INDEFINITE INTEGRAL

Definition

Let F(x) be a differentiable function of x such that $\frac{d}{dx}[F(x)] = f(x)$. Then F(x) is called the integral of f(x). Symbolically, it is written as $\int f(x)\,dx = F(x) + c$.

f(x), the function to be integrated, is called the integrand.

F(x) is also called the anti-derivative (or primitive function) of f(x).

Constant of Integration:

As the differential coefficient of a constant is zero, we have

$$\frac{d}{dx}(F(x)) = f(x) \Rightarrow \frac{d}{dx}[F(x)+c] = f(x).$$

Therefore, $\int f(x)dx = F(x) + c$.

This constant c is called the constant of integration and can take any real value.

Properties of Indefinite Integration:

(i) $\int af(x)dx = a\int f(x)dx.$ (Here 'a' is a constant)

(ii) $\int (f(x)+g(x))\,dx = \int f(x)dx + \int g(x)dx.$

(iii) If $\int f(u)du = F(u) + c$, then $\int f(ax + b)\,dx = \frac{1}{a}F(ax+b)+c, a \neq 0.$

SESSION - 1

Integration as the Inverse Process of Differentiation

Basic Formula:

Integrals of some of the common functions:

- $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right)=x^n$ $\Rightarrow \int x^n dx=\frac{x^{n+1}}{n+1}+c, n \neq -1$
- $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$ $\Rightarrow \int \frac{1}{x}dx=\ln|x|+c$
- $\frac{d}{dx}(e^x)= e^x$ $\Rightarrow \int e^x dx=e^x+c$
- $\frac{d}{dx}(a^x)=(a^x \ln a)$ $\Rightarrow \int a^x dx=\frac{a^x}{\ln a}+c$ $(a > 0)$
- $\frac{d}{dx}(\sin x)=\cos x$ $\Rightarrow \int \cos x dx=\sin x+c$
- $\frac{d}{dx}(\cos x)=-\sin x$ $\Rightarrow \int \sin x dx=-\cos x+c$
- $\frac{d}{dx}(\tan x)=\sec^2 x$ $\Rightarrow \int \sec^2 x dx=\tan x+c$
- $\frac{d}{dx}(\operatorname{cosec} x)=(-\cot x \operatorname{cosec} x)$ $\Rightarrow \int \operatorname{cosec} x \cot x dx=-\operatorname{cosec} x+c$

- $\frac{d}{dx}(\sec x)=\sec x\tan x \quad \Rightarrow \int \sec x\tan x dx=\sec x+c$
- $\frac{d}{dx}(\cot x)=-\operatorname{cosec}^2 x \quad \Rightarrow \int \operatorname{cosec}^2 x dx=-\cot x+c$
- $\frac{d}{dx}(\sin^{-1} x)=\frac{1}{\sqrt{1-x^2}} \quad \Rightarrow \int \frac{1}{\sqrt{1-x^2}}dx=\sin^{-1}x+c$
- $\frac{d}{dx}(\tan^{-1} x)=\frac{1}{1+x^2} \quad \Rightarrow \int \frac{1}{1+x^2}dx=\tan^{-1}x+c$
- $\frac{d}{dx}(\sec^{-1} x)=\frac{1}{x\sqrt{x^2-1}} \quad \Rightarrow \int \frac{1}{x\sqrt{x^2-1}}dx=\sec^{-1}x+c$

Ex. 1: Evaluate $I=\int \frac{(1+\cos 4x)}{(\cot x-\tan x)}dx$.

Solution: $I=\int \frac{2\cos^2 2x}{\cos^2 x-\sin^2 x}.\sin x.\cos x\, dx=\int 2\cos 2x\frac{1}{2}\sin 2x\; dx$

$$=\frac{1}{2}\int \sin 4x\; dx=-\frac{1}{8}\cos 4x+c.$$

Ex. 2: Evaluate $I=\int \frac{dx}{\sin^2 x.\cos^2 x}$.

Solution: Transform the integrand in the following way.

$$\frac{1}{\sin^2 x\cos^2 x}=\frac{\sin^2 x+\cos^2 x}{\sin^2 x\cos^2 x}=\sec^2 x+\operatorname{cosec}^2 x=\frac{d}{dx}[\tan x-\cot x]$$

Hence $I=\int(\sec^2 x+\operatorname{cosec}^2 x)dx=\tan x-\cot x+c$.

Ex. 3: Evaluate $\int \cos 4x\cos 7x\, dx$.

Solution: When solving such problems it is expedient to use the following trigonometric identities:

sin(mx) cos(nx) = (1/2) [sin(m – n)x + sin(m + n)x],

sin(mx) sin(nx) = (1/2) [cos(m – n)x – cos(m + n)x],

cos(mx) cos(nx) = (1/2) [cos(m – n)x + cos(m + n)x].

Here cos4x cos7x = (1/2) [cos(3x) + cos 11x]

$$\Rightarrow I=\frac{1}{2}\int(\cos 3x+\cos 11x)dx=\frac{1}{2}\left[\frac{\sin 3x}{3}+\frac{\sin 11x}{11}\right]+c.$$

Ex. 4: Evaluate $\int \frac{dx}{\sqrt{ax+b}-\sqrt{ax+c}}$

Solution: $I=\int \frac{\sqrt{ax+b}+\sqrt{ax+c}}{(ax+b)-(ax+c)}dx$

$$=\frac{1}{b-c}\int\left[(ax+b)^{\frac{1}{2}}+(ax+c)^{\frac{1}{2}}\right]dx$$

$$= \frac{1}{b-c}\left[\frac{(ax+b)^{\frac{3}{2}}}{\frac{3}{2}a} + \frac{(ax+c)^{\frac{3}{2}}}{\frac{3}{2}a}\right] + C$$

$$= \frac{2}{3a(b-c)}\left\{(ax+b)^{3/2} + (ax+c)^{3/2}\right\} + C$$

Ex. 5: $\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x} dx$

Solution: I = $\int \frac{2\cos^2 x - 1 + 2\sin^2 x}{\cos^2 x} dx$

$$= \int \frac{2(\cos^2 x + \sin^2 x) - 1}{\cos^2 x} dx = \int \sec^2 x \, dx \quad = \tan x + c$$

CLASS EXERCISE :

1. $\int \frac{\sin x + \cos x}{\sqrt{1 + \sin 2x}} dx =$

(a) $\sin x + c$ (b) $\cos x + c$ (c) $x + c$ (d) $x^2 + c$

2. $\int \frac{1 + \cos^2 x}{\sin^2 x} dx =$

(a) $-\cot x - 2x + c$ (b) $-2\cot x - 2x + c$ (c) $-2\cot x - x + c$ (d) $-2\cot x + x + c$

3. $\int \frac{x^2}{x^2 + 4} dx$ equals to

(a) $x - 2\tan^{-1}(x/2) + c$ (b) $x + 2\tan^{-1}(x/2) + c$

(c) $x - 4\tan^{-1}(x/2) + c$ (d) $x + 4\tan^{-1}(x/2) + c$

4. $\int e^{x\log a} . e^x dx$ is equal to

(a) $(ae)^x$ (b) $\frac{(ae)^x}{\log(ae)}$ (c) $\frac{e^x}{1 + \log a}$ (d) None of these

5. $\int \frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}} dx =$

(a) $e.3^{-3x} + c$ (b) $e^3 \log x + c$ (c) $\frac{x^3}{3} + c$ (d) None of these

6. $\int \frac{(1+x)^3}{\sqrt{x}} dx$ equals

(a) $\frac{2}{7}x^{7/2} + \frac{6}{5}x^{5/2} + 2x^{3/2} + 2x^{1/2} + c$ (b) $\frac{2}{5}x^{7/2} + 2x^{5/2} + 6x^{3/2} + 2x^{1/2} + c$

(c) $\frac{2}{7}x^{7/2} - \frac{6}{5}x^{5/2} + 2x^{3/2} - 2x^{1/2} + c$ (d) None of these

HOME EXERCISE :

1. $\int \frac{\cos 2x - 1}{\cos 2x + 1} dx =$

(a) $\tan x - x + c$ (b) $x + \tan x + c$ (c) $x - \tan x + c$ (d) $-x - \cot x + c$

2. If $f'(x) = \frac{1}{x} + x$ and $f(1) = \frac{5}{2}$, then $f(x) =$

(a) $\log x + \frac{x^2}{2} + 2$ (b) $\log x + \frac{x^2}{2} + 1$ (c) $\log x - \frac{x^2}{2} + 2$ (d) $\log x - \frac{x^2}{2} + 1$

3. $\int \sqrt{1 + \sin x}\, dx =$

(a) $\frac{1}{2}\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right) + c$ (b) $\frac{1}{2}\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right) + c$ (c) $2\sqrt{1 + \sin x} + c$ (d) $-2\sqrt{1 - \sin x} + c$

4. $\int \frac{dx}{1 - \sin x} =$

(a) $x + \cos x + c$ (b) $1 + \sin x + c$ (c) $\sec x - \tan x + c$ (d) $\sec x + \tan x + c$

5. $\int \frac{dx}{\sqrt{x} + \sqrt{x - 2}} =$

(a) $\frac{1}{3}\left[x^{3/2} - (x-2)^{3/2}\right] + c$ (b) $\frac{2}{3}\left[x^{3/2} - (x-2)^{3/2}\right] + c$

(c) $\frac{1}{3}\left[(x-2)^{3/2} - x^{3/2}\right] + c$ (d) $\frac{2}{3}\left[(x-2)^{3/2} - x^{3/2}\right] + c$

6. $\int \frac{5(x^6 + 1)}{x^2 + 1} dx =$

(a) $5(x^7 + x)\tan^{-1} x + c$ (b) $x^5 - \frac{5}{3}x^3 + 5x + c$

(c) $3x^4 - 5x^2 + 15x + c$ (d) $5\tan^{-1}(x^2 - 1) + \log(x^2 + 1) + c$

7. $\int \frac{dx}{a^2 - x^2}$ is equal to

(a) $\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$ (b) $\frac{1}{2a}\sin\left(\frac{a-x}{a+x}\right)$ (c) $\frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)$ (d) $\frac{1}{2a}\log\left(\frac{a-x}{a+x}\right)$

SESSION - 2

Methods of Integration

If the integrand is not a derivative of a known function, then the corresponding integrals cannot be found directly. In order to find the integrals of complex problems, generally three rules of integration are used.

- Integration by substitution or by change of the independent variable.
- Integration by parts.
- Integration by partial fractions.

Integration By Substitution

Direct Substitutions:

If the integral is of the form $\int f(g(x))\, g'(x)\, dx$, then put $g(x) = t$, provided $\int f(t)\,dt$ exists.

Ex. 1: Evaluate $\int \frac{(1+\ln x)^2}{x} dx$.

Solution: Let $1 + \ln x = t$ so that $\frac{1}{x} dx = dt$

$$\Rightarrow I = \int t^2\, dt = \frac{1}{3} t^3 + c = \frac{(1+\ln x)^3}{3} + c.$$

Ex. 2: Evaluate $\int \frac{\sin(\ln x)}{x} dx$.

Solution: Let $\ln x = t$. Then $dt = \frac{1}{x} dx$.

Hence $I = \int \sin t\, dt = -\cos t + c = -\cos(\ln x) + c$.

Ex. 3: Evaluate $\int \frac{x^2}{\sqrt{x+1}} dx$.

Solution: Put $x + 1 = t^2$ $\quad \therefore dx = 2t\, dt$ $\quad \therefore \int \frac{x^2}{\sqrt{(x+1)}} dx = \int \frac{(t^2-1)^2}{t}.2t\,dt$

$$= 2\int (t^4 - 2t^2 + 1)\, dt = 2\left[\frac{t^5}{5} - 2.\frac{t^3}{3} + t\right] + c$$

$$= 2t\left[\frac{1}{5}t^4 - \frac{2}{3}t^2 + 1\right] + c$$

$$= 2\sqrt{(x+1)}\left[\frac{1}{5}(x+1)^2 - \frac{2}{3}(x+1) + 1\right] + c$$

Ex. 4: Evaluate $\int \frac{\sec^4 x}{\sqrt{(\tan x)}} dx$.

Solution: Put $\tan x = t^2$ $\quad \therefore \sec^2 x\, dx = 2t\, dt$

Now, $\int \frac{\sec^4 x}{\sqrt{(\tan x)}} dx = \int \frac{\sec^2 x.\sec^2 x}{\sqrt{(\tan x)}} dx$

$$= \int \frac{(1+t^4)\,2t\,dt}{t} = 2\int (1+t^4)\,dt = 2\left[t + \frac{1}{5}t^5\right] + c \quad = 2\left[\sqrt{\tan x} + \frac{1}{5}(\tan x)^{5/2}\right] + c$$

Ex. 5: Evaluate $\int \frac{dx}{\sqrt{\sin^3 x \sin(x+a)}} (a \neq k\pi, k \in I)$

Solution: $\sin^3 x \sin(x + a) = \sin^3 x [\sin x \cos a + \sin a \cos x] = \sin^4 x [\cos a + \sin a \cot x]$

$$\Rightarrow \frac{1}{\sqrt{\sin^3 x \sin(x+a)}} = \frac{1}{\sin^2 x \sqrt{\cos a + \sin a \cot x}}$$

Put $\cos a + \sin a \cot x = t$, so that $\sin a\,(-\operatorname{cosec}^2 x)\,dx = dt$.

Therefore,

$$\int \frac{dx}{\sqrt{\sin^3 x \sin(x+a)}} = -\frac{1}{\sin a}\int \frac{dt}{\sqrt{t}} = -\frac{1}{\sin a}\int t^{-1/2} dt$$

$$= -\frac{1}{\sin a} \cdot \frac{t^{1/2}}{1/2} + c = -2 \operatorname{cosec} a \sqrt{t} + c$$

$= -2 \operatorname{cosec} a \sqrt{\cos a + \sin a \cot x} + c$

CLASS EXERCISE :

1. The value of $\int \frac{x^3}{\sqrt{1+x^4}} dx$ is

(a) $(1+x^4)^{1/2} + c$ (b) $-(1+x^4)^{1/2} + c$ (c) $\frac{1}{2}(1+x^4)^{1/2} + c$ (d) $-\frac{1}{2}(1+x^4)^{1/2} + c$

2. $\int \frac{x^2+1}{x(x^2-1)} dx$ is equal to

(a) $\log \frac{x^2-1}{x} + C$ (b) $-\log \frac{x^2-1}{x} + C$ (c) $\log \frac{x}{x^2+1} + C$ (d) $-\log \frac{x}{x^2+1} + C$

3. $\int \sin^2 x \cos x\, dx$ is equal to

(a) $\frac{\cos^2 x}{2}$ (b) $\frac{\sin^2 x}{3}$ (c) $\frac{\sin^3 x}{3}$ (d) $-\frac{\cos^2 x}{2}$

4. $\int \frac{t}{e^{3t^2}} dt =$

(a) $\frac{1}{6} e^{3t^2} + c$ (b) $-\frac{1}{6} e^{3t^2} + c$ (c) $\frac{1}{6} e^{-3t^2} + c$ (d) $-\frac{1}{6} e^{-3t^2} + c$

5. $\int e^{\cos^2 x} \sin 2x\, dx =$

(a) $e^{\cos^2 x} + c$ (b) $-e^{\cos^2 x} + c$ (c) $\frac{1}{2} e^{\cos^2 x} + c$ (d)None of these

6. $\int \frac{x}{\sqrt{4-x^4}} dx =$

(a) $\cos^{-1} \frac{x^2}{2}$ (b) $\frac{1}{2} \cos^{-1} \frac{x^2}{2}$ (c) $\sin^{-1} \frac{x^2}{2}$ (d) $\frac{1}{2} \sin^{-1} \frac{x^2}{2}$

HOME EXERCISE :

1. $\int e^{-x}\,\text{cosec}^2(2e^{-x}+5)\,dx =$

(a) $\frac{1}{2}\cot(2e^{-x}+5)+c$ (b) $-\frac{1}{2}\cot(2e^{-x}+5)+c$ (c) $2\cot(2e^{-x}+5)+c$ (d) $-2\cot(2e^{-x}+5)+c$

2. $\int \frac{x^3}{\sqrt{x^2+2}}\,dx =$

(a) $\frac{1}{3}(x^2+2)^{3/2}+2(x^2+2)^{1/2}+c$

(b) $\frac{1}{3}(x^2+2)^{3/2}-2(x^2+2)^{1/2}+c$

(c) $\frac{1}{3}(x^2+2)^{3/2}+(x^2+2)^{1/2}+c$

(d) $\frac{1}{3}(x^2+2)^{3/2}-(x^2+2)^{1/2}+c$

3. $\int \frac{1+\tan^2 x}{1-\tan^2 x}\,dx$ equals to

(a) $\log\left(\frac{1-\tan x}{1+\tan x}\right)+C$ (b) $\log\left(\frac{1+\tan x}{1-\tan x}\right)+C$ (c) $\frac{1}{2}\log\left(\frac{1-\tan x}{1+\tan x}\right)+C$ (d) $\frac{1}{2}\log\left(\frac{1+\tan x}{1-\tan x}\right)+C$

4. $\int \frac{a\,dx}{b+ce^x} =$

(a) $\frac{a}{b}\log\left[\frac{e^x}{b+ce^x}\right]+k$ (b) $\frac{a}{b}\log\left[\frac{b+ce^x}{e^x}\right]+k$ (c) $\frac{b}{a}\log\left[\frac{e^x}{b+ce^x}\right]+k$ (d) $\frac{b}{a}\log\left[\frac{b+ce^x}{e^x}\right]+k$

5. $\int \frac{1}{\sqrt{1-e^{2x}}}\,dx =$

(a) $x-\log[1+\sqrt{1-e^{2x}}]+c$

(b) $x+\log[1+\sqrt{1-e^{2x}}]+c$

(c) $\log[1+\sqrt{1-e^{2x}}]-x+c$

(d) None of these

6. $\int \frac{\sin 2x}{a^2+b^2\sin^2 x}\,dx =$

(a) $\frac{1}{b^2}\log(a^2+b^2\sin^2 x)+c$

(b) $\frac{1}{b}\log(a^2+b^2\sin^2 x)+c$

(c) $\log(a^2+b^2\sin^2 x)+c$

(d) $b^2\log(a^2+b^2\sin^2 x)+c$

7. $\int \frac{x+1}{\sqrt{1+x^2}}\,dx =$

(a) $\sqrt{1+x^2}+\tan^{-1}x+c$

(b) $\sqrt{1+x^2}-\log\{x+\sqrt{1+x^2}\}+c$

(c) $\sqrt{1+x^2}+\log\{x+\sqrt{1+x^2}\}+c$

(d) $\sqrt{1+x^2}+\log(\sec x+\tan x)+c$

8. $\int \frac{\sin x}{\sin(x-\alpha)}\,dx =$

(a) $x\cos\alpha-\sin\alpha\log\sin(x-\alpha)+c$

(b) $x\cos\alpha+\sin\alpha\log\sin(x-\alpha)+c$

(c) $x\sin\alpha-\sin\alpha\log\sin(x-\alpha)+c$

(d) None of these

SESSION - 3

Standard Substitutions:

- For terms of the form $x^2 + a^2$ or $\sqrt{x^2+a^2}$, put $x = a\tan\theta$ or $a\cot\theta$
- For terms of the form $x^2 - a^2$ or $\sqrt{x^2-a^2}$, put $x = a\sec\theta$ or $a\operatorname{cosec}\theta$
- For terms of the form $a^2 - x^2$ or $\sqrt{a^2-x^2}$, put $x = a\sin\theta$ or $a\cos\theta$
- If both $\sqrt{a+x}$, $\sqrt{a-x}$ are present, then put $x = a\cos\theta$
- For the type $\sqrt{(x-a)(b-x)}$, put $x = a\cos^2\theta + b\sin^2\theta$
- For the type $\left(\sqrt{x^2+a^2}\pm x\right)^n$ or $\left(x\pm\sqrt{x^2-a^2}\right)^n$, put the expression within the bracket = t.
- For the type $(x+a)^{-1-\frac{1}{n}}(x+b)^{-1+\frac{1}{n}}$ or $\left(\frac{x+b}{x+a}\right)^{\frac{1}{n}-1}\frac{1}{(x+a)^2}$ ($n \in N$, $n > 1$), put $\frac{x+b}{x+a} = t$.
- For $\frac{1}{(x+a)^{n_1}(x+b)^{n_2}}$, $n_1, n_2 \in N$ (and > 1), again put $(x + a) = t(x + b)$

Standard Formula-1:

- $\int \cot x\,dx = \int \frac{\cos x}{\sin x}dx = \ln|\sin x| + c$
- $\int \tan x\,dx = -\int \frac{-\sin x}{\cos x}dx = -\ln|\cos x| + c$ or $\ln|\sec x| + c$
- $\int \sec x\,dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}dx = \ln|\sec x + \tan x| + c$ or $\ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| + c$
- $\int \operatorname{cosec} x\,dx = \int \frac{\operatorname{cosec} x(\cot x - \operatorname{cosec} x)}{\cot x - \operatorname{cosec} x}dx = \ln|(\cot x - \operatorname{cosec} x)| + c$ or $\ln\left|\tan\frac{x}{2}\right| + c$

Standard Formula-2:

- $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left|x+\sqrt{x^2+a^2}\right| + c$ or $\sinh^{-1}\left(\frac{x}{a}\right) + c$
- $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right| + c$
- $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right) + c$
- $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$
- $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$
- $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c$
- $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right| + c$

- $\int\sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + c$
- $\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

Ex. 1: Evaluate $\int\frac{dx}{(x+1)^{6/5}(x-3)^{4/5}}$.

Solution: $I = \int\frac{dx}{(x+1)^{6/5}(x-3)^{4/5}} = \int\frac{dx}{(x+1)^2\left(\frac{(x-3)}{(x+1)}\right)^{4/5}}$.

Put $\left(\frac{x-3}{x+1}\right) = t \Rightarrow dt = \frac{4}{(x+1)^2}dx$.

Hence $I = \frac{1}{4}\int\frac{dt}{t^{4/5}} = \frac{5}{4}t^{1/5} + c = \frac{5}{4}\left(\frac{x-3}{x+1}\right)^{1/5} + c$.

Ex. 2: Evaluate $\int\frac{dx}{\sqrt{(x-a)(b-x)}}$.

Solution: Writing $x = a\cos^2\theta + b\sin^2\theta$, the given integral becomes

$$I=\int\frac{2(b-a)\sin\theta\cos\theta d\theta}{\{(a\cos^2\theta+b\sin^2\theta-a)(a\cos^2\theta+b\sin^2\theta-b\}^{\frac{1}{2}}}$$

$$= \int\frac{2(b-a)\sin\theta\cos\theta d\theta}{(b-a)\sin\theta\cos\theta} = \left(\frac{b-a}{b-a}\right)\int 2d\theta$$

$$= 2\theta + c = 2\sin^{-1}\sqrt{\left(\frac{x-a}{b-a}\right)} + c.$$

Ex. 3: Evaluate $\int\frac{\sin x+\cos x}{9+16\sin 2x}dx$.

Solution: $I = \int\frac{\sin x+\cos x}{25-16(\sin x-\cos x)^2}dx$

Put $t = \sin x - \cos x \Rightarrow dt = (\cos x + \sin x)dx$

$\Rightarrow I = \int\frac{dt}{25-16t^2} = \frac{1}{40}\ln\left|\frac{5+4t}{5-4t}\right| + c = \frac{1}{40}\ln\left|\frac{5+4(\sin x-\cos x)}{5-4(\sin x-\cos x)}\right| + c$.

Derived Substitutions:

Some times it is useful to write the integral as a sum of two related integrals, which can be evaluated by making suitable substitutions.

Examples of such integrals are:

A. Algebraic Twins

$$\int\frac{2x^2}{x^4+1}dx = \int\frac{x^2+1}{x^4+1}dx + \int\frac{x^2-1}{x^4+1}dx$$

$$\int\frac{2}{x^4+1}dx = \int\frac{x^2+1}{x^4+1}dx - \int\frac{x^2-1}{x^4+1}dx$$

$$\int \frac{2x^2}{(x^4+1+kx^2)}dx, \int \frac{2}{(x^4+1+kx^2)}dx$$

B. Trigonomatric Twins

$$\int \sqrt{\tan x}\,dx, \int \sqrt{\cot x}\,dx, \int \frac{1}{(\sin^4 x+\cos^4 x)}dx, \int \frac{1}{(\sin^6 x+\cos^6 x)}dx, \int \frac{\pm \sin x \pm \cos x}{a+b\sin x\cos x}dx.$$

Method of evaluating these integrals are illustrated by mean of the following examples:

Ex. 4: Evaluate $\int \frac{x^2+1}{x^4-x^2+1}dx$.

Solution: Let I = $\int \frac{x^2+1}{x^4-x^2+1}dx = \int \frac{\left(1+\frac{1}{x^2}\right)}{\left(x^2-1+\frac{1}{x^2}\right)}dx$.

Put $x-\frac{1}{x}=t \Rightarrow \left(1+\frac{1}{x^2}\right)dx = dt$

$$\Rightarrow I = \int \frac{dt}{1+t^2} = \tan^{-1} t + c = \tan^{-1}\left(\frac{x^2-1}{x}\right)+c.$$

Ex. 5: Evaluate $\int \sqrt{\tan x}\;dx$.

Solution: Put $\tan x = t^2 \Rightarrow \sec^2 x\,dx = 2t\,dt \Rightarrow dx = \frac{2tdt}{1+t^4}$

$$\Rightarrow I = \int \sqrt{\tan x}\,dx = \int t\frac{2tdt}{1+t^4} = \int \frac{2t^2dt}{1+t^4} = \int \frac{t^2+1}{t^4+1}dt + \int \frac{t^2-1}{t^4+1}dt.$$

This can be solved by the method used in Illustration -4.

CLASS EXERCISE :

1. $\int \sec^p x \tan x\,dx =$

(a) $\frac{\sec^{p+1} x}{p+1}+c$ (b) $\frac{\sec^p x}{p}+c$ (c) $\int \frac{\tan^{p+1} x}{p+1}+c$ (d) $\frac{\tan^p x}{p}+c$

2. $\int \sqrt{\frac{a-x}{x}}dx =$

(a) $a\left[in^{-1}\sqrt{\frac{x}{a}}+\sqrt{\frac{x}{a}}.\sqrt{\frac{a-x}{a}}\right]+C$ (b) $a\left[\sin^{-1}\sqrt{\frac{x}{a}}-\sqrt{\frac{x}{a}}\sqrt{\frac{a-x}{a}}\right]+C$

(c) $-a\left[\sin^{-1}\sqrt{\frac{x}{a}}+\sqrt{\frac{x}{a}}\sqrt{\frac{a-x}{a}}\right]+C$ (d) None of these

3. $\int \frac{1+x^2}{\sqrt{1-x^2}}dx =$

(a) $\frac{3}{2}\sin^{-1}x-\frac{1}{2}x\sqrt{1-x^2}+c$ (b) $\frac{3}{2}\sin^{-1}x+\frac{1}{2}x\sqrt{1-x^2}+c$

(c) $\frac{3}{2}\cos^{-1}x-\frac{1}{2}x\sqrt{1-x^2}+c$ (d) $\frac{3}{2}\cos^{-1}x+\frac{1}{2}x\sqrt{1-x^2}+c$

4. Let the equation of a curve passing through the point (0, 1) be given by $y=\int x^2.e^{x^3}dx$. If the equation of the curve is written in the form $x=f(y)$ then $f(y)$ is

(a) $\sqrt{\log_e(3y-2)}$ (b) $\sqrt[3]{\log_e(3y-2)}$ (c) $\sqrt[3]{\log_e(2-3y)}$ (d)None of these

5. $\int\frac{\sin 2x}{\sin^4 x+\cos^4 x}dx=$

(a) $\cot^{-1}(\tan^2 x)+c$ (b) $\tan^{-1}(\tan^2 x)+c$ (c) $\cot^{-1}(\cot^2 x)+c$ (d) $\tan^{-1}(\cot^2 x)+c$

6. The value of $\int\frac{\sqrt{(x^2-a^2)}}{x}dx$ will be

(a) $\sqrt{(x^2-a^2)}-\tan^{-1}\left[\frac{\sqrt{(x^2-a^2)}}{a}\right]$ (b) $\sqrt{(x^2-a^2)}+\tan^{-1}\left[\frac{\sqrt{(x^2-a^2)}}{a}\right]$

(c) $\sqrt{(x^2-a^2)}+a^2\tan^{-1}\left[\sqrt{x^2-a^2}\right]$ (d) $\tan^{-1}\frac{x}{a}+c$

HOME EXERCISE:

1. $\int\frac{\sin x}{\sin x-\cos x}dx=$

(a) $\frac{1}{2}\log(\sin x-\cos x)+x+c$ (b) $\frac{1}{2}[\log(\sin x-\cos x)+x]+c$

(c) $\frac{1}{2}\log(\cos x-\sin x)+x+c$ (d) $\frac{1}{2}[\log(\cos x-\sin x)+x]+c$

2. If $I=\int\sec^4 x\,\mathrm{cosec}^2 x\,dx=K\tan^3 x+L\tan x+M\cot x+\text{constant}$, then

(a) $K=\frac{1}{3},L=1,M=2$ (b) $K=\frac{1}{3},L=2,M=-1$

(c) $K=-1,L=0,M=1$ (d)None of these

3. $\int\frac{dx}{\sqrt{1+\sin x}}=$

(a) $\sqrt{2}\log\tan(x/4+\pi/8)$ (b) $\sqrt{2}\log[\mathrm{cosec}(x/2+\pi/4)-\cot(x/2+\pi/4)]$

(c) $\sqrt{2}\log[\sec(x/2-\pi/4)+\tan(x/2-\pi/4)]$ (d)All (a), (b) and (c)

4. $\int\frac{x+1}{x(1+xe^x)^2}dx=\log|1-f(x)|+f(x)+c$, then $f(x)=$

(a) $\frac{1}{x+e^x}$ (b) $\frac{1}{1+xe^x}$ (c) $\frac{1}{(1+xe^x)^2}$ (d)None of these

5. $\int \cos\left\{2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right\}dx$ is equal to

(a) $\frac{1}{8}(x^2-1)+k$ (b) $\frac{1}{2}x^2+k$ (c) $\frac{1}{2}x+k$ (d) None of these

6. $\int \sqrt{\frac{\cos x-\cos^3 x}{1-\cos^3 x}}dx$ is equal to

(a) $\frac{2}{3}\sin^{-1}(\cos^{3/2}x)+c$ (b) $\frac{3}{2}\sin^{-1}(\cos^{3/2}x)+c$ (c) $\frac{2}{3}\cos^{-1}(\cos^{3/2}x)+c$ (d) None of these

7. $\int \sqrt{\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)}dx =$

(a) $\cos^{-1}\sqrt{x}+\sqrt{1-x}.(\sqrt{x}-2)+c$

(b) $\cos^{-1}\sqrt{x}-\sqrt{1-x}.(\sqrt{x}-2)+c$

(c) $\cos^{-1}\sqrt{x}+\sqrt{1-x}.(\sqrt{x-2})+c$

(d) None of these

8. $\int \tan^3 2x \sec 2x\, dx =$

(a) $\frac{1}{6}\sec^3 2x-\frac{1}{2}\sec 2x+c$

(b) $\frac{1}{6}\sec^3 2x+\frac{1}{2}\sec 2x+c$

(c) $\frac{1}{9}\sec^2 2x-\frac{1}{3}\sec 2x+c$

(d) None of these

SESSION - 4

Integration by Parts

If u and v be two functions of x, then integral of the product of these two functions is given by:

$$\int uv\,dx = u\int v\,dx - \int\left[\frac{du}{dx}\int v\,dx\right]dx.$$

Note: In applying the above rule care has to be taken in the selection of the first function(u) and the second function (v). Normally we use the following methods:

(i) If in the product of the two functions, one of the functions is not directly integrable (e.g. lnx, $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ etc.) then we take it as the first function and the remaining function is taken as the second function. e.g. In the integration of $\int x\tan^{-1}x\,dx$, $\tan^{-1}x$ is taken as the first function and x as the second function.

(ii) If there is no other function, then unity is taken as the second function e.g. In the integration of $\int\tan^{-1}x\,dx$, $\tan^{-1}x$ is taken as the first function and 1 as the second function.

(iii) If both of the function are directly integrable then the first function is chosen in such a way that the derivative of the function thus obtained under integral sign is easily integrable. Usually we use the following preference order for the first function

(Inverse, Logarithmic, Algebraic, Trigonometric, Exponent)

In the above stated order, the function on the left is always chosen as the first function. This rule is called as **ILATE** e.g. In the integration of $\int x\sin x dx$, x is taken as the first function and sinx is taken as the second function.

Ex. 1: Evaluate $\int\sqrt{x^2-a^2}\,dx$.

Solution: Here $\int\sqrt{x^2-a^2}dx = \sqrt{x^2-a^2}\int 1dx - \int\frac{x(2x)}{2\sqrt{x^2-a^2}}dx$

$$= x\sqrt{x^2-a^2} - \left[\int\frac{x^2-a^2}{\sqrt{x^2-a^2}}dx + \int\frac{a^2}{\sqrt{x^2-a^2}}dx\right]$$

$$= \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left(x+\sqrt{x^2-a^2}\right) + c.$$

Ex. 2: Evaluate $\int\sec^3\theta d\theta$.

Solution: Let $I = \int\sec^3\theta\,d\theta = \sec\theta\int\sec^2\theta d\theta - \int\tan\theta(\sec\theta\tan\theta)d\theta$

$$= \sec\theta\tan\theta - \int(\sec^2\theta-1)\sec\theta\,d\theta = \sec\theta\tan\theta - I + \ln|\sec\theta+\tan\theta|$$

$$\Rightarrow I = \frac{1}{2}[\sec\theta\tan\theta] + \frac{1}{2}\ln|\sec\theta+\tan\theta| + c.$$

Ex. 3: Evaluate $\int\ln\left(\sqrt{1-x}+\sqrt{1+x}\right)dx.$

Solution: Here we have only one function. If we take $u = \ln\left(\sqrt{1-x}+\sqrt{1+x}\right)$ as the first function and v = 1 as the second function then

$$I = \ln\left(\sqrt{1-x}+\sqrt{1+x}\right)\int 1dx - \int\left(\frac{d}{dx}\left(\ln\left(\sqrt{1-x}+\sqrt{1+x}\right)\right)\int 1dx\right)dx$$

$$= x\ \ln\left(\sqrt{1-x}+\sqrt{1+x}\right)-\int\frac{1}{\left(\sqrt{1-x}+\sqrt{1+x}\right)}\left(-\frac{1}{2\sqrt{1-x}}+\frac{1}{2\sqrt{1+x}}\right)xdx$$

$$= x\ln\left(\sqrt{1-x}+\sqrt{1+x}\right)-\frac{1}{2}\int x\frac{\sqrt{1-x^2}-1}{x\sqrt{1-x^2}}dx$$

$$= x\ln\left(\sqrt{1-x}+\sqrt{1+x}\right)-\frac{1}{2}\int dx+\frac{1}{2}\int\frac{1}{\sqrt{1-x^2}}dx$$

$$= x\ln\left(\sqrt{1-x}+\sqrt{1+x}\right)-\frac{x}{2}+\frac{1}{2}\sin^{-1}x+c.$$

An important result: In the integral $\int g(x)e^x dx$, if g(x) can be expressed as g(x) = f(x) + f′(x) then $\int e^x[f(x)+f'(x)]dx$ **= e^x f(x) + c.**

Examples:(i) $\int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)dx=\frac{e^x}{x}+c$,

(ii) $\int e^x(\sin x+\cos x)dx=e^x\sin x+c$,

(iii) $\int e^x(\cos x-\sin x)dx=e^x\cos x+c$.

Ex. 4: Evaluate $\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx$.

Solution: Here $\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx=\int e^x\frac{(1-2x+x^2)}{(1+x^2)^2}dx$

$$= \int e^x\left(\frac{1}{(1+x^2)}-\frac{2x}{(1+x^2)^2}\right)dx = \frac{e^x}{1+x^2}+c.$$

CLASS EXERCISE:

1. $\int\frac{\log x}{x^3}dx =$

(a) $\frac{1}{4x^2}(2\log x-1)+C$ (b) $-\frac{1}{4x^2}(2\log x+1)+C$ (c) $\frac{1}{4x^2}(2\log x+1)+C$ (d) $\frac{1}{4x^2}(1-2\log x)+C$

2. If $\int xe^{2x}dx$ is equal to $e^{2x}f(x)+c$ where c is constant of integration, then $f(x)$ is

(a) $(3x-1)/4$ (b) $(2x+1)/2$ (c) $(2x-1)/4$ (d) $(x-4)/6$

3. $\int x^2\sin 2x\,dx =$

(a) $\frac{1}{2}x^2\cos 2x+\frac{1}{2}x\sin 2x+\frac{1}{4}\cos 2x+c$ (b) $-\frac{1}{2}x^2\cos 2x+\frac{1}{2}x\sin 2x+\frac{1}{4}\cos 2x+c$

(c) $\frac{1}{2}x^2\cos 2x-\frac{1}{2}x\sin 2x+\frac{1}{4}\cos 2x+c$ (d)None of these

4. $\int\log x\,dx =$

(a) $x+x\log x+c$ (b) $x\log x-x+c$ (c) $x^2\log x+c$ (d) $\frac{1}{x}\log x+x+c$

5. If $I=\int e^x\sin 2x\,dx$, then for what value of K, $KI=e^x(\sin 2x-2\cos 2x)+\text{const.}$

(a)1 (b)3 (c)5 (d)7

6. $\int(\log x)^2dx =$

(a) $x(\log x)^2 - 2x\log x - 2x + c$ (b) $x(\log x)^2 - 2x\log x - x + c$

(c) $x(\log x)^2 - 2x\log x + 2x + c$ (d) $x(\log x)^2 - 2x\log x + x + c$

HOME EXERCISE :

1. $\int x\sin^{-1} x\,dx =$

(a) $\left(\frac{x^2}{2} - \frac{1}{4}\right)\sin^{-1} x + \frac{x}{4}\sqrt{1-x^2} + c$ (b) $\left(\frac{x^2}{2} + \frac{1}{4}\right)\sin^{-1} x + \frac{x}{4}\sqrt{1-x^2} + c$

(c) $\left(\frac{x^2}{2} - \frac{1}{4}\right)\sin^{-1} x - \frac{x}{4}\sqrt{1-x^2} + c$ (d) $\left(\frac{x^2}{2} + \frac{1}{4}\right)\sin^{-1} x - \frac{x}{4}\sqrt{1-x^2} + c$

2. $\int e^{2x}\left(\frac{\sin 4x - 2}{1-\cos 4x}\right)dx =$

(a) $\frac{1}{2}e^{2x}\cot 2x + c$ (b) $-\frac{1}{2}e^{2x}\cot 2x + c$ (c) $-2e^{2x}\cot 2x + c$ (d) $2e^{2x}\cot 2x + c$

3. $\int x\cos^2 x\,dx =$

(a) $\frac{x^4}{4} - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + c$ (b) $\frac{x^4}{4} + \frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x + c$

(c) $\frac{x^4}{4} - \frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x + c$ (d) $\frac{x^4}{4} + \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + c$

4. $\int\left[\frac{1}{\log x} - \frac{1}{(\log x)^2}\right]dx =$

(a) $\frac{1}{\log x} + c$ (b) $\frac{x}{\log x} + c$ (c) $\frac{x}{(\log x)^2}$ (d)None of these

5. The value of $\int \frac{\log x}{(x+1)^2}dx$ is

(a) $\frac{-\log x}{x+1} + \log x - \log(x+1)$ (b) $\frac{\log x}{x+1} + \log x - \log(x+1)$

(c) $\frac{\log x}{x+1} - \log x - \log(x+1)$ (d) $\frac{-\log x}{x+1} - \log x - \log(x+1)$

6. $\int \tan^{-1} x dx =$

(a) $x\tan^{-1} x + \frac{1}{2}\log(1+x^2)$ (b) $x\tan^{-1} x - \frac{1}{2}\log(1+x^2$ (c) $(x-1)\tan^{-1} x$ (d) $x\tan^{-1} x - \log(1+x^2)$

7. $\int e^x \sin x(\sin x + 2\cos x)dx =$

(a) $e^x\sin^2 x + c$ (b) $e^x\sin x + c$ (c) $e^x\sin 2x$ (d)None of these

8. $\int \sin\sqrt{x}\,dx =$

(a) $2[\sin\sqrt{x} - \cos\sqrt{x}] + c$ (b) $2[\sin\sqrt{x} - \sqrt{x}\cos\sqrt{x}] + c$

(c) $2[\sin\sqrt{x} + \cos\sqrt{x}] + c$ (d) $2[\sin\sqrt{x} + \sqrt{x}\cos\sqrt{x}] + c$

SESSION - 5 & 6

Evaluation of the Various forms of Integrals by use of Standard Results .

(1) Integral of the form $\int \frac{dx}{ax^2+bx+c}$, where ax^2+bx+c can not be resolved into factors.

(2) Integral of the form $\int \frac{px+q}{ax^2+bx+c}dx$.

(3) Integral of the form $\int \frac{dx}{\sqrt{ax^2+bx+c}}$.

(4) Integral of the form $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$.

(1) **Integrals of the form $\int \frac{dx}{ax^2+bx+c}$, where ax^2+bx+c can not be resolved into factors.**

We have, $ax^2+bx+c = a\left(x^2+\frac{b}{a}.x+\frac{c}{a}\right) = a\left[\left(x+\frac{b}{2a}\right)^2-\left(\frac{b^2}{4a^2}-\frac{c}{a}\right)\right] = a\left[\left(x+\frac{b}{2a}\right)^2-\left(\frac{b^2-4ac}{4a^2}\right)\right]$

Case (i) : When $b^2-4ac>0$

$$\therefore \int \frac{dx}{ax^2+bx+c} = \frac{1}{a}\int \frac{dx}{\left(x+\frac{b}{2a}\right)^2-\left(\frac{\sqrt{b^2-4ac}}{2a}\right)^2}, \qquad \left[\text{form} \int \frac{dx}{x^2-a^2}\right]$$

$$= \frac{1}{a}.\frac{1}{2.\frac{\sqrt{b^2-4ac}}{2a}}\log\left|\frac{x+\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}}{x+\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}}\right|+c = \frac{1}{\sqrt{b^2-4ac}}\log\left|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right|+c$$

Case (ii) : When $b^2-4ac<0$

$$\int \frac{dx}{ax^2+bx+c} = \frac{1}{a}\int \frac{dx}{\left(x+\frac{b}{2a}\right)^2+\left(\frac{\sqrt{4ac-b^2}}{2a}\right)^2}, \qquad \left[\text{form} \int \frac{dx}{x^2+a^2}\right]$$

$$= \frac{1}{a}.\frac{1}{\frac{\sqrt{4ac-b^2}}{2a}}\tan^{-1}\left[\frac{x+\frac{b}{2a}}{\frac{\sqrt{4ac-b^2}}{2a}}\right]+c = \frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\left[\frac{2ax+b}{\sqrt{4ac-b^2}}\right]+c$$

Working rule for evaluating $\int \frac{dx}{ax^2+bx+c}$: To evaluate this form of integrals proceed as follows :

(i) Make the coefficient of x^2 unity by taking 'a' common from ax^2+bx+c.

(ii) Express the terms containing x^2 and x in the form of a perfect square by adding and subtracting the square of half of the coefficient of x.

(iii) Put the linear expression in x equal to t and express the integrals in terms of t.

(iv) The resultant integrand will be either in $\int \frac{dx}{x^2+a^2}$ or $\int \frac{dx}{x^2-a^2}$ or $\int \frac{dx}{a^2-x^2}$ standard form. After using the standard formulae, express the results in terms of x.

Example: 1 $\int \frac{dx}{2x^2+x+1} =$

(a) $\frac{1}{\sqrt{7}}\tan^{-1}\left(\frac{4x+1}{\sqrt{7}}\right)+c$

(b) $\frac{1}{2\sqrt{7}}\tan^{-1}\left(\frac{4x+1}{\sqrt{7}}\right)+c$

(c) $\frac{1}{2}\tan^{-1}\left(\frac{4x+1}{\sqrt{7}}\right)+c$

(d) None of these

Solution: (d) $I = \frac{1}{2}\int \frac{dx}{x^2+\frac{x}{2}+\frac{1}{2}} \Rightarrow I = \frac{1}{2}\int \frac{dx}{\left(x+\frac{1}{4}\right)^2+\left(\frac{\sqrt{7}}{4}\right)^2}$

$$I = \frac{1}{2}\cdot\frac{4}{\sqrt{7}}.\tan^{-1}\left[\frac{x+\frac{1}{4}}{(\sqrt{7}/4)}\right]+c \Rightarrow I = \frac{2}{\sqrt{7}}\tan^{-1}\left[\frac{4x+1}{\sqrt{7}}\right]+c$$

Example: 2 $\int \frac{\cos x}{\sin^2 x+4\sin x+5}dx$ equals

(a) $\tan^{-1}(\sin x)+c$ (b) $\tan^{-1}(\sin x+2)+c$ (c) $\tan^{-1}(\sin x+1)+c$ (d) None of these

Solution: (b) $I = \int \frac{\cos x}{\sin^2 x+4\sin x+5}dx = \int \frac{\cos x}{(\sin x+2)^2+1}dx$ Put $\sin x+2=t \Rightarrow \cos x\,dx = dt$

$$I = \int \frac{dt}{t^2+1} = \tan^{-1}t+c = \tan^{-1}(\sin x+2)+c.$$

(2) **Integral of the form** $\int \frac{px+q}{ax^2+bx+c}dx$ **:** The integration of the function $\frac{px+q}{ax^2+bx+c}$ is effected by breaking $px+q$ into two parts such that one part is the differential coefficient of the denominator and the other part is a constant.

If M and N are two constants, then we express $px+q$ as

$$px+q = M\frac{d}{dx}(ax^2+bx+c)+N = M.(2ax+b)+N = (2aM)x+Mb+N.$$

Comparing the coefficients of x and constant terms on both sides, we have, $p = 2aM \Rightarrow M = \frac{p}{2a}$

and $q = Mb+N \Rightarrow N = q-Mb = q-\frac{p}{2a}b$.

Thus, M and N are known. Hence, the given integral is

$$\int \frac{px+q}{ax^2+bx+c}dx = \int \frac{\frac{p}{2a}(2ax+b)+\left(q-\frac{p}{2a}b\right)}{ax^2+bx+c}dx$$

$$= \frac{p}{2a}\int \frac{2ax+b}{ax^2+bx+c}dx + \left(q-\frac{p}{2a}b\right)\int \frac{dx}{ax^2+bx+c}$$

$$= \frac{p}{2a}\log|\ ax^2+bx+c| + \left(q-\frac{p}{2a}b\right)\int \frac{dx}{ax^2+bx+c} + C$$

The integral on R.H.S. can be evaluated by the method discussed in previous section.

(i) If $b^2 - 4ac < 0,$ then

$$\int \frac{px+q}{ax^2+bx+c}dx = \frac{p}{2a}\log|\ ax^2+bx+c| + \frac{(2aq-bp)}{a\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}} + k$$

(ii) If $b^2 - 4ac > 0,$ then

$$\int \frac{px+q}{ax^2+bx+c}dx = \frac{p}{2a}\log|\ ax^2+bx+c| + \frac{(2aq-bp)}{2a\sqrt{b^2-4ac}}\log\left|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right| + k$$

Example: 3 $\int \frac{2\sin 2\theta - \cos\theta}{6-\cos^2\theta - 4\sin\theta}d\theta =$

(a) $2\log|\ \sin^2\theta - 4\sin\theta + 5| + 7\tan^{-1}(\sin\theta - 2) + c$

(b) $2\log|\ \sin^2\theta - 4\sin\theta + 5| - 7\tan^{-1}(\sin\theta - 2) + c$

(c) $-2\log|\ \sin^2\theta - 4\sin\theta + 5| + 7\tan^{-1}(\sin\theta - 2) + c$

(d) $-2\log|\ \sin^2\theta - 4\sin\theta + 5| - 7\tan^{-1}(\sin\theta - 2) + c$

Solution: (a) $I = \int \frac{2(2\sin\theta\cos\theta) - \cos\theta}{6-(1-\sin^2\theta) - 4\sin\theta}d\theta \Rightarrow I = \int \frac{(4\sin\theta - 1)\cos\theta}{\sin^2\theta - 4\sin\theta + 5}d\theta$

Put $\sin\theta = t \Rightarrow \cos\theta\, d\theta = dt$, $\therefore I = \int \frac{4t-1}{t^2-4t+5}dt$

Let $4t-1 = M\frac{d}{dt}(t^2-4t+5) + N \Rightarrow 4t-1 = M(2t-4) + N$

Comparing the coefficient of t and constant terms on both side, then $M = 2,\ \ N = 7$

$\therefore I = \int \frac{2(2t-4)+7}{t^2-4t+5}dt$

$\Rightarrow I = 2\int \frac{2t-4}{t^2-4t+5}dt + \int \frac{7dt}{t^2-4t+5} \Rightarrow I = 2\log|\ t^2-4t+5| + 7\int \frac{dt}{(t-2)^2+1}$

$\Rightarrow I = 2\log|\ t^2-4t+5| + 7\tan^{-1}(t-2) + c$

$= 2\log|\ \sin^2\theta - 4\sin\theta + 5| + 7\tan^{-1}(\sin\theta - 2) + c$

(3) **Integral of the form** $\int \frac{dx}{\sqrt{ax^2+bx+c}}$ **:** To evaluate this form of integrals proceed as follows :

(i) Make the coefficient of x^2 unity by taking $\sqrt{a}$ common from $\sqrt{ax^2+bx+c}$.

Then, $\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{a}} \int \frac{dx}{\sqrt{x^2+\frac{b}{a}x+\frac{c}{a}}}$.

(ii) Put $x^2+\frac{b}{a}x+\frac{c}{a}$, by the method of completing the square in the form, $\sqrt{A^2-X^2}$ or $\sqrt{X^2+A^2}$ or $\sqrt{X^2-A^2}$ where, X is a linear function of x and A is a constant.

(iii) After this, use any of the following standard formulae according to the case under consideration

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)+c$$

$$\Rightarrow \int \frac{dx}{\sqrt{x^2+a^2}} = \log\left| x+\sqrt{x^2+a^2}\right| +c \text{ and } \int \frac{dx}{\sqrt{x^2-a^2}} = \log\left| x+\sqrt{x^2-a^2}\right| +c.$$

Note : ❑ If $a<0,\ b^2-4ac>0,$ then $\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{-a}}\sin^{-1}\left(\frac{2ax+b}{\sqrt{b^2-4ac}}\right)+k.$

❑ If $a>0,\ b^2-4ac<0,$ then $\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{a}}\sinh^{-1}\left[\frac{2ax+b}{\sqrt{4ac-b^2}}\right]+k$

❑ If $a>0,\ b^2-4ac>0$ $\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{a}}\cosh^{-1}\frac{2ax+b}{\sqrt{b^2-4ac}}+k$

Example: 4 $\int \frac{dx}{\sqrt{2-3x-x^2}}$ equals

(a) $\tan^{-1}\left(\frac{2x+3}{\sqrt{17}}\right)+c$ (b) $\sec^{-1}\left(\frac{2x+3}{\sqrt{17}}\right)+c$ (c) $\sin^{-1}\left(\frac{2x+3}{\sqrt{17}}\right)+c$ (d) $\cos^{-1}\left(\frac{2x+3}{\sqrt{17}}\right)+c$

Solution: (c) $I = \int \frac{dx}{\sqrt{2-3x-x^2}} = \int \frac{dx}{\sqrt{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x+\frac{3}{2}\right)^2}}$

Put $x+3/2=t \Rightarrow dx=dt$

$$I = \int \frac{dx}{\sqrt{\left(\frac{\sqrt{17}}{2}\right)^2-t^2}} = \sin^{-1}\left(\frac{t}{\sqrt{17}/2}\right)+c = \sin^{-1}\left(\frac{2x+3}{\sqrt{17}}\right)+c$$

Example: 5 $\int \frac{dx}{\sqrt{x^2-4x+2}} =$

(a) $\log|x-2+\sqrt{x^2+2-4x}|+c$ (b) $\log|x-2-\sqrt{x^2+2-4x}|+c$

(c) $\log|x-2+\sqrt{x^2+2+4x}|+c$ (d) $\log|x-2-\sqrt{x^2+2+4x}|+c$

Solution: (a) $I=\int\frac{dx}{\sqrt{(x-2)^2-2}}\Rightarrow I=\int\frac{dt}{\sqrt{t^2-(\sqrt{2})^2}}$

Put $x-2=t\ \Rightarrow dx=dt\ \Rightarrow I=\log|t+\sqrt{t^2-2}|+c\ \Rightarrow I=\log|x-2+\sqrt{x^2-4x+2}|+c$

(4) **Integral of the form** $\int\frac{px+q}{\sqrt{ax^2+bx+c}}dx$ **:** To evaluate this form of integrals, first we write,

$$px+q=M\frac{d}{dx}(ax^2+bx+c)+N\Rightarrow\ px+q=M(2ax+b)+N$$

Where M and N are constants.

By equating the coefficients of x and constant terms on both sides, we get

$p=2aM\Rightarrow M=\frac{p}{2a}$ and $q=bM+N\Rightarrow N=q-\frac{bp}{2a}$

In this way, the integral breaks up into two parts given by

$$\int\frac{px+q}{\sqrt{ax^2+bx+c}}dx=\frac{p}{2a}\int\frac{2ax+b}{\sqrt{ax^2+bx+c}}dx+\left(q-\frac{bp}{2a}\right)\int\frac{dx}{\sqrt{ax^2+bx+c}}\quad=I_1+I_2,\ \text{(say)}$$

Now, $I_1=\frac{p}{2a}\int\frac{2ax+b}{\sqrt{ax^2+bx+c}}dx$

Putting $ax^2+bx+c\Rightarrow(2ax+b)dx=dt,$ we have,

$$I_1=\frac{p}{2a}\int t^{-1/2}dt=\frac{p}{2a}.\frac{t^{1/2}}{\frac{1}{2}}+C_1=\frac{p}{a}\sqrt{ax^2+bx+c}+C_1$$

and I_2 is calculated as in the previous section.

Note : ❑ $\frac{px+q}{\sqrt{ax^2+bx+c}}dx=\frac{p}{a}\sqrt{ax^2+bx+c}+\frac{2aq-bp}{2a}\int\frac{dx}{\sqrt{ax^2+bx+c}}.$

Example: 6 $\int\frac{5-2x}{\sqrt{6+x-x^2}}dx=$

(a) $2\sqrt{6+x-x^2}-4\sin^{-1}\left(\frac{2x-1}{5}\right)+c$ (b) $2\sqrt{6+x-x^2}+4\sin^{-1}\left(\frac{2x-1}{5}\right)+c$

(c) $-2\sqrt{6+x-x^2}-4\sin^{-1}\left(\frac{2x-1}{5}\right)+c$ (d) $-2\sqrt{6+x-x^2}+4\sin^{-1}\left(\frac{2x-1}{5}\right)+c$

Solution: (b) $I=\int\frac{5-2x}{\sqrt{6+x-x^2}}dx$

Let $5-2x=M\frac{d}{dx}(6+x-x^2)+N\ \Rightarrow\ 5-2x=M(1-2x)+N$

Equating the coefficients of x and constant terms on both sides, we get

$-2 = -2M \Rightarrow M = 1$ and $5 = M + N \Rightarrow N = 5 - 1 = 4 \quad \therefore\ 5 - 2x = (1 - 2x) + 4$

Hence, $I = \int \frac{(1-2x)+4}{\sqrt{6+x-x^2}}dx = \int \frac{1-2x}{\sqrt{6+x-x^2}}dx + 4\int \frac{dx}{\sqrt{6+x-x^2}} = I_1 + 4I_2$, (say)

Now, $I_1 = \int \frac{1-2x}{\sqrt{6+x-x^2}}dx$

Putting $6 + x - x^2 = t \Rightarrow (1-2x)dx = dt$, we have,

$I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} + C_1 = 2\sqrt{6+x-x^2} + C_1$ and $I_2 = \int \frac{dx}{\sqrt{6+x-x^2}} = \int \frac{dx}{\sqrt{6-(x^2-x)}}$

$$I = \int \frac{dx}{\sqrt{6-\left(x^2 - x + \frac{1}{4}\right)+\frac{1}{4}}} = \int \frac{dx}{\sqrt{\frac{25}{4}-\left(x-\frac{1}{2}\right)^2}} = \int \frac{du}{\sqrt{\left(\frac{5}{2}\right)^2 - u^2}} \qquad \left(\text{where, } u = x - \frac{1}{2}\right)$$

$$= \sin^{-1}\left(\frac{u}{5/2}\right) + C_2 = \sin^{-1}\left[\frac{2}{5}\left(x - \frac{1}{2}\right)\right] + C_2 = \sin^{-1}\left(\frac{2x-1}{5}\right) + C_2$$

$$\therefore\ I = I_1 + 4I_2 = 2\sqrt{6+x-x^2} + 4\sin^{-1}\left(\frac{2x-1}{5}\right) + C \quad (\text{where, } C = C_1 + 4C_2)$$

Example: 7 $\int \frac{x+2}{\sqrt{x^2-2x+4}}$ equals

(a) $\sqrt{x^2-2x+4} + 3\sinh^{-1}\left[\frac{(x-1)}{\sqrt{3}}\right] + c$

(b) $\sqrt{x^2-2x+4} - 3\sinh^{-1}\left[\frac{(x-1)}{\sqrt{3}}\right] + c$

(c) $\sqrt{x^2-2x+4} + 3\cosh^{-1}\left[\frac{(x-1)}{\sqrt{3}}\right] + c$

(d) $\sqrt{x^2-2x+4} - 3\cosh^{-1}\left[\frac{(x-1)}{\sqrt{3}}\right] + c$

Solution:

(a) $\int \frac{x+2}{\sqrt{x^2-2x+4}}dx = \int \frac{x+2}{\sqrt{(x-1)^2+(\sqrt{3})^2}}dx = \int \frac{x-1+3}{\sqrt{(x-1)^2+(\sqrt{3})^2}}dx = \int \frac{x-1}{\sqrt{x^2-2x+4}}dx + \int \frac{3\,dx}{\sqrt{(x-1)^2+(\sqrt{3})^2}}$

Put $x^2 - 2x + 4 = t^2$ in the first expression $\Rightarrow 2(x-1)dx = 2tdt$

$\Rightarrow (x-1)\,dx = tdt$

$$\int \frac{x+2}{\sqrt{x^2-2x+4}}\,dx = \int \frac{tdt}{t} + 3\int \frac{dx}{\sqrt{(x-1)^2+(\sqrt{3})^2}}$$

$$\int \frac{x+2}{\sqrt{x^2-2x+4}}dx = \sqrt{x^2-2x+4} + 3\sinh^{-1}\left[\frac{x-1}{\sqrt{3}}\right] + c$$

Integrals of the form $\int \frac{dx}{a+b\cos x}, \int \frac{dx}{a+b\sin x}\quad \int \frac{dx}{a+b\cos x+c\sin x}, \int \frac{dx}{a\sin x+b\cos x}$.

To evaluate such form of integrals, proceed as follows:

(1) Put $\cos x = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and $\sin x = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$.

(2) Replace $1+\tan^2\frac{x}{2}$ in the numerator by $\sec^2\frac{x}{2}$.

(3) Put $\tan\frac{x}{2} = t$ so that $\frac{1}{2}\sec^2\frac{x}{2}dx = dt$.

(4) Now, evaluate the integral obtained which will be of the form $\int \frac{dt}{at^2+bt+c}$ by the method discussed earlier.

Example: 1 $\displaystyle\int \frac{dx}{3+4\cos x} =$

(a) $\frac{1}{\sqrt{7}}\log\left(\frac{\tan(x/2)-\sqrt{7}}{\tan(x/2)+\sqrt{7}}\right)+c$

(b) $\frac{1}{\sqrt{7}}\log\left(\frac{\tan(x/2)+\sqrt{7}}{\tan(x/2)-\sqrt{7}}\right)+c$

(c) $\frac{1}{\sqrt{7}}\log\left(\frac{\sqrt{7}+\tan(x/2)}{\sqrt{7}-\tan(x/2)}\right)+c$

(d) $\frac{1}{\sqrt{7}}\log\left(\frac{\sqrt{7}-\tan(x/2)}{\sqrt{7}+\tan(x/2)}\right)+c$

Solution: (b) If $b > a$ then, $\displaystyle\int \frac{dx}{a+b\cos x} = \frac{1}{\sqrt{b^2-a^2}}\log\left[\frac{\sqrt{b-a}\tan\frac{x}{2}+\sqrt{b+a}}{\sqrt{b-a}\tan\frac{x}{2}-\sqrt{b+a}}\right]+c$

$$\Rightarrow I = \frac{1}{\sqrt{7}}\log\left(\frac{\tan\frac{x}{2}+\sqrt{7}}{\tan\frac{x}{2}-\sqrt{7}}\right)+c$$

Example: 2 If $\displaystyle\int \frac{dx}{1+\sin x} = \tan\left(\frac{x}{2}+a\right)+b$, then

(a) $a = \frac{\pi}{4}$, $b = 3$

(b) $a = \frac{-\pi}{4}$, $b = 3$

(c) $a = \frac{\pi}{4}$, $b =$ arbitrary constant

(d) $a = \frac{-\pi}{4}$, $b =$ arbitrary constant

Solution: (d) If $a = b$, then $\displaystyle\int \frac{dx}{a+b\sin x} = -\frac{1}{a}\cot\left(\frac{\pi}{4}+\frac{x}{2}\right)+c$

$$\therefore \int \frac{dx}{1+\sin x} = -\cot\left(\frac{\pi}{4}+\frac{x}{2}\right) = -\tan\left(\frac{\pi}{2}-\frac{\pi}{4}-\frac{x}{2}\right)+c = -\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)+c = \tan\left(\frac{x}{2}-\frac{\pi}{4}\right)+c$$

Hence $a = \frac{-\pi}{4}$ and $b =$ arbitrary constant.

Example: 3 $\displaystyle\int \frac{dx}{5+4\cos x} =$

(a) $\frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan x\right)+c$ (b) $\frac{1}{3}\tan^{-1}\left(\frac{1}{3}\tan x\right)+c$

(c) $\frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{x}{2}\right)+c$ (d) $\frac{1}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{x}{2}\right)+c$

Solution: (c) If $a>b$, then $\int\frac{dx}{a+b\cos x}=\frac{2}{\sqrt{a^2-b^2}}.\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)+c$

$\therefore \int\frac{dx}{5+4\cos x}=\frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{x}{2}\right)+c$.

Example: 4 $\int\frac{dx}{\sin x+\cos x}=$

(a) $\log\tan(\pi/8+x/2)+c$ (b) $\log\tan(\pi/8-x/2)+c$

(c) $\frac{1}{\sqrt{2}}\log\tan(\pi/8+x/2)+c$ (d) None of these

Solution: (c) $\int\frac{dx}{\frac{2\tan x/2}{1+\tan^2 x/2}+\frac{1-\tan^2 x/2}{1+\tan^2 x/2}} \Rightarrow \int\frac{\sec^2 x/2}{2\tan x/2+1-\tan^2 x/2}dx$

Put $\tan x/2=t \Rightarrow \frac{1}{2}\sec^2 x/2\,dx=dt$

$\therefore \quad I=2\int\frac{dt}{2t+1-t^2}=2\int\frac{dt}{2-(t^2-2t+1)}$

$\Rightarrow \quad I=2\int\frac{dt}{(\sqrt{2})^2-(t-1)^2}=\frac{2}{2\sqrt{2}}\log\left|\frac{\sqrt{2}+t-1}{\sqrt{2}-t+1}\right|+c$

$\Rightarrow \quad I=\frac{1}{\sqrt{2}}\log\frac{[(\sqrt{2}-1)+\tan x/2][\sqrt{2}-1]}{[(\sqrt{2}+1)-\tan x/2][\sqrt{2}-1]}+c$

$\Rightarrow \quad I=\frac{1}{\sqrt{2}}\log\frac{\tan\pi/8+\tan x/2}{1-(\sqrt{2}-1)\tan x/2}+\frac{1}{\sqrt{2}}\log(\sqrt{2}-1)+c$

$\Rightarrow \quad I=\frac{1}{\sqrt{2}}\log\tan(\pi/8+x/2)+c_1$ where $c_1=\frac{1}{\sqrt{2}}\log(\sqrt{2}-1)+c$

Example: 5 $\int\frac{dx}{1-\cos x-\sin x}=$

(a) $\log\left|1+\cot\frac{x}{2}\right|+c$ (b) $\log\left|1-\tan\frac{x}{2}\right|+c$ (c) $\log\left|1-\cot\frac{x}{2}\right|+c$ (d) $\log\left|1+\tan\frac{x}{2}\right|+c$

Solution: (c) Given $I=\int\frac{dx}{1-\cos x-\sin x}$

$$I=\int \frac{dx}{1-\frac{(1-\tan^2 x/2)}{(1+\tan^2 x/2)}-\frac{2\tan x/2}{1+\tan^2 x/2}} \Rightarrow I=\int \frac{\sec^2 x/2\,.\,dx}{1+\tan^2 x/2-1+\tan^2 x/2-2\tan x/2}$$

$$I=\int \frac{\sec^2 x/2\,.\,dx}{2\tan^2 x/2-2\tan x/2} \Rightarrow \int \frac{1/2\,.\sec^2 x/2\,.dx}{\tan^2 x/2-\tan x/2}$$

Put $\tan x/2=t \Rightarrow \frac{1}{2}\sec^2 x/2.dx=dt$ therefore $I=\int \frac{dt}{t^2-t} \Rightarrow I=\int \frac{dt}{t(t-1)}$

$$\Rightarrow \int\left[-\frac{1}{t}+\frac{1}{t-1}\right]dt \Rightarrow I=\int \frac{dt}{t-1}-\int \frac{dt}{t}$$

$$I=\log(t-1)-\log t+c \Rightarrow I=\log\left|\frac{t-1}{t}\right|+c \Rightarrow I=\log\left|\frac{\tan x/2-1}{\tan x/2}\right|+c$$

$$\Rightarrow I=\log|1-\cot x/2|+c$$

Example: 6 The antiderivative of $f(x)=\frac{1}{3+5\sin x+3\cos x}$ whose graph passes through the point $(0,0)$ is

(a) $\frac{1}{5}\left(\log\left|1-\frac{5}{3}\tan x/2\right|\right)$ (b) $\frac{1}{5}\left(\log\left|1+\frac{5}{3}\tan x/2\right|\right)$

(c) $\frac{1}{5}\left(\log\left|1+\frac{5}{3}\cot x/2\right|\right)$ (d) None of these

Solution: (b) $y=\int \frac{dx}{(3+5\sin x+3\cos x)}=\int \frac{\sec^2 x/2\,dx}{10\tan x/2+6}=\frac{1}{5}\log(5\tan x/2+3)+c$

$$=\frac{1}{5}\log\left|\frac{5}{3}\tan x/2+1\right|+c$$

Passes through $(0,0)$

$\therefore c=0$ then $y=\frac{1}{5}\log\left|1+\frac{5}{3}\tan x/2\right|$.

Integrals of the form

$$\int \frac{dx}{a+b\cos^2 x}, \int \frac{dx}{a+b\sin^2 x}, \int \frac{dx}{a\sin^2 x+b\cos^2 x}, \int \frac{dx}{(a\sin x+b\cos x)^2}, \int \frac{dx}{a+b\sin^2 x+c\cos^2 x}$$

To evaluate the above forms of integrals proceed as follows:

(1) Divide both the numerator and denominator by $\cos^2 x$.

(2) Replace $\sec^2 x$ in the denominator, if any by $(1+\tan^2 x)$.

(3) Put $\tan x=t \Rightarrow \sec^2 xdx=dt$.

(4) Now, evaluate the integral thus obtained, by the method discussed earlier.

Example: 7 $\int \frac{dx}{1+3\sin^2 x}=$

(a) $\frac{1}{3}\tan^{-1}(3\tan^2 x)+c$ (b) $\frac{1}{2}\tan^{-1}(2\tan x)+c$ (c) $\tan^{-1}(\tan x)+c$ (d) None of these

Solution: (b) $\int \frac{\sec^2 x\,dx}{\sec^2 x+3\tan^2 x}=\int \frac{\sec^2 x\,dx}{1+4\tan^2 x}$

Put $2\tan x=t$ $\qquad \sec^2 x\,dx=\frac{dt}{2}$

$\therefore I=\frac{1}{2}\int \frac{dt}{1+t^2}=\frac{1}{2}\tan^{-1}t+c \Rightarrow I=\frac{1}{2}\tan^{-1}(2\tan x)+c$.

Example: 8 $\int \frac{dx}{4\sin^2 x+5\cos^2 x}$

(a) $\frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{2\tan x}{\sqrt{5}}\right)+c$ (b) $\frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{\tan x}{\sqrt{5}}\right)+c$

(c) $\frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{2\tan x}{\sqrt{5}}\right)+c$ (d)None of these

Solution: (c) $\int \frac{\sec^2 x\,dx}{4\tan^2 x+5}$ $\qquad \left(\begin{array}{l}\text{Put } 2\tan x=t\\ 2\sec^2 x\,dx=dt\end{array}\right)$

$\Rightarrow \frac{1}{2}\int \frac{dt}{t^2+(\sqrt{5})^2}=\frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{2\tan x}{\sqrt{5}}\right)+c$

Example: 9 $\int \frac{dx}{(2\sin x+\cos x)^2}$

(a) $\frac{1}{2}\left(\frac{1}{2\tan x+1}\right)$ (b) $\frac{1}{2}\log(2\tan x+1)+c$

(c) $\frac{1}{2+\cot x}+c$ (d) $-\frac{1}{2}\left(\frac{1}{2\tan x-1}\right)+c$

Solution: (c) $\int \frac{dx}{(2\sin x+\cos x)^2}=\int \frac{dx}{\sin^2 x(2+\cot x)^2}=\int \frac{\operatorname{cosec}^2 x\,dx}{(2+\cot x)^2}$

Put $(2+\cot x)=t \Rightarrow -\operatorname{cosec}^2 x\,dx=dt$ $=\int \frac{-dt}{t^2}=\frac{1}{t}+c=\frac{1}{2+\cot x}+c$

Example: 10 $\int \frac{\cos x\,dx}{\cos 3x}=$

(a) $\frac{1}{2\sqrt{3}}\log\frac{1+\sqrt{3}\tan x}{1-\sqrt{3}\tan x}+c$ (b) $\frac{-1}{2\sqrt{3}}\log\frac{1+\sqrt{3}\tan x}{1-\sqrt{3}\tan x}+c$

(c) $\frac{1}{\sqrt{6}}\tan^{-1}(\sqrt{3}\tan x)+c$ (d) None of these

Solution: (a) Let $I=\int \frac{\cos x}{\cos 3x}dx=\int \frac{\cos x}{4\cos^3 x-3\cos x}dx$

Dividing the numerator and denominator by $\cos^2 x$, we have $I=\int \frac{dx}{4\cos^2 x-3}$

$$I=\int\frac{dt}{1-3t^2}=\frac{1}{3}\int\frac{dt}{\frac{1}{3}-t^2}=\frac{1}{3}\int\frac{dt}{\left(\frac{1}{\sqrt{3}}\right)^2-t^2}=\frac{1}{3}\cdot\frac{1}{2.\frac{1}{\sqrt{3}}}\log\left|\frac{\frac{1}{\sqrt{3}}+t}{\frac{1}{\sqrt{3}}-t}\right|+c=\frac{1}{2\sqrt{3}}\log\left|\frac{1+\sqrt{3}t}{1-\sqrt{3}t}\right|+c$$

$$I=\frac{1}{2\sqrt{3}}\log\left|\frac{1+\sqrt{3}\tan x}{1-\sqrt{3}\tan x}\right|+c.$$

CLASS EXERCISE :

1. $\int\frac{dx}{x^2+8x+20}=$

(a) $\tan^{-1}\left(\frac{x+4}{2}\right)+c$

(b) $\frac{1}{2}\tan^{-1}\left(\frac{x+4}{2}\right)+c$

(c) $-\tan^{-1}\left(\frac{x+4}{2}\right)+c$

(d) $-\frac{1}{2}\tan^{-1}\left(\frac{x+4}{2}\right)+c$

2. $\int\frac{x\,dx}{x^2+4x+5}$

(a) $\frac{1}{2}\log(x^2+4x+5)+2\tan^{-1}x+c$

(b) $\frac{1}{2}\log(x^2+4x+5)-\tan^{-1}(x+2)+c$

(c) $\frac{1}{2}\log(x^2+4x+5)+\tan^{-1}(x+2)+c$

(d) $\frac{1}{2}\log(x^2+4x+5)-2\tan^{-1}(x+2)+c$

3. $\int\sqrt{x^2-8x+7}\,dx=$

(a) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}+9\log[x-4+\sqrt{x^2-8x+7}]+c$

(b) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}-3\sqrt{2}\log[x-4+\sqrt{x^2-8x+7}]+c$

(c) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}-\frac{9}{2}\log[x-4+\sqrt{x^2-8x+7}]+c$

(d) None of these

4. $\int\frac{dx}{7+5\cos x}=$

(a) $\frac{1}{\sqrt{6}}\tan^{-1}\left(\frac{1}{\sqrt{6}}\tan\frac{x}{2}\right)+c$

(b) $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{x}{2}\right)+c$

(c) $\frac{1}{4}\tan^{-1}\left(\tan\frac{x}{2}\right)+c$

(d) $\frac{1}{7}\tan^{-1}\left(\tan\frac{x}{2}\right)+c$

5. $\int \frac{c^2 \sin 2x}{a^2 + b^2 \sin^2 x} dx =$

(a) $\frac{c^2}{b^2} \log(a^2 + b^2 \sin^2 x) + k$

(b) $\frac{c^2}{a^2} \log(a^2 + b^2 \sin^2 x) + k$

(c) $\frac{b^2}{c^2} \log(a^2 + b^2 \sin^2 x) + k$

(d) None of these

6. $\int \frac{dx}{1 - \tan x} =$

(a) $\frac{1}{2} x - \frac{1}{2} \log|\cos x - \sin x| + c$

(b) $\frac{1}{2} x + \frac{1}{2} \log|\cos x + \sin x| + c$

(c) $\frac{1}{2} x + \frac{1}{2} \log|\cos x - \sin x| + c$

(d) None of these

HOME EXERCISE :

1. The value of $\int \frac{dx}{3 - 2x - x^2}$ will be **[UPSEAT 1999]**

(a) $\frac{1}{4} \log\left(\frac{3+x}{1-x}\right)$

(b) $\frac{1}{3} \log\left(\frac{3+x}{1-x}\right)$

(c) $\frac{1}{2} \log\left(\frac{3+x}{1-x}\right)$

(d) $\log\left(\frac{1-x}{3+x}\right)$

2. $\int \frac{2x - 3}{x^2 + 3x - 18} dx =$

(a) $\log|\ x^2 + 3x - 18| - \frac{2}{3} \log\left|\frac{x-3}{x+6}\right| + c$

(b) $\log|\ x^2 + 3x - 18| + \frac{2}{3} \log\left|\frac{x-3}{x+6}\right| + c$

(c) $-\log|\ x^2 + 3x - 18| - \frac{2}{3} \log\left|\frac{x-3}{x+6}\right| + c$

(d) $-\log|\ x^2 + 3x - 18| + \frac{2}{3} \log\left|\frac{x-3}{x+6}\right| + c$

3. $\int \frac{dx}{\sqrt{2x - x^2}} =$

(a) $\cos^{-1}(x-1) + c$

(b) $\sin^{-1}(x-1) + c$

(c) $\cos^{-1}(1+x) + c$

(d) $\sin^{-1}(1-x) + c$

4. $\int \frac{x^4 + x^2 + 1}{x^2 - x + 1} dx =$

(a) $\frac{1}{3} x^3 + \frac{1}{2} x^2 + x + c$

(b) $\frac{1}{3} x^3 - \frac{1}{2} x^2 + x + c$

(c) $\frac{1}{3} x^3 + \frac{1}{2} x^2 - x + c$

(d) None of these

5. $\displaystyle\int \frac{dx}{1-\sin x+\cos x}=$

(a) $\log\left|1-\tan\frac{x}{2}\right|+c$ (b) $-\log\left|1-\tan\frac{x}{2}\right|+c$ (c) $\log\left|1+\tan\frac{x}{2}\right|+c$ (d) None of these

6. $\displaystyle\int \frac{1}{1+\sin^2 x}\,dx=$

(a) $\frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}\tan x)+k$ (b) $\sqrt{2}\tan^{-1}(\sqrt{2}\tan x)+k$

(c) $-\frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}\tan x)+k$ (d) $-\sqrt{2}\tan^{-1}(\sqrt{2}\tan x)+k$

7. $\displaystyle\int \frac{dx}{(a\sin x+b\cos x)^2}=$

(a) $\frac{1}{a(a\tan x+b)}+c$ (b) $\frac{-1}{a(a\tan x+b)}+c$ (c) $\frac{1}{a\tan x+b}+c$ (d) $\frac{-1}{a\tan x+b}+c$

8. $\displaystyle\int \frac{dx}{\cos x-\sin x}$ is equal to

(a) $\frac{1}{\sqrt{2}}\log\left|\tan\left(\frac{x}{2}+\frac{3\pi}{8}\right)\right|+c$ (b) $\frac{1}{\sqrt{2}}\log\left|\cot\left(\frac{x}{2}\right)\right|+c$

(c) $\frac{1}{\sqrt{2}}\log\left|\tan\left(\frac{x}{2}-\frac{3\pi}{8}\right)\right|+c$ (d) $\frac{1}{\sqrt{2}}\log\left|\tan\left(\frac{x}{2}-\frac{\pi}{8}\right)\right|+c$

9. $\displaystyle\int \frac{6x+7}{\sqrt{(x-5)(x-4)}}\,dx=$

(a) $6\sqrt{x^2-9x+20}+34\log\left|x-\frac{9}{2}+\sqrt{x^2-9x+20}\right|+c$

(b) $6\sqrt{x^2+9x+20}+34\log\left|x-\frac{9}{2}+\sqrt{x^2+9x+20}\right|+c$

(c) $6\sqrt{x^2-9x+20}-34\log\left|x-\frac{9}{2}+\sqrt{x^2-9x+20}\right|+c$

(d) $6\sqrt{x^2+9x+20}-34\log\left|x-\frac{9}{2}+\sqrt{x^2+9x+20}\right|+c$

SESSION - 7 & 8

5.12 Integration of Rational Functions by using Partial Fractions.

(1) **Proper rational functions:** Functions of the form $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomial and $g(x) \neq 0$, are called rational functions of x.

If degree of $f(x)$ is less than degree of $g(x)$, then $\frac{f(x)}{g(x)}$ is called a proper rational function.

(2) **Improper rational function :** If degree of $f(x)$ is greater than or equal to degree of $g(x)$, then $\frac{f(x)}{g(x)}$, is called an improper rational function and every improper rational function can be transformed to a proper rational function by dividing the numerator by the denominator.

For example, $\frac{x^3}{x^2 - 5x + 6}$ is an improper rational function and can be expressed as $(x+5) + \frac{19x - 30}{x^2 - 5x + 6}$, which is the sum of a polynomial $(x + 5)$ and a proper function $\frac{19x - 30}{x^2 - 5x + 6}$.

(3) **Partial fractions:** Any proper rational function can be broken up into a group of different rational fractions, each having a simple factor of the denominator of the original rational function. Each such fraction is called a partial fraction.

If by some process, we can break a given rational function $\frac{f(x)}{g(x)}$ into different fractions, whose denominators are the factors of $g(x)$, then the process of obtaining them is called the resolution or decomposition of $\frac{f(x)}{g(x)}$ into its partial fractions.

Depending on the nature of the factors of the denominator, the following cases arise.

Case I: When the denominator consists of non-repeated linear factors: To each linear factor $(x - a)$ occurring once in the denominator of a proper fraction, there corresponds a single partial fraction of the form $\frac{A}{x - a}$, where A is a constant to be determined.

Case II: When the denominator consists of linear factors, some repeated: To each linear factor $(x - a)$ occurring r times in the denominator of a proper rational function, there corresponds a sum of r partial fractions of the form.

$$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \dots\dots\dots. + \frac{A_r}{(x-a)^r}$$

Where $A's$ are constants to be determined. Of course, A_r is not equal to zero.

Case III: When the denominator consists of quadratic factors: To each irreducible non repeated quadratic factor $ax^2 + bx + c$, there corresponds a partial fraction of the form $\frac{Ax + B}{ax^2 + bx + c}$, where A and B are constants to be determined.

To each irreducible quadratic factor ax^2+bx+c occurring r times in the denominator of a proper rational fraction there corresponds a sum of r partial fractions of the form

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\dots\dots\dots+\frac{A_rx+B_r}{(ax^2+bx+c)^r}$$

Where, A's and B's are constants to be determined.

(4) **General methods of finding the constants**

(i) In the given proper fraction, first of all factorize the denominator.

(ii) Express the given proper fraction into its partial fractions according to rules given above and multiply both the sides by the denominator of the given fraction.

(iii) Equate the coefficients of like powers of x in the resulting identity and solve the equations so obtained simultaneously to find the various constant is short method. Sometimes, we substitute particular values of the variable x in the identity obtained after clearing of fractions to find some or all the constants. For non-repeated linear factors, the values of x used as those for which the denominator of the corresponding partial fractions become zero.

Note : ❑ If the given fraction is improper, then before finding partial fractions, the given fraction must be expressed as sum of a polynomial and a proper fraction by division.

(5) **Special cases:** Some times a suitable substitution transform the given function to a rational fraction which can be integrated by breaking it into partial fractions.

Example: 1 $\int \frac{\cos x}{(1+\sin x)(2+\sin x)}dx=$

(a) $\log[(1+\sin x)(2+\sin x)]+c$

(b) $\log\left[\frac{2+\sin x}{1+\sin x}\right]+c$

(c) $\log\left[\frac{1+\sin x}{2+\sin x}\right]+c$

(d) None of these

Solution: (c) Put $\sin x=t \Rightarrow \cos x\,dx=dt$

$$\int\frac{\cos x\,dx}{(1+\sin x)(2+\sin x)}=\int\frac{dt}{(1+t)(2+t)}=\int\left[\frac{1}{t+1}-\frac{1}{t+2}\right]dt=\log(t+1)-\log(t+2)+c=$$

$$\log\left[\frac{t+1}{t+2}\right]+c=\log\left[\frac{1+\sin x}{2+\sin x}\right]+c.$$

Example: 2 $\int\frac{3x+1}{(x-2)^2(x+2)}=$

(a) $\frac{5}{16}\log\left|\frac{x+2}{x-2}\right|+\frac{7}{4(x-2)}+c$

(b) $\frac{5}{16}\log\left|\frac{x-2}{x+2}\right|+\frac{7}{4(x-2)}+c$

(c) $\frac{16}{5}\log\left|\frac{x-2}{x+2}\right|-\frac{7}{4(x-2)}+c$

(d) None of these

Solution: (b) We have, $\frac{3x+1}{(x-2)^2(x+2)}=\frac{A}{(x-2)}+\frac{B}{(x-2)^2}+\frac{C}{(x+2)}$

$$3x+1=A(x-2)(x+2)+B(x+2)+C(x-2)^2 \quad ....(i)$$

Putting $x=2$ and - 2 successively in equation (i), we get $B=\frac{7}{4}$, $C=\frac{-5}{16}$

Now, we put $x=0$ and get $A=\frac{5}{16}$

$$\int \frac{3x+1}{(x-2)^2(x+2)}dx = \frac{5}{16}\int \frac{dx}{x-2}+\frac{7}{4}\int \frac{dx}{(x-2)^2}+\frac{-5}{16}\int \frac{dx}{x+2}$$

$$=\frac{5}{16}\log(x-2)-\frac{7}{4}\frac{1}{(x-2)}-\frac{5}{16}\log(x+2)+c = \frac{5}{16}\log\left|\frac{x-2}{x+2}\right|-\frac{7}{4(x-2)}+c$$

Example: 3 $\int \frac{x^2+1}{x(x^2-1)}dx =$

(a) $\log \frac{x^2-1}{x}+c$ (b) $-\log \frac{x^2-1}{x}+c$

(c) $\log \frac{x}{x^2+1}+c$ (d) $-\log \frac{x}{x^2+1}+c$

Solution: (a) $I=\int \left(\frac{2x}{x^2-1}-\frac{1}{x}\right)dx \Rightarrow I=\log(x^2-1)-\log x+c \Rightarrow I=\log \frac{x^2-1}{x}+c$.

Example: 4 If $\int \frac{2x^2+3}{(x^2-1)(x^2+4)}dx = a\log\left(\frac{x-1}{x+1}\right)+b\tan^{-1}\left(\frac{x}{2}\right)+c$ then values of a and b are

(a) $1, -1$ (b) $-1, 1$ (c) $1/2, -1/2$ (d) $1/2, 1/2$

Solution: (d) Put $x^2=y$

$$\therefore \quad \frac{2x^2+3}{(x^2-1)(x^2+4)}=\frac{2y+3}{(y-1)(y+4)} \Rightarrow \frac{2y+3}{(y-1)(y+4)}=\frac{A}{(y-1)}+\frac{B}{(y+4)}$$

$$\therefore \quad 2y+3=A(y+4)+B(y-1)$$

Comparing the coefficient of y and constant terms

$\Rightarrow A+B=2$ and $4A-B=3$

$\therefore \quad A=1$ and $B=1$

$$\therefore \quad I=\int \frac{1}{y-1}dx+\int \frac{1}{y+4}dx \Rightarrow I=\int \frac{1}{x^2-1}dx+\int \frac{1}{x^2+4}dx$$

$$\Rightarrow I=\frac{1}{2}\log\left|\frac{x-1}{x+1}\right|+\frac{1}{2}\tan^{-1}\frac{x}{2}+c \qquad \therefore a=\frac{1}{2} \text{ and } b=\frac{1}{2}.$$

Example: 5 $\int \frac{x^4}{(x-1)(x^2+1)}dx =$

(a) $\frac{x(x+2)}{2}+\frac{\log(x-1)}{2}-\frac{\log(x^2+1)}{4}-\frac{\tan^{-1}x}{2}+c$

(b) $\frac{x(x+2)}{2}+\frac{\log(x-1)}{2}+\frac{\log(x^2+1)}{4}-\frac{\tan^{-1}x}{2}+c$

(c) $\frac{x(x+2)}{2}+\frac{\log(x-1)}{2}+\frac{\log(x^2+1)}{4}+\frac{\tan^{-1}x}{2}+c$

(d) None of these

Solution: (a) $\int \frac{x^4}{(x-1)(x^2+1)}dx = \int \frac{x^4-1}{(x-1)(x^2+1)}dx + \int \frac{1}{(x-1)(x^2+1)}dx$

$$= \int \frac{(x+1)(x-1)(x^2+1)}{(x-1)(x^2+1)}dx + \int \frac{dx}{(x-1)(x^2+1)}$$

$$= \int (x+1)\,dx + \int \frac{dx}{(x-1)(x^2+1)} = \frac{x^2}{2}+x+\left[\frac{1}{2}\log(x-1)-\frac{1}{4}\log(x^2+1)-\frac{1}{2}\tan^{-1}x\right]+c$$

$$= \frac{x(x+2)}{2}+\frac{\log(x-1)}{2}-\frac{\log(x^2+1)}{4}-\frac{\tan^{-1}x}{2}+c.$$

Example: 6 $\int \frac{dx}{x(x^n+1)} =$

(a) $\frac{1}{n}\log\frac{x^n}{x^n+1}+c$ (b) $n\log\frac{x^n+1}{x^n}+c$ (c) $\frac{-1}{n}\log\frac{x^n}{x^n+1}+c$ (d) $-n\log\frac{x^n+1}{x^n}+c$

Solution: (a) Let $I=\int \frac{dx}{x(x^n+1)} = \int \frac{x^{n-1}}{x^n(x^n+1)}dx$

Putting $x^n = t \Rightarrow nx^{n-1}dx = dt,$ we have

$$I=\frac{1}{n}\int \frac{dt}{t(t+1)} = \frac{1}{n}\int\left[\frac{1}{t}-\frac{1}{t+1}\right]dt,$$ (by resolving into partial fractions)

$$=\frac{1}{n}[\log t-\log(t+1)]+c = \frac{1}{n}\log\left|\frac{t}{t+1}\right|+c = \frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right|+c.$$

Integration of Trigonometric Functions

(1) **Integral of the form $\int \sin^m x\cos^n x\,dx$:** (i) To evaluate the integrals of the form $I=\int \sin^m x\cos^n x\,dx$, where m and n are rational numbers.

(a) Substitute $\sin x = t,$ if n is odd;

(b) Substitute $\cos x = t$, if m is odd;

(c) Substitute $\tan x = t,$ if $m+n$ is a negative even integer; and

(d) Substitute $\cot x = t,$ if $\frac{1}{2}(n-1)$ is an integer.

(e) If m and n are rational numbers and $\left(\frac{m+n-2}{2}\right)$ is a negative integer, then substitution $\cos x = t$ or $\tan x = t$ is found suitable.

(ii) Integrals of the form $\int R(\sin x, \cos x)dx$, where R is a rational function of $\sin x$ and $\cos x$, are transformed into integrals of a rational function by the substitution $\tan\frac{x}{2} = t$, where $-\pi < x < \pi$. This is the so called universal substitution. Sometimes it is more convenient to make the substitution $\cot\frac{x}{2} = t$ for $0 < x < 2\pi$.

The above substitution enables us to integrate any function of the form $R(\sin x, \cos x)$. However, in practice, it sometimes leads to extremely complex rational function. In some cases, the integral can be simplified by:

(a) Substituting $\sin x = t$, if the integral is of the form $\int R(\sin x)\cos x\, dx$.

(b) Substituting $\cos x = t$, if the integral is of the form $\int R(\cos x)\sin x\, dx$.

(c) Substituting $\tan x = t$, *i.e.*, $dx = \frac{dt}{1+t^2}$, if the integral is dependent only on $\tan x$.

(d) Substituting $\cos x = t$, if $R(-\sin x, \cos x) = -R(\sin x, \cos x)$

(e) Substituting $\sin x = t$, if $R(\sin x, -\cos x) = -R(\sin x, \cos x)$

(f) Substituting $\tan x = t$, if $R(-\sin x, -\cos x) = -R(\sin x, \cos x)$

Example: 7 $\int \sin^3 x\, cos^2 x\, dx$

(a) $\frac{\cos^2 x}{5} - \frac{\cos^3 x}{3} + c$ (b) $\frac{\cos^5}{5} + \frac{\cos^3}{3} + c$ (c) $\frac{\sin^5}{5} - \frac{\sin^3}{3} + c$ (d) $\frac{\sin^5}{5} + \frac{\sin^3}{3} + c$

Solution: (a) $I = \int \sin x(1-\cos^2 x)\cos^2 x\, dx$

Put $\cos x = t \Rightarrow -\sin x\, dx = dt \Rightarrow I = -\int (t^2 - t^4)\, dt = \frac{t^5}{5} - \frac{t^3}{3} + c = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + c.$

Example: 8 $\int \frac{d\theta}{\sin\theta . \cos^3\theta} =$

(a) $\log\tan\theta + \tan^2\theta + c$

(b) $\log\tan\theta - \frac{1}{2}\tan^2\theta + c$

(c) $\log\tan\theta + \frac{1}{2}\tan^2\theta + c$

(d) None of these

Solution: (c) $\int \frac{d\theta}{\sin\theta\cos^3\theta} = \int \frac{\sec^2\theta\, d\theta}{\sin\theta\cos\theta} = \int \frac{\sec^2\theta(1+\tan^2\theta)d\theta}{\tan\theta\, d\theta}$

Put $t = \tan\theta \Rightarrow dt = \sec^2\theta\, d\theta$ then it reduces to $\Rightarrow \int \frac{1+t^2}{t}\, dt = \int\left(\frac{1}{t} + t\right)dt$

$= \log t + \frac{t^2}{2} + c = \log\tan\theta + \frac{1}{2}\tan^2\theta + c.$

Example: 9 $\int \frac{\sin^3 2x}{\cos^5 2x} dx =$

(a) $\tan^4 x + c$ (b) $\tan 4x + c$ (c) $\tan^4 2x + x + c$ (d) $\frac{1}{8}\tan^4 2x + c$

Solution: (d) Given, $I = \int \frac{\sin^3 2x}{\cos^5 2x} dx$. The given equation may be written as

$$\int \frac{\sin^3 2x}{\cos^3 2x} \cdot \frac{1}{\cos^2 2x} dx = \int \tan^3 2x . \sec^2 2x\, dx.$$

Put $\tan 2x = t$ and $2\sec^2 2x\, dx = dt$ $I = \frac{1}{2}\int t^3\, dt = \frac{t^4}{8} + c = \frac{\tan^4 2x}{8} + c.$

Example: 10 If $I_n = \int \frac{\sin nx}{\sin x} dx,$ where $n > 1,$ then $I_n - I_{n-2} =$

(a) $\frac{2}{(n-1)}\cos(n-1)x$ (b) $\frac{2}{n-1}\sin(n-1)x$ (c) $\frac{2}{n}\cos nx$ (d) $\frac{2}{n}\sin nx$

Solution: (b) $I_n = \int \frac{\sin nx}{\sin x} dx$

$$I_{n-2} = \int \frac{\sin(n-2)x}{\sin x} dx \Rightarrow I_n - I_{n-2} = \int \frac{\sin nx - \sin(n-2)x}{\sin x} dx = \int \frac{2\cos(n-1)x . \sin x}{\sin x} dx$$

$$I_n - I_{n-2} = \frac{2\sin(n-1)x}{(n-1)}$$

Integral of the type $f[x,(ax+b)^{m_1/n_1},(ax+b)^{m_2/n_2}\ldots]$ where f is a rational function and m_1, n_1, m_2, n_2 are Integers

To evaluate such type of integral, we transform it into an integral of rational function by putting $(ax+b) = t^s$, where s is the least common multiple ($L.C.M.$) of the numbers n_1, n_2.

Integrals of the form $\int x^m (a+bx^n)^p\, dx$

Case I : If $p \in N$ (Natural number). We expand the integral with the help of binomial theorem and integrate.

Example: 11 Evaluate $\int x^{1/3}(2+x^{1/2})dx$

(a) $3x^{4/3} + \frac{7}{3}x^{7/3} + c$ (b) $x^{4/3} + \frac{5}{3}x^{5/3} + c$

(c) $3x^{4/3} + \frac{3}{5}x^{5/3} + c$ (d) $3x^{4/3} + \frac{3}{7}x^{7/3} + \frac{24}{11}x^{11/6} + c$

Solution: (d) $I = \int x^{1/3}(2+x^{1/2})^2 dx$ Since P is natural number,

$$I = \int x^{1/3}(4 + x + 4x^{1/2})dx = \int (4x^{1/3} + x^{4/3} + 4x^{5/6})dx = \frac{4x^{4/3}}{4/3} + \frac{x^{7/3}}{7/3} + \frac{4.x^{11/6}}{11/6} + c$$

$$= 3x^{4/3} + \frac{3}{7}x^{7/3} + \frac{24}{11}x^{11/6} + c.$$

Case II: If $p \in I^-$ (*i.e.* negative integer). Write $x = t^k$, where k is the *L.C.M.* of denominator of m and n.

Example: 12 Evaluate $\int x^{-2/3}(1+x^{2/3})^{-1}dx$

(a) $3\tan^{-1}(x^{1/3})+c$ (b) $3\tan^{-1}x+c$ (c) $3\tan^{-1}(x^{2/3})+c$ (d) None of these

Solution: (a) If we substitute $x = t^3$ (as we know $P \in$ negative integer)

$\therefore$ *Let* $x = t^k$, where k is *L.C.M.* of denominator m and n.

$$\therefore x = t^3 \Rightarrow dx = 3t^2 dt \text{ or } I = \int \frac{3t^2 dt}{t^2(1+t^2)} = 3\int \frac{dt}{t^2+1} = 3\tan^{-1}(t)+c \Rightarrow I = 3\tan^{-1}(x^{1/3})+c.$$

Case III: If $\frac{m+1}{n}$ is an integer and $p \in$ fraction put $(a+bx^n) = t^k$, where k is the denominator of the fraction p.

Example: 13 Evaluate $\int x^{-2/3}(1+x^{1/3})^{1/2}dx$

(a) $2(1+x^{1/3})^{2/3}+c$ (b) $2(1+x^{1/3})^{3/2}+c$

(c) $2(1+x^{2/3})^{3/2}+c$ (d) $2(1+x^{2/3})^{2/3}+c$

Solution: (b) If we substitute $1 = x^{1/3} = t^2$ then, $\frac{1}{3x^{2/3}}dx = dt$

$$\therefore I = \int \frac{t.6t\,dt}{1} = 6\int t^2\,dt = 2t^3 + c \text{ or } I = 2(1+x^{1/3})^{3/2}+c.$$

Case IV: If $\left(\frac{m+1}{n}+P\right)$ is an integer and $P \in$ fraction. We put $(a+bx^n) = t^k.x^n$, where k is the denominator of the fraction P.

Example: 14 Evaluate $\int x^{-11}(1+x^4)^{-1/2}dx$

(a) $\frac{1}{2}(t^5+t^3+t)+c$ (b) $-\frac{1}{2}\left[\frac{t^5}{5}-\frac{2t^3}{3}+t\right]+c$ (c) $\frac{1}{2}\left[\frac{t^4}{4}+\frac{2t^3}{3}+t\right]+c$ (d) None of these

where $t = \sqrt{1+\frac{1}{x^4}}$

Solution: (b) Here $\left(\frac{m+1}{n}+P\right) = \left[\frac{-11+1}{4}+\frac{1}{2}\right] = -3$

If we substitute then

$$1+\frac{1}{x^4} = t^2 \text{ and } \frac{-4}{x^5}dx = 2t\,dt \Rightarrow I = \int \frac{dx}{x^{11}(1+x^4)^{1/2}} = \int \frac{dx}{x^{11}.x^2(1+1/x^4)^{1/2}}$$

$$I=\int \frac{dx}{x^{13}(1+1/x^4)^{1/2}}=\frac{1}{4}\int \frac{2t\,dt}{x^8 t}=-\frac{1}{2}\int (t^2-1)^2\,dt=\frac{-1}{2}\int (t^4-2t^2+1)\,dt$$

$$=\frac{-1}{2}\left[\frac{t^5}{5}-\frac{2t^3}{3}+t\right]+c,$$

where $t=\sqrt{1+\frac{1}{x^4}}$ $\therefore \int \frac{x\,dx}{(\sqrt{7x-10-x^2})^3}=\frac{-2}{9}\left(\frac{-5}{t}+2t\right)+c$, where $t=\frac{\sqrt{7x-10-x^2}}{x-2}$

Some Integrals which can not be Found

Any function continuous on interval (a, b) has an antiderivative in that interval. In other words, there exists a function $F(x)$ such that $F'(x)=f(x)$.

However not every antiderivative $F(x)$, even when it exists is expressible in closed form in terms of elementary functions such as polynomials, trigonometric, logarithmic, exponential etc. function. Then we say that such antiderivatives or integrals "can not be found." Some typical examples are:

(i) $\int \frac{dx}{\log x}$ (ii) $\int e^{x^2}dx$ (iii) $\int \frac{x^2}{1+x^5}dx$

(iv) $\int \sqrt[3]{1+x^2}\,dx$ (v) $\int \sqrt{1+x^3}\,dx$ (vi) $\sqrt{1-k^2\sin^2 x}\,dx$

(vii) $\int e^{-x^2}dx$ (viii) $\int \frac{\sin x}{x}dx$ (ix) $\int \frac{\cos x}{x}dx$

CLASS EXERCISE :

1. Correct evaluation of $\int \frac{x}{(x-2)(x-1)}dx$ is

(a) $\log_e \frac{(x-2)^2}{(x-1)}+p$ (b) $\log_e \frac{(x-1)}{(x-2)}+p$ (c) $\frac{x-1}{x-2}+p$ (d) $2\log_e \frac{(x-2)}{(x-1)}+p$

(where p is an aribitrary constant)

2. $\int \frac{dx}{(x-1)^2(x-2)}=$

(a) $\log\left|\frac{x-2}{x-1}\right|+\frac{1}{x-1}+c$ (b) $\log\left|\frac{x-1}{x-2}\right|+\frac{1}{x-1}+c$

(c) $\log\left|\frac{x-2}{x-1}\right|-\frac{1}{x-1}+c$ (d) $\log\left|\frac{x-1}{x-2}\right|-\frac{1}{x-1}+c$

3. $\int \frac{x^2}{(x^2+2)(x^2+3)}dx=$

(a) $-\sqrt{2}\tan^{-1}x+\sqrt{3}\tan^{-1}x+c$ (b) $-\sqrt{2}\tan^{-1}\frac{x}{\sqrt{2}}+\sqrt{3}\tan^{-1}\frac{x}{\sqrt{3}}+c$

(c) $\sqrt{2}\tan^{-1}\frac{x}{\sqrt{2}}+\sqrt{3}\tan^{-1}\frac{x}{\sqrt{3}}+c$ (d) None of these

4. $\int \frac{dx}{1+x+x^2+x^3} =$

(a) $\log\sqrt{1+x} - \frac{1}{2}\log\sqrt{1+x^2} + \frac{1}{2}\tan^{-1} x + c$ (b) $\log\sqrt{1+x} - \log\sqrt{1+x^2} + \tan^{-1} x + c$

(c) $\log\sqrt{1+x^2} - \log\sqrt{1+x} + \frac{1}{2}\tan^{-1} x + c$ (d) $\log\sqrt{1+x} + \tan^{-1} x + \log\sqrt{1+x^2} + c$

5. If $\int \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}} dx = Ax + B\log(9e^{2x} - 4) + C$ then A, B and C are

(a) $A = \frac{3}{2}$, $B = \frac{36}{35}$, $C = \frac{3}{2}\log 3 +$ constant (b) $A = \frac{3}{2}$, $B = \frac{35}{36}$, $C = \frac{3}{2}\log 3 +$ constant

(c) $A = -\frac{3}{2}$, $B = -\frac{35}{36}$, $C = -\frac{3}{2}\log 3 +$ constant (d)None of these

6. $\int \frac{dx}{\sin x + \sin 2x} =$

(a) $\frac{1}{6}\log(1-\cos x) + \frac{1}{2}\log(1+\cos x) - \frac{2}{3}\log(1+2\cos x) + c$

(b) $6\log(1-\cos x) + 2\log(1+\cos x) - \frac{2}{3}\log(1+2\cos x) + c$

(c) $6\log(1-\cos x) + \frac{1}{2}\log(1+\cos x) + \frac{2}{3}\log(1+2\cos x) + c$

(d) None of these

7. $\int \frac{x^{5/2}}{\sqrt{1+x^7}} dx$ is

(a) $\frac{2}{7}\log(x^{7/2} + \sqrt{x^7+1}) + c$ (b) $\frac{1}{2}\log\frac{x^7+1}{x^7-1} + c$ (c) $2\sqrt{1+x^7} + c$ (d) None of these

8. If $\int \frac{1}{x\sqrt{1-x^3}} dx = a\log\left|\frac{\sqrt{1-x^2}-1}{\sqrt{1-x^2}+1}\right| + b,$ then a is equal to

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $-\frac{1}{3}$ (d) $-\frac{2}{3}$

HOME EXERCISE :

1. $\int \frac{1}{x-x^3} dx =$

(a) $\frac{1}{2}\log\frac{(1-x^2)}{x^2} + c$ (b) $\log\frac{(1-x)}{x(1+x)} + c$ (c) $\log x(1-x^2) + c$ (d) $\frac{1}{2}\log\frac{x^2}{(1-x^2)} + c$

2. If $\int \frac{1}{(\sin x+4)(\sin x-1)} dx = A\frac{1}{\tan\frac{x}{2}-1} + B\tan^{-1} f(x) + c$, then

(a) $A = \frac{1}{5}$, $B = \frac{-2}{5\sqrt{15}}$, $f(x) = \frac{4\tan x + 3}{\sqrt{15}}$ (b) $A = -\frac{1}{5}$, $B = \frac{1}{\sqrt{15}}$, $f(x) = \frac{4\tan\left(\frac{x}{2}\right)+1}{\sqrt{15}}$

(c) $A=\frac{2}{5},\ B=\frac{-2}{5},\ f(x)=\frac{4\tan x+1}{5}$ (d) $A=\frac{2}{5},\ B=\frac{-2}{5\sqrt{15}},\ f(x)=\frac{4\tan\frac{x}{2}+1}{\sqrt{15}}$

3. If $\int\frac{2x+3}{(x-1)(x^2+1)}dx=\log_e\left\{(x-1)^{\frac{5}{2}}(x^2+1)^a\right\}-\frac{1}{2}\tan^{-1}x+A$, Where A is any arbitrary constant, then the value of 'a' is

(a) $\frac{5}{4}$ (b) $-\frac{5}{3}$ (c) $-\frac{5}{6}$ (d) $-\frac{5}{4}$

4. If $\int\frac{(2x^2+1)dx}{(x^2-4)(x^2-1)}=\log\left[\left(\frac{x+1}{x-1}\right)^a\left(\frac{x-2}{x+2}\right)\right]+c$, then the values of a and b are respectively

(a) $\frac{1}{2},\frac{3}{4}$ (b) $-1,\frac{3}{2}$ (c) $1,\frac{3}{2}$ (d) $\frac{-1}{2},\frac{3}{4}$

5. $\int\frac{dx}{e^x+1-2e^x}=$

(a) $\log(e^x-1)-\log(e^x+2)+c$ (b) $\frac{1}{2}\log(e^x-1)-\frac{1}{3}\log(e^x+2)+c$

(c) $\frac{1}{3}\log(e^x-1)-\frac{1}{3}\log(e^x+2)+c$ (d) $\frac{1}{3}\log(e^x-1)+\frac{1}{3}\log(e^x+2)+c$

6. $\int\frac{(1+x)^3}{(1-x)^3}dx=$

(a) $x+6\log|1-x|+\frac{12}{1-x}-\frac{4}{(1-x)^2}+c$ (b) $-x+6\log|1-x|+\frac{12}{1-x}-\frac{4}{(1-x)^2}+c$

(c) $-x-6\log|1-x|-\frac{12}{1-x}-\frac{4}{(1-x)^2}+c$ (d) None of these

7. $\int\frac{dx}{x(x^5+1)}=$

(a) $\frac{1}{5}\log\left|\frac{x^5}{x^5+1}\right|+c$ (b) $5\log\left|\frac{x^5}{x^5+1}\right|+c$ (c) $-\frac{1}{5}\log\left|\frac{x^5}{x^5+1}\right|+c$ (d) $-5\log\left|\frac{x^5}{x^5+1}\right|+c$

8. $\int x^{-\frac{2}{3}}(1+x^{\frac{1}{2}})^{-\frac{5}{3}}dx$ is equal to

(a) $3(1+x^{-1/2})^{-1/3}+c$ (b) $3(1+x^{-1/2})^{-2/3}+c$ (c) $3(1+x^{1/2})^{-2/3}+c$ (d)None of these

9. Let $\int\frac{x^{1/2}}{\sqrt{1-x^3}}dx=\frac{2}{3}gof(x)+c$, then

(a) $f(x)=\sqrt{x}$ (b) $f(x)=x^{3/2}$ (c) $f(x)=x^{2/3}$ (d) $g(x)=\sin^{-1}x$ (e) Both b and d

10. $\int\frac{dx}{(x+\alpha)^{8/7}(x-\beta)^{6/7}}=$

(a) $\frac{6}{\alpha+\beta}\left(\frac{x-\beta}{x+\alpha}\right)^{\frac{1}{6}}$ (b) $\frac{6}{\alpha+\beta}\left(\frac{x+\alpha}{x-\beta}\right)^{\frac{1}{6}}$ (c) $\frac{7}{\alpha+\beta}\left(\frac{x+\alpha}{x-\beta}\right)^{\frac{1}{7}}$ (d) $\frac{7}{\alpha+\beta}\left(\frac{x-\beta}{x+\alpha}\right)^{\frac{1}{7}}$

SUBJECTIVE

Problem 1:Evaluate $\int \frac{\sin x}{\sqrt{3\sin^2 x + 4\cos^2 x}}dx$.

Solution: $\int \frac{\sin x}{\sqrt{3\sin^2 x + 4\cos^2 x}}dx$

$= \int \frac{\sin x}{\sqrt{3+\cos^2 x}}dx$ (for $\cos x = p$, $-\sin x\, dx = dp$)

$= \int \frac{-dp}{\sqrt{3+p^2}} = -\ln\left(p+\sqrt{p^2+3}\right)$

$= -\ln\left(\cos x + \sqrt{\cos^2 x + 3}\right) + c$.

Problem 2: Evaluate $\int \frac{dx}{(x-\beta)\sqrt{(x-\alpha)(\beta-x)}}$.

Solution: Put $x = \alpha\sin^2\theta + \beta\cos^2\theta$. Then $dx = 2(\alpha-\beta)\sin\theta\cos\theta\, d\theta$

and $(x-\alpha) = (\beta-\alpha)\cos^2\theta$, and $(x-\beta) = (\alpha-\beta)\sin^2\theta$

$\Rightarrow I = \int \frac{4(\alpha-\beta)\sin\theta\cos\theta\, d\theta}{(\alpha-\beta)\sin 2\theta(\sin^2\theta)(\beta-\alpha)}$

$= \frac{2}{\beta-\alpha}\int \operatorname{cosec}^2\theta d\theta = -\frac{2}{\beta-\alpha}\cot\theta + c = -\frac{2}{(\beta-\alpha)}\sqrt{\frac{(x-\alpha)}{(\beta-x)}} + c$.

Problem 3: Prove that $\int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}}dx = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{3}\sqrt{x+1}}{-x}\right) + c$.

Solution: Let $f(x) = \int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}}dx$.

Substituting $x + 1 = p^2$, $dx = 2p\,dp$, we get

$$f(x) = \int \frac{(p^2+1)2p dp}{p\{(p^2-1)^2+3(p)^2\}} = 2\int \frac{(p^2+1)}{p^4+1-2p^2+3p^2}dp = 2\int \frac{(p^2+1)dp}{(p^4+p^2+1)}$$

$$= 2\int \frac{\left(1+\frac{1}{p^2}\right)dp}{\left(p^2+1+\frac{1}{p^2}\right)} = 2\int \frac{\left(1+\frac{1}{p^2}\right)dp}{\left(p-\frac{1}{p}\right)^2+3} = \frac{2}{\sqrt{3}}\tan^{-1}\frac{1}{\sqrt{3}}\left(p-\frac{1}{p}\right) + c$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{p^2-1}{\sqrt{3}p}\right) + c = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}\sqrt{x+1}}\right) + c$$

$$= \frac{2}{\sqrt{3}}\left[-\tan^{-1}\frac{-x}{\sqrt{3}\sqrt{x+1}}\right] + c = \frac{2}{\sqrt{3}}\left[\cot^{-1}\frac{-x}{\sqrt{3}\sqrt{x+1}} - \frac{\pi}{2}\right] + c$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{-\sqrt{3}\sqrt{x+1}}{x}\right\} + c_1.$$

Problem 4: Evaluate $\int \frac{\sec x dx}{\sqrt{\sin(2x+A)+\sin A}}$.

Solution: Let I = $\int \frac{\sec x dx}{\sqrt{2\sin(x+A)\cos x}} = \int \frac{\sec^2 x dx}{\sqrt{\frac{2\sin(x+A)}{\cos x}}}$

$$= \int \frac{1}{\sqrt{2}} \int \frac{\sec^2 x\, dx}{\sqrt{\tan x \cos A + \sin A}} = \frac{\sec A}{\sqrt{2}} \int \frac{2pdp}{p}$$

(If tan x cos A + sin A = p^2, then cosA $\sec^2$ x d x = 2pdp)

i.e. I = $\sqrt{2}\sec A \int dp$

$= \sqrt{2}\sec A\sqrt{\tan x \cos A + \sin A} + c$.

Problem 5: Evaluate $\int \sin^4 x dx$.

Solution: Let I = $\int \sin^3 x \sin x dx = \sin^3 x(-\cos x) + \int 3\sin^2 x \cos^2 xdx$

$= -\sin^3 x\cos x + \int 3\sin^2 x(1-\sin^2 x)dx$

$= -\sin^3 x\cos x + 3\int \sin^2 x dx - 3\int \sin^4 xdx$

4I = $-\sin^3 x\cos x + 3\int \left(\frac{1-\cos 2x}{2}\right)dx$

I = $\frac{1}{4}\left[-\sin^3 x\cos x + \frac{3}{2}\left[x - \frac{\sin 2x}{2}\right]\right] + c$.

Alternative Method:

I = $\int \sin^4 xdx = \frac{1}{4}\int (1-\cos 2x)^2 dx = \frac{1}{4}\int (1-2\cos 2x + \cos^2 2x)dx$

$= \frac{1}{4}[x - \sin 2x] + \frac{1}{4}\int \left(\frac{\cos 4x + 1}{2}\right)dx = -\frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + \frac{3x}{8} + c$.

Problem 6: If I_n = $\int (\ln x)^n dx$ then prove that $I_n + nI_{n-1} = x(\ln x)^n$.

Solution: $I_n = x(\ln x)^n - \int \frac{x(n)(\ln x)^{n-1}}{x}dx = x(\ln x)^n - n I_{(n-1)}$

$\Rightarrow I_n + n I_{n-1} = x(\ln x)^n$.

Problem 7: Evaluate $\int \cos 2\theta \ln\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right)d\theta$.

Solution: Let I = $\int \cos 2\theta \ln\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right)d\theta = \int \cos 2\theta \ln\left(\frac{1+\sin 2\theta}{\cos 2\theta}\right)d\theta$

$= \ln\frac{1+\sin 2\theta}{\cos 2\theta}\cdot\frac{\sin 2\theta}{2} - \int \frac{\sin 2\theta}{2}\left\{\frac{2\cos 2\theta}{1+\sin 2\theta} + \frac{2\sin 2\theta}{\cos 2\theta}\right\}d\theta$

$= \frac{\sin 2\theta}{2}\ln\left(\frac{1+\sin 2\theta}{\cos 2\theta}\right) - \int \tan 2\theta\, d\theta$

$$= \frac{\sin 2\theta}{2}\ln\left(\frac{1+\sin 2\theta}{\cos 2\theta}\right) - \frac{1}{2}\ln|\sec 2\theta| + c\,.$$

Problem 8: Evaluate $I = \int \frac{(1+x)}{x(1+xe^x)^2}\,dx$**.**

Solution: Let $I = \int \frac{(1+x)}{x(1+xe^x)^2}\,dx$

$= \int \frac{(1+x)e^x}{(xe^x)(1+xe^x)^2}\,dx$ $(1+ xe^x = p,\ e^x\,(1+x)\,dx = dp)$

$= \int \frac{dp}{(p-1)p^2}$. Let $\frac{1}{(p-1)p^2} = \frac{A}{(p-1)} + \frac{B}{p} + \frac{C}{p^2}$

$\Rightarrow 1 = Ap^2 + B\,(p)\,(p-1) + C\,(p-1)$.

For p = 1, p = 0, and p = -1, A = 1, C= -1 and B= -1.

$\Rightarrow I = \int \frac{1}{(p-1)}\,dp - \int \frac{dp}{p} - \int \frac{dp}{p^2}$

$= \ln\frac{(p-1)}{p} + \frac{1}{p} + c = \ln\left(\frac{xe^x}{1+xe^x}\right) + \left(\frac{1}{1+xe^x}\right) + c\,.$

Problem 9: Evaluate $\int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)dx$ ***;*** $|x| < 1$**.**

Solution: Let $I = \int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)dx$

$= \int \sin^{-1}\left(\frac{2x+2}{(2x+2)^2+3^2}\right)dx = \int \tan^{-1}\left(\frac{2x+2}{3}\right)dx\,.$

Let $\frac{2x+2}{3} = t \Rightarrow dx = \frac{3}{2}dt$, so that

$I = \frac{3}{2}\int \tan^{-1} t\,dt = \frac{3}{2}\left[t\cdot\tan^{-1} t - \int\frac{t\,dt}{1+t^2}\right] = \frac{3}{2}\left[t\cdot\tan^{-1} t - \frac{1}{2}\ln(1+t^2)\right] + c$

$= (x+1)\tan^{-1}\left(\frac{2x+2}{3}\right) - \frac{3}{4}\ln\left(1+\left(\frac{2x+2}{3}\right)^2\right) + c\,.$

Problem 10:Evaluate $\int \frac{\sqrt{x-1}}{x\sqrt{x+1}}\,dx$**.**

Solution:Let $I = \int \frac{\sqrt{x-1}}{x\sqrt{x+1}}\,dx = \int \frac{(x-1)\,dx}{x\sqrt{x^2-1}} = \int \frac{dx}{\sqrt{x^2-1}} - \int \frac{dx}{x\sqrt{x^2-1}}$; $= \ln\left(x+\sqrt{x^2-1}\right) - \sec^{-1} x + c\,.$

OBJECTIVE

Problem 1: $\int \frac{dx}{\sqrt{2x-x^2}}$ ***is equal to***

(A) $\sin^{-1}(1-x)+c$ (B) $-\cos^{-1}(1-x)+c$

(C) $\sin^{-1}(x-1)+c$ (D) $\cos^{-1}(x-1)+c$

Solution: Let $I=\int \frac{dx}{\sqrt{1-(x^2-2x+1)}}=\int \frac{dx}{\sqrt{1-(x-1)^2}}=\sin^{-1}\left(\frac{x-1}{1}\right)+c$.

Hence (C) is the correct answer.

Problem 2: If $I=\int(\sqrt{\tan x}+\sqrt{\cot x})dx$ ***= f (x) + c then f (x) is equal to***

(A) $\sqrt{2}\sin^{-1}(\sin x-\cos x)$ (B) $\frac{\pi}{\sqrt{2}}-\sqrt{2}\cos^{-1}(\sin x-\cos x)$

(C) $\sqrt{2}\tan^{-1}\left(\frac{\tan x-1}{\sqrt{2}\sqrt{\tan x}}\right)$ (D) none of these

Solution: Let $I=\int(\sqrt{\tan x}+\sqrt{\cot x})dx=\int\sqrt{2}\frac{\sin x+\cos x}{\sqrt{2\sin x\cos x}}dx$.

If $\sin x-\cos x=p$ then $(\cos x+\sin x)$
$dx = dp$ so that

$$I=\sqrt{2}\int\frac{dp}{\sqrt{1-p^2}}=\sqrt{2}\sin^{-1}p+c=\sqrt{2}\sin^{-1}(\sin x-\cos x)+c$$

$$=\frac{\pi}{\sqrt{2}}-\sqrt{2}\cos^{-1}(\sin x-\cos x)$$

$$=\sqrt{2}\tan^{-1}\frac{\sin x-\cos x}{\sqrt{1-(\sin x-\cos x)^2}}=\sqrt{2}\tan^{-1}\frac{\sin x-\cos x}{\sqrt{2\sin x\cos x}}$$

$=\sqrt{2}\tan^{-1}\left(\frac{\tan x-1}{\sqrt{2\tan x}}\right)$. Hence (A), (B), (C) are correct answers.

Problem 3: If I = $\int\frac{1}{2p}\sqrt{\frac{p-1}{p+1}}dp=f(p)+c$ ***, then f(p) is equal to***

(A) $\frac{1}{2}\ln\left(p-\sqrt{p^2-1}\right)$ (B) $\left(\frac{1}{2}\cos^{-1}p+\frac{1}{2}\sec^{-1}p\right)$

(C) $\ln\sqrt{p+\sqrt{p^2-1}}-\frac{1}{2}\sec^{-1}p$ (D) none of these

Solution: Let $I=\int\frac{1}{2p}\sqrt{\frac{p-1}{p+1}}dp$ I

$$=\frac{1}{2}\int\frac{p-1}{p\sqrt{(p+1)(p-1)}}dp=\frac{1}{2}\int\frac{pdp}{p\sqrt{p^2-1}}-\frac{1}{2}\int\frac{dp}{p\sqrt{p^2-1}}$$

$=\frac{1}{2}\log_e(p+\sqrt{p^2-1})$ - $\frac{1}{2}\sec^{-1}p$. Hence (C) is the correct answer.

Problem 4: If $\int \sqrt{\dfrac{\cos x - \cos^3 x}{(1-\cos^3 x)}}dx$ ***= f (x) + c, then f (x) is equal to***

(A) $\dfrac{2}{3}\sin^{-1}\left(\cos^{\frac{3}{2}} x\right)$ (B) $\dfrac{3}{2}\sin^{-1}\left(\cos^{\frac{3}{2}} x\right)$ (C) $\dfrac{2}{3}\cos^{-1}\left(\cos^{\frac{3}{2}} x\right)$ (D) $-\dfrac{2}{3}\sin^{-1}\left(\cos^{\frac{3}{2}} x\right)$

Solution: Let I = $\int \sqrt{\dfrac{\cos x - \cos^3 x}{(1-\cos^3 x)}}dx = \int \dfrac{\sqrt{\cos x}\left(\sqrt{1-\cos^2 x}\right)}{\sqrt{1-\left(\cos^{\frac{3}{2}} x\right)^2}}dx$

If $\cos^{\frac{3}{2}} x = p$, then $\left(-\dfrac{3}{2}\cos^{\frac{1}{2}} x \sin x\right)dx = dp$

I = $-\dfrac{2}{3}\int \dfrac{dp}{\sqrt{1-p^2}}$, = $-\dfrac{2}{3}\sin^{-1}\left(\cos^{\frac{3}{2}} x\right)$ + c = $-\dfrac{2}{3}\cos^{-1}\left(\cos^{\frac{3}{2}} x\right)$ + c_1

Hence (C) and (D) are the correct answers.

Problem 5: $\int \dfrac{dx}{x(x^n+1)}$ ***is equal to***

(A) $\dfrac{1}{n}\log_e\left(\dfrac{x^n}{x^n+1}\right)+c$

(B) $-\dfrac{1}{n}\log_e\left(\dfrac{x^n+1}{x^n}\right)+c$

(C) $\log_e\left(\dfrac{x^n}{x^n+1}\right)+c$

(D) none of these

Solution: Let I = $\int \dfrac{dx}{x(x^n+1)} = \int \dfrac{dx}{x^{n+1}\left(1+\dfrac{1}{x^n}\right)}$.

If $\left(1+\dfrac{1}{x^n}\right) = p$, then $-\dfrac{n}{x^{n+1}}dx = dp$

$\Rightarrow$ I = $-\dfrac{1}{n}\int \dfrac{dp}{p} = \dfrac{-1}{n}\log_e p = -\dfrac{1}{n}\log_e\left(\dfrac{x^n+1}{x^n}\right)$.

Hence (B) is the correct answer.

Problem 6: $\int \dfrac{dx}{(1+\sqrt{x})\sqrt{(x-x^2)}}$ ***is equal to***

(A) $\dfrac{1+\sqrt{x}}{(1-x)^2}+c$ (B) $\dfrac{1+\sqrt{x}}{(1+x)^2}+c$ (C) $\dfrac{1-\sqrt{x}}{(1-x)^2}+c$ (D) $\dfrac{2(\sqrt{x}-1)}{\sqrt{(1-x)}}+c$

Solution: Let I = $\int \dfrac{dx}{(1+\sqrt{x})\sqrt{(x-x^2)}}$

If $\sqrt{x} = \sin p$, then $\dfrac{1}{2\sqrt{x}}dx = \cos p\, dp$

$$\Rightarrow I = \int \frac{2\sin p\cos p\,dp}{(1+\sin p)\sin p\cos p} = 2\int \frac{dp}{(1+\sin p)} = 2\int \frac{(1-\sin p)dp}{\cos^2 p}$$

$$= 2\left\{\int \sec^2 p\,dp - \int (\tan p\sec p)dp\right\}$$

$$= 2\,(\tan p - \sec p) = 2\left(\sqrt{\frac{x}{(1-x)}} - \frac{1}{\sqrt{(1-x)}}\right) + c = \frac{2\left(\sqrt{x}-1\right)}{\sqrt{(1-x)}} + c\,.$$

Hence (D) is the correct answer.

Problem 7: $\int \frac{x^4-1}{x^2(x^4+x^2+1)^{1/2}}dx$ ***is equal to***

(A) $\sqrt{\frac{x^4+x^2+1}{x}}+c$

(B) $\sqrt{x^2+1+\frac{1}{x^2}}+c$

(C) $\sqrt{\frac{x^4+x^2+1}{x^2}}+c$

(D) none of these

Solution: Let I = $\int \frac{x^4-1}{x^2(x^4+x^2+1)^{1/2}}dx$

$$= \int \frac{x^3\left(x-\frac{1}{x^3}\right)dx}{x^3\sqrt{x^2+\frac{1}{x^2}+1}}\,.\quad \text{If } x^2+\frac{1}{x^2}+1 = p^2, \text{ then} \left(2x-\frac{2}{x^3}\right)dx = 2p\,dp$$

$$\Rightarrow\ I = \int \frac{p\,dp}{p} = p + c = \sqrt{\frac{x^4+x^2+1}{x^2}} + c\,.$$

Hence (B) and (C) are correct answers.

Problem 8: $\int \frac{dx}{\cos^6 x+\sin^6 x}$ ***is equal to***

(A) $\log_e(\tan x - \cot x) + c$

(B) $\log_e(\cot x - \tan x) + c$

(C) $\tan^{-1}(\tan x - \cot x) + c$

(D) $\tan^{-1}(-2\cot 2x) + c$

Solution: Let I = $\int \frac{dx}{\cos^6 x+\sin^6 x} = \int \frac{\sec^6 x}{1+\tan^6 x}dx = \int \frac{(1+\tan^2 x)^2\sec^2 x\,dx}{1+\tan^6 x}$.

If $\tan x = p$, then $\sec^2 x\,dx = dp$

$$\Rightarrow I = \int \frac{(1+p^2)^2\,dp}{1+p^6} = \int \frac{(1+p^2)}{p^4-p^2+1}dp = \int \frac{p^2\left(1+\frac{1}{p^2}\right)}{p^2\left(p^2+\frac{1}{p^2}-1\right)}dp$$

$$= \int \frac{dk}{k^2+1} = \tan^{-1}(k) + c \quad \left(p-\frac{1}{p} = k,\ \left(1+\frac{1}{p^2}\right)dp = dk\right)$$

$$= \tan^{-1}\left(p-\frac{1}{p}\right)+c = \tan^{-1}(\tan x - \cot x) + c = \tan^{-1}(-2\cot 2x) + c\,.$$

Hence (C) and (D) are correct answers.

Problem 9: $\int \frac{dx}{(x+a)^{\frac{8}{7}}(x-b)^{\frac{6}{7}}}$ ***is equal to***

(A) $\left(\frac{7}{a+b}\right)\left(\frac{x+a}{x-b}\right)^{\frac{1}{7}}+c$ (B) $\left(\frac{7}{a+b}\right)\left(\frac{x-b}{x+a}\right)^{\frac{1}{7}}+c$

(C) $\frac{6}{a+b}\left(\frac{x-b}{x+a}\right)^{\frac{1}{6}}+c$ (D) $\frac{6}{a+b}\left(\frac{x+a}{x-b}\right)^{\frac{1}{6}}+c$

Solution: Let $I=\int \frac{dx}{(x+a)^{\frac{8}{7}}(x-b)^{\frac{6}{7}}}=\int \frac{dx}{(x+a)^2\left(\frac{x-b}{x+a}\right)^{\frac{6}{7}}}$.

If $\left(\frac{x-b}{x+a}\right)=p$, then $\frac{a+b}{(x+a)^2}dx=dp$ $\Rightarrow$ $I=\frac{1}{a+b}\int \frac{dp}{p^{\frac{6}{7}}}$

$=\frac{7}{a+b}\left(p^{\frac{1}{7}}\right)=\left(\frac{7}{a+b}\right)\left(\frac{x-b}{x+a}\right)^{\frac{1}{7}}+c$. Hence (B) is the correct answer.

Problem 10: $\int \frac{3+2\cos x}{(2+3\cos x)^2}dx$ ***is equal to***

(A) $\left(\frac{\sin x}{3\cos x+2}\right)+c$ (B) $\left(\frac{2\cos x}{3\sin x+2}\right)+c$ (C) $\left(\frac{2\cos x}{3\cos x+2}\right)+c$ (D) $\left(\frac{2\sin x}{3\sin x+2}\right)+c$

Solution: Let $I=\int \frac{3+2\cos x}{(2+3\cos x)^2}dx$.

Multiplying Nr. & Dr. by $\operatorname{cosec}^2 x$

$$\Rightarrow I=\int \frac{(3\operatorname{cosec}^2 x+2\cot x\operatorname{cosec} x)}{(2\operatorname{cosec} x+3\cot x)^2}dx=-\int \frac{-3\operatorname{cosec}^2 x-2\cot x\operatorname{cosec} x}{(2\operatorname{cosec} x+3\cot x)^2}dx$$

$$=\frac{1}{2\operatorname{cosec} x+3\cot x}=\left(\frac{\sin x}{2+3\cos x}\right)+c\ .$$

Hence (A) is the correct answer.

Alternative Solution:

$$I=\int \frac{3\sin^2 x+3\cos^2 x+2\cos x}{(2+3\cos x)^2}dx$$

$$=\int \frac{\cos x}{(2+3\cos x)}dx+\int \frac{3\sin x\cdot\sin x}{(2+3\cos x)^2}dx$$

$$=\int \frac{\cos x}{2+3\cos x}dx+\frac{\sin x}{2+3\cos x}-\int \frac{\cos x}{2+3\cos x}dx=\frac{\sin x}{2+3\cos x}+c\ .$$

Problem 11: $\int \left(x+\frac{1}{x}\right)^{n+5}\left(\frac{x^2-1}{x^2}\right)dx$ ***is equal to***

(A) $\dfrac{\left(x+\dfrac{1}{x}\right)^{n+6}}{n+6}+c$ (B) $\left(\dfrac{x^2+1}{x^2}\right)^{n+6}(n+6)+c$ (C) $\left(\dfrac{x}{x^2+1}\right)^{n+6}(n+6)+c$ (D) none of these

Solution: Here I $=\int p^{n+5}\,dp$.

If $x+\dfrac{1}{x}=p$ then, $\left(1-\dfrac{1}{x^2}\right)dx=dp$,

$$\Rightarrow I=\int\left(x+\frac{1}{x}\right)^{n+5}\left(\frac{x^2-1}{x^2}\right)dx=\int p^{n+5}dp$$

$=\dfrac{p^{n+6}}{n+6}+c=\dfrac{\left(x+\dfrac{1}{x}\right)^{n+6}}{n+6}+c$. Hence (A) is the correct answer.

Problem 12: $\displaystyle\int\frac{dx}{x+\sqrt{a^2-x^2}}$ ***is equal to***

(A) $\dfrac{1}{2}\sin^{-1}\dfrac{x}{a}+\ln\sqrt{x+\sqrt{a^2-x^2}}+c$

(B) $\dfrac{1}{2}\sin^{-1}\dfrac{x}{a}+\ln(x\sqrt{a^2-x^2})+c$

(C) $\dfrac{1}{2}\sin^{-1}\dfrac{x}{a}+\ln(\sqrt{a+\sqrt{a^2-x^2}})+c$

(D) none of these.

Solution: Here $I=\displaystyle\int\frac{\cos\theta}{\sin\theta+\cos\theta}d\theta$ ($x = a\sin\theta$, $dx = a\cos\theta\, d\theta$)

$$=\frac{1}{2}\int d\theta+\frac{1}{2}\int\frac{\cos\theta-\sin\theta}{\sin\theta+\cos\theta}d\theta$$

$$=\frac{1}{2}\theta+\frac{1}{2}\ln(\sin\theta+\cos\theta)\quad=\frac{1}{2}\sin^{-1}\frac{x}{a}+\ln\sqrt{x+\sqrt{a^2-x^2}}+c.$$

Hence (A) is the correct answer.

Problem 13: $\displaystyle\int\frac{dx}{\sqrt{x+1}+(x+1)^{\frac{1}{3}}}$ ***is equal to***

(A) $2p^3-3P^2+6p-6\ln(1+p)+c$, where $p=(x+1)^{\frac{1}{6}}$

(B) $2p^3+6p-6\ln(1+p)+c$, where $p=(x+1)^{\frac{1}{6}}$

(C) $2p^3+3p^2+6p-6\ln(1+p)+c$, where $p=(x+1)^{\frac{1}{6}}$

(D) none of these

Solution: Here I $=\displaystyle\int\frac{6p^5dp}{p^3+p^2}$ ($(x+1)=p^6$, then $dx=6p^5\,dp$).

$$=6\int p^2dp-6\int pdp+6\int dp\;-6\int\frac{p^2}{p^2(p+1)}dp$$

$$=\frac{6p^3}{3}-\frac{6p^2}{2}+6p-6\ln(p+1)+c$$

$$=2p^3-3p^2+6p-6\ln(1+p)+c,\quad\text{where}\quad p=(x+1)^{\frac{1}{6}}.$$

Hence (A) is the correct answer.

Problem 14: $\int \frac{(x^2-1)}{(x^2+1)\sqrt{x^4+1}}dx$ ***is equal to***

(A) $\sec^{-1}\left(\frac{x^2+1}{\sqrt{2}\,x}\right)+c$ (B) $\frac{1}{\sqrt{2}}\sec^{-1}\left(\frac{x^2+1}{\sqrt{2}\,x}\right)+c$ (C) $\frac{1}{\sqrt{2}}\sec^{-1}\left(\frac{x^2+1}{\sqrt{2}}\right)+c$ (D) none of these

Solution: Let I = $\int \frac{x^2\left(1-\frac{1}{x^2}\right)dx}{x^2\left(x+\frac{1}{x}\right)\left(x^2+\frac{1}{x^2}\right)^{1/2}}$. Let $x+\frac{1}{x}=p \Rightarrow \left(1-\frac{1}{x^2}\right)dx=dp$

$\Rightarrow$ I = $\int \frac{dp}{p\sqrt{p^2-2}}=\frac{1}{\sqrt{2}}\sec^{-1}\frac{p}{\sqrt{2}}=\frac{1}{\sqrt{2}}\sec^{-1}\left(\frac{x^2+1}{\sqrt{2}x}\right)+c$.

Hence (B) is the correct answer.

Problem 15: If $\int xe^x \cos x dx. = f(x)+c$***, then f (x) is equal to***

(A) $\frac{e^x}{2}\{(1-x)\sin x - x\cos x\}+c$

(B) $\frac{e^x}{2}\{(x-1)\sin x + x\cos x\}+c$

(C) $\frac{e^x}{2}\{(1+x)\sin x - x\cos x\}+c$

(D) none of these

Solution: Here I = Rear part of $\int xe^{(1+i)x}dx$

$= \frac{xe^{(1+i)x}}{(1+i)}-\int\frac{e^{(1+i)x}}{(1+i)}dx \quad = \frac{xe^{(1+i)x}}{(1+i)}-\frac{e^{(1+i)x}}{(1+i)^2}$

$= e^{(1+i)x}\left\{\frac{x(1+i)-1}{(1+i)^2}\right\}$

$= e^x[\cos x + i\sin x]\left[\frac{(x-1)+ix}{1+2i-1}\right] = \frac{e^x}{-2}\{i\cos x-\sin x\}\{(x-1)+ix\}$

$= \frac{e^x}{-2}[(1-x)\sin x - x\cos x]+c = \frac{e^x}{2}[x\cos x+(x-1)\sin x]+c.$

Hence (B) is the correct answer.

Practice Questions

VERY SHORT ANSWER TYPE QUESTIONS

1. $\int \frac{x^3}{1+x^4}dx$

2. $\int \frac{e^x.x}{(1+x)^2}dx$

3. $\int \frac{\sin^{-1}x}{\sqrt{1-x^2}}dx$

4. $\int \frac{3\cos 3x - 2\sin 2x}{\cos 2x + \sin 3x}dx$

5. $\int \frac{1}{x^2+2x+5}dx$

6. $\int \sqrt{5+x^2}\,dx$

7. $\int \frac{1}{x\log x}dx$

8. $\int \frac{dx}{e^x+e^{-x}}$

9. $\int \frac{dx}{\sqrt{2-x^2}}$

10. $\int x.2^x dx$

11. $\int e^{ax}.\cos(bx)dx$

12. $\int \frac{1+\sin^2 x}{1+\cos 2x}dx$

13. $\int \frac{x}{\sqrt{1+x^2}}dx$

14. $\int xe^x dx$

15. $\int \cot 2x\,dx$

16. $\int e^{ax}\sin(bx)\,dx$

17. $\int e^x(\cos x - \sin x)\,dx$

18. $\int \frac{dx}{9x^2+16}$

19. $\int x\log x\,dx$

20. $\int e^{\log x}.\cos x\,dx$

21. $\int x^2 e^x\,dx$

22. $\int \frac{\sin x + \cos x}{\sqrt{1+\sin 2x}}dx$

23. $\int \frac{dx}{x^2+6x+10}$

24. $\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx$

25. $\int e^x \sec x(1+\tan x)\,dx$

26. $\int \log x\,dx$

27. $\int \sqrt{16+x^2}\,dx$

28. $\int \sin^{-1}x\,dx$

29. $\int e^{3x}.\cos 2x\,dx$

30. $\int \tan^2 x\,dx$

31. $\int \frac{dx}{\sqrt{x^2+16}}$

32. $\int \frac{dx}{(x+2)\sqrt{x+1}}$

33. $\int \frac{dx}{1+\cos x}$

34. $\int x\cos 2x\,dx$

35. $\int e^{2\log\cot x}\,dx$

36. $\int \sqrt{6-x^2}\,dx$

37. $\int (\log x)^2\ dx$

38. $\int \frac{\log(1+x)}{1+x}\ dx$

39. $\int \frac{e^{\ln x}}{x}\ dx$

40. $\int \frac{x^8}{1+x^{18}}\ dx$

41. $\int \frac{dx}{1+x^4}$

42. $\int \frac{x^2-1}{x^4+1}\ dx$

SHORT ANSWER TYPE QUESTIONS

1. $\int \frac{dx}{x+x^3}$

2. $\int \frac{xdx}{(x^2+a^2)(x^2+b^2)}$

3. $\int \frac{dx}{5+4\cos x}$

4. $\int \frac{2\sin x+3\cos x}{3\sin x+4\cos x}dx$

5. $\int \frac{dx}{5+4\sin x}$

6. $\int \frac{x^8}{x^6+1}dx$

7. $\int \frac{x^3dx}{1+x^8}$

8. $\int \frac{dx}{1+\sin x+\cos x}$

9. $\int \frac{1-x^4}{1-x}dx$

10. $\int \frac{dx}{3+4\cos x}$

11. $\int \frac{\cos x}{3\cos x+4\sin x}dx$

12. $\int \frac{dx}{\sqrt{5x-6-x^2}}$

13. $\int \frac{3x-1}{2x^2-4x+3}dx$

14. $\int \sqrt{3x^2-4x+2}\ dx$

15. $\int e^x \frac{(x+2)}{(x+3)^2}dx$

16. $\int \frac{\tan^{-1}x}{x^2}dx$

17. $\int \frac{x^2}{x^6+2x^3+2}dx$

18. $\int \sqrt{\frac{5-x}{x-2}}dx$

19. $\int \frac{1+x^2}{\sqrt{1-x^2}}\ dx$

20. $\int (5x-3)\ \sqrt{x^2-6x+13}\,dx$

Level - I

1. If $\int \frac{\cos 4x + 1}{\cot x - \tan x} dx$ = k cos 4x + c, then

(A) $k = -\frac{1}{2}$ (B) $k = -\frac{1}{8}$ (C) $k = -\frac{1}{4}$ (D) none of these

2. $\int \frac{e^x(1+\sin x)}{1+\cos x}$ dx is equal to

(A) log |tan x| + c (B) $e^x \tan\left(\frac{x}{2}\right)$ + c (C) sin e^x cot x + c (D) e^x cot x + c

3. $\int \cos\sqrt{x}\, dx$ is equal to

(A) $2[\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x}] + c$ (B) $\sin\sqrt{x} + c$

(C) $2[\sqrt{x}\cos\sqrt{x} - \sin\sqrt{x}] + c$ (D) none of these

4. $\int \frac{dx}{a^2\cos^2 x + b^2 \sin^2 x}$ is equal to

(A) $\tan^{-1}\left(\frac{a}{b}\tan x\right) + c$ (B) $\frac{1}{ab}\tan^{-1}\left(\frac{b}{a}\cot x\right) + c$

(C) $\frac{1}{ab}\tan^{-1}\left(\frac{b}{a}\tan x\right) + c$ (D) $\tan^{-1}\left(\frac{b}{a}\tan x\right) + c$

5. $\int e^x \sec x(1+\tan x)\,dx$ is equal to

(A) e^x sec x + c (B) e^x sec x tan x + c (C) e^x tan x + c (D) none of these

6. $\int \left(x + \frac{1}{x}\right)^{3/2}\left(\frac{x^2-1}{x^2}\right)dx$ is equal to

(A) $\frac{5}{2}\left(x+\frac{1}{x}\right)^{5/2} + c$ (B) $\frac{2}{5}\left(x+\frac{1}{x}\right)^{5/2} + c$ (C) $2\left(x+\frac{1}{x}\right)^{1/2} + c$ (D) none of these

7. $\int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}} =$

(A) $\frac{2}{3}\cdot\frac{1}{(a-b)}\left[(x+a)^{3/2} - (x+b)^{3/2}\right] + c$ (B) $\frac{1}{2}\cdot\frac{1}{(a-b)}\left[(x+a)^{1/2} - (x+b)^{1/2}\right] + c$

(C) $\frac{3}{2}\cdot\frac{1}{(a-b)}\left[(x+a)^{3/2} + (x+b)^{3/2}\right] + c$ (D) none of these

8. $\int \frac{x^2+1}{\sqrt[3]{(x^3+3x+6)}}dx =$

(A) $\frac{1}{2}(x^3+3x+6)^{-1/2} + c$ (B) $-\frac{1}{2}(x^3+3x+6)^{1/2} + c$

(C) $\frac{1}{2}(x^3+3x+6)^{2/3} + c$ (D) none of these

9. $\int \sec^4 x\, dx =$

(A) $\tan x + \frac{\tan^2 x}{3} + c$ (B) $\tan x + \frac{\tan^3 x}{3} + c$

(C) $\tan x + \frac{\tan^4 x}{3} + c$ (D) $\frac{\tan^4 x}{4} + c$

10. $\int_0^{\pi/2} \sin^6\theta\cos^3\theta\, d\theta =$

(A) $\frac{2}{65}$ (B) $\frac{2}{63}$ (C) $\frac{1}{63}$ (D) $\frac{3}{130}$

11. If $\int \frac{\sqrt{\cot x}}{\sin x\cos x} dx = A\sqrt{\cot x} + B$, then A =

(A) 1 (B) 2 (C) −1 (D) −2

12. If $\int \frac{(x^2-1)}{(x^4+3x^2+1)\tan^{-1}\left(\frac{x^2+1}{x}\right)} dx = k \log\left|\tan^{-1}\frac{x^2+1}{x}\right| + c$, then k is equal to

(A) 1 (B) 2 (C) 3 (D) 5

13. $\int \frac{\cos 2x}{\cos x} dx$ is equal to

(A) 2sinx + log|(secx – tanx)| + c (B) 2sinx – log |(secx – tanx)| + c

(C) 2sinx + log |(secx + tanx)| + c (D) 2sinx – log|(secx + tanx)| + c

14. $\int e^x \frac{1+\sin x}{1+\cos x} dx$ is

(A) $\frac{e^x}{1+\cos x} + c$ (B) $e^x \cot\frac{x}{2} + c$ (C) $e^x \tan\frac{x}{2} + c$ (D) None of these

15. $\int x^{13/2}\sqrt{1+x^{5/2}}\, dx$ is equal to

(A) $\frac{4}{5}\left[\frac{1}{7}(1+x^{5/2})^{7/2} - \frac{2}{5}(1+x^{5/2})^{5/2} + \frac{1}{3}(1+x^{5/2})^{3/2} + c\right]$

(B) $\frac{4}{5}\left[\frac{1}{7}(1+x^{5/2})^{7/2} - \frac{1}{5}(1+x^{5/2})^{5/2} + (1+x^{5/2})^{3/2} + c\right]$

(C) $\frac{4}{5}\left[(1+x^{5/2})^{7/2} - \frac{2}{5}(1+x^{5/2})^{5/2} + (1+x^{5/2})^{3/2} + c\right]$

(D) none of these

16. If $\int f(x)\cos x\,dx = \frac{1}{2}f^2(x) + c$, then f(x) can be

(A) x (B) 1 (C) cosx (D) sinx

17. The value of the integral $\int e^{\sin^2 x}(\cos x + \cos^3 x)$ sinx dx is

(A) $\frac{1}{2}e^{\sin^2 x}(3-\sin^2 x) + c$ (B) $e^{\sin^2 x}\left(1+\frac{1}{2}\cos^2 x\right) + c$

(C) $e^{\sin^2 x}(3\cos^2 x + 2\sin^2 x) + c$ (D) $e^{\sin^2 x}(2\cos^2 x + 3\sin^2 x) + c$

18. $\int \frac{dx}{\sqrt{2x-x^2}}$ is equal to

(A) $\sin^{-1}(1-x)+c$
(B) $-\cos^{-1}(1-x)+P$
(C) $\sin^{-1}(x-1)+c$
(D) $\cos^{-1}(x-1)+P$

19. $I = \int \frac{dx}{1+e^x}dx$ is equal to

(A) $\log_e\left(\frac{1+e^x}{e^x}\right)+c$
(B) $\log_e\left(\frac{e^x}{1+e^x}\right)+c$
(C) $\log_e (e^x)(e^x+1)+c$
(D) $\log_e\left(e^{2x}+1\right)+c$

20. $I = \int e^{\tan^{-1}x}\left(\frac{1+x^2+x}{1+x^2}\right)dx$ is equal to

(A) $xe^{\tan^{-1}x}+c$
(B) $x^2 e^{\tan^{-1}x}+c$
(C) $\frac{1}{x}e^{\tan^{-1}x}+c$
(D) none of these

21. The antiderivative of $\frac{2^x}{\sqrt{1-4^x}}$ w.r.t x is

(A) $\log_2 e\cdot\sin^{-1}\left(2^x\right)+k$
(B) $\sin^{-1}\left(2^x\right)+k$
(C) $\cos^{-1}\left(2^x\right)\log_2 e+k$
(D) none of these

22. The antiderivative of a periodic function is a

(A) periodic function always
(B) non periodic function always
(C) some times periodic and sometimes non periodic
(D) none of these

23. $\int \frac{dx}{x\sqrt{x^4-1}}$ is equal to

(A) $\sec^{-1}x^2+c$
(B) $\frac{1}{2}\sec^{-1}x^2+c$
(C) $\tan^{-1}x^2+c$
(D) $\operatorname{cosec}^{-1}x^2+c$

24. $\int \frac{dx}{\sec^2 x+\tan^2 x}$ is equal to

(A) $\sqrt{2}\tan^{-1}(\sqrt{2}\tan x)+x+c$
(B) $\sqrt{2}\tan^{-1}(\sqrt{2}\tan x)-x+c$
(C) $\sqrt{2}\tan^{-1}(2\tan x)+c$
(D) none of these

25. $\int \frac{x^2+\sin^2 x}{1+x^2}\sec^2 x dx$ is equal to

(A) tan x +c
(B) $\tan x-\tan^{-1}x+c$
(C) $\tan x+\tan^{-1}x+c$
(D) none of these

*****

Level - II

1. $\int \frac{dx}{\sqrt{1+x}+\sqrt{x}}$ is equal to

(A) $\frac{2}{3}(1+x)^{2/3}-\frac{2}{3}x^{2/3}+c$
(B) $\frac{3}{2}(1+x)^{2/3}+\frac{3}{2}x^{3/2}+c$
(C) $\frac{3}{2}(1+x)^{3/2}+\frac{3}{2}x^{3/2}+c$
(D) $\frac{2}{3}(1+x)^{3/2}-\frac{2}{3}x^{3/2}+c$

2. The value of $\int \frac{dx}{(e^x+1)(2e^x+3)}$ is equal to

(A) $x+\ln(e^x+1)-\frac{2}{3}\ln(2e^x+3)+c$
(B) $\frac{1}{3}x-\ln(e^x+1)+\frac{2}{3}\ln(2e^x+3)+c$
(C) $x-\frac{2}{3}\ln(e^x+1)+\ln(2e^x+3)+c$
(D) none of these

3. The value of $\int \frac{\cos^3 x dx}{\sin^2 x+\sin x}$ is equal to

(A) log sin x – sin x + c
(B) log |sin x| – sin x + c
(C) log |sin x| + c
(D) none of these

4. The value of $\int \frac{dx}{x(x^n+1)}$ is equal to

(A) $\log\left|\frac{x^n}{1+x^n}\right|+c$
(B) $\frac{1}{n}\log\left|\frac{x^n+1}{x^n}\right|+c$
(C) $\log\left|\frac{x^n+1}{x^n}\right|+c$
(D) $\frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right|+c$

5. The value of $\int \frac{1+x+x^2}{1+x^2}e^{\tan^{-1}x}dx$ is equal to

(A) $x^2e^{\tan^{-1}x}$
(B) $e^{\tan^{-1}x}+c$
(C) $xe^{\tan^{-1}x}+c$
(D) none of these

6. $\int \frac{\tan x}{\sqrt{\cos x}}dx$ is equal to

(A) $\frac{2}{\sqrt{\sin x}}+c$
(B) $\frac{2}{\sqrt{\cos x}}+c$
(C) $\frac{2}{\sqrt{\tan x}}+c$
(D) $\frac{2}{(\sin x)^{3/2}}+c$

7. $\int \frac{dx}{x\ln x\ln(\ln x)}$ is equal to

(A) ln |(ln (ln x))| + c
(B) |ln x| + c
(C) $\ln\left|\ln\left(\frac{1}{x}\right)\right|+c$
(D) ln |ln x| + c

8. Value of the integral $\int e^x\left[\frac{1+\sqrt{1-x^2}\sin^{-1}x}{\sqrt{1-x^2}}\right]dx$ equals to

(A) $e^x\sin^{-1}x+c$
(B) $\frac{e^x}{\sqrt{1-x^2}}+c$
(C) $e^x\sqrt{1-x^2}+c$
(D) $\sqrt{1-x^2}\sin^{-1}x+c$

9. If f(x) = x – [x] : for every real number of x, when [x] is the integral point of x. Then $\int_{-1}^{1} f(x)dx$ is equal to

(A) 1
(B) 2
(C) 0
(D) $-\frac{1}{2}$

10. Let $f(x)=\int_1^x \sqrt{2-t^2}dt$. Then the real roots of the equation $x^2-f'(x)=0$ are

(A) (0, 1)
(B) $\pm\frac{1}{\sqrt{2}}$
(C) $\pm\frac{1}{2}$
(D) ± 1

11. Let T > 0, be a fixed real number. Suppose f is a continuous function such that for all $x \in R$, $f(x+T) = f(x)$. If $I = \int_0^T f(x)dx$, then value of $\int_3^{3+3T} f(2x)dx$ is

(A) $-\frac{3}{2}I$ (B) 2I (C) 3I (D) 6I

12. The value of $\int_0^{100} a^{x-[x]}dx$ is

(A) $\frac{100(a-1)}{\log a}$ (B) $\frac{100(a+1)}{\log a}$ (C) 100(a -1) (D) none of these

13. Let f: $(0, \infty) \to R$ and $F(x^2) = \int_0^{x^2} f(t)dt$. If $F(x^2) = x^2(1+x)$, then f(4) is equal to

(A) $\frac{5}{4}$ (B) 4 (C) 7 (D) 2

14. The value of $\int \frac{x^3}{1+x^8}dx$ is

(A) $\frac{1}{4}\tan^{-1}x^4 + c$ (B) $\frac{1}{2}\tan^{-1}x^4 + c$ (C) $\frac{1}{4}\cot^{-1}x^2 + c$ (D) none of these

15. The value of $\int xe^x dx$ is

(A) $xe^x + e^x + c$ (B) $xe^x - e^x + c$ (C) $-xe^x + e^x + c$ (D) none of these

16. $\int \frac{dx}{\sqrt{2x-x^2}}$ is equal to

(A) $\sin^{-1}(1-x)+c$ (B) $-\cos^{-1}(1-x)+P$ (C) $\sin^{-1}(x-1)+c$ (D) $\cos^{-1}(x-1)+P$

17. If $I = \int \frac{1}{2p}\sqrt{\frac{p-1}{p+1}}dp = f(p) + c$, then f(p) is equal to

(A) $\frac{1}{2}\ln\left(p-\sqrt{p^2-1}\right)$ (B) $\left(\frac{1}{2}\cos^{-1}p + \frac{1}{2}\sec^{-1}p\right)$

(C) $\ln\sqrt{p+\sqrt{p^2-1}} - \frac{1}{2}\sec^{-1}p$ (D) none of these

18. $\int \frac{\sin\theta+\cos\theta}{\sqrt{(\sin 2\theta)}}d\theta$ is equal to

(A) $\sin^{-1}(\sin\theta+\cos\theta)$ (B) $\sin^{-1}\theta(\sin\theta-\cos\theta)$ (C) $\sin^{-1}(\cos\theta-\sin\theta)$ (D) none of these

19. If $\int x^6 \sin(5x^7)dx = \frac{k}{5}\cos(5x^7)$, $x \neq 0$ then

(A) k = 7 (B) k = –7 (C) $k = \frac{1}{7}$ (D) $k = -\frac{1}{7}$

20. $\int x\log x(\log x - 1)\, dx$ is equal to

(A) $2(x\log x - x)^2 + c$ (B) $\frac{1}{2}(x\log x - x)^2 + c$ (C) $(x\log x)^2 + c$ (D) $\frac{1}{2}(x\log x)^2 + c$

21. $\int \sqrt{\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)}dx$ is equal to

(A) $\cos^{-1}\sqrt{x} + \sqrt{1-x}.(\sqrt{x}-2)+c$ (B) $\cos^{-1}\sqrt{x} - \sqrt{1-x}.(\sqrt{x}-2)+c$

(C) $\cos^{-1}\sqrt{x} + \sqrt{1-x}.(\sqrt{x-2})+c$ (D) None of these

22. $\int \frac{xe^x}{(x+1)^2}dx$ is equal to

(A) $\frac{2e^x}{x+1}$ (B) $\frac{e^x}{(x+1)^2}$ (C) $-\frac{e^x}{(x+1)^3}$ (D) $\frac{e^x}{x+1}$

23. $\int \frac{dx}{\sqrt{2x-x^2}}$ is equal to

(A) $\sin^{-1}(1-x)+c$ (B) $-\cos^{-1}(1-x)+P$ (C) $\sin^{-1}(x-1)+c$ (D) $\cos^{-1}(x-1)P$

24. $\int e^x(4x^2+8x+3)dx$ equals

(A) $(2x+1)^2 e^x + k$ (B) $(x+1)^2 e^x + k$ (C) $(4x+2)e^x + k$ (D) none of these

25. $\int \frac{\{f(x)\cdot\phi'(x)-f'(x)\cdot\phi(x)\}}{f(x)\cdot\phi(x)}\cdot\{\log\phi(x)-\log f(x)\}dx$ is equal to

(A) $\log\frac{\phi(x)}{f(x)}+k$ (B) $\frac{1}{2}\left\{\log\frac{\phi(x)}{f(x)}\right\}^2+k$ (C) $\frac{\phi(x)}{f(x)}\log\frac{\phi(x)}{f(x)}+k$ (D) none of these

KEY

SESSION – 1

CLASS EXERCISE :

1.

HOME EXERCISE :

1.c	2.a	3.d	4.d	5.a	6.b	7.c

SESSION – 2

CLASS EXERCISE :

1.c	2.a	3.c	4.d	5.b	6.d

HOME EXERCISE :

1.a	2.b	3.d	4.a	5.a	6.a	7.c
8.b						

SESSION – 3

CLASS EXERCISE :

1.b	2.b	3.a	4.b	5.b	6.a

HOME EXERCISE :

1.a	2.b	3.d	4.b	5.b	6.c	7.a
8.a						

SESSION – 4

CLASS EXERCISE :

1.b	2.c	3.b	4.b	5.c	6.c

HOME EXERCISE :

1.a	2.a	3.b	4.b	5.a	6.b	7.a
8.b						

SESSION – 5 & 6

CLASS EXERCISE :

1.b	2.a	3.b	4.a	5.a	6.a	

HOME EXERCISE :

1.c	2.c	3.a	4.a	5.b	6.a	7.b
8.a	9.a					

SESSION – 7 & 8

CLASS EXERCISE :

1.a	2.a	3.b	4.a	5.d	6.a	7.a
8.a						

HOME EXERCISE :

1.d	2.d	3.d	4.a	5.c	6.c	7.a
8.b	9.e	10.d				

Level - I

1.	B	2.	B	3.	A	4.	C
5.	A	6.	B	7.	A	8.	C
9.	B	10.	A	11.	D	12.	A
13.	D	14.	C	15.	A	16.	D
17.	AB	18.	C	19.	B	20.	B
21.	A	22.	C	23.	B	24.	B
25.	B						

Level - II

1.	D	2.	B	3.	B	4.	D
5.	C	6.	B	7.	D	8.	A
9.	A	10.	D	11.	C	12.	A
13.	B	14.	A	15.	B	16.	C
17.	C	18.	C	19.	D	20.	A
21.	D	22.	B	23.	C	24.	D
25.	B						

*****