Planetary Motion & Kepler's Laws

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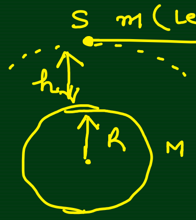
Q 2016 (Mains) :→ A satellite is revolving in a circular orbit at a height h above the Earth's surface where $h <<< R_e$. find the minimum increase in its orbital velocity such that the satellite could escape from Earth's gravity.

a) $\sqrt{2gR}$ b) $\sqrt{gR}$

c) $\sqrt{\frac{gR}{2}}$ d) $\sqrt{gR}\cdot(\sqrt{2}-1)$

Soln :→



$$v_o = \sqrt{\frac{GM}{R+h}} \quad \because R >> h \quad \therefore R+h \approx R$$

$$= \sqrt{\frac{GM}{R}} = \sqrt{\left(\frac{GM}{R^2}\right)\times R}$$

$$v_o = \sqrt{gR} \text{ — ① : initial orbital speed}$$

$\because K_S + U_S = E_\infty$ (for the escape of the satellite)

$$\frac{1}{2}mv_e^2 + \left(-\frac{GMm}{R+h}\right) = 0$$

$$\Rightarrow \frac{v_e^2}{2} = \frac{GM}{R} \quad \because h << R$$

$$\Rightarrow v_e^2 = 2\left(\frac{GM}{R^2}\right)\times R = 2\cdot g\cdot R$$

escape speed $v_e = \sqrt{2gR}$ — ②

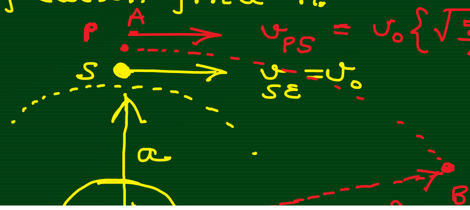
finally given

$\therefore$ rise in speed required

$$\Delta v = v_e - v_o = \sqrt{gR}\times(\sqrt{2}-1) \text{ m/s.}$$

Q: A satellite is revolving around earth at a circular orbit of radius 'a', with a velocity v_0. A particle is projected from satellite in the forward direction with a velocity $v_0\cdot\left\{\sqrt{\frac{5}{4}}-1\right\}$. It is found that the particle reaches a maximum distance $\frac{n\cdot a}{5}$ from the center of earth. find n.

Soln :→



$v_{PS} = v_0\left\{\sqrt{\frac{5}{4}}-1\right\}$

$v_{SE} = v_0$

$v_B = 0$

$\because \vec{v}_{PS} = \vec{v}_{PE} - \vec{v}_{SE}$

$\therefore \vec{v}_{PE} = \vec{v}_{PS} + \vec{v}_{SE}$

$\therefore \vec{v}_{PE} = \vec{v}_{PS} + \vec{v}_{SE}$

$= v_0\left\{\sqrt{\frac{5}{4}} - 1\right\} + v_0$

initial velocity of projection of the particle from the satellite w.r.t. earth. $v_{PE} = v_0 \cdot \sqrt{\frac{5}{4}}$ — ①

from conservation of mechanical energy b/w A & B

$K_A + U_A = U_B + K_B$

$\frac{1}{2}m \cdot \left(v_0\sqrt{\frac{5}{4}}\right)^2 + \left(-\frac{GMm}{a}\right) = \left(-\frac{GMm}{r_{max}}\right) + 0$

$\Rightarrow \frac{5}{8}v_0^2 = GM \cdot \left\{\frac{1}{a} - \frac{1}{na} \times 3\right\}$

$\Rightarrow \frac{5}{8} \cdot v_0^2 = \frac{GM}{a} \cdot \left(1 - \frac{3}{n}\right)$ — ②

$\because$ orbital speed at a distance 'a' from the center of planet

$v_0 = \sqrt{\frac{GM}{a}}$

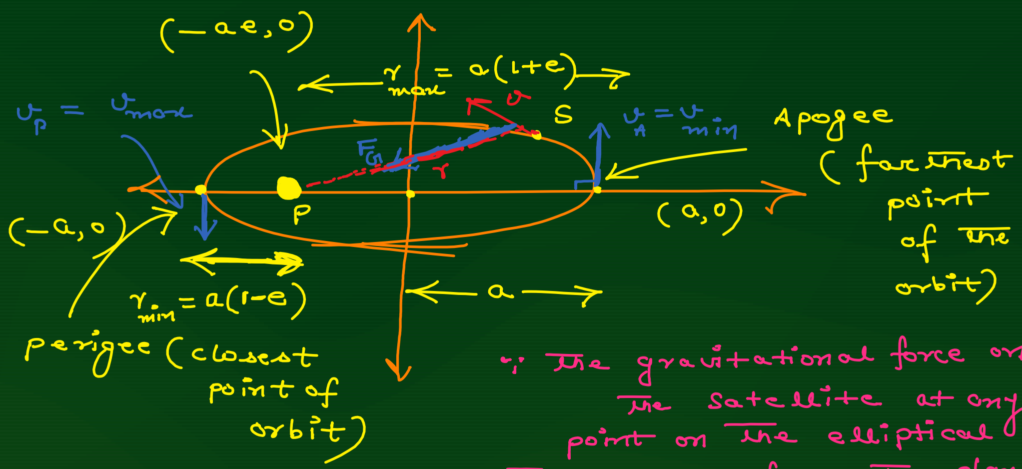
from eqn ②:

$\frac{5}{8} \times \cancel{\frac{GM}{a}} = \cancel{\frac{GM}{a}}\left(1 - \frac{3}{n}\right) \Rightarrow \frac{5}{8} = 1 - \frac{3}{n}$

$\Rightarrow \frac{3}{n} = 1 - \frac{5}{8} = \frac{3}{8}$

$\Rightarrow \boxed{n = 8}$

imp concept: if a satellite is in elliptical orbit then the planet will be at one of the 2 focii of the elliptical orbit.

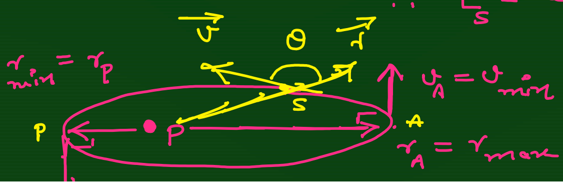


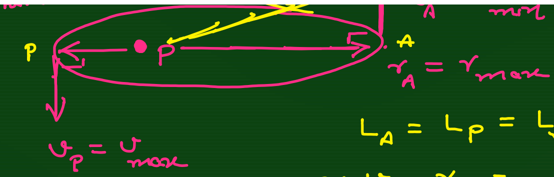
$\because$ the gravitational force on the satellite at any point on the elliptical path will pass from the planet

$\therefore$ this will provide no torque to satellite about the axis passing from the planet.

$\therefore \tau_S = 0$

$\therefore L_S = \text{const.}$ (ie: angular momentum of the satellite will remain conserve.)





$$L_A = L_P = L_S$$

$$m\cdot v_A\cdot r_A = m\cdot v_p\cdot r_p = m\cdot v\times r\times\sin\theta$$

$$v_A\cdot r_A = v_p\cdot r_p \Rightarrow v\times r = \text{const}$$

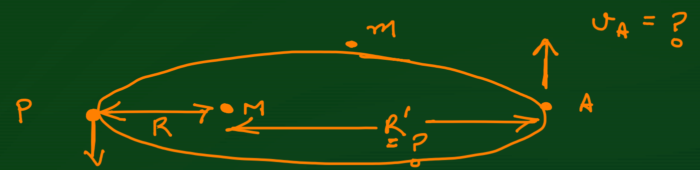
$$\downarrow v \propto \frac{1}{r}\uparrow$$

Note

*) in elliptical orbit, both the mechanical Energy as well as the angular momentum will remain conserved.

Q: A satellite of mass 'm' is in an elliptical orbit around any planet of mass M (where M >> m). The speed of the satellite at perigee is found $\sqrt{\frac{6GM}{5R}}$: here R is the distance of perigee from the planet. If the increase in speed of the satellite w.r.t. the speed at Apogee is $\sqrt{\frac{GM}{K\cdot R}}$ then find K.

Sol^n →



$v_p = \sqrt{\frac{6GM}{5R}}$ ∵ from C.O.A.M.

$$L_A = L_P$$

$$\Rightarrow m\cdot v_A\times R' = m\times v_p\times R$$

$$\Rightarrow R' = \frac{v_p\times R}{v_A} \quad \text{—(1)}$$

from C.O.M.E.

$$K_P + U_P = K_A + U_A$$

$$\frac{1}{2}mv_p^2 + \left(-\frac{GMm}{R}\right) = \frac{1}{2}mv_A^2 + \left(-\frac{GMm}{R'}\right)$$

$$\Rightarrow \frac{v_p^2}{2} - \frac{GM}{R} = \frac{v_A^2}{2} - \frac{GM\cdot v_A}{v_p\cdot R}$$

$$\Rightarrow \frac{\cancel{6}^{3}GM}{\cancel{2}\times 5R} - \frac{GM}{R} = \frac{v_A^2}{2} - \frac{GM}{R}\cdot\frac{\sqrt{5R}}{\sqrt{6GM}}\cdot v_A$$

$$\Rightarrow -\frac{2GM}{5R} = \frac{v_A^2}{2} - \sqrt{\frac{5GM}{6R}}\cdot v_A$$

$$\Rightarrow \frac{v_A^2}{2} - \sqrt{\frac{5GM}{6R}}\cdot v_A + \frac{2GM}{5R} = 0$$

$$\Rightarrow v_A = \frac{\sqrt{\frac{5GM}{6R}} \pm \sqrt{\frac{5GM}{6R} - \frac{4GM}{5R}}}{1}$$

$$\Rightarrow v_A = \left\{\sqrt{\frac{5GM}{6R}} \pm \sqrt{\frac{GM}{30R}}\right\}$$

change in speed $(\Delta v) = v_p - v_A$

$$= \sqrt{\frac{6GM}{5R}} - \left\{\sqrt{\frac{5GM}{6R}} - \sqrt{\frac{GM}{30R}}\right\}$$

$$= \sqrt{\frac{GM}{R}}\cdot\left\{\sqrt{\frac{6}{5}} - \sqrt{\frac{5}{6}} + \frac{1}{\sqrt{30}}\right\}$$

$$\{6 - 5 + 1\}$$

$$= \sqrt{\frac{GM}{30R}} \cdot \{6 - 5 + 1\}$$

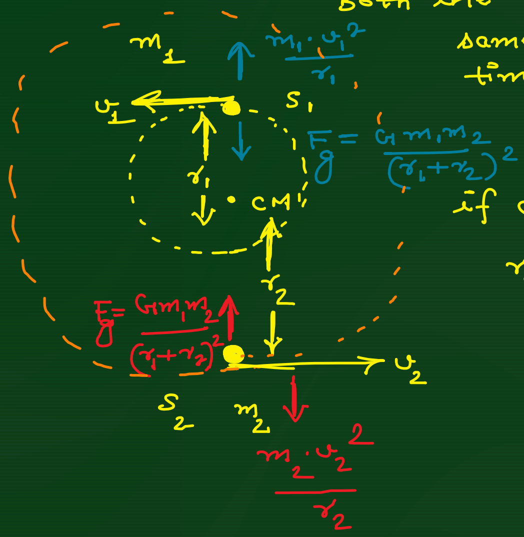
$$= 2 \cdot \sqrt{\frac{GM}{30R}}$$

$$\omega = \sqrt{\frac{4GM}{30R}}$$

$$\therefore K = \frac{30}{4} = 7.5 \quad \underline{\text{Ans.}}$$

Binary star system or double star system:

it is a pair of 2 stars which revolves around their common C.M.



Both the stars lie along a same line, with same time period & angular speed.

if C.M. is taken at origin

$$r_{CM} = 0 = \frac{m_1 \cdot r_1 + m_2 \cdot (-r_2)}{m_1 + m_2}$$

$$0 = m_1 r_1 - m_2 \cdot r_2$$

$$\boxed{m_1 r_1 = m_2 \cdot r_2} \Rightarrow \frac{r_1}{r_2} = \frac{m_2}{m_1} \quad \text{—— (1)}$$

$$\because \frac{Gm_1m_2}{(r_1+r_2)^2} = \frac{m_1 \cdot v_1^2}{r_1} = \frac{m_2 \cdot v_2^2}{r_2} \quad \text{—— (2)}$$

$$\because T = \frac{2\pi r}{v}$$

$$\therefore v = \frac{2\pi r}{T}$$

$$\text{from (2)}: \frac{m_1}{r_1} \cdot \frac{4\pi^2 r_1^2}{T_1^2} = \frac{m_2}{r_2} \cdot \frac{4\pi^2 r_2^2}{T_2^2}$$

$$\Rightarrow \frac{m_1 r_1}{T_1^2} = \frac{m_2 \cdot r_2}{T_2^2}$$

from eqn (1)

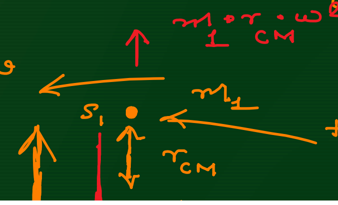
$$\frac{1}{T_1^2} = \frac{1}{T_2^2} \Rightarrow \boxed{T_1 = T_2}$$

$$\because \omega = \frac{2\pi}{T}$$

$$\Rightarrow \boxed{\omega_1 = \omega_2}$$

Time period of a Binary star system

$$\frac{2\pi}{T} = \omega$$



taking the S_1 at origin

$$r_{CM} = \frac{m_1 \cdot y_1 + m_2 \cdot y_2}{}$$

S_1 • ← taking the S_1 at origin

r_{CM}

• CM

$r_1 + r_2 = d$ $\qquad \frac{Gm_1m_2}{d^2}$

S_2 • m_2

$$r_{CM} = \frac{m_1 \cdot y_1 + m_2 \cdot y_2}{m_1 + m_2}$$

$$r_{CM} = \frac{m_1 \times 0 + m_2 \times d}{m_1 + m_2}$$

$$r_{CM} = \frac{m_2 \cdot d}{m_1 + m_2} \quad \text{—(1)}$$

position of C.M w.r.t. S_1 or radius of the circular orbit of S_1.

for S_1: $\frac{G\cancel{m_1}m_2}{d^2} = \cancel{m_1} \cdot r_{CM} \cdot \omega^2$

$$\Rightarrow \frac{Gm_2}{d^2} = \frac{m_2 \cdot d}{(m_1 + m_2)} \cdot \omega^2$$

$$\omega^2 = \frac{G(m_1 + m_2)}{d^3}$$

$$\omega = \sqrt{\frac{G(m_1 + m_2)}{d^3}} \quad \text{—(2)} \quad \text{rad/s}$$

ang. speed of each star

$\because T = \frac{2\pi}{\omega}$

$$\Rightarrow T = \frac{2\pi \times d^{3/2}}{\sqrt{G(m_1 + m_2)}} \quad \text{—(3)}$$

time period each star

$T \propto d^{3/2} \Rightarrow T^2 \propto d^3$