16. Find a point on the curve $x^2 + 2y^2 = 6$ whose distance from the line $x + y = 7$ is minimum.

[IIT-JEE, 2003]

17. Tangent to the curve $y = x^2 + 6$ at a point $P(1, 7)$ touches the curve $x^2 + y^2 + 16x + 12y + c = 0$ at a point Q. Then the co-ordinates of Q are

(a) $(-9, -13)$ (b) $(-10, -15)$

(c) $(-6, -7)$ (d) $(6, -7)$

[IIT-JEE, 2005]

18. If a function $f(x)$ satisfies the condition $|f(x) - f(y)| \le (x - y)^2$, $\forall x, y \in R$. Find an equation of tangent to the curve $y = f(x)$ at the point (1, 2).

[IIT-JEE, 2005]

19. No questions asked in 2006.

20. The tangent to the curve $y = e^x$ drawn at the point (c, e^c) intersects the line joining the points $(c - 1, e^{c-1})$, $(c + 1, e^{c+1})$

(a) on the left of $x = c$

(b) on the right of $x = c$

(c) at no point

(d) at all points.

[IIT-JEE, 2007]

21. No questions asked in between 2008 to 2010.

22. Tangents are drawn to the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ parallel to the straight line $2x - y = 1$, The points of contact to the tangent and the hyperbola are

(a) $\left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ (b) $\left(-\frac{9}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

(c) $(3\sqrt{3}, -2\sqrt{2})$ (d) $(-3\sqrt{3}, 2\sqrt{2})$

[IIT-JEE, 2011]

23. No questions asked in between 2012 to 2013.

24. The slope of the tangent to the curve $(y - x^5)^2 = x(1 + x^2)^2$ at the point (1, 3) is...

[IIT-JEE, 2014]

ANSWERS

LEVEL II

1. (a)	2. (b)	3. (a)	4. (a)	5. (b)
6. (a)	7. (a)	8. (d)	9. (d)	10. (c)
11. (d)	12. (a)	13. (b)	14. (a)	15. (c)
16. (a)	17. (c)	18. (b)	19. (c)	20. (c)
21. (a)	22. (c)	23. (a)	24. (b)	25. (c)
26. (b)	27. (d)	28. (a)	29. (c)	30. (a)
31. (b)	32. (c)	33. (c)	34. (c)	35. (b)
36. (d)	37. (b,d)	38. (c)	39. (b)	40. (a)
41. (c)	42. (d)	43. (d)	44. (a)	45. (b)
46. (d)	47. (d)	48. (b)	49. (b)	50. (c)
51. (a,b,c)	52. (c,c)	53. (a,b,c)	54. (a,c)	55. (a,c)

INTEGER TYPE QUESTIONS

1. 1	2. 2	3. 3	4. 1	5. 5
6. 4	7. 2	8. 4	9. 4	10. 5
11. 5	12. 6	13. 4	14. 5	15. 5

COMPREHENSIVE LINK PASSAGES

Passage I:	1. (b),	2. (a),	3. (a)
Passage II:	1. (c),	2. (c),	3. (a)
Passage III:	1. (a),	2. (b),	3. (a)
Passage IV:	1. (c),	2. (b),	3. (b)
Passage V:	1. (a),	2. (c),	3. (a)
Passage VI:	1. (b),	2. (a),	3. (a)

MATRIX MATCH

1. (A)→(Q), (B)→(P), (C)→(S), (D)→(R)
2. (A)→(P), (B)→(Q), (C)→(S), (D)→(R).
3. (A)→(S), (B)→(R), (C)→(Q), (D)→(P).
4. (A)→(Q), (B)→(Q), (C)→(R), (D)→(S).
5. (A)→(S), (B)→(R), (C)→(Q), (D)→(P).
6. (A)→(P), (B)→(Q), (C)→(R), (D)→(S).
7. (A)→(P), (B)→(Q), (C)→(R), (D)→(R).

HINTS AND SOLUTIONS

Level I

1. The given curve is

$$y = x^3 + 3x^2 + 3x - 10$$

$$\frac{dy}{dx} = 3x^3 + 6x + 3$$

Thus, $m = \left(\frac{dy}{dx}\right)_{x=2} = 3.4 + 6.2 + 3 = 27$

Hence, the slope of the tangent is 27.

2. The given curve is $y = x^x + 1$

$$\frac{dy}{dx} = x^x(\log x + 1)$$

Thus, $m = \left(\frac{dy}{dx}\right)_{x=2} = 2^2(\log 2 + 1) = 4(\log 2 + 1)$

Hence, the slope of the normal is $= -\frac{1}{4(\log 2 + 1)}$

3. The given curve is $y = \frac{ax}{b - x}$

Since the point (1, 1) lies on the curve,

so $1 = \dfrac{a}{b-1}$

$\Rightarrow \quad a = b - 1$

Also, $\dfrac{dy}{dx} = \dfrac{ab}{(b-x)^2}$

Now, $\left(\dfrac{dy}{dx}\right)_{(1,1)} = 2$

$\Rightarrow \quad \dfrac{ab}{(b-1)^2} = 2$

$\Rightarrow \quad a(a+1) = 2a^2$

$\Rightarrow \quad a^2 + a = 2a^2$

$\Rightarrow \quad a^2 = a$

$\Rightarrow \quad a = 0, 1$

when $a = 0, b = 1$

Then $a + b + 10 = 0 + 1 + 10 = 11$

when $a = 1, b = 2$

Then $a + b + 10 = 1 + 2 + 10 = 13.$

4. when $x = 0, y = 1$

Thus, the point is (0, 1)

Now, $\dfrac{dy}{dx} = 2.e^{2x}$

Thus, $m = \left(\dfrac{dy}{dx}\right)_{x=0} = 2.1 = 2$

Hence, the equation of the tangent is

$$y - 1 = 2(x - 0)$$

$\Rightarrow \quad y = 2x + 1$

5. when $x = 1, y = 0$

Thus, the point is (1, 0)

Now, $\dfrac{dy}{dx} = \dfrac{1}{3}(x-1)^{-\frac{2}{3}}$

Thus, $m = \left(\dfrac{dy}{dx}\right)_{x=1} = \infty$

Hence, the equation of the tangent to the curve is

$$y - 0 = \infty(x - 1)$$

$\Rightarrow \quad (x-1) = \dfrac{y}{\infty} = 0$

$\Rightarrow \quad x = 1$

6. The equation of the given curve is $y = be^{-x/a}$

put $x = 0$, then $y = b$

So, the point is $(0, b)$

Now, $\dfrac{dy}{dx} = -\dfrac{b}{a}e^{-x/a}$

Thus, $m = \left(\dfrac{dy}{dx}\right)_{(0,b)} = -\dfrac{b}{a}$

Hence, the equation of the tangent is

$$y - b = -\frac{b}{a}(x - 0)$$

$\Rightarrow \quad ay - ab = -bx$

$\Rightarrow \quad bx + ay = ab$

$\Rightarrow \quad \dfrac{bx}{ab} + \dfrac{ay}{ab} = \dfrac{ab}{ab}$

$\Rightarrow \quad \dfrac{x}{a} + \dfrac{y}{b} = 1$

7. when $x = 0, y = 2$

So, the point is (0, 2)

Now, $\dfrac{dy}{dx} = -\dfrac{8x}{(x^2+2)^2}$

Thus, $m = \left(\dfrac{dy}{dx}\right)_{x=0} = -\dfrac{8}{4} = -2$

Hence, the equation of the tangent is

$$y - 2 = -2(x - 0)$$

$\Rightarrow \quad y = 2 - 2x.$

8. The given curve is $x^3 + y^3 = 3xy$

$\Rightarrow \quad 3x^2 + 3y^2\dfrac{dy}{dx} = 6\left(x\dfrac{dy}{dx} + y\cdot 1\right)$

$\Rightarrow \quad (y^2 - 2x)\dfrac{dy}{dx} = (2y - x^2)$

$\Rightarrow \quad \dfrac{dy}{dx} = \left(\dfrac{2y - x^2}{y^2 - 2x}\right)$

Thus, $m = \left(\dfrac{dy}{dx}\right)_{(3,3)} = \dfrac{6-9}{9-6} = -1$

Hence, the equation of the normal is

$$y - 3 = 1(x - 3)$$

$\Rightarrow \quad x - y = 0$

9. when $x = 0, y = 14$

Hence, the point is (0, 14)

Now, $\dfrac{dy}{dx} = 6x + 2\cos x - 4\sin x$

Thus, $m = \left(\dfrac{dy}{dx}\right)_{x=0} = 2$

Hence, the equation of the normal is

$$y - 14 = -\frac{1}{2}(x - 0)$$

$\Rightarrow \quad 2y - 28 = -x$

$\Rightarrow \quad x + 2y = 28.$

10. The equation of the given curve is $x + y = x^y$ put $y = 0$, then $x = 1$.

So, the point is (1, 0)

Now, $x + y = x^y$

$\Rightarrow \log(x + y) = y \log x$

$\Rightarrow \frac{1}{(x+y)}\left(1 + \frac{dy}{dx}\right) = y \cdot \frac{1}{x} + \log x \cdot \frac{dy}{dy}$

put $x = 1, y = 0, \left(1 + \frac{dy}{dx}\right) = 0$

$\Rightarrow \left(\frac{dy}{dx}\right) = -1$

$\Rightarrow$ Slope of normal = 1

Hence, the equation of the normal is

$y - 0 = 1\ (x - 1)$

$\Rightarrow y = x - 1.$

11. The equation of the given curve is

$y = |x^2 - |x||$

$\Rightarrow y = |x^2 - (-x)|$ (since at $x = -2$, $|x| = -x$)

$\Rightarrow y = |x^2 + x|$

$\Rightarrow y = x^2 + x$ (at $x = -2$, $x^2 + x > 0$)

when $x = -2, y = 2$

So, the point is $(-2, 2)$

Now, $\frac{dy}{dx} = 2x + 1$

Thus, $m = \left(\frac{dy}{dx}\right)_{x=-2} = -3$

So, slope of the normal = 1/3

Hence, the equation of the normal is

$y - 2 = \frac{1}{3}\ (x + 2)$

$\Rightarrow 3y - 6 = x + 2$

$\Rightarrow 3y = x + 8$

12. The equation of the given curve is

$y^2 - 2x^3 - 4y + 8 = 0$

$\Rightarrow \frac{dy}{dx} = \frac{3x^2}{y-2}$

Let the point (α, β) lies on the curve

Thus, $m = \left(\frac{dy}{dx}\right)_{(\alpha,\beta)} = \frac{3\alpha^2}{\beta - 2}$

Therefore, the equation of the tangent at

(α, β) is $y - \beta = \left(\frac{3\alpha^2}{\beta-2}\right)(x - \alpha)$ ...(1)

which is passing through (1, 2)

So, $(2 - \beta) = \left(\frac{3\alpha^2}{\beta-2}\right)(1 - \alpha)$ ...(2)

Also, the point (α, β) lies on the curve

$y^2 - 2x^3 - 4y + 8 = 0$

So, $\beta^2 - 2\alpha^2 - 4\beta + 8 = 0$ ...(3)

From (2) and (3), we get,

$2(\alpha^3 - 2) = 3\alpha^2(\alpha - 1)$

$\Rightarrow \alpha^3 - 3\alpha^2 + 4 = 0$

$\Rightarrow \alpha = 2$

when $\alpha = 2, \beta = \pm 2\sqrt{3}$

Hence, the equation of the tangents are

$(y - (2 \pm 2\sqrt{3})) = \pm 2\sqrt{3}(x - 2)$

13. The equation of the given curve is $x^2 = 4y$

$\Rightarrow 2x = 4\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{x}{2}$

Let the point on the given curve be (α, β)

Now, $m = \left(\frac{dy}{dx}\right)_{(\alpha,\beta)} = \frac{\alpha}{2}$

Therefore the equation of normal at (α, β) is

$y - \beta = -\frac{2}{\alpha}(x - \alpha)$ ...(1)

which is passing through (1, 2).

So, $2 - \beta = -\frac{2}{\alpha}(1 - \alpha)$ ...(2)

Also, the point (α, β) lies on the curve

$\alpha^2 = 4\beta$ ...(3)

Solving (2) and (3), we get, $\alpha = 2, \beta = 1$

Hence, the equation of the normal is

$y - 1 = -\frac{2}{2}(x - 2)$

$\Rightarrow y - 1 = -x + 2$

$\Rightarrow x + y = 3.$

14. The equation of the given curve is $y = x - e^{xy}$

$\Rightarrow \frac{dy}{dx} = 1 - e^{xy}\left(y \cdot 1 + x \cdot \frac{dy}{dx}\right)$

$\Rightarrow (1 - xe^{xy})\frac{dy}{dx} = (ye^{xy} - 1)$

$\Rightarrow \frac{dy}{dx} = \frac{ye^{xy} - 1}{1 - xe^{xy}}$

For a vertical tangent, $\frac{dx}{dy} = 0$

$\Rightarrow 1 - xe^{xy} = 0$

$\Rightarrow x = 1, y = 0$

Therefore, the curve $y = x - e^{xy}$ has a vertical tangent at (1, 0).

15. The given curve is $x + y - \log(x + y) = 2x + 5$

$\Rightarrow 1 + \frac{dy}{dx} - \frac{1}{x+y}\left(1 + \frac{dy}{dx}\right) = 2$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{x + y + 1}{x + y - 1}$

Now, $\quad \left(\dfrac{dy}{dx}\right)_{(p,q)} = \dfrac{p + q + 1}{p + q - 1}$

Since, the tangent is vertical, so $\dfrac{dx}{dy} = 0$

$\Rightarrow \quad p + q = 1$

The value of $p + q + 10 = 11$.

16. Since the curve has horizontal tangent, so $\dfrac{dx}{dy} = 0$

$\Rightarrow \quad 6x^2 + 6x - 12 = 0$

$\Rightarrow \quad x^2 + x - 2 = 0$

$\Rightarrow \quad (x + 2)(x - 1) = 0$

$\Rightarrow \quad x = 1, -2$

When $\quad x = 1, y = 2 + 3 - 12 + 1 = -6$

So, the point is $(1, -6)$

when $x = -2$, $y = 16 + 12 - 26 + 1 = 3$

So, the point is $(-2, 3)$

Hence, the points are $(1, -6)$ and $(-2, 3)$.

17. Since the tangents are parallel to x-axis, so $\dfrac{dy}{dx} = 0$

$\Rightarrow \quad 3x^2 - 2x - 1 = 0$

$\Rightarrow \quad (3x + 1)(x - 1) = 0$

$\Rightarrow \quad x = 1, -1/3$

when $\quad x = 1, y = 1 - 1 - 1 + 3 = 2$

when $\quad x = -1/3, y = 70/27$

Hence, the points are $(1, 2)$ and $(-1/3, 70/27)$

18. The given curve is $y = \dfrac{x^3}{3} + \dfrac{x^2}{2}$

$\Rightarrow \quad \dfrac{dy}{dx} = 2x^2 + x$

Since the tangents make equal angles with the axes, so $\dfrac{dy}{dx} = \pm 1$

$\Rightarrow \quad 2x^2 + x = \pm 1$

$\Rightarrow \quad 2x^2 + x - 1 = 0, 2x^2 + x + 1 = 0$

$\Rightarrow \quad (2x - 1)(x + 1) = 0$

$\Rightarrow \quad x = 1/2, -1$

when $\quad x = 1/2, y = 5/24$

and when $x = -1$, $y = 1/6$

Hence, the points are $(1/2, 5/24)$ & $(-1, 1/6)$

19. The equation of the tangent to the curve

$y^2 = 4ax$ is $yy_1 = 2a(x + x_1)$

$\Rightarrow \quad y \cdot 2at = 2a(x + at^2)$

$\Rightarrow \quad yt = x + at^2$

20. The equation of the tangent to the curve $x^2 + y^2 + x + y = 0$ is

$xx_1 + yy_1 + \left(\dfrac{x + x_1}{2}\right) + \left(\dfrac{y + y_1}{2}\right) = 0$

$\Rightarrow \quad x \cdot 1 - y \cdot 1 + \left(\dfrac{x + 1}{2}\right) + \left(\dfrac{y - 1}{2}\right) = 0$

$\Rightarrow \quad 2x - 2y + x + 1 + y - 1 = 0$

$\Rightarrow \quad 3x - y = 0$

21. Equation of the normal to the curve

$x^2 + y^2 = 10$ is $\dfrac{x}{x_1} = \dfrac{y}{y_1}$

$\Rightarrow \quad \dfrac{x}{3} = \dfrac{y}{1}$

$\Rightarrow \quad x - 3y = 0$

22. The equation of the normal to the curve $x^2 + y^2 + 4x + 6y + 9 = 0$ at (x_1, y_1) is

$\dfrac{x - x_1}{x_1 + 2} = \dfrac{y - y_1}{y_1 + 3}$

$\Rightarrow \quad \dfrac{x + 4}{-4 + 2} = \dfrac{y + 3}{-3 + 3}$

$\Rightarrow \quad y + 3 = 0.$

23. Equation of the tangent to the curve is

$\dfrac{xx_1}{9} + \dfrac{yy_1}{4} = 2$

$\Rightarrow \quad \dfrac{3x}{9} + \dfrac{2y}{4} = 2$

$\Rightarrow \quad \dfrac{x}{3} + \dfrac{y}{2} = 2$

$\Rightarrow \quad 2x + 3y = 6$

Thus, slope of the tangent is $= -2/3$

Therefore, the slope of the normal is $= 3/2$

Equation of the normal to the curve at $(3, 2)$ is

$(y - 2) = \dfrac{3}{2}(x - 3)$

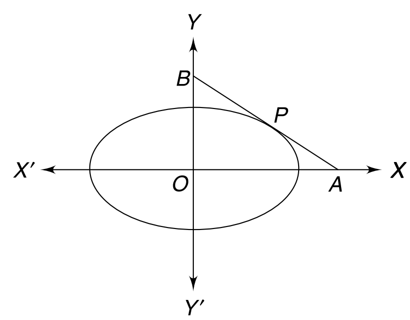
$\Rightarrow \quad 2y - 4 = 3x - 9$

$\Rightarrow \quad 3x - 2y = 5.$

24. Clearly, the point $(1, 2)$ lies outside of the curve $y^2 - 2x^2 - 4y + 8 = 0$. (as $4 - 2 - 4 + 8 = 12 - 6 = 6 > 0$)

Since the point lies outside of the given curve, so the number of tangent will be 2.

25.



Let the point P be $(2\sqrt{2}\cos\theta, 3\sqrt{2}\sin\theta)$

Equation of the tangent at P is

$$\frac{xx_1}{8} + \frac{yy_1}{18} = 1$$

$$\Rightarrow \quad \frac{x \cdot 2\sqrt{2}\cos\theta}{8} + \frac{y \cdot 3\sqrt{2}\sin\theta}{18} = 1$$

$$\Rightarrow \quad \frac{x}{2\sqrt{2}\sec\theta} + \frac{y}{3\sqrt{2}\operatorname{cosec}\theta} = 1$$

Thus, the co-ordinates of A and B are

$(2\sqrt{2}\sec\theta, 0)$ and $(0, 3\sqrt{2}\operatorname{cosec}\theta)$

Now, $ar(\Delta OAB) = \frac{1}{2} \times OA \times OB$

$$= \frac{1}{2} \times 2\sqrt{2}\sec\theta \times 3\sqrt{2}\operatorname{cosec}\theta$$

$$= \frac{12}{2\sin\theta\cos\theta}$$

$$= \frac{12}{\sin 2\theta}$$

The area of the triangle is maximum, when $\sin 2\theta = 1$

$$\Rightarrow \quad 2\theta = \frac{\pi}{2}$$

$$\Rightarrow \quad \theta = \frac{\pi}{4}$$

Hence, the point P is (2, 3).

26. Let any point on the curve be $M(x_1, y_1)$

The equation of the given curve is

$$\sqrt{x} + \sqrt{y} = \sqrt{a}$$

$$\Rightarrow \quad \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_M = -\frac{\sqrt{y_1}}{\sqrt{x_1}}$$

Now, x-intercept $= OP = x_1 - \frac{y_1}{dy/dx}$

$$= x_1 - \frac{y_1}{\left(-\frac{\sqrt{y_1}}{\sqrt{x_1}}\right)} = x_1 + \sqrt{x_1 y_1}$$

y-intercept $= y_1 - x_1 \frac{dy}{dx}$

$$= y_1 - x_1\left(-\frac{\sqrt{y_1}}{\sqrt{x_1}}\right) = y_1 + \sqrt{x_1 y_1}$$

Thus, $OP + OQ$

$$= x_1 + \sqrt{x_1 y_1} + y_1 + \sqrt{x_1 y_1}$$

$$= x_1 + y_1 + 2\sqrt{x_1 y_1}$$

$$= (\sqrt{x_1} + \sqrt{y_1})^2$$

$$= (\sqrt{a})^2 = a$$

27. Let the point on the curve be $P(\alpha, \beta)$

The equation of the given curve is $\frac{a}{x^2} + \frac{b}{y^2} = 1$

$$\Rightarrow \quad -\frac{2a}{x^3} - \frac{2b}{y^3} \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{ay^3}{bx^3}$$

Now, $\left(\frac{dy}{dx}\right)_P = -\frac{a\beta^3}{b\alpha^3}$

Therefore, x-intercept

$$= x - \frac{y}{dy/dx}$$

$$= \alpha - \frac{\beta}{\left(-\frac{a\beta^3}{b\alpha^3}\right)} = \alpha + \frac{b\alpha^3}{a\beta^2}$$

$$= \alpha + \frac{\alpha^3}{a} \times \frac{b}{\beta^2}$$

$$= \alpha + \frac{\alpha^3}{a} \times \left(1 - \frac{a}{\alpha^2}\right)$$

$$= \alpha + \frac{\alpha^3}{a} - \alpha$$

$$= \frac{\alpha^3}{a}$$

$\Rightarrow$ x-intercept is proportional to α^3.

28. The equation of the tangent to the origin is

$xy = 0$.

$\Rightarrow \quad x = 0$ and $y = 0$

29. The equation of the tangent at the origin is

$ax + by = 0$.

30. The equation of the tangent to the curve at the origin is

$x^2 - y^2 = 0$

$\Rightarrow \quad x + y = 0$ and $x - y = 0$.

31. The equation of the tangent to the curve at the origin is $2012x - 2013y = 0$.

32. The given curves are $x^2 = y$ and $y^2 = x$

We have, $x = x^4$

$\Rightarrow \quad x(1 - x^3) = 0$

$\Rightarrow \quad x = 0, 1$

when $x = 0$, $y = 0$ and when $x = 1$, $y = 1$

So, the point of intersections are (0, 0), (1, 1)

At the point of intersection (0, 0)

Now, $y = x^2$

$$\Rightarrow \quad \frac{dy}{dx} = 1$$

$$m_1 = \left(\frac{dy}{dx}\right)_{(0,0)} = 1$$

Also, $y^2 = x$

$\Rightarrow \quad 2y \dfrac{dy}{dx} = 1$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{1}{2y}$

$$m_2 = \left(\frac{dy}{dx}\right)_{(0,0)} = \frac{1}{0} = \infty$$

Let θ be the angle between them

Then, $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right| = \infty$

$\Rightarrow \quad \theta = \dfrac{\pi}{2}$

Now, consider the point of intersection (1, 1)

$$m_1 = \left(\frac{dy}{dx}\right)_{(1,1)} \text{ and } m_2 = \left(\frac{dy}{dx}\right)_{(1,1)} = \frac{1}{2}$$

Let ϕ be the angle between them

Then $\tan(\varphi) = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\dfrac{1 - \frac{1}{2}}{1 + 1\cdot\frac{1}{2}}\right| = \dfrac{1}{3}$

$\Rightarrow \quad \varphi = \tan^{-1}\left(\dfrac{1}{3}\right).$

33. The given curves are $y = 4 - x^2$ and $y = x^2$
On solving, we get, $x^2 = 4 - x^2$

$\Rightarrow \quad x = \pm\sqrt{2}$

when $x = \pm\sqrt{2}$, $y = 2$

So, the points of intersections are

$(\sqrt{2}, 2)$ & $(-\sqrt{2}, 2)$

Now, $y = 4 - x^2$

$\Rightarrow \quad \dfrac{dy}{dx} = -2x$

$$m_1 = \left(\frac{dy}{dx}\right)_{(\sqrt{2},\,2)} = -2\sqrt{2}$$

Also, $y = x^2$

$\Rightarrow \quad \dfrac{dy}{dx} = 2x$

$$m_2 = \left(\frac{dy}{dx}\right)_{(\sqrt{2},\,2)} = 2\sqrt{2}$$

Let θ be the angle between them

Then,

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\frac{2\sqrt{2} + 2\sqrt{2}}{1 - 8}\right| = \left(\frac{4\sqrt{2}}{7}\right)$$

$\Rightarrow \quad \theta = \tan^{-1}\left(\dfrac{4\sqrt{2}}{7}\right)$

Similarly, we can find, $\varphi = \tan^{-1}\left(\dfrac{4\sqrt{2}}{7}\right)$

34. The given curves are $y^2 = 4x$ and $y = e^{-x/2}$
Let the point of intersection be (x_1, y_1)

Now, $y^2 = 4x$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{2}{y}$

$\Rightarrow \quad m_1 = \left(\dfrac{dy}{dx}\right)_{(x_1,\,y_1)} = \dfrac{2}{y_1}$

Also, $y = e^{-x/2}$

$$\frac{dy}{dx} = e^{-x/2} \times -\frac{1}{2}$$

$\Rightarrow \quad m_2 = \left(\dfrac{dy}{dx}\right)_{(x_1,\,y_1)} = -\dfrac{1}{2} \times e^{\frac{-x_1}{2}} = -\dfrac{1}{2}y_1$

Let θ be the angle between them

Then $\tan\theta = \left|\dfrac{m_2 - m_1}{1 + m_1 m_2}\right|$

$\Rightarrow \quad \tan\theta = \left|\dfrac{-\frac{y_1}{2} - \frac{2}{y_1}}{1 + \left(\frac{-y_1}{2} \times \frac{2}{y_1}\right)}\right| = \infty$

$\Rightarrow \quad \theta = \dfrac{\pi}{2}$

35. The given curves are $y = \sin x$ and $y = \cos x$.
On solving, we get, the point of intersection is

$$\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right).$$

Now, $y = \sin x$

$\Rightarrow \quad \dfrac{dy}{dx} = \cos x$

$\Rightarrow \quad m_1 = \left(\dfrac{dy}{dx}\right)_{\left(\frac{\pi}{4},\,\frac{1}{\sqrt{2}}\right)} = \dfrac{1}{\sqrt{2}}$

Also, $y = \cos x$

$\Rightarrow \quad \dfrac{dy}{dx} = -\sin x$

$\Rightarrow \quad m_2 = \left(\dfrac{dy}{dx}\right)_{\left(\frac{\pi}{4},\,\frac{1}{\sqrt{2}}\right)} = -\dfrac{1}{\sqrt{2}}$

Let θ be the angle between them

Then $\tan\theta = \left|\dfrac{m_2 - m_1}{1 + m_1, m_2}\right|$

$\Rightarrow \quad \tan\theta = \left|\dfrac{-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}}{1 - \frac{1}{2}}\right| = 2\sqrt{2}$

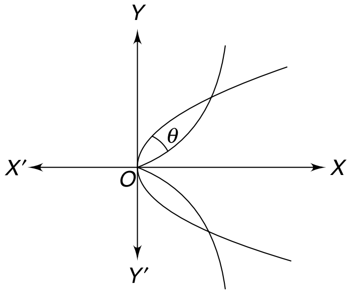
$\Rightarrow \qquad \theta = \tan^{-1}(2\sqrt{2})$

36. The given curves are $2y^2 = x^3$ and $y^2 = 32x$

On solving we get, the point of intersections are $O(0, 0)$, $P(8, 16)$ and $Q(8, -16)$.

From the diagram, it is clear that, the angle of intersection at (0, 0) is $\frac{\pi}{2}$.

Now, $\qquad 2y^2 = x^3$



From the diagram, it is clear that, the angle of intersection at (0, 0) is $\frac{\pi}{2}$.

Now, $\qquad 2y^2 = x^3$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{3x^2}{4y}$$

$$\Rightarrow \qquad m_1 = \left(\frac{dy}{dx}\right)_P = \frac{3 \times 64}{4 \times 16} = 3$$

Also, $\qquad y^2 = 32x$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{32}{2y} = \frac{16}{y}$$

$$\Rightarrow \qquad m_2 = \left(\frac{dy}{dx}\right)_P = \frac{16}{16} = 1$$

Let θ be the angle between them

Then, $\qquad \tan\theta = \left|\frac{m_2 - m_1}{1 + m_1\, m_2}\right|$

$$\Rightarrow \qquad \tan\theta = \left|\frac{1-3}{1+1.3}\right| = \frac{1}{2}$$

$$\Rightarrow \qquad \theta = \tan^{-1}\left(\frac{1}{2}\right)$$

Thus, the angle of intersection at P and Q is $\tan^{-1}\left(\frac{1}{2}\right)$.

37. The given curves are $y^2 = 4x$ and

$$x^2 + y^2 - 6x + 1 = 0$$

Now, $\qquad y^2 = 4x$

$$\Rightarrow \qquad 2y\frac{dy}{dx} = 4$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{2}{y}$$

$$\Rightarrow \qquad m_1 = \left(\frac{dy}{dx}\right)_{(1,2)} = \frac{2}{2} = 1$$

Also, $\qquad x^2 + y^2 - 6x + 1 = 0$

$$\Rightarrow \qquad 2x + 2y\cdot\frac{dy}{dx} - 6 = 0$$

$$x + y\frac{dy}{dx} - 3 = 0$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{3-x}{y}$$

$$\Rightarrow \qquad m_2 = \left(\frac{dy}{dx}\right)_{(1,2)} = \frac{3-1}{2} = 1$$

Let θ be the angle between them

Then $\qquad \tan\theta = \left|\frac{m_2 - m_1}{1 + m_1\cdot m_2}\right| = \left|\frac{1-1}{1+1.1}\right| = 0$

$$\Rightarrow \qquad \theta = 0$$

Hence, the curves touch each other.

38. The given curves are $y = 6 - x + x^2$

and $\qquad y(x-1) = x + 2.$

Now, $\qquad y = 6 - x + x^2$

$$\frac{dy}{dx} = -1 + 2x$$

$$m_1 = \left(\frac{dy}{dx}\right)_{(2,4)} = -1 + 4 = 3$$

Also, $\qquad y = (x-1)(x+2)$

$$\Rightarrow \qquad \frac{dy}{dx} = x + 2 + x - 1 = 2x + 1$$

$$\Rightarrow \qquad m_2 = \left(\frac{dy}{dx}\right)_{(2,4)} = 3$$

Let θ be the angle between them

Then, $\qquad \tan\theta = \left|\frac{m_2 - m_1}{1 + m_1\cdot m_2}\right|$

$$\Rightarrow \qquad \tan = \theta = \left|\frac{3-3}{1+9}\right| = 0$$

$$\Rightarrow \qquad \theta = 0$$

Hence, the curves touch each other

39. The given curves are $x = y^2$ and $x\,y = k$

On solving we get, $y = k^{1/3}$, $x = k^{2/3}$

So, the point of intersection is $P\,(k^{2/3}, k^{1/3})$

Now, $\qquad y^2 = x$

$$\Rightarrow \qquad 2y\frac{dy}{dx} = 1$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2y}$$

$$\Rightarrow \quad m_1 = \left(\frac{dy}{dx}\right)_P = \frac{1}{2k^{1/3}}$$

Also, $\quad xy = k \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{k}{x^2}$

$$\Rightarrow \quad m_2 = \left(\frac{dy}{dx}\right)_P = \frac{k}{k^{4/3}} = -\frac{1}{k^{1/3}}$$

since the given curves cut at right angles, so

$$m_1 \times m_2 = -1$$

$$\Rightarrow \quad \frac{1}{2k^{1/3}} \times -\frac{1}{k^{1/3}} = -1$$

$$\Rightarrow \quad \frac{1}{2k^{2/3}} = 1$$

$$\Rightarrow \quad \frac{1}{8k^2} = 1$$

$$\Rightarrow \quad 8k^2 = 1$$

40. The given curves are $\frac{x^2}{a^2} + \frac{y^2}{4} = 1$ and $y^3 = 16x$

Let the point of intersection be (x_1, y_1)

Now, $\quad \frac{x^2}{a^2} + \frac{y^2}{4} = 1$

$$\Rightarrow \quad \frac{2x}{a^2} + \frac{2y}{4}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{x}{a^2} + \frac{y}{2}\cdot\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{2x}{a^2 y}$$

$$\Rightarrow \quad m_1 = \left(\frac{dy}{dx}\right)_P = -\frac{2x_1}{a^2 y_1}$$

Also, $\quad y^3 = 16x$

$$\Rightarrow \quad 3y^2\frac{dy}{dx} = 16$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{16}{3y^2}$$

$$\Rightarrow \quad m_2 = \left(\frac{dy}{dx}\right)_P = \frac{16}{3{y_1}^2}$$

Since two curves are orthogonal, so

$$m_1 \times m_2 = -1$$

$$\Rightarrow \quad -\frac{2x_1}{a^2 y_1} \times \frac{16}{3{y_1}^2} = -1$$

$$\Rightarrow \quad \frac{32x_1}{3a^2 {y_1}^3} = 1$$

$$\Rightarrow \quad \frac{32x_1}{3a^2 \cdot 16x_1} = 1$$

$$\Rightarrow \quad a^2 = \frac{2}{3}$$

$$\Rightarrow \quad a = \pm\sqrt{\frac{2}{3}}$$

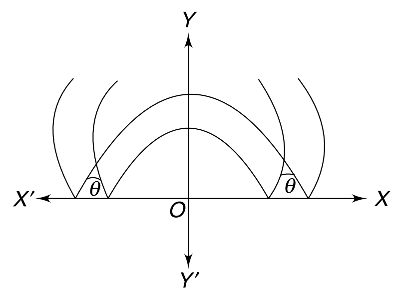
41. The given curves are $y = |x^2 - 1|$ and

$$y = |x^2 - 3|$$

On solving, we get, $x^2 - 1 = -x^2 + 3$

$$\Rightarrow \quad 2x^2 = 4$$

$$\Rightarrow \quad x^2 = 2$$



$$\Rightarrow \quad x = \pm\sqrt{2}$$

when $\quad x = \pm\sqrt{2}$, then $y = 2 - 1 = 1$

so, the point of intersection is $(\pm\sqrt{2}, 1)$

Now, consider the point of intersection is $P(\sqrt{2}, 1)$

Now, $\quad y = |x^2 - 1|$

$$\Rightarrow \quad y = x^2 - 1$$

$$\Rightarrow \quad \frac{dy}{dx} = 2x$$

$$\Rightarrow \quad m_1 = \left(\frac{dy}{dx}\right)_P = 2\sqrt{2}$$

Also, $\quad y = |x^2 - 3|$

$$\Rightarrow \quad y = -x^2 + 3$$

$$\Rightarrow \quad \frac{dy}{dx} = -2x$$

$$\Rightarrow \quad m_2 = \left(\frac{dy}{dx}\right)_P = -2\sqrt{2}$$

Let θ be the angle between them

Then $\quad \tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\frac{-2\sqrt{2} - 2\sqrt{2}}{1 + 2\sqrt{2}\cdot(-2\sqrt{2})}\right|$

$$\Rightarrow \quad \tan\theta = \frac{4\sqrt{2}}{7}$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{4\sqrt{2}}{7}\right).$$

42. We have, $y = [|\sin x| + |\cos x|]$

$\Rightarrow \quad y = 1$, since $1 \le |\sin x| + |\cos x| \le \sqrt{2}$

when $\quad y = 1, x = \pm 2$

So, the point of intersection are (2, 1) and (–2, 1)

Now, $\quad y = 1 \Rightarrow \dfrac{dy}{dx} = 0$

$$\Rightarrow \quad m_1 = \left(\frac{dy}{dx}\right)_p = 0$$

Also, $\quad x^2 + y^2 = 5$

$$\Rightarrow \quad 2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}$$

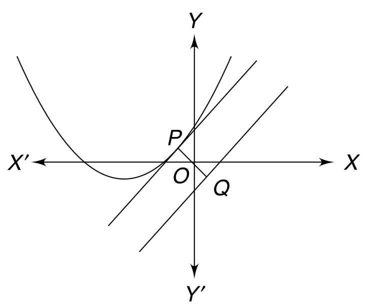
$$\Rightarrow \quad m_2 = \left(\frac{dy}{dx}\right)_p = -2$$

Let θ be the angle between them

Then $\quad \tan\theta = \left|\dfrac{m_2 - m_1}{1 + m_1 \cdot m_2}\right| = \left|\dfrac{-2-0}{1+0}\right| - 2$

$$\Rightarrow \quad \theta = \tan^{-1}(2)$$

43.



The given curves are $y = x^2 + 3x + 2$ and

$y = x - 2$

$$\Rightarrow \quad \frac{dy}{dx} = 2x + 3 \text{ and } \frac{dy}{dx} = 1$$

Since the tangents are parallel, so their slopes are same.

Therefore, $2x + 3 = 1$

$\Rightarrow \quad x = 1$

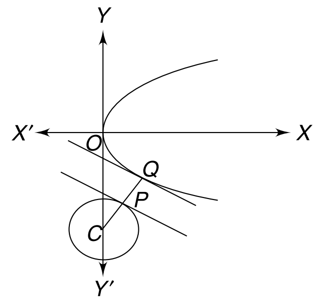
when $x = 1$, then y is 6.

Thus, the point lies on the curve is (1, 6)

Hence, the length of the shortest distance

$$= \left|\frac{1 - 6 + 2}{\sqrt{1^2 + 1^2}}\right| = \frac{3}{\sqrt{2}}$$

44.



The given curves are $y^2 = 4x$ and

$x^2 + y^2 - 12x + 31 = 0.$

In this case, tangent at P on the parabola is parallel to the tangent at Q on the circle.

So their slopes are same.

Thus, $\quad \dfrac{4}{2y} = \dfrac{6-x}{y}$

$\Rightarrow \quad x = 4$

when $\quad x = 4$, then $y = -4$

So, the point on the parabola is (4, –4)

Now shortest distance = PQ

$= CP - CQ$

$= 2\sqrt{5} - \sqrt{5}$

$= \sqrt{5}$

45. The given curves are $y^2 = x^3$ and

$9x^2 + 9y^2 - 30y + 16 = 0.$

In this case, tangent at P on the curve $y^2 = x^3$

So their slopes are same.

Now, $\quad \dfrac{3x^2}{2y} = -\dfrac{3x}{3y-5}$

$$\Rightarrow \quad y = \frac{5x}{3x + 2}$$

Put the value of y in $y^2 = x^3$, we get,

$$\frac{25x^2}{(3x+2)^2} = x^3$$

$\Rightarrow \quad 9x^3 + 12x^2 + 4x - 25 = 0$

$\Rightarrow \quad x = 1.$

when $\quad x = 1$, then $y = 1$

So, the point is (1, 1)

Now, the equation of the normal to the curve

$y^2 = x^3$ at (1, 1) is $y - 1 = -\dfrac{2}{3}(x - 1)$

$\Rightarrow \quad 2x + 3y = 9$

On solving, we get, the point on the ellipse is

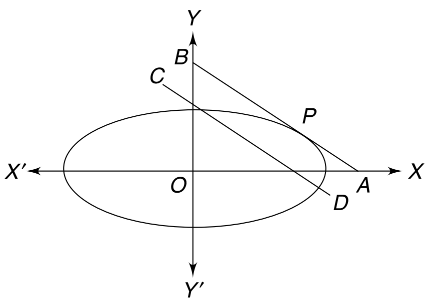
$$\left(\frac{3}{\sqrt{13}}, \frac{5\sqrt{13}-6}{3\sqrt{13}}\right)$$

Hence, the required shortest distance

$$= \sqrt{\left(\frac{3}{\sqrt{13}} - 1\right)^2 + \left(\frac{5\sqrt{13}-6}{3\sqrt{13}}\right)^2}$$

$$= \sqrt{\frac{1}{13}(110 - 30\sqrt{3})}.$$

46.



The given curve is $x^2 + 2y^2 = 6$

$\Rightarrow \quad \frac{x^2}{6} + \frac{y^2}{3} = 1$

since the tangent at P to the ellipse is parallel to the given line, so their slopes are same.

Now, $\quad -\frac{x}{2y} = -1$

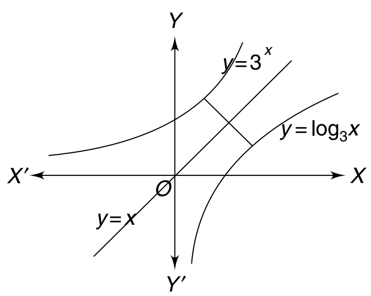
On solving, we get, $6y^2 = 6$

$\Rightarrow \quad y^2 = 1$

$\Rightarrow \quad y = \pm 1$

So, the point can be either (2, 1) or (–2, –1).

47.



The given curves are $y = 3^x$ and $y = \log_3 x$.

Clearly, $y = \log_3 x$ is the image of the curve $y = 3^x$ with respect to the line $y = x$.

Therefore, $3^x \cdot \log 3 = 1$

$\Rightarrow \quad 3^x = \frac{1}{\log 3} = (\log 3)^{-1}$

$\Rightarrow \quad x = \log_3(\log 3)^{-1} = -\log_3(\log 3)$

when $\quad x = -\log_3(\log 3)$, then $y = \frac{1}{\log 3}$

Thus, the point $\left(-\log_3(\log 3), \frac{1}{\log 3}\right)$ lies on the curve $y = 3^x$.

Since the curve $y = \log_3 x$ is the image of the curve $y = 3^x$ with respect to the line $y = x$, so the point on the curve $y = \log_3 x$ is

$$\left(\frac{1}{\log 3}, -\log_3(\log 3)\right)$$

Hence, the shortest distance

$$= \sqrt{\left(\frac{1}{\log 3} + \log_3(\log 3)\right)^2 + \left(\log_3(\log 3) + \frac{1}{\log 3}\right)^2}$$

$$= \sqrt{2}\left(\frac{1}{\log 3} + \log_3(\log 3)\right)$$

$$= \sqrt{2}\left(\frac{1 + \log(\log 3)}{\log 3}\right)$$

$$= \sqrt{\left(\frac{1}{\log 3} + \log_3(\log 3)\right)^2 + \left(\log_3(\log 3) + \frac{1}{\log 3}\right)^2}$$

$$= \sqrt{2}\left(\frac{1}{\log 3} + \log_3(\log 3)\right)$$

$$= \sqrt{2}\left(\frac{1 + \log(\log 3)}{\log 3}\right)$$

48. The given curves are $y = x^2 + x + 1$ and

$$y = x^2 - 5x + 6$$

Let the common tangent be $y = ax + b$.

On solving with both the given curves, we have, $ax + b = x^2 + x + 1$ and $ax + b = x^2 - 5x + 6$

$\Rightarrow \quad x^2 + (1 - a)x + (1 - b) = 0$

and $\quad x^2 + (5 + a)x + (6 - b) = 0$

Since they have a common tangent, so the given equations have equal roots.

Thus, $\quad D = 0$

$\Rightarrow \quad (1 - a)^2 - 4(1 - b) = 0$

and $\quad (5 + a)^2 - 4(6 - b) = 0$

$\Rightarrow \quad a^2 - 2a + 4b - 3 = 0$

and $\quad a^2 + 10a + 4b + 1 = 0$

$\Rightarrow \quad a = -1/3$ and $b = 5/9$.

Hence, the equation of the common tangent is

$$3x + 9y = 5.$$

49. The given curves are $y = 3x^2$ and $y = 2x^3 + 1$

$$\frac{dy}{dx} = 6x \text{ and } \frac{dy}{dx} = 6x^2$$

Since the given curves have common tangent, so their slopes are same.

Thus, $6x = 6x^2$.

$\Rightarrow \quad x = 0$ and 1.

when $x = 0$, $y = 0$ and when $x = 1$, $y = 3$.

So, the points are $(0, 0)$ and $(1, 3)$.

Hence, the equations of the common tangents are

$$y = 0 \text{ and } y - 1 = 6(x - 1) = 6x - 6$$

$$\Rightarrow \quad y = 0 \text{ and } y = 6x - 5.$$

50. The given curves are $y = 6 - x - x^2$ and $y = 1 + \frac{3}{x}$

$$\Rightarrow \quad \frac{dy}{dx} = -1 - 2x \text{ and } \frac{dy}{dx} = -\frac{3}{x^2}$$

Since the given curves have their common tangent, so $\quad -\frac{3}{x^2} = -1 - 2x$

$$\Rightarrow \quad 2x^3 + x^2 - 3 = 0$$

$$\Rightarrow \quad x = 1$$

when $\quad x = 1, y = 6 - 1 - 1 = 4$

So, the point is $(1, 4)$

Hence, the equation of the common tangent is

$$y - 4 = -3(x - 1)$$

$$\Rightarrow \quad y - 4 = -3x + 3$$

$$\Rightarrow \quad 3x + y = 7.$$

51. The given curve is $y^2 = x(2 - x)^2$

$$\Rightarrow \quad 2y\frac{dy}{dx} = (2 - x)^2 - 2x(2 - x)$$

$$\Rightarrow \quad 2y\frac{dy}{dx} = 4 - 4x + x^2 - 4x + 2x^2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3x^2 - 8x + 4}{2y}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(1,1)} = \frac{3 - 8 + 4}{2} = -\frac{1}{2}$$

Hence, the equation of the tangent is

$$y - 1 = -\frac{1}{2}(x - 1)$$

$$\Rightarrow \quad 2y - 2 = -x + 1$$

$$\Rightarrow \quad x + 2y = 3$$

On solving, the given curve and the tangent, we get,

$$y^2 = (3 - 2y)(2 - 3 + 2y)^2$$

$$\Rightarrow \quad y^2 = (3 - 2y)(2y - 1)^2$$

$$\Rightarrow \quad 8y^3 - 19y^2 + 14y - 3 = 0$$

$$\Rightarrow \quad (y - 1)(8y^2 - 11y + 3) = 0$$

$$\Rightarrow \quad (y - 1)^2(8y - 3) = 0$$

$$\Rightarrow \quad y = 1, 3/8$$

when $y = 3/8$, $x = 3 - 2y = 3 - \frac{3}{4} = \frac{9}{4}$

Hence, the point is $\left(\frac{9}{4}, \frac{3}{8}\right)$.

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53.

54. Given $\quad x = \sec^2\theta$ and $y = \cot\theta$

$$\frac{dx}{d\theta} = 2\sec^2\theta\tan\theta \text{ and } \frac{dy}{d\theta} = -\operatorname{cosec}^2\theta$$

Thus, $\quad \frac{dy}{dx} = -\frac{\operatorname{cosec}^2\theta}{2\sec^2\theta\tan\theta} = -\frac{1}{2}\cot^3\theta$

Now, $\quad \left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{4}} = -\frac{1}{2}$

So, the point P is $(2, 1)$

Equation of the tangent at P is

$$y - 1 = -\frac{1}{2}(x - 2)$$

$$\Rightarrow \quad 2y - 2 = -(x - 2) = -x + 2$$

$$\Rightarrow \quad x + 2y = 4 \qquad \text{...(i)}$$

Eliminating 'θ' between $x = \sec^2\theta$ & $y = \cot\theta$

we get, $x - \frac{1}{y^2} = 1 \qquad$ ...(ii)

On solving (i) and (ii), we get,

$$\Rightarrow \quad 4 - 2y - \frac{1}{y^2} = 1$$

$$\Rightarrow \quad 4y^2 - 2y^3 - 1 = y^2$$

$$\Rightarrow \quad 3y^2 - 2y^3 - 1 = 0$$

$$\Rightarrow \quad 2y^3 - 3y^2 + 1 = 0$$

$$\Rightarrow \quad 2y^3 - 2y^2 - y^2 + y - y + 1 = 0$$

$$\Rightarrow \quad 2y^2(y - 1) - y(y - 1) - (y - 1) = 0$$

$$\Rightarrow \quad (y - 1)(2y^2 - y - 1) = 0$$

$$\Rightarrow \quad (y - 1) = 0, (2y^2 - y - 1) = 0$$

$$\Rightarrow \quad (y - 1) = 0, (2y^2 - 2y + y - 1) = 0$$

$$\Rightarrow \quad (y - 1) = 0, (2y(y - 1) + (y - 1)) = 0$$

$$\Rightarrow \quad (y - 1) = 0, (y - 1)(2y + 1) = 0$$

$$\Rightarrow \quad y = 1, -\frac{1}{2}$$

$$\Rightarrow \quad y = -\frac{1}{2}$$

when $y = -\frac{1}{2}$, then $x = 4 - 2y = 4 + 2\cdot\frac{1}{2} = 5$

Thus, the point Q is $\left(5, -\frac{1}{2}\right)$

Now, the length of PQ

$$= \sqrt{(2-5)^2 + \left(1 + \frac{1}{2}\right)^2}$$

$$= \sqrt{9 + \frac{9}{4}}$$

$$= \sqrt{\frac{45}{4}}$$

$$= \frac{3\sqrt{5}}{2}$$

55. The given curve is $y^2 = 4ax$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{4a}{2y} = \frac{2a}{y}$$

Now, $\left(\frac{dy}{dx}\right)_P = \frac{2a}{2at} = \frac{1}{t}$

(i) The length of the tangent

$$= y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$= 2at\sqrt{1 + t^2}$$

(ii) The length of the sub-tangent

$$= y \cdot \frac{dx}{dy}$$

$$= 2at \times t = 2at^2$$

(iii) The length of the normal

$$= y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$= 2at \times \sqrt{1 + \frac{1}{t^2}}$$

$$= 2a\sqrt{t^2 + 1}$$

(iv) The length of the sub-normal

$$= y \cdot \frac{dy}{dx}$$

$$= 2at \times \frac{1}{t}$$

$$= 2a.$$

56. The given curve is $y = be^{x/a}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{b}{a}e^{x/a}$$

Let the point be (x, y)

Now, the length of the sub-tangent

$$= y \cdot \frac{dx}{dy}$$

$$= be^{x/a} \times \frac{a}{be^{x/a}}$$

$$= a$$

$$= \text{constant.}$$

57. The given curves are $x = a(\theta - \sin\theta)$,

$$y = a(1 - \cos\theta)$$

$$\Rightarrow \quad \frac{dx}{d\theta} = a(1 - \cos\theta) \text{ and } \frac{dy}{d\theta} = a\sin\theta$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{a\sin\theta}{a(1 - \cos\theta)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2\sin^2\left(\frac{\theta}{2}\right)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \cot\left(\frac{\theta}{2}\right)$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{2}} = \cos\left(\frac{\pi}{4}\right) = 1$$

when $\theta = \frac{\pi}{4}$, $y = a\left(\frac{1-1}{\sqrt{2}}\right)$

The length of the tangent

$$= y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}$$

$$= a\left(1 - \frac{1}{\sqrt{2}}\right) \times \sqrt{2}$$

$$= a(\sqrt{2} - 1)$$

The length of the normal

$$= y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$= a\left(1 - \frac{1}{\sqrt{2}}\right) \times \sqrt{2}$$

$$= a(\sqrt{2} - 1)$$

58. The given curve is $y^2 = x^3$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3x^2}{2y}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(4,8)} = \frac{3.16}{2.8} = 3$$

Hence, the length of the sub-normal

$$= y \cdot \frac{dy}{dx} = 8.3 = 24$$

59. The given curve is $y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right)$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{c}{2}\left(\frac{1}{c}e^{\frac{x}{c}} - \frac{1}{c}e^{-\frac{x}{c}}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right)$$

Hence, the length of the sub-tangent

$$= y \cdot \frac{dx}{dy}$$

$$= \frac{\frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right)}{\frac{1}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right)}$$

$$= \frac{c}{2}\left(\frac{e^{\frac{2x}{c}} + 1}{e^{\frac{2x}{c}} - 1}\right).$$

Level III

1. The straight line $\frac{x}{a} + \frac{y}{b} = 2$ ...(i)

touches the curve $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 2$ ...(ii)

if the intersection of (i) and (ii) is a unique point

From (i) and (ii), we get,

$$\left(2 - \frac{y}{b}\right)^2 + \left(\frac{y}{b}\right)^2 = 2$$

$$\Rightarrow \quad 4 - \frac{4y}{b} + \frac{y^2}{b^2} + \left(\frac{y}{b}\right)^2 = 2$$

$$\Rightarrow \quad \frac{2y^2}{b^2} - \frac{4y}{b} + 2 = 0$$

$$\Rightarrow \quad \frac{y^2}{b^2} - \frac{2y}{b} + 1 = 0$$

$$\Rightarrow \quad \left(\frac{y}{b} - 1\right)^2 = 0$$

$$\Rightarrow \quad \left(\frac{y}{b} - 1\right) = 0$$

$$\Rightarrow \quad y = b$$

Thus, the straight line (i) is a tangent to the curve (ii)

Also, the point of contact is (a, b).

2. Let the tangent from the origin to the curve $y = \sin x$ meet the curve again at (x_1, y_1)

Equation of tangent at (x_1, y_1) is

$$y - y_1 = \cos(x_1)(x - x_1)$$

since it passes through the origin, so

$$y_1 = x_1 \cos(x_1) \quad \text{...(i)}$$

Also, the point (x_1, y_1) lies on the curve, so

$$y_1 = \sin(x_1) \quad \text{...(ii)}$$

From (i) and (ii), we get,

$$\sin(x_1) = x_1 \cos(x_1)$$

$$\Rightarrow \quad x_1 = \tan(x_1)$$

$$\Rightarrow \quad x_1^2 = \tan^2(x_1)$$

$$\Rightarrow \quad x_1^2 = \sec^2(x_1) - 1$$

$$\Rightarrow \quad x_1^2 = \left(\frac{x_1^2}{y_1^2}\right) - 1$$

$$\Rightarrow \quad x_1^2 y_1^2 = x_1^2 - y_1^2$$

Hence, the locus of (x_1, y_1) is

$$x^2y^2 = x^2 - y^2$$

3. Let $P(x_1, y_1)$ be the point of contact of the tangent.

Given $\quad x^a y^b = k^{a+b}$

$$\Rightarrow \quad a\log x + b\log y = (a + b)\log k$$

$$\Rightarrow \quad \frac{a}{x} + \frac{b}{y}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{ay}{bx}$$

Equation of the tangent at P is

$$y - y_1 = \left(-\frac{ay_1}{bx_1}\right)(x - x_1) \quad \text{...(i)}$$

Put $\quad y = 0$ in (i), we get, $x = \left(\frac{a + b}{a}\right)x_1$

Thus, $\quad A = \left(\left(\frac{a + b}{a}\right)x_1, 0\right)$

put $\quad x = 0$ in (i), we get, $y = \left(\frac{a + b}{b}\right)y_1$

Thus, $\quad B = \left(0, \left(\frac{a + b}{b}\right)y_1\right)$

Let P divide AB in the ratio $\lambda : 1$

Thus,

$$P = \left(\frac{\lambda.0 + 1\cdot\left(\frac{a + b}{a}\right)x_1}{\lambda + 1}, \frac{\lambda.\left(\frac{a + b}{b}\right)y_1 + 1.0}{\lambda + 1}\right)$$

$$= \left(\frac{\left(\frac{a + b}{a}\right)x_1}{\lambda + 1}, \frac{\lambda.\left(\frac{a + b}{b}\right)y_1}{\lambda + 1}\right)$$

Thus, $\quad x_1 = \frac{\left(\frac{a + b}{a}\right)x_1}{\lambda + 1}, \; y_1 = \frac{\lambda.\left(\frac{a + b}{b}\right)y_1}{\lambda + 1}$

$$\Rightarrow \quad \lambda + 1 = \left(\frac{a + b}{a}\right), \; \lambda + 1 = \left(\frac{a + b}{b}\right)$$

$$\Rightarrow \quad \lambda = \frac{b}{a} \text{ or } \frac{a}{b}$$

Therefore P divides AB in the ratio $a : b$.

4. Given curve is $y^2 - 2x^3 - 4y + 8 = 0$

$$\Rightarrow \quad 2y\frac{dy}{dx} - 6x^2 - 4\frac{dy}{dx} = 0$$

$$\Rightarrow \quad y\frac{dy}{dx} - 3x^2 - 2\frac{dy}{dx} = 0$$