Higher Order Derivatives:

$y = f(x)$

$1^{st} \Rightarrow \frac{dy}{dx} = f'(x) = y_1 = y'$

$2^{nd} \Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = y_2 = y'' = f''(x)$

$3^{rd} \Rightarrow \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = f'''(x) = y_3 = y'''$

n^{th} Derivative : $\frac{d^ny}{dx^n} = f^{\overbrace{'''''}^{n\text{ times}}}(x) = y_n = y^{\overbrace{'''''}^{n\text{ times}}}$

e.g $f(x) = \sin x = y$

find y_3

$\Rightarrow y_1 = \cos x$

$\Rightarrow y_2 = -\sin x$

$\boxed{y_3 = -\cos x}$

Q) If $e^y(x+1) = 1$ prove that $y_2 = y_1^2$

$\frac{d}{dx}\left(e^y(x+1)\right) = 0$ ✓

$\Rightarrow e^y \frac{dy}{dx}(x+1) + e^y \cdot 1 = 0$

$\Rightarrow \frac{dy}{dx}(x+1) + 1 = 0$

$y_1 = \frac{dy}{dx} = \frac{-1}{x+1}$

Diff wrt x

$y_2 = \frac{1}{(1+x)^2} = \left(\frac{-1}{1+x}\right)^2$

$\boxed{y_2 = y_1^2}$

$\left(\frac{dy}{dx}\right)^2 = \left(f'(x)\right)^2$ ✓

$\left(\frac{d^2y}{dx^2}\right) = f''(x)$ ✓

$\frac{d^2y}{d^2x} =$ ✗

$\left(\frac{d^2y}{dx^2}\right)^2 = \left(f''(x)\right)^2$

$\frac{d^n(y)}{dx^n} \rightarrow n^{th}$ Derivative

Q) $y^{1/m} = \left(x + \sqrt{1+x^2}\right)$ P.T $(1+x^2)y_2 + xy_1 = m^2y$

$\Rightarrow y = \left(x + \sqrt{1+x^2}\right)^m$

Diff w.r.t x

$\Rightarrow y_1 = m\left(x+\sqrt{1+x^2}\right)^{m-1}\left(1 + \frac{\cancel{2}x}{\cancel{2}\sqrt{1+x^2}}\right)$

$y_1 = m\left(x+\sqrt{1+x^2}\right)^{m-1}\frac{\left(\sqrt{1+x^2}+x\right)^1}{\sqrt{1+x^2}}$

$y_1\sqrt{1+x^2} = my$

Diff w.r.t x

$y_2\sqrt{1+x^2} + y_1 \frac{\cancel{2}x}{\cancel{2}\sqrt{1+x^2}} = my_1$

$\Rightarrow y_2(1+x^2) + y_1 x = m\, y_1\sqrt{1+x^2}$

$y_2(1+x^2) + y_1 x = m.my = m^2y$

Q) If $y = \cos^{-1}x \longrightarrow x = \cos y \longrightarrow$ Diff w.r.t y

find $y_2 = \frac{d^2y}{dx^2} = -\cot y \,\mathrm{cosec}^2 y$

$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}} = -1(1-x^2)^{-1/2}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-(1-x^2)^{-1/2}\right)$

$= \frac{1}{\cancel{2}}(1-x^2)^{-3/2}(-\cancel{2}x) = \frac{-x}{(1-x^2)^{3/2}} \Rightarrow \frac{-\cos y}{(1-\cos^2 y)^{3/2}} = \frac{-\cos y}{\sin^3 y} = -\cot y \cdot \mathrm{cosec}^2 y$

$\frac{dx}{dy} = -\sin y$

$\frac{dy}{dx} = -\mathrm{cosec}\, y$

Diff wrt y

$\boxed{\frac{d}{dy}\left(\frac{dx}{dy}\right) = -\cos y}$

$\frac{d^2x}{dy^2} = -\cos y$

$\frac{d}{dx}\left(\frac{dy}{dx}\right) = -\frac{d}{dx}(\mathrm{cosec}\, y)$

$\boxed{= \mathrm{cosec}\, y \cot y \left(\frac{dy}{dx}\right)}$

$\boxed{\frac{d^2y}{dx^2} = -\mathrm{cosec}^2 y \cdot \cot y}$

Higher order Derivative for Parametric functions:

① $y = f(t), \quad x = g(t)$

$$\Rightarrow \frac{dy}{dx} = \frac{f'(t)}{g'(t)}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{f'(t)}{g'(t)}\right)\cdot\frac{dt}{dt}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{f'(t)}{g'(t)}\right)\frac{dt}{dx}$$

② e.g. $x = at^2, \; y = 2at$ $\qquad \frac{dy}{dt} = 2a, \quad \frac{dx}{dt} = 2at$

$$\frac{dy}{dx} = \frac{2a}{2at}$$

$$\frac{dy}{dx} = \frac{1}{t}$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(t^{-1}\right)$$

$$= -t^{-2}\frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{-1}{t^2}\cdot\frac{1}{2at} = \frac{-1}{2at^3}$$

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{-1}{2at^3}\right)\frac{dt}{dt}$$

$$= \frac{d}{dt}\left(\frac{-1}{2a}t^{-3}\right)\frac{dt}{dx}$$

$$= \frac{3t^{-4}}{2a}\cdot\frac{1}{2at}$$

$$\frac{d^3y}{dx^3} = \frac{3}{4a^2t^5}$$

Q) $x = \phi(t), \; y = \psi(t)$ $\qquad \frac{dx}{dt} = \phi'(t) \qquad \frac{dt}{dx} = \frac{1}{\phi'(t)}$

then prove that $\frac{d^2y}{dx^2} = \frac{\phi'\psi'' - \psi'\phi''}{(\phi')^3}$

$$\frac{dy}{dx} = \frac{\psi'(t)}{\phi'(t)}$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{\psi'(t)}{\phi'(t)}\right)\cdot\frac{dt}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{\psi'(t)}{\phi'(t)}\right)\cdot\frac{dt}{dx}$$

$$= \left(\frac{\phi'(t)\psi''(t) - \phi''(t)\psi'(t)}{(\phi'(t))^2}\right)$$

$$= \frac{\phi'\psi'' - \phi''\psi'}{(\phi')^3}$$

H.w

If $x = a\left(\cos\theta + \log\tan\frac{\theta}{2}\right)$

and $y = a\sin\theta$

find $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{4}$.

Differentiation of Inverse functions:

If $f(x)$ & $g(x)$ are Inverse of each other;

$$f(g(x)) = x = g(f(x))$$

$$g(x) = f^{-1}(x)$$

Diff. wrt x.

$$\underbrace{f'(g(x))\, g'(x)}_{\text{Chain Rule}} = 1 \Rightarrow \boxed{g'(x) = \frac{1}{f'(g(x))}}$$

Q) $f(x) = x^3 + e^{x/2}$ and $g(x) = f^{-1}(x)$

$x=0$ then find $g'(1)$.

A) 0 C) $1/2$

B) 1 D) 2 ✓

$$f'(x) = 3x^2 + \frac{1}{2}e^{x/2}$$

$$f'(0) = 0 + \frac{1}{2}$$

$$g'(x) = \frac{1}{f'(g(x))}$$

$x=1$

$$g'(1) = \frac{1}{f'(g(1))} = \frac{1}{f'(0)} = \frac{1}{1/2} = 2$$

$$\Rightarrow \boxed{f(0) = 0+1 = 1}$$

$$\Rightarrow g(1) = 0$$

Q) If g is the inverse of function f

and $f'(x) = \dfrac{1}{1+x^5}$ then $g'(x)$ is equals to

A) $1+x^5$

B) $5x^4$

C) $\dfrac{1}{1+(g(x))^5}$

D) $1+(g(x))^5$ ✓

$$f'(g(x)) = \frac{1}{1+(g(x))^5}$$

$$\Rightarrow g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\frac{1}{1+(g(x))^5}} = (1+g(x))^5$$