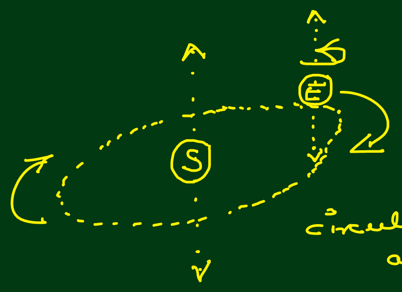
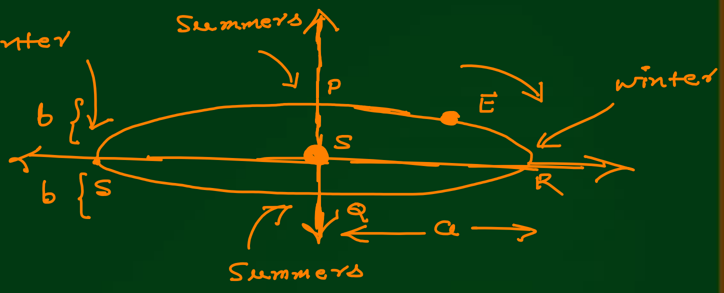
Kepler's Laws

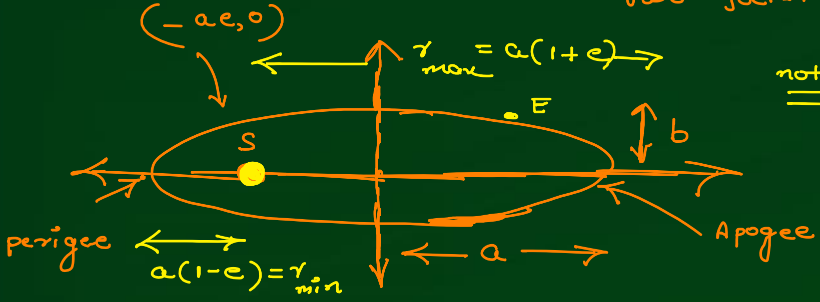


circular orbits are not possible



Sun cannot be present at the center of the elliptical orbit.

1st Law (Law of orbit) :→ Earth Revolves around the sun in an elliptical orbit havin sun at one of its two foci.



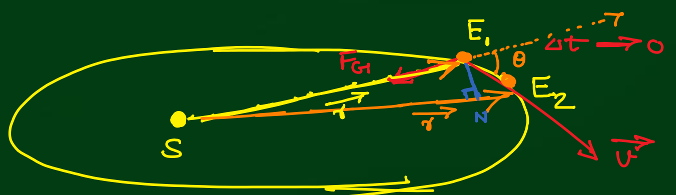
note: if elliptical orbit is compared with a circular orbit the radius of the circular orbit will taken equal to the semi-major axis of the ellipse.

$r_{av} = a$.

Most imp

II Law (Law of Area) :→ According to this law the line joining the Earth & sun sweeps equal area in equal time intervals.

ie: the areal velocity $\left(\frac{d\vec{A}}{dt}\right)$ of earth around sun is constant.



$E_1E_2 = \Delta l = v \times \Delta t$

$\therefore E_1 \cdot N = \Delta l \sin\theta$

or

$\Rightarrow E_1N = v \cdot \Delta t \cdot \sin\theta$ —— ①

in $\Delta S E_1 N$

$\Delta A = \frac{1}{2} \times SE_1 \times E_1N$

area swept $\quad \Delta A = \frac{1}{2} \cdot r \cdot v \cdot \Delta t \cdot \sin\theta$ —— ②

$\Delta A = \frac{1}{2} \ldots$

area swept by the radius vector in Δt time $\quad \Delta A = \frac{1}{2}\cdot r\cdot v\cdot \Delta t\cdot \sin\theta$ —— (2)

$$\Rightarrow \frac{\Delta A}{\Delta t} = \frac{1}{2}\cdot r\cdot v\cdot \sin\theta$$

$$\Rightarrow \lim_{\Delta t\to 0} \frac{\Delta A}{\Delta t} = \frac{1}{2}\cdot r\cdot v\cdot \sin\theta$$

$\because$ F_G will pass from sun

$\therefore$ $\tau = 0$

$\Rightarrow$ $L = \text{const}$

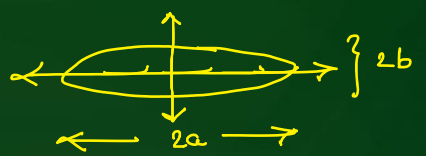
$$\Rightarrow \frac{dA}{dt} = \frac{1}{2} r\cdot v\cdot \sin\theta$$

$$\Rightarrow \frac{dA}{dt} = \frac{m\cdot v\cdot r\cdot \sin\theta}{2m} = \frac{p\cdot r\cdot \sin\theta}{2m}$$

areal speed of Earth $\quad \boxed{\frac{dA}{dt} = \frac{L}{2m} = \text{const}}$

Eg: Suppose a planet is revolving around the sun in an elliptical orbit given by eqn. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Find the time period of revolution in terms of the angular momentum L of the planet about sun.

Soln :→



$$\because \frac{dA}{dt} = \frac{L}{2m}$$

$A = \pi ab$

$$\Rightarrow dA = \frac{L}{2m}\cdot dt$$

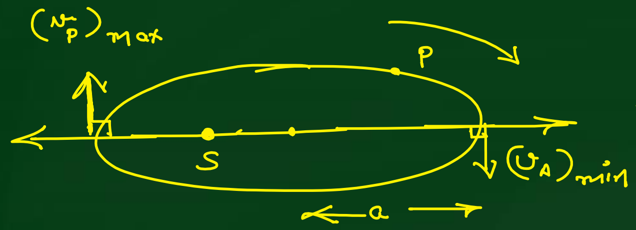
$$\Rightarrow \int_0^{\pi ab} dA = \frac{L}{2m}\cdot \int_0^{T} dt$$

$$\Rightarrow (A)_0^{\pi ab} = \frac{L}{2m}\cdot (t)_0^{T}$$

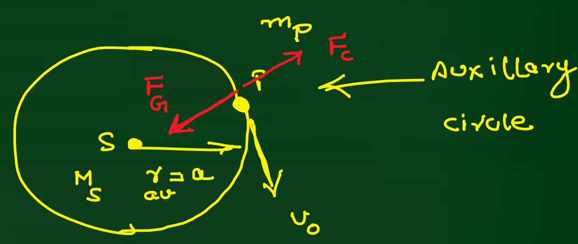
$$\therefore \pi ab = \frac{L}{2m}\cdot T$$

$$\therefore T = \frac{2\pi m ab}{L}$$

III Law (Law of Period) :→ According to this law the square of the time period of any planet around sun is directly proportional to cube of the average distance b/w them.



$\Rightarrow$



$F_G = F_C$

$$\frac{G M_S \cdot \cancel{m_P}}{a^2} = \frac{\cancel{m_P}\cdot v_0^2}{\cancel{a}}$$

Average

$$\Rightarrow v_0 = \sqrt{\frac{GM_S}{a}} \quad \text{--- (1)} \quad : \text{orbital speed}$$

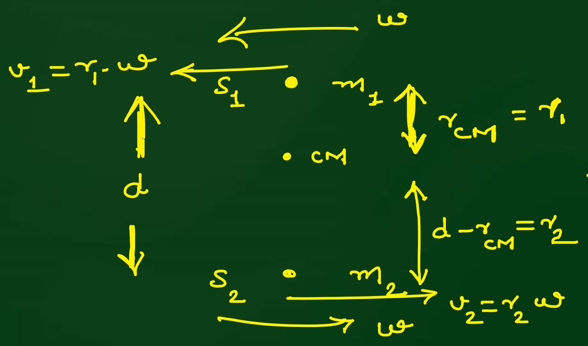
$$\because \quad T = \frac{2\pi r_{cir}}{v_0} = \frac{2\pi \cdot a}{\sqrt{\frac{GM_S}{a}}}$$

$$\therefore \quad T = \frac{2\pi}{\sqrt{GM_S}} \cdot a^{3/2} \quad \text{--- (2)} \quad : \text{time period of the planet}$$

$$\Rightarrow \quad T \propto a^{3/2}$$

$$\Rightarrow \quad T^2 \propto a^3$$

Extra Topic: Total Energy of a Binary star system



$$r_{CM} = \frac{m_2 \cdot d}{(m_1 + m_2)} \quad ; \quad \omega = \sqrt{\frac{G(m_1+m_2)}{d^3}} \text{ rad/s}$$

total mechanical energy of the system

$$E = K_1 + K_2 + U_{12}$$

$$= \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 \cdot v_2^2 - \frac{G m_1 m_2}{d}$$

$$= \frac{1}{2} m_1 \cdot r_{CM}^2 \cdot \omega^2 + \frac{1}{2} m_2 \cdot (d - r_{CM})^2 \cdot \omega^2 - \frac{G m_1 m_2}{d}$$

$$= \frac{1}{2} m_1 \cdot \frac{m_2^2 \cdot \cancel{d^2}}{(m_1+m_2)^{\cancel{2}}} \times \frac{G \cancel{(m_1+m_2)}}{d^{\cancel{3}}}$$

$$+ \frac{1}{2} \cdot m_2 \cdot \frac{m_1^2 \cdot \cancel{d^2}}{(m_1+m_2)^{\cancel{2}}} \times \frac{G \cancel{(m_1+m_2)}}{d^{\cancel{3}}} - \frac{G m_1 m_2}{d}$$

$$= \frac{G m_1 m_2^2}{2d(m_1+m_2)} + \frac{G m_2^2 m_1}{2d(m_1+m_2)} - \frac{G m_1 m_2}{d}$$

$$= \frac{G m_1 m_2 \cdot \cancel{(m_2 + m_1)}}{2d \cancel{(m_1+m_2)}} - \frac{G m_1 m_2}{d}$$

$$E = -\frac{G m_1 m_2}{2d} \text{ Joule.}$$