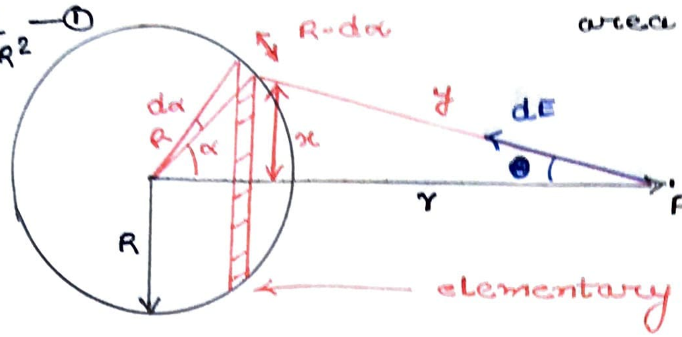
Gravitational Field due to a thin spherical shell

case 1: outside the shell:

M ; $\sigma = \frac{M}{4\pi R^2}$ —①



area of the elementary Ring

$dA = 2\pi x \cdot R\cdot d\alpha$

$dA = 2\pi R^2 \sin\alpha \, d\alpha$ —②

so mass of the element

$dm = \sigma \cdot dA$

$\Rightarrow dm = \frac{M \cdot \sin\alpha \cdot d\alpha}{2}$ —③

field at point P due to the considered element

$$dE_x = \frac{G\,dm\cdot r}{(r^2+x^2)^{\frac{3}{2}}} = \frac{G\,dm}{(\sqrt{r^2+x^2})^2}\cdot\frac{r}{\sqrt{r^2+x^2}}$$

$$\Rightarrow dE_x = \frac{G\,dm\cdot\cos\theta}{y^2}$$

from ③: $dE_x = \frac{GM\cdot\sin\alpha\cdot d\alpha\cdot\cos\theta}{2y^2}$ —④

$\because \cos\alpha = \frac{R^2+r^2-y^2}{2R\cdot r}$

$\Rightarrow y^2 = R^2+r^2-2R\cdot r\cdot\cos\alpha$

Differentiating both sides;

$2y\cdot dy = 0+0-2\cdot R\cdot r\cdot(-\sin\alpha\cdot d\alpha)$

so $\sin\alpha\cdot d\alpha = \frac{y\cdot dy}{r\cdot R}$ —⑤

from ④ & ⑤:

$$dE_x = \frac{GM}{2y^2}\cdot\frac{y\cdot dy}{r\cdot R}\cdot\cos\theta = \frac{GM}{2yr\cdot R}\cdot dy\cdot\cos\theta \quad —⑥$$

$\because \cos\theta = \frac{r^2+y^2-R^2}{2ry}$

$$\therefore dE_x = \frac{GM\cdot dy}{2yr\cdot R}\cdot\left(\frac{r^2+y^2-R^2}{2ry}\right)\cdot dy$$

$$\Rightarrow dE_x = \frac{GM}{4r^2R}\cdot\left[\frac{y^2+r^2-R^2}{y^2}\right]\cdot dy$$

$$\Rightarrow \int_0^{E_{out}} dE_x = \frac{GM}{4r^2R}\cdot\int_{r-R}^{r+R}\left[1+\frac{(r^2-R^2)}{y^2}\right]\cdot dy$$

$$\Rightarrow (E_x)_0^{E_{out}} = \frac{GM}{4r^2R}\cdot\left[y-\frac{(r^2-R^2)}{y}\right]_{r-R}^{r+R}$$

$$\Rightarrow (E_x - 0) = \frac{GM}{4r^2R}\cdot\left[\left\{(r+R) - \frac{(r^2-R^2)}{(r+R)}\right\} - \left\{(r-R) - \frac{(r^2-R^2)}{(r-R)}\right\}\right]$$

$$= \frac{GM}{4r^2R}\cdot\left[\left\{(r+R) - (r-R)\right\} - \left\{(r-R) - (r+R)\right\}\right]$$

$$= \frac{GM}{4r^2 R}\cdot\left[\cancel{r} + R - \cancel{r} + R - \cancel{r} + R + \cancel{r} + R\right]$$

$$= \frac{GM}{4r^2 R}\cdot 4R$$

$$\therefore E_{out} = \frac{GM}{r^2}\ m.s^{-2} \quad (\text{at } r > R) \quad —①^*$$

case 2: on the surface;

$$r = R$$

$$\text{so } E_S = \frac{GM}{R^2}\ m s^{-2} \quad (\text{at } r = R) \quad —②^*$$

case ③: inside the shell: integration will be done from;

$$L_1 = (R-r) \text{ to } L_2 = (R+r)$$

$$\int_0^{E_x} dE_x = \frac{GM}{4r^2R}\cdot\int_{(R-r)}^{(R+r)} \left\{1 + \frac{(r^2-R^2)}{y^2}\right\}\cdot dy$$

$$\Rightarrow (E_x)_0^{E_x} = \frac{GM}{4r^2R}\cdot\left[y - \frac{(r^2-R^2)}{y}\right]_{R-r}^{R+r}$$

$$= \frac{GM}{4r^2R}\cdot\left[\left\{(R+r) - \frac{(r^2-R^2)}{(R+r)}\right\} - \left\{(R-r) - \frac{(r^2-R^2)}{(R-r)}\right\}\right]$$

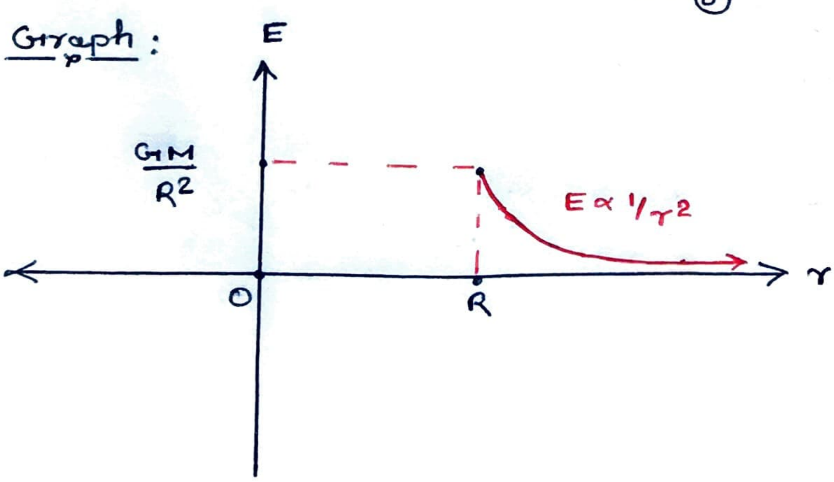
$$= \frac{GM}{4r^2R}\cdot\left[\left\{(R+r) - (r-R)\right\} - \left\{(R-r) + (R+r)\right\}\right]$$

$$= \frac{GM}{4r^2R}\cdot\left[\cancel{R}+\cancel{r} - \cancel{r} + \cancel{R} - \cancel{R} + \cancel{r} - \cancel{R} - \cancel{r}\right]$$

$$= \frac{GM}{4r^2R}\cdot 0$$

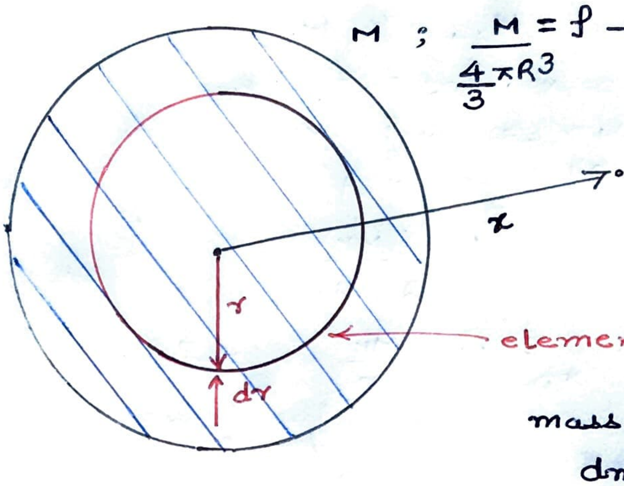
$$E_{in} = 0\ m s^{-2} \quad (\text{at } r < R) \quad —③^*$$

Graph :



Gravitational field due to uniform solid sphere.

M ; $\frac{M}{\frac{4}{3}\pi R^3} = \rho$ — ⊛



elementary shell of thickness dr & radius r.

mass of the elementary shell

$dm = \rho . dv$

$\Rightarrow dm = \frac{M}{\frac{4}{3}\pi R^3} . 4\pi r^2 . dr$ — ①

gravitational field due to the shell.

$$dE = \frac{G\,dm}{x^2} = \frac{3G . r^2 . dr . M}{R^3 \; x^2}$$

case 1: outside the shell : $x > R$

$$\int_0^{E_{out}} dE = \frac{3GM}{R^3 . x^2} . \int_0^R r^2 . dr$$

$$\Rightarrow (E)_0^{E_{out}} = \frac{3GM}{R^3 . x^2} . \left(\frac{r^3}{3}\right)_0^R$$

$$\Rightarrow (E_{out} - 0) = \frac{3GM}{R^3 . x^2} . \left(\frac{R^3}{3} - 0\right)$$

$$\therefore E_{out} = \frac{GM}{x^2} \; ; \; (x > R)$$

— ②

case ②: on the surface : $x = R$

$$\int_0^{E_S} dE = \frac{3GM}{R^3 \times R^2} . \int_0^R r^2 . dr$$

$$\Rightarrow E_S = \frac{3GM}{R^5} . \frac{R^3}{3}$$

$$\therefore E_S = \frac{GM}{R^2} \text{ — ③ } \quad (x = R)$$

$= max.$

case 3: inside the sphere : $x < R$

$$\int_0^{E_{in}} dE = \int_0^x \frac{3GM}{R^3 . x^2} . r^2 . dr$$

$$(E)_0^{E_{in}} = \frac{3GM}{R^3 . x^2} . \left(\frac{r^3}{3}\right)_0^x$$

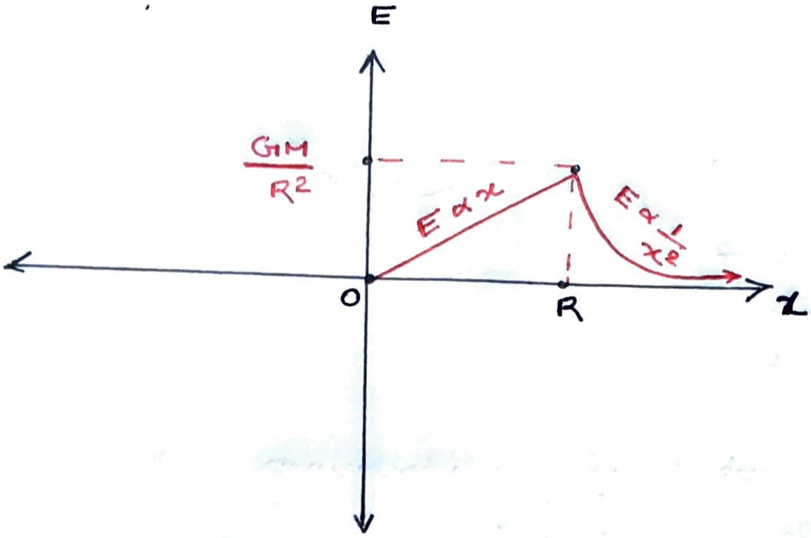
$$E_{in} = \frac{3GM}{R^3 . x^2} . \frac{x^3}{3}$$

$$\Rightarrow E_{in} = \frac{GMx}{R^3} \quad \text{— ④} \quad (x<R)$$

at the center; $x=0$

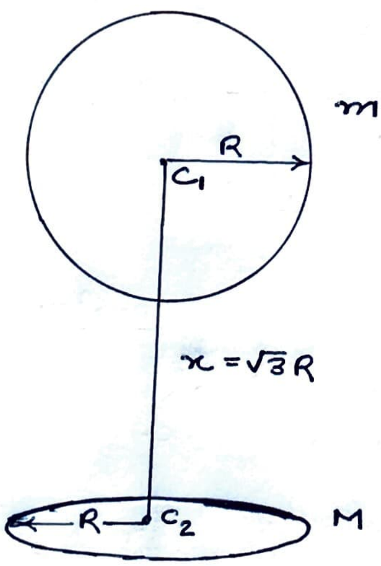
Graph:

$$(E_{in})_{center} = 0 \quad \text{— ⑤} \quad (x=0)$$



Eg: A uniform Ring of mass M is lying at a distance $\sqrt{3}R$ from the center of a uniform sphere of mass m just below the sphere as shown in the figure, radius of both the ring & sphere is R. The gravitational force exerted by the sphere on the ring is $K\frac{GMm}{R^2}$. Find K.

Solⁿ:→



as gravitational force follows the Newton's 3rd Law

force applied by sphere on ring = force applied by the ring on sphere.

$$F = m \times E_{C_1}$$

$$= m . \frac{GMx}{[R^2+x^2]^{\frac{3}{2}}}$$

$$= \frac{GMm . \sqrt{3}R}{[R^2+3R^2]^{\frac{3}{2}}}$$

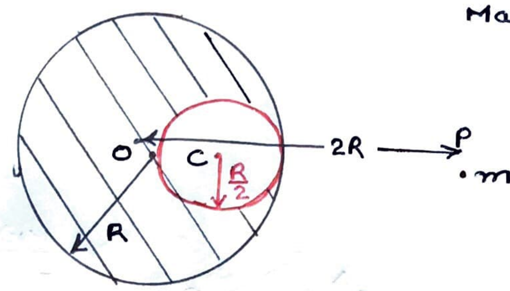
$$= \frac{\sqrt{3}\, GMm . R}{(2\sqrt{2})^2 R^3}$$

$$\Rightarrow F = \frac{\sqrt{3}}{8} \frac{GMm . R}{R^3} = \frac{\sqrt{3}}{8} \frac{GMm}{R^2}$$

so $K = \frac{\sqrt{3}}{8}$

Eg: find the Gravitational field interaction b/w the sphere and the point mass. The mass of the sphere before the cavity was made, was M.

Sol^n:



Mass of the complete sphere is M

mass of the removed portion

$$m_2 = \rho \times V_{cavity}$$

$$= \frac{M}{\frac{4}{3}\pi R^3} \times \frac{4}{3}\pi\left(\frac{R}{2}\right)^3$$

$$m_2 = \frac{M}{8} \quad \text{—①}$$

so mass of the remaing portion

$$m_1 = M - m_2 = \frac{7M}{8}$$

Net force on the point mass Due to the given sphere of remaining Mass (F) = Force due to complete sphere − force due to removed sphere

$$= m \cdot E_{complete} - m \cdot E_{removed}$$

$$= m \cdot \frac{GM}{(2R)^2} - m \cdot \frac{Gm_2}{\left(\frac{3R}{2}\right)^2}$$

$$= \frac{GMm}{4R^2} - \frac{4}{9} G \frac{M}{8} \cdot \frac{m}{R^2}$$

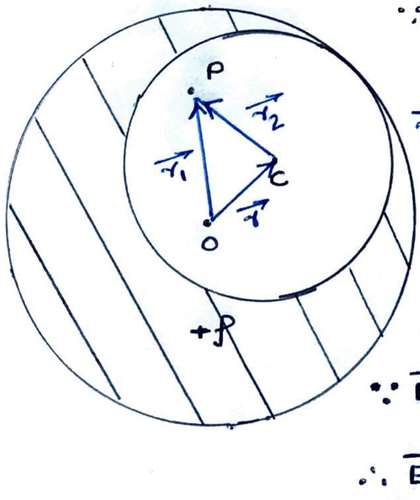
$$= \frac{GMm}{R^2} \cdot \left[\frac{1}{4} - \frac{1}{18}\right]$$

$$= \frac{GMm}{R^2} \cdot \left(\frac{9-2}{36}\right)$$

$$F = \frac{7}{36} \cdot \frac{GMm}{R^2}$$

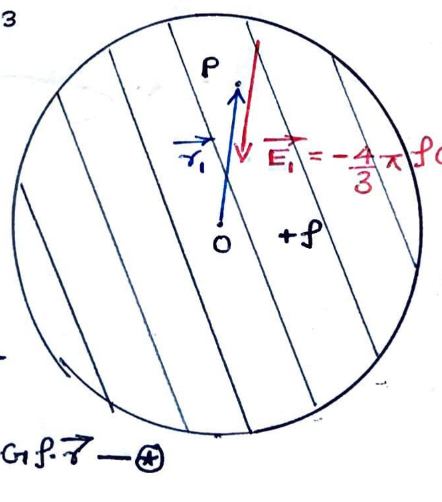
Gravitational Field inside a cavity of sphere:

$$\because \rho = \frac{M}{\frac{4}{3}\pi R^3}$$



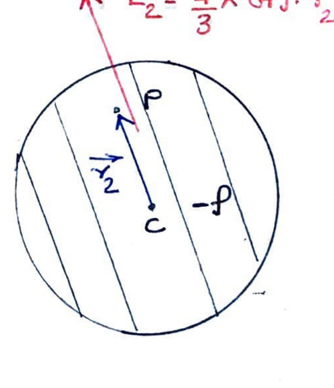
$$\vec{r} = \vec{r_1} - \vec{r_2}$$

=



$$\vec{E_1} = -\frac{4}{3}\pi\rho G\vec{r_1}$$

+



$$\vec{E_2} = \frac{4}{3}\pi G\rho\cdot\vec{r_2}$$

$$\because \vec{E}_{in} = \frac{-GM\cdot\vec{r}}{R^3}$$

$$\therefore \vec{E}_{in} = -\frac{4}{3}\pi G\rho\cdot\vec{r} \quad \text{—} \circledast$$

so $\vec{E} = \vec{E_1} + \vec{E_2}$

$$\Rightarrow \vec{E} = \frac{4}{3}\pi\rho G\cdot\left[-\vec{r_1} + \vec{r_2}\right] = -\frac{4}{3}\pi\rho G\vec{r}$$

so $\vec{E}_{cavity} = -\frac{4}{3}\pi G\rho\cdot\vec{r} = \text{const}$

ie, const throught the cavity