DPP (Definite Integral)

DPP # 1

Theory :

Basic definition, Geometrical meaning
Properties of definite integration

P-1 $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$,   P-2 $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$,   P-3 $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Examples

1(e) Evaluate : $\int_1^2 \ell n\, x\, dx$ **Ans.** $\frac{4}{e}$

2(e) Evaluate : $\int_1^2 x^2\, d(\ell n\, x)$ **Ans.** $\frac{e^4 - e^2}{2}$

3(e) If 2a + 3b + 6c = 0 then equation $ax^2 + bx + c = 0$ will have
(A*) atleast one root in (0, 1)   (B) atleast one root in (0, 2)
(C) exactly one root in (0, 1)   (D) exactly one root in (0, 2)

4(e) If f(x) = $\begin{cases} x+3 & : x<3 \\ 3x^2+1 & : x\ge 3 \end{cases}$ then find $\int_2^5 f(x)\,dx$ **Ans.** $\frac{211}{2}$

Evaluate the following :

5(e) $\int_0^3 [x]\,dx$ **Ans.** 3

6(e) $\int_0^2 |2x-3|\,dx$ **Ans.** $\frac{5}{2}$

7(e) $\int_0^{2\pi} |1+2\cos x|\,dx$ **Ans.** $4\sqrt{3} + \frac{2\pi}{3}$

8(e) $\int_0^9 [\sqrt{t}]\,dt$ **Ans.** 13

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9. $\int_{-1}^3 \left[x + \frac{1}{2}\right] dx$ **Ans.** 4

10. $\int_0^{\pi} \sqrt{\frac{1+\cos 2x}{2}}\,dx$ **Ans.** 2

11. $\int_0^2 |x^2 + 2x - 3|\,dx$ **Ans.** 4

12. $\int_0^2 [x^2 - x + 1]\,dx$ **Ans.** $\frac{5-\sqrt{5}}{2}$

13. $\int_{-\pi/2}^{2\pi} \sin^{-1}(\sin x)\,dx$ **Ans.** $-\frac{\pi^2}{8}$

Theory :

P-4 $$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} (f(x)+f(-x))\,dx = \begin{cases} 2\int_{0}^{a} f(x)\,dx & \text{If } f(x) \text{ is even} \\ 0 & \text{if } f(x) \text{ is odd} \end{cases}$$

Examples of definite integration by parts
Examples of definite integration by substitution

Examples
Evaluate the following :

1(e) $\int_{-a}^{a} x\sqrt{a^2-x^2}\,dx$ **Ans.** 0

2(e) $\int_{-1/2}^{1/2} \sec x\,\ell n\left(\frac{1-x}{1+x}\right)dx$ **Ans.** 0

3(e) $\int_{-\pi/4}^{\pi/4} \frac{x^7-3x^5+3x^3-x+1}{\cos^2 x}\,dx$ **Ans.** 2

4(e) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{1+e^x}\,dx$ **Ans.** 1

5. $\int_{-1}^{1} \frac{e^x+e^{-x}}{1+e^x}\,dx$ **Ans.** $\frac{e^2-1}{e}$

6. $\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \left(\frac{x^4}{1-x^4}\right)\cos^{-1}\left(\frac{2x}{1+x^2}\right)dx$ **Ans.** $\pi\left\{\frac{\pi}{12}-\frac{1}{\sqrt{3}}+\frac{1}{4}\,\ell n\left|2+\sqrt{3}\right|\right\}$

Evaluate the following :

1(e) $\int_{0}^{1} \ell n\,x\,dx$ **Ans.** -1

2(e) $\int_{0}^{\pi} x^2\cos x\,dx$ **Ans.** -2π

3. $\int_{0}^{2} |(1-x)\ln x|\,dx$ **Ans.** 1

Evaluate the following :

1(e) $\int_{1}^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}}$ **Ans.** $\pi/6$

2(e) $\int_{0}^{\pi} \frac{dx}{1+2\sin^2 x}$ **Ans.** $\pi/\sqrt{3}$

Note : subsititution tan x = t is not valid because of discontinuity at $x = \pi/2$

3. $\int_{-2}^{2} \frac{dx}{4+x^2}$ **Ans.** $\pi/4$ **Note :** substitution $x = \frac{1}{t}$ is not valid because of discontinuity at x = 0

4. $\int_{0}^{2\pi} \frac{dx}{5-2\cos x}$ **Ans.** $\frac{2\pi}{\sqrt{21}}$

Note :- Substitution $\tan\frac{x}{2} = t$ is not valid because of discontinuity at $x = \pi$

DPP– 2

Theory : P-5 $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

P-6 $\int_0^{2a} f(x)\,dx = \int_0^a (f(x)+f(2a-x))\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a-x)=f(x) \\ 0, & \text{if } f(2a-x)=-f(x) \end{cases}$

Examples :

Evaluate the following :

1(e) $\int_0^{\pi/2} \sin^2 x\,dx$ **Ans.** $\pi/4$

2(e) $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$ **Ans.** $\pi/12$

3(e) $\int_0^1 x(1-x)^{99}\,dx$ **Ans.** $\frac{1}{10100}$

4(e) $\int_0^{\pi} \frac{x^2 \sin x\,dx}{(2x-\pi)(1+\cos^2 x)}$ **Ans.** $\pi^2/4$

5. $\int_0^{\pi/4} \ell n\,(1+\tan x)\,dx$ **Ans.** $\pi/8\ \ell n\ 2$

6. $\int_0^{\pi} \frac{dx}{1+2^{\tan x}}$ **Ans.** $\pi/2$

7. $\int_2^3 \frac{x^2\,dx}{2x^2-10x+25}$ **Ans.** $1/2$

Evaluate the following :

8(e) $\int_0^{\pi} \sin^3 x \cos^3 x\,dx$ **Ans.** 0

9(e) $\int_0^{2\pi} \cos^5 x\,dx$ **Ans.** 0

10(e) $\int_0^{2\pi} \sin^4 x\,dx$ **Ans.** $3\pi/4$

11(e) $\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}\,dx$ **Ans.** π^2

12(e) $\int_0^{\pi/2} \ell n \sin x\,dx = \int_0^{\pi/2} \ell n \cos x\,dx = \int_0^{\pi/2} \ell n \sin 2x\,dx = -\frac{\pi}{2}\ell n 2$

13. $\int_0^{\infty} \ell n\left(x+\frac{1}{x}\right)\frac{dx}{1+x^2}$ **Ans.** $\pi\ \ell n2$

14. $\int_0^1 \frac{\sin^{-1} x}{x}\,dx$ **Ans.** $\pi/2\ \ell n2$

15. $\int_0^{\pi} x \ell n \sin x\,dx$ **Ans.** $-\frac{\pi^2}{2}\ell n2$

DPP– 3

Theory :

If f(x) is periodic function with period T then

P – 7 (i) $\int_0^{nT} f(x)\,dx = n\int_0^{T} f(x)\,dx$, $n \in z$ (ii) $\int_a^{a+nT} f(x)\,dx = n\int_0^{T} f(x)\,dx$ $n \in z,\ a \in R$

(iii) $\int_{mT}^{nT} f(x)\,dx = (n-m)\int_0^{T} f(x)\,dx$, $m, n \in z$ (iv) $\int_{nT}^{a+nT} f(x)\,dx = \int_0^{a} f(x)\,dx$, $n \in z,\ a \in R$

(v) $\int_{a+nT}^{b+nT} f(x)\,dx = \int_a^{b} f(x)\,dx$, $n \in z,\ a, b \in R.$

Examples :

Evaluate the following :

1(e) $\int_{-1}^{2} e^{\{x\}}\,dx$ **Ans.** $3(e-1)$

2(e) $\int_{-1}^{2} e^{\{3x\}}\,dx$ **Ans.** $3(e-1)$

3(e) $\int_0^{x\pi+v} |\cos x|\,dx$, $\pi/2 < v < \pi,\ x \in z$ **Ans.** $(2n+2)\,v - \sin v$

4(e) $\int_0^{2000\pi} \frac{dx}{1+e^{\sin x}}$ **Ans.** $1000\,\pi$

5. $\int_{\pi}^{5\pi/4} \frac{\sin 2x}{\sin^4 x + \cos^4 x}\,dx$ **Ans.** $\pi/4$

6. $\int_{-\pi}^{199\pi} \sqrt{\frac{1-\cos 2x}{2}}\,dx$ **Ans.** 400

DPP– 4

Theory :

Leibnitz Theorem

If $F(x) = \int_{g(x)}^{h(x)} f(t)\, dt$ then $\frac{dF(x)}{dx} = f(h(x))\, h'(x) - f(g(x))\, g'(x)$

Examples :

1(e) If $f(x) = \int_{x}^{x^2} \sqrt{\sin t}\, dt$ then find $f'(x)$ **Ans.** $\sqrt{\sin x^2}\ .\ 2x - \sqrt{\sin x}$

2(e) If $f(x) = \int_{e^{2x}}^{e^{3x}} \frac{t}{\ell n t}\, dt$, then find first and second derivative of f(x) with respect to ℓnx at $x = \ell n2$.

Ans. $48,\ 5.2^6.\ell n2$

3(e) Evaluate : $\lim\limits_{x\to 0} \frac{\int_0^{x^2} \cos t^2\, dt}{x \sin x}$ **Ans.** 1

4(e) Evaluate : $\lim\limits_{x\to\infty} \frac{\left[\int_0^x e^{t^2} dt\right]^2}{\int_0^x e^{2t^2} dt}$ **Ans.** 0

5(e) If $f(x) = x \int_{x}^{x^2} x \sin t\, dt$ then find $f'(x)$ **Ans.** $x^2 (2x \sin x^2 - \sin x) + (\cos x - \cos x^2)\, 2x$

6(e) If $d(x) = \cos x - \int_0^x (x - t)\, d(t)\, dt$ then find the value of $d''(x) + d(x)$ **Ans.** $-\cos x$

7. If $f(x) = e^{g(x)}$ and $g(x) = \int_2^x \frac{t\, dt}{1 + t^4}$ then find $f'(2)$ **Ans.** $\frac{2}{17}$

8. If $x = \int_0^y \frac{dt}{\sqrt{1 + 4t^2}}$ and $\frac{d^2y}{dx^2} = Ry$ then find R **Ans.** 4

9. The value of the function $f(x) = 1 + x + \int_1^x (\ell n^2 t + 2\ell n t)\, dt$ where $f'(x)$ venishes is –

Ans. $1 + 2e^{-1}$

10. If $f(x) = \int_{\ell n x}^{x} \frac{dt}{x + t}$ then find $f'(x)$ **Ans.** $-\left(\frac{1}{x + \ell n x}\right)\left(1 + \frac{1}{x}\right) + \frac{1}{x}$

DPP– 5

Theory :

Estimation of integral

Walli's formula

Examples :

1(e) Estimate the value of $I = \int_0^1 e^{x^2-x}\,dx$ **Ans.** $\frac{1}{e^{1/4}} < I < 1$

2(e) Prove that $0 < \int_0^2 \frac{x}{16+x^3}\,dx < \frac{1}{6}$

3(e) Estimate the value of $I = \int_0^{\pi/2} \frac{\sin x}{x}\,dx$ **Ans.** $1 < I < \pi/2$

4(e) If $I_1 = \int_0^1 \frac{dx}{\sqrt{4-x^2}}$, $I_2 = \int_0^1 \frac{dx}{\sqrt{4-2x^2}}$ and $I_3 = \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}}$ then using I_1 and I_2 estimate the value of I_3

Ans. $\pi/6 < I_3 < \frac{1}{4\sqrt{2}}$

5. Prove that : $0 < \int_0^1 \frac{x^7dx}{\sqrt[3]{(1+x^8)}} < \frac{1}{8}$

6. Prove that : $\ell n\, 2 < \int_0^1 \frac{dx}{\sqrt{(1+x^6)}} < \frac{\pi}{2}$

Evaluate the following :

7(e) $\int_0^{\pi/2} \sin^5 x\,dx$ **Ans.** $\frac{8}{15}$

8(e) $\int_0^{\pi/2} \sin^4 x \cos^6 x\,dx$ **Ans.** $\frac{3\pi}{512}$

9(e) $\int_0^{\pi} x \sin^5 x \cos^6 x\,dx$ **Ans.** $\frac{8\pi}{693}$

10. $\int_0^{\pi} \sin^5 x \cos^4 x\,dx$ **Ans.** $\frac{16}{315}$

11. $\int_0^1 x^3 (1-x)^5\,dx$ **Ans.** $\frac{1}{504}$

Theory : *Definite integral as a limit of sum, Miscellaneous Problems :*

Examples :

1. Find the value of definite integrals by ab initio method or first principle method.

(i)**(e)** $\int_0^1 x\,dx$ **Ans.** 1/2 (ii) $\int_0^2 e^x\,dx$ **Ans.** $e^2 - 1$ (iii)**(e)** $\int_0^{\pi/2} \sin x\,dx$ **Ans.** 1

2(e) Evaluate : $\lim_{n\to\infty}\left(\frac{1}{1+n}+\frac{1}{2+n}+\frac{1}{3+n}+..........+\frac{1}{2n}\right)$ **Ans.** $\ell n2$

3(e) Evaluate : $\lim_{n\to\infty}\left[\frac{n+1}{n^2+1^2}+\frac{n+2}{n^2+2^2}+.........+\frac{n+2n}{n^2+4n^2}\right]$ **Ans.** $\tan^{-1}2 + 1/2\,\ell n5$

4. Evaluate : $\lim_{n\to\infty}\left(\frac{n!}{n^n}\right)^{\frac{1}{n}}$ **Ans.** 1/e

5. Evaluate : $\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2}}+\frac{1}{\sqrt{n^2+n}}+\frac{1}{\sqrt{n^2+2n}}+.......+\frac{1}{\sqrt{n^2+n^2}}\right)$ **Ans.** $2(\sqrt{2}-1)$

6. Evaluate : $\lim_{n\to\infty}\frac{\sqrt{1}+\sqrt{2}+..........+2\sqrt{n}}{n\sqrt{n}}$ **Ans.** 16/3

7(e) Show that the sum of two integrals $\int_{-4}^{-5} e^{(x+5)^2}dx + 3\int_{1/3}^{2/3} e^{9(x-2/3)^2}dx$ is zero. **Ans.** $= -\int_0^1 e^{x^2}dx + \int_0^1 e^{x^2}dx = 0$

8(e) Prove that $F(x) = \int_0^x \ell n\left(\frac{1-t}{1+t}\right)dt$ is an even function.

9(e) Find f(x) if $f'(x) = f(x) + \int_0^1 f(x)\,dx$ given $f(0) = 1$. **Ans.** $f(x)\ \frac{2e^x}{(3-e)}+\left(\frac{1-e}{3-e}\right)$

10(e) Given an even function f defined and integrable everywhere and periodic with period 2,

let $g(x) = \int_0^x f(t)\,dt$ and $g(1) = A$.

(a) Prove that g is odd and that $g(x + 2) - g(x) = g(2)$

(b) Compute g(2) and g(5) in terms of A **Ans.** g(2) = 2A, g(5) = 5A

(c) For what A, will g be periodic with period 2 ? **Ans.** A = 0

11. Prove that $F(x) = \int_0^x \left(\frac{t}{e^t-1}+\frac{t}{2}+1\right)dt$ is an odd function.

12. If $f(x) = \int_\alpha^x \frac{1}{f(x)}dx$ and $\int_\alpha^1 \frac{1}{f(x)}dx = \sqrt{2}$ then find f(x). **Ans.** $f(x) = \sqrt{2x}$

13. If $f(x) = x + \int_0^1 (xy^2 + x^2y)\ f(y)\,dy$, find f(x). **Ans.** $\frac{180}{119}x + \frac{80}{119}x^2$