and charge 'q' is fired towards a uniformly charged non-conducting sphere of radius 'R' & charge 'q'. If it strikes the surface of the sphere with a speed 'u', find the min. value of u so that it may go through the sphere. (Neglect all other forces except the electrostatic force).



Sol^n:→ if the bullet anyhow reaches & crosses the center even with a negligible velocity, it will be expelled by the sphere's positive charge after that:

from conservation of energy b/w A & C.

$$K_A + U_A = K_C + U_C$$

$$\Rightarrow \frac{1}{2}mu^2 + q\cdot V_A = \frac{1}{2}m\cdot v_C^2 + q\cdot V_C$$

$\because$ for u_{min} : $v_C \to 0$

$$\Rightarrow \frac{1}{2}mu^2 + q\cdot\frac{K\cdot q}{R} = 0 + q\cdot\frac{3}{2}\cdot\frac{Kq}{R}$$

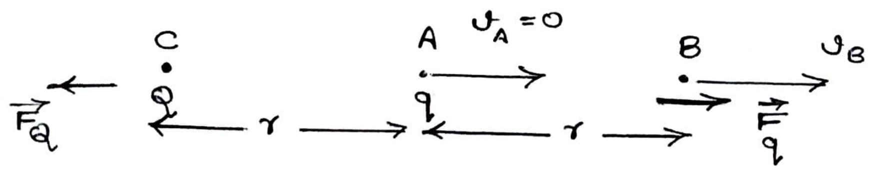
$$\Rightarrow \frac{1}{\cancel{2}}mu^2 = \frac{K\cdot q^2}{\cancel{2}R}$$

$$\Rightarrow mu^2 = \frac{q^2}{4\pi\varepsilon_0 R}$$

$$\text{so } u_{min} = \frac{q}{\sqrt{4\pi\varepsilon_0 mR}} \text{ m/s.}$$

eg: A charge particle 'Q' is held fixed, another charge particle of mass 'm' & charge 'q' is released from a distance 'r'. Find the force exerted by the external agent on the fixed charge by the time distance b/w them becomes 2r.

Sol^n:→



from conservation of energy:

$$K_A + U_A = K_B + U_B$$

$$\Rightarrow 0 + \frac{KQ\cdot q}{r} = \frac{1}{2}mv_B^2 + \frac{K\cdot Q\cdot q}{2r}$$

$$\Rightarrow \frac{K\cdot Q\cdot q}{\cancel{2}r} = \frac{1}{\cancel{2}}m\cdot v_B^2$$

$$\text{speed at } B \Rightarrow v_B = \sqrt{\frac{Q\cdot q}{4\pi\varepsilon_0 rm}} \text{ m/s} \quad \text{--- (1)}$$

$\because \quad \vec{F_Q} = -\vec{F_q}$

$\Rightarrow \quad \vec{F_Q}\cdot t = -\vec{F_q}\cdot t$

$\Rightarrow \quad \vec{J_Q} = -\vec{J_q}$

$= -\Delta\vec{p_q}$

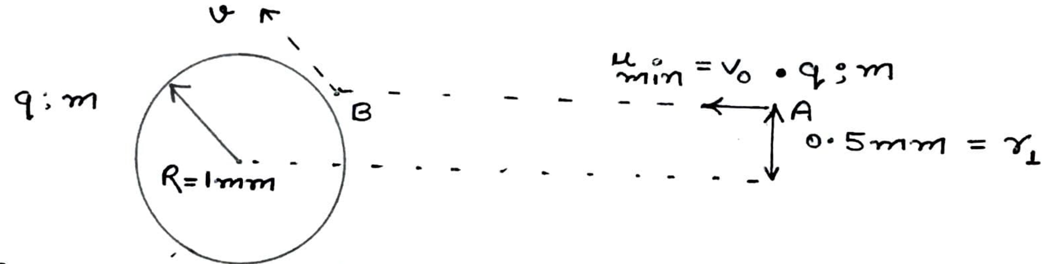
$= -[m\cdot\vec{v_B} - m\cdot\vec{v_A}]$

$= -m\sqrt{\frac{Q\cdot q}{4\pi\epsilon_0 mR}}$

$= -\sqrt{\frac{Q\cdot q\cdot m^2}{4\pi\epsilon_0 m r}}$

so $J_Q = \sqrt{\frac{Q\cdot q\cdot m}{4\pi\epsilon_0 r}}$ (along -x-axis)

eg: A particle of mass 1kg & charge $\frac{1}{3}\mu C$ is projected towards a non conducting fixed spherical shell having same charge uniformly distributed over its surface. Find the minimum initial speed of projection required if the particle just grazes the shell.



Sol^n ⇒

<u>from conservation of energy b/w A & B</u>

$K_A + U_A = K_B + U_B$

$\frac{1}{2}mv_0^2 + 0 = \frac{1}{2}mv^2 + q\cdot V_B$

$\Rightarrow \frac{mV_0^2}{2} = \frac{mv^2}{2} + \frac{K\cdot q^2}{R}$ — ①

as the electric force on the particle is central force;

so $\tau = 0$

$\Rightarrow L = \text{const}$

$\Rightarrow L_A = L_B$

$\Rightarrow \cancel{m}\cdot V_0\cdot r_\perp = \cancel{m}\cdot v\cdot R$

$\Rightarrow V_0 \times 0.5 \cancel{\times 10^{-3}} = v \times \cancel{10^{-3}}$

so $v = \frac{V_0}{2}$ — ②

<u>from ① & ②</u>

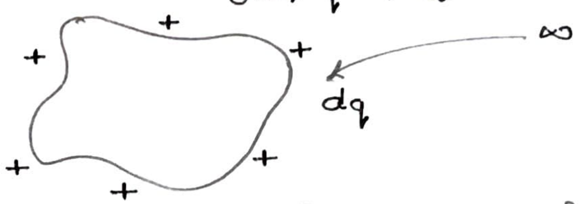
$\frac{mV_0^2}{2} = \frac{mV_0^2}{8} + \frac{K\cdot q^2}{R}$

$\Rightarrow \frac{3}{8}mV_0^2 = \frac{K\cdot q^2}{R}$

$\Rightarrow \frac{3}{8}\times 1\times V_0^2 = \frac{\cancel{9}\times 10^9}{10^{-3}} \times \frac{1}{\cancel{9}} \times \cancel{10^{-12}}$ so $V_0 = 2\cdot\sqrt{\frac{2}{3}}$ m/s.

ii) **Electro-static self energy:** it is equal to the work needed to charge a body from bringing electric charge from ∞ to the body.

$0 \rightarrow q \rightarrow Q$

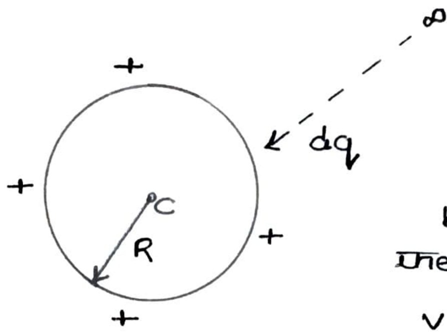


$dW_{ext} = dq \cdot V$

so $U = \int dW_{ext} = \int dq \cdot V$

a) **Electro-static self energy of a charged conducting sphere**

$0 \rightarrow q \rightarrow Q$



Let their is an isolated charged spherical shell of radius R and charge Q.

when q charge was already brought on the sphere surface, the instantaneous potential over it:

$V = \frac{K \cdot q}{R}$ volts.

$\therefore$ change in potential energy to bring dq charge on the surface from ∞ at this instant:

$$du = dq \cdot V$$

$$du = \frac{K \cdot q \cdot dq}{R}$$

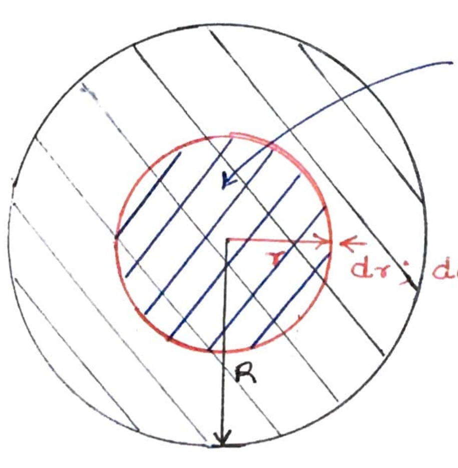
$$\Rightarrow \int_0^U dU = \frac{K}{R} \cdot \int_0^Q q \cdot dq$$

$$\Rightarrow (U)_0^U = \frac{K}{R} \cdot \left[\frac{q^2}{2}\right]$$

so $\boxed{U = \frac{K \cdot Q^2}{2R} = \frac{Q^2}{8\pi\varepsilon_0 R}}$ Joule

b) **Electro-static self Energy of a non-conducting uniformly charged sphere.**

Q; $\rho = \frac{Q}{\frac{4}{3}\pi R^3}$



$q = \rho \cdot V = \frac{Q}{\frac{4}{3}\pi R^3} \times \frac{4}{3}\pi r^3 = \frac{Q \cdot r^3}{R^3}$

(already brought charge)

dr; $dq = \rho \cdot dv = \frac{Q}{\frac{4}{3}\pi R^3} \times 4\pi r^2 \times dr = \frac{3Qr^2 \cdot dr}{R^3}$

(further brought charge)

so increase in E.P.E during dq charge has been brought from ∞ to radius r.

$$du = dq \cdot V$$

$$= dq \cdot \frac{K \cdot q}{r}$$

$$= \frac{3r^2 Q\, dr}{R^3} \cdot \frac{K}{r} \cdot \frac{Q}{R^3} \cdot r^3$$

$$du = \frac{3 \cdot K \cdot Q^2}{R^6} \cdot r^4 \cdot dr$$

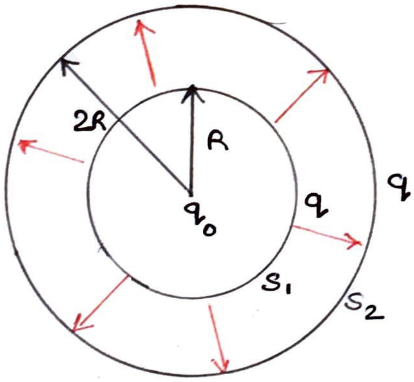
$$\Rightarrow \int_0^U du = \frac{3KQ^2}{R^6} \cdot \int_0^R r^4 \cdot dr$$

$$\Rightarrow (U)_0^U = \frac{3K \cdot Q^2}{R^6} \left(\frac{r^5}{5}\right)_0^R$$

$$\Rightarrow \boxed{U = \frac{3}{5} \cdot \frac{K \cdot Q^2}{R} = \frac{3 \cdot Q^2}{20\pi \varepsilon_0 R}} \quad J$$

Eg: A spherical shell of radius R with a uniformly distributed charge q has a point charge q_0 kept at its center. find the work done by the electric forces to change its radius from R to 2R.

Sol^n :→



Expansion of the shell will be a result of electro-static repulsion.

$$\because (W_{elect})_{conservative} = -\Delta U_{sys}$$

$$= -(U_f - U_i)$$

$$\therefore W_{elect} = (U_i - U_f) \quad \text{—} \circledast$$

initial potential energy of the system (U_i)

= interaction P.E + self energy

$$U_i = \left(q_0 \cdot \frac{K \cdot q}{R} + \frac{K \cdot q^2}{2R}\right) \text{— ①}$$

final potential energy of the system (U_f) = interaction E. + Self E.

$$U_f = \left(q_0 \cdot \frac{K \cdot q}{2R} + \frac{Kq^2}{4R}\right) \text{— ②}$$

$$\text{so } W_{elect} = \frac{K \cdot q \cdot q_0}{R} \cdot \left(1 - \frac{1}{2}\right) + \frac{Kq^2}{2R} \cdot \left(1 - \frac{1}{2}\right)$$

$$\Rightarrow W_{elect} = \frac{+Kqq_0}{2R} + \frac{Kq^2}{4R} = +\frac{Kq}{4R} \cdot (2q_0 + q) \; J$$